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Row and Column-Max-Average Norm and Max-Min Norm of Fuzzy Matrices

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Abstract. In this paper, we define two new type of operators of fuzzy matrices denoted by the symbol $\oplus$ and $\otimes$. Using these operators of fuzzy matrices we define row-max-average norm, column-max-average norm. Here instead of addition of fuzzy matrices we use the operator $\oplus$ and instead of multiplication of fuzzy matrices we use the operator $\otimes$. We also define Pseudo norm of fuzzy matrices and max-min norm.

Keywords: *fuzzy matrices, row-max-average norm, column-max-average norm, pseudo norm of fuzzy matrices, max-min norm.*

1. Introduction

The study of linear algebra has become more and more popular in the last few decades. People are attracted to this subject because of its beauty and its connection with many other pure and applied areas. In theoretical development of the subject as well as in many application, one often needs to measure the length of vectors. For this purpose, norm functions are consider on a vector space.

A norm on a real vector space V is a function $\|.\|: V \to R$ satisfying

1. $\|u\| > 0$ for any nonzero $u \in V$.
2. $\|ru\| = |r|\,\|u\|$ for any $r \in R$ and $u \in$ V.
3. $\|u+v\| \le \|u\| + \|v\|$ for any $u, v \in V$.

The norm is a measure of the size of the vector u where condition (1) requires the size to be positive, condition (2) requires the size to be scaled as the vector is scaled, and condition (3) is known as the triangle inequality and has its origin in the notion of distance in R^3. The condition (2) is called homogeneous condition and this condition ensure that the norm of the zero vector in V is 0; this condition is often included in the definition of a norm.

Common example of norms on R^n are the l_p norms,where $1 \le p \le \infty$, defined by

$l_p(u) = \{\sum_{j=1}^{n} |u_j|^p\}^{\frac{1}{p}}$ if $1 \le p < \infty$ and $l_p(u) = \max_{1 \le j \le n} |u_j|$ if $p = \infty$

for any $u = (u_1, u_2, ..., u_n)^t \in R^n$. Note that if one define an l_p function on R^n as define above with $0 < p < 1$, then it does not satisfy the triangle inequality, hence is not a norm. Given a norm on a real vector space V, one can compare the norms of vectors, discuss convergence of sequence of vectors, study limits and continuity of transformations, and consider approximation problems such as finding the nearest element in a subset or a subspace of V to a given vector. These problems arise naturally in analysis, numerical analysis, differential equations, Markov chains, etc.

The norm of a matrix is a measure of how large its elements are. It is a way of determining the "size" of a matrix that is necessarily related to how many rows or columns the matrix has. The norm of a square matrix A is a non negative real number denoted by $\|A\|$. There are several different ways of defining a matrix norm but they all share the following properties:

1. $\|A\| \ge 0$ for any square matrix A.
2. $\|A\| = 0$ iff the matrix $A = 0$.
3. $\|KA\| = |K| \|A\|$ for any scaler K.
4. $\|A + B\| \le \|A\| + \|B\|$ for any square matrix A, B.
5. $\|AB\| \le \|A\| \|B\|$

Different types of matrix norm:

The 1-norm

$$\|A\|_1 = \max_{1 \le j \le n} (\sum_{i=1}^{n} |a_{ij}|)$$

The infinity norm

$$\|A\|_\infty = \max_{1 \le i \le n} (\sum_{j=1}^{n} |a_{ij}|)$$

The infinity norm of a square matrix is the maximum of the absolute row sum. Simply we sum the absolute values along each row and then take the biggest answer.

Euclidean norm

$$\|A\|_E = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})^2}$$

The Euclidean norm of a square matrix is the square root of the sum of all the squares of the elements. This is similar to ordinary "Pythagorean" length where the size of a vector is found by taking the square root of the sum of the squares of all the elements.

Any definition you can define of which satisfies the five condition mentioned at the beginning of this section is a definition of a norm. There are many many possibilities, but

the three given above are among the most commonly used.
Like vector norm and matrix norm, norm of a fuzzy matrix is also a function $\|.\|: M_n(F) \to [0,1]$ which satisfies the following properties

1. $\|A\| \ge 0$ for any fuzzy matrix A.
2. $\|A\| = 0$ iff the fuzzy matrix A=0.
3. $\|KA\| = |K| \|A\|$ for any scaler $K \in [0,1]$.
4. $\|A+B\| \le \|A\| + \|B\|$ for any two fuzzy matrix A and B.
5. $\|AB\| \le \|A\|\|B\|$ for any two fuzzy matrix A and B.

