Intuitive Curvature: No Relation to the Riemann Tensor

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Abstract

Merriam-Webster's Collegiate Dictionary, Eleventh Edition, gives a technical definition of curvature, "the rate of change of the angle through which the tangent to a curve turns in moving along the curve and which for a circle is equal to the reciprocal of the radius". That precisely describes a curve's intuitive curvature, but the Riemann "curvature" tensor is zero for all curves! We work out the natural extension of intuitive curvature to hypersurfaces, based on the rates that their tangents develop components which are orthogonal to the local tangent hyperplane. Intuitive curvature is seen to have the form of a second-rank symmetric tensor which cannot be algebraically expressed in terms of the metric tensor and a finite number of its partial derivatives. The Riemann "curvature" tensor contrariwise is a fourth-rank tensor with both antisymmetric and symmetric properties that famously is algebraically expressed in terms of the metric tensor and its first and second partial derivatives. Thus use of the word "curvature" with regard to the Riemann tensor is misleading, and since it can't encompass intuitive curvature, Gauss-Riemann "geometry" oughtn't be termed differential geometry either. That "geometry" is no more than the class of the algebraic functions of the metric and any finite number of the metric's partial derivatives, which it is convenient to organize into generally covariant entities such as the Riemann tensor because those potentially play a role in generally-covariant metric-based field theories.

Introduction

The fourth-rank Riemann "curvature" tensor is antisymmetric in certain pairs of its four indices [1], which implies that it vanishes altogether in one dimension. Thus the Riemann tensor, notwithstanding the customary inclusion of the word "curvature" in its name, is intrinsically *incapable* of registering the curvature of any *curve*, such as a circle.

Since the Riemann "curvature" tensor thus demonstrably fails to *consistently register* the curvature of hypersurfaces, we seek an intuitively sensible definition of hypersurface curvature.

We begin by reviewing the pre-Riemann local osculating circle approach to the local curvature of planar curves.

Since three noncollinear points in a plane determine a circle, three points of a planar curve $\mathbf{r}(t)$ permit a circle to be crudely fitted to that curve. In the limit that the *separation* between those three points becomes *infinitesimal*, the curve's *local osculating circle* that corresponds to a particular one of its parameter points t is produced.

For an *infinitesimal traversal* along the curve from $\mathbf{r}(t)$ to $\mathbf{r}(t+\delta t)$, the *center* $\mathbf{r}_c(t)$ of the local osculating circle must be *stationary*, as must be *the distance from the curve to that center*. Therefore the relation,

$$|\mathbf{r}(t) - \mathbf{r}_c(t)|^2 = |\mathbf{r}(t+\delta t) - \mathbf{r}_c(t)|^2, \tag{1a}$$

is required to hold *through at least first order in* δt. Thus the equivalent equation,

$$2\mathbf{r}_c(t) \cdot (\mathbf{r}(t+\delta t) - \mathbf{r}(t)) = |\mathbf{r}(t+\delta t)|^2 - |\mathbf{r}(t)|^2, \tag{1b}$$

is as well required to hold through at least first order in δt. It is apparent, however, that if Eq. (1b) is required to hold *only through first order in* δt, then we will *only* obtain information about the component of $\mathbf{r}_c(t)$ which is *parallel to* $\dot{\mathbf{r}}(t)$. Since $\mathbf{r}_c(t)$ has *two* components, we *must* require Eq. (1b) *to hold through second order in* δt.

We therefore insert into Eq. (1b) the expansions through second order in δt of $(\mathbf{r}(t+\delta t) - \mathbf{r}(t))$ and $(|\mathbf{r}(t+\delta t)|^2 - |\mathbf{r}(t)|^2)$, which are,

$$\mathbf{r}(t+\delta t) - \mathbf{r}(t) = \dot{\mathbf{r}}(t)\delta t + \tfrac{1}{2}\ddot{\mathbf{r}}(t)(\delta t)^2 + O((\delta t)^3),$$

and,

$$|\mathbf{r}(t+\delta t)|^2 - |\mathbf{r}(t)|^2 = 2\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)\delta t + (\mathbf{r}(t) \cdot \ddot{\mathbf{r}}(t) + |\dot{\mathbf{r}}(t)|^2)(\delta t)^2 + O((\delta t)^3).$$

