A functional recurrence to obtain the prime numbers using the Smarandache prime function.

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Theorem: We are considering the function:

For n integer:

$$F(n) = n + 1 + \sum_{m=n+1}^{2n} \prod_{i=n+1}^{m} \left\lfloor -\left\lfloor -\frac{\sum_{j=1}^{i}\left(\left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor\right) - 2}{i} \right\rfloor \right\rfloor$$

one has: $p_{k+1} = F(p_k)$ for all $k \geq 1$ where $\{p_k\}_{k\geq 1}$ are the prime numbers and $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Observe that the knowledge of p_{k+1} only depends on knowledge of p_k and the knowledge of the fore primes is unnecessary.

Proof:

Suppose that we have found a function $P(i)$ with the following property:

$$P(i) = \begin{cases} 1 \; if \;\; i \; is \; composite \\ 0 \; if \;\; i \; is \;\; prime \end{cases}$$

This function is called Smarandache prime function.(Ref.)

Consider the following product:

$$\prod_{i=p_k+1}^{m} P(i)$$

If $p_k < m < p_{k+1}$ $\prod_{i=p_k+1}^{m} P(i) = 1$ since $i : p_{k+1} \leq i \leq m$ are all composites.

If $m \geq p_{k+1}$ $\prod_{i=p_k+1}^{m} P(i) = 0$ since $P(p_{k+1}) = 0$

Here is the sum:

$$\sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} P(i) = \sum_{m=p_k+1}^{p_{k+1}-1} \prod_{i=p_k+1}^{m} P(i) + \sum_{m=p_{k+1}}^{2p_k} \prod_{i=p_k+1}^{m} P(i) = \sum_{m=p_k+1}^{p_{k+1}-1} 1 =$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

The second sum is zero since all products have the factor $P(p_{k+1}) = 0$.

Therefore we have the following recurrence relation:

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} P(i)$$

Let's now see we can find $P(i)$ with the asked property.

Consider:

$$\left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor = \begin{cases} 1 & si \quad j \mid i \\ 0 & si \quad j\ not \mid i \end{cases} \qquad j = 1,2,\cdots,i \quad i \geq 1$$

We deduce of this relation:

$$d(i) = \sum_{j=1}^{i} \left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor$$

where $d(i)$ is the number of divisors of i.

If i is prime $d(i)=2$ therefore:

$$-\left\lfloor -\frac{d(i)-2}{i} \right\rfloor = 0$$

If i is composite $d(i)>2$ therefore:

$$0<\frac{d(i)-2}{i}<1 \Rightarrow -\left\lfloor -\frac{d(i)-2}{i} \right\rfloor = 1$$

Therefore we have obtained the Smarandache Prime Function $P(i)$ which is:

$$P(i) = -\left\lfloor -\frac{\sum_{j=1}^{i}\left(\left\lfloor \frac{i}{j} \right\rfloor - \left\lfloor \frac{i-1}{j} \right\rfloor\right) - 2}{i} \right\rfloor \qquad i \geq 2 \quad \text{integer}$$

With this, the theorem is already proved .

References:

[1] E. Burton, “Smarandache Prime and Coprime functions”.
www.gallup.unm.edu/~Smarandache/primfnct.txt
[2]F. Smarandache, “Collected Papers”, Vol II 200 p.p. 137, Kishinev University Press, Kishinev, 1997.