In this project paper we will define different type of norm on fuzzy matrices.

2. Fuzzy matrix

We know that matrices play an important role in various areas such as mathematics, physics, statistics, engineering, social sciences and many others. Several works on classical matrices are available in different journals even in books also. But in our real life problems in social science, medical science, environment etc. do not always involve crisp data. Consequently, we can not successfully use traditional classical matrices because of various types of uncertainties present in our daily life problems. Nowa days probability, fuzzy sets, intuitionistic fuzzy sets, vague sets, rough sets are used as mathematical tools for dealing uncertainties. Fuzzy matrices arise in many application, one of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have important applications in pattern classification and in handing fuzziness in knowledge based systems.

Fuzzy matrices were introduce for the first time by Thomason [42], who discussed the convergence of powers of fuzzy matrix. Ragab et al. [33,34] presented some properties of the min-max composition of fuzzy matrices. Hashimoto [18,19] studied the canonical form of a transitive fuzzy matrix. Hemashina et al. [20] Investigated iterates of fuzzy circulant matrices. Powers and nilpotent conditions of matrices over a distributive lattice are consider by Tan [41]. After that Pal, Bhowmik, Adak, Shyamal, Mondal have done lot of works on fuzzy, intuitionistic fuzzy, interval-valued fuzzy, etc. matrices [1-12,25-32,35-39].

The elements of a fuzzy matrix having values in the closed interval [0,1]. We can still see that all fuzzy matrices are matrices but every matrix in general is not a fuzzy matrix. We see the fuzzy interval, i.e. the unit interval is a subset of reals. Thus a matrix in general is not a fuzzy matrix since the unit interval [0,1] is contained in the set of reals. The big question is can we add two fuzzy matrices A and B and get the sum of them to be fuzzy matrix. The answer in general is not possible for the sum of two fuzzy matrices may turn out to be a matrix which is not a fuzzy matrix. If we add above two fuzzy matrix A and B then all entries in A+B will not lie in [0,1], hence A+B is only just a matrix and not a fuzzy matrix.

So only in case of fuzzy matrices the max or min operation are defined. Clearly under the max or min operation the resultant matrix is again a fuzzy matrix. In general to add two matrix we use max operation.

We see the product of two fuzzy matrices under usual matrix multiplication is not a fuzzy matrix. So we need to define a compatible operation analogous to product so that the product again happens to be a fuzzy matrix. However even for this new operation if the product XY is to be defined we need the number of columns of X is equal to the number of rows of Y. The two types of operation which we can have are max-min operation and min-max operation.

In [23], we introduced max-norm and square-max norm of fuzzy matrices and some properties of this two norm.

In this paper, we we have introduced two new operators on fuzzu matrices denoted by the symbol $\oplus$ and $\otimes$. Using these operators we define different types of norm of fuzzy matrices.

Definition 1. [41] *A **fuzzy matrix** (FM) of order $m\times n$ is defined as $A=<a_{ij}, a_{ij\mu}>$ where $a_{ij\mu}$ is the membership value of the ij-th element a_{ij} in A.*
An $n\times n$ fuzzy matrix R is called reflexive iff $r_{ii}=1$ for all i=1,2,...,n. It is called α-reflexive iff $r_{ii}\geq\alpha$ for all i=1,2,...,n where $\alpha\in[0,1]$. It is called weakly reflexive iff $r_{ii}\geq r_{ij}$ for all i,j=1,2,...,n. An $n\times n$ fuzzy matrix R is called irreflexive iff $r_{ii}=0$ for all i=1,2,...,n.

Definition 2. *An $n\times n$ fuzzy matrix S is called symmetric iff $s_{ij}=s_{ji}$ for all i,j=1,2,....,n. It is called antisymmetric iff $S\wedge S'\leq I_n$ where I_n is the usual unit matrix.*

Note that the condition $S\wedge S'\leq I_n$, means that $s_{ij}\wedge s_{ji}=0$ for all $i\neq j$ and $s_{ii}\leq 1$ for all i. So if $S_{ij}=1$ then $s_{ji}=0$, which the crisp case.