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The *two results* of requiring Eq. (1b) to hold in orders δt and $(\delta t)^2$ can both be conveniently stated in terms of $\mathbf{R}(t) \stackrel{\text{def}}{=} (\mathbf{r}_c(t) - \mathbf{r}(t))$, namely the vector from the curve to its local osculating circle's center; these two results are respectively,

$$\dot{\mathbf{r}}(t) \cdot \mathbf{R}(t) = 0, \tag{1c}$$

and,

$$\ddot{\mathbf{r}}(t) \cdot \mathbf{R}(t) = |\dot{\mathbf{r}}(t)|^2. \tag{1d}$$

From Eq. (1c) we see that $\mathbf{R}(t)$ *is perpendicular to the curve's tangent* $\dot{\mathbf{r}}(t)$, namely it is *normal* to the curve. Because we are working in the two-dimensional plane, we can construct the unit vector which is normal to the curve as a linear combination of the two vectors $\ddot{\mathbf{r}}(t)$ and $\dot{\mathbf{r}}(t)$, namely,

$$\widehat{\mathbf{n}}(t) = (\ddot{\mathbf{r}}|\dot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}}) / \left[|\dot{\mathbf{r}}|(|\ddot{\mathbf{r}}|^2|\dot{\mathbf{r}}|^2 - (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2)^{\frac{1}{2}}\right].$$

It is readily verified that $\widehat{\mathbf{n}}(t)$ is indeed a unit vector which is perpendicular to $\dot{\mathbf{r}}(t)$. Since $\mathbf{R}(t)$ is perpendicular to $\dot{\mathbf{r}}(t)$, and we are working in the two-dimensional plane, it must be the case that,

$$\mathbf{R}(t) = R(t)\widehat{\mathbf{n}}(t),$$

where $R(t)$ is a scalar that satisfies $|R(t)| = |\mathbf{R}(t)|$ because $\widehat{\mathbf{n}}(t)$ is a unit vector. Putting this representation of $\mathbf{R}(t)$, together with the above-given explicit form of the normal unit vector $\widehat{\mathbf{n}}(t)$, into Eq. (1d) allows us to solve for $R(t)$,

$$R(t) = |\dot{\mathbf{r}}|^3 / (|\ddot{\mathbf{r}}|^2|\dot{\mathbf{r}}|^2 - (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2)^{\frac{1}{2}}.$$

Since $\mathbf{R}(t) = R(t)\widehat{\mathbf{n}}(t)$, this result for $R(t)$, together with the above explicit form for $\widehat{\mathbf{n}}(t)$, yields,

$$\mathbf{R}(t) = |\dot{\mathbf{r}}|^2(\ddot{\mathbf{r}}|\dot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}}) / (|\ddot{\mathbf{r}}|^2|\dot{\mathbf{r}}|^2 - (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2). \tag{1e}$$

From $\mathbf{R}(t)$, the vector from the curve to the center of its local osculating circle, which vector is normal to the curve, we can construct the curve's *normally-directed curvature vector* $\mathbf{C}(t)$ as simply,

$$\mathbf{C}(t) \stackrel{\text{def}}{=} \mathbf{R}(t)/|\mathbf{R}(t)|^2, \tag{1f}$$

which has the two key properties that $\mathbf{C}(t) \cdot \mathbf{R}(t) = 1$ and $\mathbf{C}(t) \cdot \dot{\mathbf{r}}(t) = 0$. The detailed result for $\mathbf{C}(t)$ is,

$$\mathbf{C}(t) = (\ddot{\mathbf{r}}|\dot{\mathbf{r}}|^2 - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}})/|\dot{\mathbf{r}}|^4. \tag{1g}$$

Without restricting oneself to planar curves or explicitly constructing the curve's local osculating circle, one can attempt to conceptualize curvature in terms of the instantaneous rate at which the curve's tangent vector develops a component that is perpendicular to its current direction; the curve's local curvature is clearly zero if its tangent vector's direction isn't instantaneously changing.