Definition 3. *An $n\times n$ fuzzy matrix N is called nilpotent iff $N^n=0$ (the zero matrix). If $N^m=0$ and $N^{m-1}\neq 0$; $1\leq m\leq n$ then N is called nilpotent of degree m. An $n\times n$ fuzzy matrix E is called idempotent iff $E^2=E$. It is called transitive iff $E^2\leq E$. It is called compact iff $E^2\geq E$.*

3. New opertors of fuzzy matrices

We already discussed addition and multiplication of fuzzy matrices in introduction. We used max operation to add fuzzy matrices and min-max operation to multiply fuzzy matrices till now. But here we will define new type of operators of fuzzy matrices denoted by the symbol $\oplus$ and $\otimes$. Instead of addition of fuzzy matrices we will use the operator $\oplus$ and instead of multiplication we will use the operator $\otimes$. This two new operators are define by the following way.

$$\text{If } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ and } B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

then $A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & \cdots\cdots & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & \cdots\cdots & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & \cdots\cdots & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$

and $A \otimes B = \begin{bmatrix} \wedge\{a_{11}, b_{11}\} & \wedge\{a_{12}, b_{12}\} & \cdots\cdots & \wedge\{a_{1n}, b_{1n}\} \\ \wedge\{a_{21}, b_{21}\} & \wedge\{a_{22}, b_{22}\} & \cdots\cdots & \wedge\{a_{2n}, b_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1}, b_{n1}\} & \wedge\{a_{n2}, b_{n2}\} & \cdots\cdots & \wedge\{a_{nn}, b_{nn}\} \end{bmatrix}$.

Must be remember that in this type of multiplication, fuzzy matrices will be of same order.

Proposition 1. [23] $\bigvee(a_1+b_1, a_2+b_2) \le \bigvee(a_1, a_2) + \bigvee(b_1, b_2)$

4. Row-max-average Norm

Here we will define a new type of norm called Row-Max-Average norm. We will used new type of operators of fuzzy matrices for this norm. Here, at first we will find maximum element in each row. Then we will determine the average of the maximum element. Row-max-average norm of a fuzzy matrix A is denoted by $\|A\|_{RMA}$ and define by

$$\|A\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$$

Lemma 1. *All the conditions of norm are satisfied by* $\|A\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$.

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$\therefore \|A\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$ and $\|B\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})$

(i) As all $a_{ij} \ge 0$ so according to the definition of Row-max-average norm obviously $\|A\|_{RMA} \ge 0$.

Now $\|A\|_{RMA} = 0 \quad\quad \Leftrightarrow \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) = 0$

$\Leftrightarrow \bigvee_{j=1}^{n} a_{ij} = 0$ for all *i=1,2,...,n.*

$\Leftrightarrow a_{i1} = a_{i2} = ...a_{in} = 0$ for all *i=1,2,...,n.*

$\Leftrightarrow a_{ij} = 0$ for all $i, j = 1,2,..., n.$ $\Leftrightarrow A = 0$

So, $\|A\|_{RMA} = 0$ iff $A = 0$.

(ii) Here we define a new type of scaler multiplication as follows

$$\alpha a_{ij} = \begin{cases} |\alpha| & if \quad |\alpha| \le \|A\|_{RMA} \\ \|A\|_{RMA} & if \quad |\alpha| > \|A\|_{RMA} \end{cases}$$

So, if $|\alpha| \le \|A\|_{RMA}$ then $\|\alpha A\|_{RMA} = |\alpha| = |\alpha|\,\|A\|_{RMA}$

and if $|\alpha| > \|A\|_{RMA}$ then $\|\alpha A\|_{RMA} = \|A\|_{RMA} = |\alpha|\,\|A\|_{RMA}$.

Therefore $\|\alpha A\|_{RMA} = |\alpha|\,\|A\|_{RMA}$ for all $\alpha \in [0,1]$.

(iii)

$$A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & \cdots & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & \cdots & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & \cdots & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$$

$\therefore \|A \oplus B\|_{RMA}$

$$= \frac{\bigvee_{i=1}^{n}(\frac{a_{1i}+b_{1i}}{2}) + \bigvee_{i=1}^{n}(\frac{a_{2i}+b_{2i}}{2}) + ...+ \bigvee_{i=1}^{n}(\frac{a_{ni}+b_{ni}}{2})}{n}$$

$$\le \frac{(\bigvee_{i=1}^{n} a_{1i} + \bigvee_{i=1}^{n} b_{1i}) + (\bigvee_{i=1}^{n} a_{2i} + \bigvee_{i=1}^{n} b_{2i}) + ...+ (\bigvee_{i=1}^{n} a_{ni} + \bigvee_{i=1}^{n} b_{ni})}{2n}$$

$$= \frac{\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) + \sum_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})}{2n} = \frac{\frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) + \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})}{2} = \frac{\|A\|_{RMA} + \|B\|_{RMA}}{2}$$

$= \|A\|_{RMA} \oplus \|B\|_{RMA}$

So, $\|A \oplus B\|_{RMA} \le \|A\|_{RMA} \oplus \|B\|_{RMA}$.

(iv)

$$A \otimes B = \begin{bmatrix} \wedge\{a_{11}, b_{11}\} & \wedge\{a_{12}, b_{12}\} & \dots & \wedge\{a_{1n}, b_{1n}\} \\ \wedge\{a_{21}, b_{21}\} & \wedge\{a_{22}, b_{22}\} & \dots & \wedge\{a_{2n}, b_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1}, b_{n1}\} & \wedge\{a_{n2}, b_{n2}\} & \dots & \wedge\{a_{nn}, b_{nn}\} \end{bmatrix}$$

Now $\wedge\{a_{ij}, b_{ij}\} \leq a_{ij}$ and b_{ij} for all i, j .

$$\therefore \bigvee_{j=1}^{n} \{\wedge(a_{ij}, b_{ij})\} \leq \bigvee_{j=1}^{n} a_{ij} \text{ and } \bigvee_{j=1}^{n} b_{ij} \text{ for all } i.$$

$$\therefore \frac{1}{n}\sum_{i=1}^{n}[\bigvee_{j=1}^{n}\{\wedge(a_{ij}, b_{ij})\}] \leq \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) \text{ and } \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$$

$$\Rightarrow \|A \otimes B\|_{RMA} \leq \|A\|_{RMA} \otimes \|B\|_{RMA}$$

Hence all the conditions of norm are satisfied by Row-max-average.

5. Properties of row-max-average Norm

Properties 1. *If* A *and* B *are two fuzzy matrices then* $\|(A \oplus B)^T\|_{RMA} \leq \|A^T\|_{RMA} \oplus \|B^T\|_{RMA}$.

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}.$$

Then
$$A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & \dots & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & \dots & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & \dots & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$$

and $(A\oplus B)^T = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{21}+b_{21}}{2} & ..... & \frac{a_{n1}+b_{n1}}{2} \\ \frac{a_{12}+b_{12}}{2} & \frac{a_{22}+b_{22}}{2} & ..... & \frac{a_{n2}+b_{n2}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{1n}+b_{1n}}{2} & \frac{a_{2n}+b_{2n}}{2} & ..... & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$.

$$\therefore \left\|(A\oplus B)^T\right\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}\frac{a_{ji}+b_{ji}}{2}\Big) \le \frac{1}{2n}\sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}a_{ji}+\bigvee_{j=1}^{n}b_{ji}\Big)$$

$$\le \frac{\frac{1}{n}\sum_{i=1}^{n}\big(\bigvee_{j=1}^{n}a_{ji}\big)+\frac{1}{n}\sum_{i=1}^{n}\big(\bigvee_{j=1}^{n}b_{ji}\big)}{2} = \frac{\left\|A^T\right\|_{RMA}+\left\|B^T\right\|_{RMA}}{2} = \left\|A^T\right\|_{RMA}\oplus\left\|B^T\right\|_{RMA}$$

Properties 2. *If A and B are two fuzzy matrices and $A\le B$ then $\|A\|_{RMA}\le\|B\|_{RMA}$.*

Proof: As $A\le B$ so, $a_{ij}\le b_{ij}$ for all i, j.

$$\Rightarrow \bigvee_{j=1}^{n}a_{ij}\le\bigvee_{j=1}^{n}b_{ij} \qquad \text{for all } i.$$

$$\Rightarrow \sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}a_{ij}\Big)\le\sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}b_{ij}\Big) \Rightarrow \frac{1}{n}\sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}a_{ij}\Big)\le\frac{1}{n}\sum_{i=1}^{n}\Big(\bigvee_{j=1}^{n}b_{ij}\Big) \Rightarrow \|A\|_{RMA}\le\|B\|_{RMA}$$

Properties 3. *If A and B are two fuzzy matrices and $A\le B$ then $\|A\otimes C\|_{RMA}\le\|B\otimes C\|_{RMA}$ hold.*