At the parameter value t, the curve's tangent vector is $\dot{\mathbf{r}}(t)$, and at the infinitesimally different parameter value $t + \delta t$, its tangent vector is $\dot{\mathbf{r}}(t + \delta t)$. The component perpendicular to $\dot{\mathbf{r}}(t)$ which $\dot{\mathbf{r}}(t + \delta t)$ has developed is equal to $\dot{\mathbf{r}}(t+\delta t)$ minus the projection of $\dot{\mathbf{r}}(t+\delta t)$ into the direction of $\dot{\mathbf{r}}(t)$. That projection of $\dot{\mathbf{r}}(t+\delta t)$ into the direction of $\dot{\mathbf{r}}(t)$ consists of the unit vector $(\dot{\mathbf{r}}(t)/|\dot{\mathbf{r}}(t)|)$ multiplied by the dot product of the vector $\dot{\mathbf{r}}(t + \delta t)$ with that *same* unit vector $(\dot{\mathbf{r}}(t)/|\dot{\mathbf{r}}(t)|)$. Putting these statements together, the component perpendicular to $\dot{\mathbf{r}}(t)$ which $\dot{\mathbf{r}}(t + \delta t)$ has developed is equal to,

$$\dot{\mathbf{r}}(t + \delta t) - \dot{\mathbf{r}}(t)[(\dot{\mathbf{r}}(t + \delta t) \cdot \dot{\mathbf{r}}(t))/|\dot{\mathbf{r}}(t)|^2].$$

For the purpose of calculating curvature, we in fact require the *dimensionless fraction of* $\dot{\mathbf{r}}(t + \delta t)$ that its component perpendicular to $\dot{\mathbf{r}}(t)$ represents, i.e., we require that the above formula for the component of $\dot{\mathbf{r}}(t + \delta t)$ that is perpendicular to $\dot{\mathbf{r}}(t)$ *be divided by* $|\dot{\mathbf{r}}(t + \delta t)|$, producing the dimensionless entity,

$$\{\dot{\mathbf{r}}(t + \delta t) - \dot{\mathbf{r}}(t)[(\dot{\mathbf{r}}(t + \delta t) \cdot \dot{\mathbf{r}}(t))/|\dot{\mathbf{r}}(t)|^2]\}/|\dot{\mathbf{r}}(t + \delta t)|.$$

The curvature $\mathbf{C}(t)$ presented in Eqs. (1f) and (1g) however has dimensions of inverse length. Related to that, *the infinitesimal change in the curve's length which corresponds to the infinitesimal parameter interval* δt is equal to $(|\dot{\mathbf{r}}(t)|\delta t)$. To attempt to calculate curvature, we therefore need to divide the above dimensionless vector by this infinitesimal change in the curve's length, followed by taking the limit $\delta t \to 0$,

$$\mathbf{C}'(t) = \lim_{\delta t \to 0} \{\dot{\mathbf{r}}(t+\delta t) - \dot{\mathbf{r}}(t)[(\dot{\mathbf{r}}(t+\delta t) \cdot \dot{\mathbf{r}}(t))/|\dot{\mathbf{r}}(t)|^2]\}/(|\dot{\mathbf{r}}(t+\delta t)||\dot{\mathbf{r}}(t)|\delta t). \tag{2a}$$

To calculate the limit $\delta t \to 0$ in Eq. (2a), we note that,

$$\dot{\mathbf{r}}(t+\delta t) = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\delta t + O((\delta t)^2),$$

which together with Eq. (2a) readily yields,

$$\mathbf{C}'(t) = \{\ddot{\mathbf{r}}(t) - \dot{\mathbf{r}}(t)[(\ddot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t))/|\dot{\mathbf{r}}(t)|^2]\}/|\dot{\mathbf{r}}(t)|^2. \tag{2b}$$

A straightforward rewriting of Eq. (2b) shows that it has reproduced Eq. (1g),

$$\mathbf{C}'(t) = (\ddot{\mathbf{r}}|\dot{\mathbf{r}}|^2 - (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}})\dot{\mathbf{r}})/|\dot{\mathbf{r}}|^4 = \mathbf{C}(t). \tag{2c}$$

It is also of interest to take the dot product with itself of the *vector* curvature $\mathbf{C}(t)$ that is given by Eqs. (2c) and (1g) to obtain the *scalar* curvature presentation $|\mathbf{C}(t)|^2$,

$$|\mathbf{C}(t)|^2 = (|\ddot{\mathbf{r}}|^2|\dot{\mathbf{r}}|^2 - (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2)/|\dot{\mathbf{r}}|^6. \tag{2d}$$

It is furthermore of interest to note that in this one-dimensional situation the one-by-metric tensor $g(t)$ is given by $|\dot{\mathbf{r}}(t)|^2$. Therefore *almost all* of the factors which occur in the Eq. (2d) expression for the scalar curvature presentation $|\mathbf{C}(t)|^2$ can be *replaced* by the metric tensor $g(t)$ and its parameter derivatives; particularly notable in that regard is the derivative relation,

$$\tfrac{1}{2}\dot{g}(t) = (\ddot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t)).$$

The *one exception* to the possibility of replacing the factors of the scalar curvature presentation $|\mathbf{C}(t)|^2$ given in Eq. (2d) by the metric tensor $g(t)$ and its parameter derivatives is that the factor $|\ddot{\mathbf{r}}(t)|^2$ *cannot* thus be replaced by an algebraic function of a finite number of parameter derivatives of the metric $g(t) = |\dot{\mathbf{r}}(t)|^2$.

That *exception* is actually *important* in light of the fact that a *hallmark* of Riemann "curvature" is that it is written *entirely* as an algebraic function of the metric tensor and its first and second parameter derivatives [2]. Therefore the presence of the factor $|\ddot{\mathbf{r}}(t)|^2$ in Eq. (2d) reveals an insuperable disconnect between hypersurface curvature and Riemann "curvature" in one dimension; to be sure that isn't exactly *news* because we have seen that Riemann "curvature" vanishes identically in one dimension. However, we shall see that an insuperable disconnect between hypersurface curvature and Riemann "curvature" which is very closely analogous to the fact that $|\ddot{\mathbf{r}}(t)|^2$ can't be expressed as an algebraic function of a finite number of derivatives of the metric *persists in higher dimensions*.

Finally, it is of interest to point out that both the metric $g(t)$ and the curvature $\mathbf{C}(t)$ become extremely simple in the case that the curve's *parameter* t is transformed to its intermediate *arc length* $s(t)$, i.e.,

$$s(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} dt'\,|\dot{\mathbf{r}}(t')|, \text{ which implies that } \dot{s}(t) = |\dot{\mathbf{r}}(t)|.$$

The arc-length parameter transformed curve $\bar{\mathbf{r}}(s)$ is, of course, *defined* as,

$$\bar{\mathbf{r}}(s) \stackrel{\text{def}}{=} \mathbf{r}(t(s)),$$

and it therefore clearly has *exactly the same locus of vector points as that of* $\mathbf{r}(t)$. However, its tangent $d\bar{\mathbf{r}}(s)/ds$ *has the special property of always being a unit vector* because,

$$d\bar{\mathbf{r}}(s)/ds = d\mathbf{r}(t(s))/ds = \dot{\mathbf{r}}(t(s))\,(dt(s)/ds) = \dot{\mathbf{r}}(t(s))/\dot{s}(t(s)) = \dot{\mathbf{r}}(t(s))/|\dot{\mathbf{r}}(t(s))|,$$

and $\dot{\mathbf{r}}(t(s))/|\dot{\mathbf{r}}(t(s)|$ is obviously always a unit vector. Because $|d\bar{\mathbf{r}}(s)/ds| = 1$, the arc-length parameter transformed *metric* $\bar{g}(s)$ which corresponds to $\bar{\mathbf{r}}(s)$ *is everywhere equal to unity*, i.e.,

$$\bar{g}(s) = |d\bar{\mathbf{r}}(s)/ds|^2 = 1.$$

One consequence of this fact is that,

$$(d^2\bar{\mathbf{r}}(s)/ds^2) \cdot (d\bar{\mathbf{r}}(s)/ds) = \tfrac{1}{2} d\bar{g}(s)/ds = 0,$$

which, together with $|d\bar{\mathbf{r}}(s)/ds| = 1$ and Eq. (2c) or (1g), implies that the arc-length parameter transformed *curvature* $\bar{\mathbf{C}}(s)$ which corresponds to $\bar{\mathbf{r}}(s)$ *has the extremely simple form*,

$$\bar{\mathbf{C}}(s) = d^2\bar{\mathbf{r}}(s)/ds^2,$$

or in the scalar curvature presentation, $|\bar{\mathbf{C}}(s)|^2 = |d^2\bar{\mathbf{r}}(s)/ds^2|^2$.

The fact that the arc-length parameter transformed *metric* $\bar{g}(s) = 1$ for *all* curves makes it clear *why* the Riemann tensor, *which is an algebraic creature of exclusively the metric and the metric's first and second derivatives with respect to the parameter* [2], vanishes for *all* curves.

However, the fact that there is no reason whatsoever that the arc-length parameter transformed *curvature* in the scalar curvature presentation $|\bar{\mathbf{C}}(s)|^2 = |d^2\bar{\mathbf{r}}(s)/ds^2|^2$ need vanish makes it apparent that the Riemann tensor *has no relation to curvature*.