Proof: Let us consider

$$A=\begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & ..... & a_{nn} \end{bmatrix} B=\begin{bmatrix} b_{11} & b_{12} & ..... & b_{1n} \\ b_{21} & b_{22} & ..... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ..... & b_{nn} \end{bmatrix} \text{and } C=\begin{bmatrix} c_{11} & c_{12} & ..... & c_{1n} \\ c_{21} & c_{22} & ..... & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & ..... & c_{nn} \end{bmatrix}.$$

Then $A\otimes C = \begin{bmatrix} \wedge\{a_{11},c_{11}\} & \wedge\{a_{12},c_{12}\} & ..... & \wedge\{a_{1n},c_{1n}\} \\ \wedge\{a_{21},c_{21}\} & \wedge\{a_{22},c_{22}\} & ..... & \wedge\{a_{2n},c_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1},c_{n1}\} & \wedge\{a_{n2},c_{n2}\} & ..... & \wedge\{a_{nn},c_{nn}\} \end{bmatrix}$

and $B \otimes C = \begin{bmatrix} \wedge\{b_{11}, c_{11}\} & \wedge\{b_{12}, c_{12}\} & ..... & \wedge\{b_{1n}, c_{1n}\} \\ \wedge\{b_{21}, c_{21}\} & \wedge\{b_{22}, c_{22}\} & ..... & \wedge\{b_{2n}, c_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{b_{n1}, c_{n1}\} & \wedge\{b_{n2}, c_{n2}\} & ..... & \wedge\{b_{nn}, c_{nn}\} \end{bmatrix}$.

$\therefore \|A \otimes C\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}[\bigvee_{j=1}^{n}\{\wedge(a_{ij}, c_{ij})\}]$ and $\|B \otimes C\|_{RMA} = \frac{1}{n}\sum_{i=1}^{n}[\bigvee_{j=1}^{n}\{\wedge(b_{ij}, c_{ij})\}]$

Now $A \le B \Rightarrow a_{ij} \le b_{ij}$ for all i, j.

$\Rightarrow \wedge\{a_{ij}, c_{ij}\} \le \wedge\{b_{ij}, c_{ij}\}$ for all i, j.

$\Rightarrow \bigvee_{j=1}^{n}\{\wedge(a_{ij}, c_{ij})\} \le \bigvee_{j=1}^{n}\{\wedge(b_{ij}, c_{ij})\}$ for all i.

$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}[\bigvee_{j=1}^{n}\{\wedge(a_{ij}, c_{ij})\}] \le \frac{1}{n}\sum_{i=1}^{n}[\bigvee_{j=1}^{n}\{\wedge(b_{ij}, c_{ij})\}]$

$\Rightarrow \|A \otimes C\|_{RMA} \le \|B \otimes C\|_{RMA}$

6. Column-max-average norm

Like Row-max-average norm we will define Column-max-average norm. Here we will find maximum element in each column and then average of the maximum elements. Here we will also use the new type of operators of fuzzy matrices. The Column-max-average norm of a fuzzy matrix A is denoted by $\|A\|_{CMA}$ and define by $\|A\|_{CMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$.

Lemma 2. *All the conditions of norm are satisfied by* $\|A\|_{CMA} = \frac{1}{n}\sum_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$.

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix}$$

$\therefore \|A\|_{CMA} = \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij})$ and $\|B\|_{CMA} = \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} b_{ij})$

(i) As all $a_{ij} \ge 0$ so according to the definition of Column-max-average norm obviously $\|A\|_{CMA} \ge 0$.

Now $\|A\|_{CMA} = 0$

$\Leftrightarrow \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij}) = 0 \qquad \Leftrightarrow \bigvee_{i=1}^{n} a_{ij} = 0$ for all $j=1,2,...,n$.

$\Leftrightarrow a_{1j} = a_{2j} = ... = a_{nj} = 0$ for all $j=1,2,...,n$.

$\Leftrightarrow a_{ij} = 0$ for all $i, j = 1,2,..., n$.

$\Leftrightarrow A = 0$. So, $\|A\|_{CMA} = 0$ iff $A = 0$.

(ii) Here we will use the same type of scaler multiplication which we used in Row-max-average norm and that is

$$\alpha a_{ij} = \begin{cases} |\alpha| & if \quad |\alpha| \le \|A\|_{CMA} \\ \|A\|_{CMA} & if \quad |\alpha| > \|A\|_{CMA} \end{cases}$$

So if $|\alpha| \le \|A\|_{CMA}$ then $\|\alpha A\|_{CMA} = |\alpha| = |\alpha|\,\|A\|_{CMA}$

and if $|\alpha| > \|A\|_{CMA}$ then $\|\alpha A\|_{CMA} = \|A\|_{CMA} = |\alpha|\,\|A\|_{CMA}$.