Having become comfortable with the details of the curvature of one-dimensional curves, we now turn our attention to the curvature of n-dimensional hypersurfaces. The basic idea is that curvature is a measure of the instantaneous rates at which tangent vectors develop components that are perpendicular to the local tangent hyperplane. That is nothing more than *a straightforward extension of the idea of the curvature of a one-dimensional curve*, but the need to project vectors into the hyperplane spanned by the n tangent vectors, where n is an *arbitrary* positive integer, is technically very much more burdensome than was the special case wherein $n = 1$. One needs to solve n^2 simultaneous linear equations to construct the needed projector; fortunately the result of that can be *formally* quite elegantly stated in terms of the inverse of the metric tensor.

The curvature coefficients of n-dimensional hypersurfaces

An n-dimensional hypersurface $\mathbf{r}(t^1, \ldots, t^n) = \mathbf{r}(\mathbf{t})$ of course takes an n-dimensional parameter vector $\mathbf{t} \stackrel{\text{def}}{=} (t^1, \ldots, t^n)$ as its argument, and at each parameter vector value $\mathbf{t}$ locally has the set of n tangent vectors $\{\partial\mathbf{r}(\mathbf{t})/\partial t^1, \ldots, \partial\mathbf{r}(\mathbf{t})/\partial t^n\}$. The set of n^2 *scalar dot products* that can be formed from the n tangent vectors comprise the *metric tensor* of the hypersurface,

$$g_{ij}(\mathbf{t}) \stackrel{\text{def}}{=} ((\partial\mathbf{r}(\mathbf{t})/\partial t^i) \cdot (\partial\mathbf{r}(\mathbf{t})/\partial t^j)), \tag{3a}$$

which, because of the commutative property of the dot product, is *symmetric*,

$$g_{ji}(\mathbf{t}) = g_{ij}(\mathbf{t}). \tag{3b}$$

It is thus obvious that in ordinary differential geometry the metric tensor is less fundamental than the tangent vectors from which it is defined, and we have seen that in one dimension the concept of intuitive curvature simply cannot be algebraically expressed in terms of the metric and a finite number of its derivatives. The *specialized* ostensible "geometry" of Gauss and Riemann, however, *excludes by fiat* anything which cannot be algebraically expressed in terms of the metric tensor and a finite number of its partial derivatives [3]. The *cogency* of the reasons put forward to justify the imposition of such a draconian restriction is by far strongest when the metric tensor is regarded to be a physically measurable and theoretically complete field, analogous in those respects to the electromagnetic field, and therefore to be handled accordingly. From an ordinary differential geometry standpoint, however, *sytematically ignoring the existence of the tangent vectors* and being *unable* to exposit on intuitive curvature are truly confounding. It seems fair to contend that the drastic restriction on differential geometry imposed by Gauss and Riemann actually marked an *exit* from the realm of differential geometry and an *entry* into a very different realm that is closely related to field theories of the metric.

Changes of the n-dimensional parameter vector $\mathbf{t}$ to another n-dimensional vector parameter $\bar{\mathbf{t}}$ are made using *everywhere well-defined and sufficiently-smooth parameter-vector transformation functions* $\bar{\mathbf{t}}(\mathbf{t})$ which have *everywhere well-defined and sufficiently-smooth inverses* $\mathbf{t}(\bar{\mathbf{t}})$. (In practice, gravitational physicists all too frequently aggressively ignore the injunction against ill-defined or insufficiently smooth behavior anywhere in the parameter-vector transformation function or its inverse, with the consequence that fundamental theorems which they take for granted, such as those regarding the the transformation properties of contracted tensors, actually fail.)

Under such transformations of the parameter vector, i.e., $\mathbf{t} \to \bar{\mathbf{t}} = \bar{\mathbf{t}}(\mathbf{t})$, the calculus chain rule implies that the transformation of the tangent vectors is given by,

$$\frac{\partial \bar{\mathbf{r}}(\bar{\mathbf{t}})}{\partial \bar{t}^k} \stackrel{\text{def}}{=} \frac{\partial \mathbf{r}(\mathbf{t}(\bar{\mathbf{t}}))}{\partial \bar{t}^k} = \sum_{i=1}^{n} \frac{\partial t^i(\bar{\mathbf{t}})}{\partial \bar{t}^k} \frac{\partial \mathbf{r}(\mathbf{t}(\bar{\mathbf{t}}))}{\partial t^i(\bar{\mathbf{t}})}, \tag{3c}$$

and consequently the transformation of the metric tensor is given by,

$$\bar{g}_{kl}(\bar{\mathbf{t}}) \stackrel{\text{def}}{=} \left(\frac{\partial \bar{\mathbf{r}}(\bar{\mathbf{t}})}{\partial \bar{t}^k}\right) \cdot \left(\frac{\partial \bar{\mathbf{r}}(\bar{\mathbf{t}})}{\partial \bar{t}^l}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial t^i(\bar{\mathbf{t}})}{\partial \bar{t}^k} \frac{\partial t^j(\bar{\mathbf{t}})}{\partial \bar{t}^l}\, g_{ij}(\mathbf{t}(\bar{\mathbf{t}})). \tag{3d}$$

Clearly the application of the calculus chain rule in Eq. (3c) is *invalidated* wherever the *inverse* $\mathbf{t}(\bar{\mathbf{t}})$ of the parameter-vector transformation function $\bar{\mathbf{t}}(\mathbf{t})$ is ill-defined or insufficiently smooth, and that issue consequently *also* impacts *the standard covariant transformation rule for the metric tensor* which is delineated in Eq. (3d). Gravitational physicists who aggressively introduce parameter-vector transformations, such as that of Kruskal and Szekeres, which feature points where their Jacobians vanish, unfortunately almost always ignore the consequences thereof.

Besides the covariant transformation rules exemplified by Eqs. (3c) and (3d), there exist the "mirror image" contravariant transformation rules; the transformation of *the matrix inverse of the metric tensor* illustrates those rules, the straightforward details of which the reader can readily work out from this information. Summing over any *paired* covariant and contravariant tensor indices effectively removes them *both* as actors in parameter-vector transformations. This "contraction theorem" is fundamental to gravitational physics, but its proof is *also* rooted in the calculus chain rule, *and that proof is as well invalidated by ill-conditioned parameter-vector transformations*.

The intuitive curvature of the n-dimensional hypersurface $\mathbf{r}(\mathbf{t})$ manifests itself when any of its tangent vectors has a non-vanishing instaneous rate of development of a component perpendicular to the local tangent hyperplane. At the local vector parameter value $\mathbf{t}$ the ith tangent vector is $\partial\mathbf{r}(\mathbf{t})/\partial t^i$, $i = 1, \ldots, n$, while at one of the n infinitesimally displaced vector parameter values $\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j$, $j = 1, \ldots, n$, where $\widehat{\mathbf{u}}_j$ *is the* jth *unit vector* of the n-dimensional vector parameter space, that ith tangent vector is $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$. To capture the ij *coefficient of curvature* of the hypersurface at $\mathbf{t}$, we need to obtain the component of $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$ *which is perpendicular to the hypersurface's local tangent hyperplane at* $\mathbf{t}$; that local hyperplane is of course spanned by the set of n local tangents at $\mathbf{t}$, which is $\{\partial\mathbf{r}(\mathbf{t})/\partial t^1, \ldots, \partial\mathbf{r}(\mathbf{t})/\partial t^n\}$. We write the desired perpendicular component of $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$ in the schematic form,

$$\frac{\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)}{\partial t^i} - \mathbf{P}_{\left\{\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^1}, \ldots, \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^n}\right\}}\left(\frac{\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)}{\partial t^i}\right),$$

where the second term denotes the *projection* of the vector $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$ *into the hypersurface's local tangent hyperplane at* $\mathbf{t}$ which is spanned by the set of n tangents $\{\partial\mathbf{r}(\mathbf{t})/\partial t^1, \ldots, \partial\mathbf{r}(\mathbf{t})/\partial t^n\}$. The *hyperplane projector* that appears in the above formula is of course *defined* by the fact that it satisfies the following n vector equations pertaining to the n tangents at $\mathbf{t}$,

$$\mathbf{P}_{\left\{\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^1}, \ldots, \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^n}\right\}}\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^k}\right) = \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^k} \quad \text{for } k = 1, \ldots, n.$$