Therefore $\|\alpha A\|_{CMA} = |\alpha|\,\|A\|_{CMA}$ for all $\alpha \in [0,1]$.

(iii) $$A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & ..... & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & ..... & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & ..... & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$$

$\therefore \|A \oplus B\|_{CMA}$

$$= \frac{\bigvee_{i=1}^{n}(\frac{a_{i1}+b_{i1}}{2}) + \bigvee_{i=1}^{n}(\frac{a_{i2}+b_{i2}}{2}) + ... + \bigvee_{i=1}^{n}(\frac{a_{in}+b_{in}}{2})}{n}$$

$$\le \frac{(\bigvee_{i=1}^{n} a_{i1} + \bigvee_{i=1}^{n} b_{i1}) + (\bigvee_{i=1}^{n} a_{i2} + \bigvee_{i=1}^{n} b_{i2}) + ... + (\bigvee_{i=1}^{n} a_{in} + \bigvee_{i=1}^{n} b_{in})}{2n}$$

$$= \frac{\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij}) + \sum_{j=1}^{n}(\bigvee_{i=1}^{n} b_{ij})}{2n} = \frac{\frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij}) + \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} b_{ij})}{2} = \frac{\|A\|_{CMA} + \|B\|_{CMA}}{2}$$

$= \|A\|_{CMA} \oplus \|B\|_{CMA}$

So, $\|A \oplus B\|_{CMA} \le \|A\|_{CMA} \oplus \|B\|_{CMA}$.

(iv)

$$A \otimes B = \begin{bmatrix} \wedge\{a_{11}, b_{11}\} & \wedge\{a_{12}, b_{12}\} & \ldots\ldots & \wedge\{a_{1n}, b_{1n}\} \\ \wedge\{a_{21}, b_{21}\} & \wedge\{a_{22}, b_{22}\} & \ldots\ldots & \wedge\{a_{2n}, b_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1}, b_{n1}\} & \wedge\{a_{n2}, b_{n2}\} & \ldots\ldots & \wedge\{a_{nn}, b_{nn}\} \end{bmatrix}$$

Now $\wedge\{a_{ij}, b_{ij}\} \le a_{ij}$ and b_{ij} for all i, j.

$\therefore \bigvee_{i=1}^{n} \{\wedge(a_{ij}, b_{ij})\} \le \bigvee_{i=1}^{n} a_{1i}$ and $\bigvee_{i=1}^{n} b_{ij}$ for all j.

$\therefore \frac{1}{n}\sum_{j=1}^{n}[\bigvee_{i=1}^{n}\{\wedge(a_{ij}, b_{ij})\}] \le \frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij})$ and $\frac{1}{n}\sum_{j=1}^{n}(\bigvee_{i=1}^{n} a_{ij})$

$\Rightarrow \|A \otimes B\|_{CMA} \le \|A\|_{CMA} \otimes \|B\|_{CMA}$

Hence all the conditions of norm are satisfied by Column-max-average norm.

Note 1. *Relation between Row-max-average norm and Column-max-average norm* is $\|A\|_{RMA} = \|A^T\|_{CMA}$.

Note 2. *If* A *is symmetric i.e* $A = A^T$ *then* $\|A\|_{RMA} = \|A\|_{CMA}$.

7. Pseudo norm on fuzzy matrix

Pseudo norm on fuzzy matrices is a one type of norm but there is a difference between norm on fuzzy matrix and pseudo norm on fuzzy matrix. Pseudo norm of a fuzzy matrix fulfill the following conditions

1. $\|A\| \ge 0$ for any fuzzy matrix A.
2. if $A = 0$ then $\|A\| = 0$.
3. $\|kA\| = |k|\,\|A\|$ for any scaler $k \in [0,1]$.
4. $\|A + B\| \le \|A\| + \|B\|$ for any two fuzzy matrix A and B.
5. $\|AB\| \le \|A\|\|B\|$ for any two fuzzy matrix A and B.

Clearly except condition-2 all the condition of norm on fuzzy matrix and pseudo norm on fuzzy matrix are same.