To solve the above set of n vector equations for the tangent hyperplane projector, we assume that this projector's effect on an arbitrary vector $\mathbf{a}$ is achieved by a form which is *bilinear* in the n tangents, and of course is *linear* in $\mathbf{a}$,

$$\mathbf{P}_{\left\{\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^1}, \ldots, \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^n}\right\}}(\mathbf{a}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^i} h_{ij}\left(\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^j}\right)\cdot\mathbf{a}\right),$$

where the n^2 entities h_{ij} are treated as *unknowns* to be determined by the n vector equations given by the previous displayed formula. We see that when $\mathbf{a}$ is set equal to one of the tangents $\partial\mathbf{r}(\mathbf{t})/\partial t^k$, $k = 1, \ldots, n$, the consequent dot products *produce components of the metric tensor* $(g(\mathbf{t}))_{jk}$. Therefore if the *unknowns* h_{ij} are taken to be the components of the *inverse* of the metric tensor, i.e., $h_{ij} = ((g(\mathbf{t}))^{-1})_{ij}$, the set of n vector equations above, which the projector must satisfy, are seen to have in fact been solved. Adhering now to the standard convention for denoting the *contravariant components* of the *inverse* of the metric tensor, we write $((g(\mathbf{t}))^{-1})_{ij}$ as simply $(g(\mathbf{t}))^{ij}$, and therefore,

$$\mathbf{P}_{\left\{\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^1}, \ldots, \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^n}\right\}}(\mathbf{a}) = \sum_{i,j=1}^{n} \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^i}(g(\mathbf{t}))^{ij}\left(\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^j}\right)\cdot\mathbf{a}\right).$$

With this explicit result for the projector into the local tangent hyperplane at $\mathbf{t}$ we have for the component perpendicular to that hyperplane of $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$, which is the ith tangent vector at the infinitesimally different parameter vector $\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j$, the expression,

$$\frac{\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)}{\partial t^i} - \sum_{l,m=1}^{n} \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^l}(g(\mathbf{t}))^{lm}\left(\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^m}\right)\cdot\left(\frac{\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)}{\partial t^i}\right)\right).$$

Just as in the $n=1$ case, we must change the above raw vector component perpendicular to the local tangent hyperplane at $\mathbf{t}$ into a dimensionless *fraction* of the *complete vector* $\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i$ by *dividing* that raw perpendicular component by *the complete vector's length* $|\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)/\partial t^i|$.

To obtain the *ij curvature coefficient* from this dimensionless intermediate result, we must (also just as in the $n=1$ case) *further divide* by the infinitesimal distance along the hypersurface which the infinitesimal parameter vector change $\delta t_j\widehat{\mathbf{u}}_j$ implies. That particular infinitesimal parameter vector change implies a traversal along the jth tangent vector of the hypersurface whose infinitesimal distance is $\delta t_j|\partial\mathbf{r}(\mathbf{t})/\partial t^j|$.

After *the two divisions just described* into the above raw vector component perpendicular to the local tangent hyperplane at $\mathbf{t}$, we must, of course, *take the limit* of the resulting expression as $\delta t_j\to 0$. Taking that limit becomes straightforward upon making the following expansion in powers of δt_j,

$$\frac{\partial\mathbf{r}(\mathbf{t}+\delta t_j\widehat{\mathbf{u}}_j)}{\partial t^i} = \frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^i} + \left(\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j}\right)\delta t_j + O((\delta t_j)^2),$$

and simultaneously bearing firmly in mind the tangent-vector projective nature of the second term of the above raw vector component perpendicular to the local tangent hyperplane at $\mathbf{t}$.

The result of this limit is the ij local vector curvature coefficient $\mathbf{C}_{ij}(\mathbf{t})$ of the hypersurface $\mathbf{r}(\mathbf{t})$,

$$\mathbf{C}_{ij}(\mathbf{t}) = (g_{ii}(\mathbf{t})g_{jj}(\mathbf{t}))^{-\frac{1}{2}}\left[\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j} - \sum_{l,m=1}^{n}\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^l}(g(\mathbf{t}))^{lm}\left(\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^m}\right)\cdot\left(\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j}\right)\right)\right], \tag{3e}$$

where we have used the fact that $|\partial\mathbf{r}(\mathbf{t})/\partial t^k|^2 = g_{kk}(\mathbf{t})$. A description in words of this ij local vector curvature coefficient $\mathbf{C}_{ij}(\mathbf{t})$ of the hypersurface $\mathbf{r}(\mathbf{t})$ is that it is the projection orthogonal to the hypersurface's local tangent hyperplane at $\mathbf{t}$ of the instantaneous rate of change of the hypersurface's ith unit tangent vector at $\mathbf{t}$ with respect to distance traversed along the hypersurface in the direction of its jth unit tangent vector at $\mathbf{t}$, where we note that the hypersurface's kth unit tangent vector at $\mathbf{t}$ is given by,

$$\left|\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^k}\right|^{-1}\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^k}\right) = (g_{kk}(\mathbf{t}))^{-\frac{1}{2}}\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^k}\right),\ k=1,\ldots,n.$$

We further note that in the $n=1$ one-dimensional case, $\mathbf{C}_{11}(t)$ *is the same as* $\mathbf{C}'(t)$ of Eq. (2b), which by Eq. (2c) is equal, of course, to $\mathbf{C}(t)$ of Eq. (1g).

In the more interesting scalar curvature presentation, the ij local curvature coefficient $|\mathbf{C}_{ij}(\mathbf{t})|^2$ of the hypersurface $\mathbf{r}(\mathbf{t})$ is,

$$|\mathbf{C}_{ij}(\mathbf{t})|^2 = (g_{ii}(\mathbf{t})g_{jj}(\mathbf{t}))^{-1}\left[\left|\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j}\right|^2 - \sum_{l,m=1}^{n}\left(\left(\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j}\right)\cdot\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^l}\right)\right)(g(\mathbf{t}))^{lm}\left(\left(\frac{\partial\mathbf{r}(\mathbf{t})}{\partial t^m}\right)\cdot\left(\frac{\partial^2\mathbf{r}(\mathbf{t})}{\partial t^i\partial t^j}\right)\right)\right]. \tag{3f}$$

It is apparent that the term $|\partial^2\mathbf{r}(\mathbf{t})/(\partial t^i\partial t^j)|^2$ which occurs in Eq. (3f) cannot be an algebraic function of the metric tensor and a finite number of its partial derivatives. Therefore the coefficients of curvature $|\mathbf{C}_{ij}(\mathbf{t})|^2$, an integral part of ordinary differential geometry (in which the tangent vectors are fundamental and the metric tensor is incidental), are *completely unrelated* to the Riemann tensor, which famously *is algebraically expressed entirely in terms of the metric tensor and its first and second partial derivatives* [2]. Completely unrelated as it is to intuitive curvature, the Riemann tensor can't reasonably be termed curvature.

The ostensible "geometry" of Gauss and Riemann, which is *prohibited* to have any ingredients that can't be algebraically expressed in terms of the metric tensor and a finite number of its partial derivatives [3], is *likewise* unable to describe intuitive curvature, and therefore can't reasonably be termed differential geometry. However, after it is systematically organized into a repository of entities which are generally covariant, such as the Riemann tensor, Gauss-Riemann (misnamed) "geometry" becomes a comprehensive source of general covariants which potentially play a role in generally-covariant metric-based field theories.

Thirring and Feynman obtained the Einstein equation *from field-theoretic physical reasoning alone, without reference to any "geometry"-related concept whatsoever* [4, 5], but Einstein's path of formulating the Einstein tensor from the well-studied Riemann tensor of Gauss-Riemann (misnamed) "geometry" is technically vastly simpler and more advantageous than the highly intricate successive approximation scheme to the Einstein equation which is impelled by field-theoretic physical reasoning alone. (That iteration is needed to fully take into account the fact that the the gravitational field *itself* contributes stress-energy which is part of the gravitational field's *source*, a recipe for an endless iteration loop that is spared electromagnetic field theory because the electromagnetic field itself doesn't contribute any charge-current density to its source.) Indeed, the intricate *explicit iteration* of the Lorentz-covariant weak-field *linear* gravitational equation (that Thirring and Feynman obtained from field-theoretic physical reasoning about special-relativistic upgrading of Newtonian gravitational theory) can be entirely *avoided* via the *realization* that the *result* of the iteration loop *must*, in light of Einstein's principle of equivalence, *be a generally covariant gravitational equation*. Therefore *the entire iteration process can be effectively replaced by some algebraic manipulation*: the *exact relation* of the metric-tensor dependent left-hand side of the weak-field linear gravitational equation to the weak-field linear form of the Riemann tensor and its weak-field contractions *is worked out*, following which the insistence on imposing general covariance yields the full Einstein tensor offshoot of the full Riemann tensor and also the full Einstein equation.

This major example makes it clear that the great utility of Gauss-Riemann (misnamed) "geometry" lies *in nothing more* than its being a comprehensive source of generally-covariant entities constructed as algebraic functions of the metric tensor and a finite number of that tensor's partial derivatives, which potentially play a role in generally-covariant metric-based field theories—the Riemann tensor being one of the simplest examples of these general covariants that actually involves partial derivatives of the metric tensor.

References

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