8. Max-min Norm

Max-min norm is an example of pseudo norm on fuzzy matrix. Here, first we will find the maximum element in each row and then minimum of the maximum elements. In this norm, we will use the new type of addition and multiplication of fuzzy matrices which already we use in case of Row-Max-Average norm. Max-Min norm of a fuzzy matrix A is denoted by $\|A\|_{MM}$ and define by $\|A\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$.

Lemma 3. *All the conditions of pseudo norm of fuzzy matrix are satisfied b*

$\|A\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$.

Proof: Let us consider

$$A = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & ..... & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & ..... & b_{1n} \\ b_{21} & b_{22} & ..... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ..... & b_{nn} \end{bmatrix}$$

$\therefore \|A\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij})$ and $\|B\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})$

(i) Clearly $\|A\|_{MM} \geq 0$ and if $A = 0$ then $\|A\|_{MM} = 0$.

(ii) According to the definition of max-min norm if $|\alpha| > \|A\|_{MM}$ then

$\|\alpha A\|_{MM} = \|A\|_{MM} = |\alpha|\,\|A\|_{MM}$ and if $|\alpha| < \|A\|_{MM}$ then $\|\alpha A\|_{MM} = |\alpha| = |\alpha|\,\|A\|_{MM}$.

Therefore $\|\alpha A\|_{MM} = |\alpha|\,\|A\|_{MM}$ for all $\alpha \in [0,1]$.

(iii) Now

$$A \oplus B = \begin{bmatrix} \frac{a_{11}+b_{11}}{2} & \frac{a_{12}+b_{12}}{2} & ..... & \frac{a_{1n}+b_{1n}}{2} \\ \frac{a_{21}+b_{21}}{2} & \frac{a_{22}+b_{22}}{2} & ..... & \frac{a_{2n}+b_{2n}}{2} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}+b_{n1}}{2} & \frac{a_{n2}+b_{n2}}{2} & ..... & \frac{a_{nn}+b_{nn}}{2} \end{bmatrix}$$

$$\therefore \|A \oplus B\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} \frac{a_{ij}+b_{ij}}{2}) < \bigwedge_{i=1}^{n}\{\bigvee_{j=1}^{n} \frac{a_{ij}}{2} + \bigvee_{j=1}^{n} \frac{b_{ij}}{2}\} = \frac{1}{2}\{\bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) + \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})\}$$

$$= \frac{1}{2}[\|A\|_{MM} + \|B\|_{MM}] = \|A\|_{MM} \oplus \|B\|_{MM}$$

Therefore $\|A \oplus B\|_{MM} \leq \|A\|_{MM} \oplus \|B\|_{MM}$.

(iv) Now

$$A \otimes B = \begin{bmatrix} \wedge\{a_{11}, b_{11}\} & \wedge\{a_{12}, b_{12}\} & ..... & \wedge\{a_{1n}, b_{1n}\} \\ \wedge\{a_{21}, b_{21}\} & \wedge\{a_{22}, b_{22}\} & ..... & \wedge\{a_{2n}, b_{2n}\} \\ \vdots & \vdots & & \vdots \\ \wedge\{a_{n1}, b_{n1}\} & \wedge\{a_{n2}, b_{n2}\} & ..... & \wedge\{a_{nn}, b_{nn}\} \end{bmatrix}$$

If we denote $\wedge\{a_{ij}, b_{ij}\}$ as $a_{ij}b_{ij}$ then $\|A\otimes B\|_{MM} = \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}b_{ij})$.

Now $a_{ij}b_{ij} \le a_{ij}$ and b_{ij} for all *i*,*j*.

$$\Rightarrow \bigvee_{j=1}^{n} a_{ij}b_{ij} \le \bigvee_{j=1}^{n} a_{ij} \text{ and } \bigvee_{j=1}^{n} b_{ij} \text{ for all } i.$$

$$\Rightarrow \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}b_{ij}) \le \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} a_{ij}) \text{ and } \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{n} b_{ij})$$

$$\Rightarrow \|A\otimes B\|_{MM} \le \|A\|_{MM} \otimes \|B\|_{MM}$$

9. Conclusion

In this paper, we define two new types of operators on fuzzy matrices. Using this operators we define different types of norm such as row-max-average norm, column-max-average norm. Using these norm we can define conditional number to check whether a system of linear equation is ill posed or well posed. Norm of fuzzy matrices can take a effective contribution to solve a fuzzy system of linear equation.

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