Nine steps transforming incompressible Navier-Stokes equations into parabolic partial differential equations

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Abstract

It was shown that using spatial transform obtained by applying the difference to spatial global coordinates and time integral of velocities non linear Navier-Stokes equation transforms into parabolic equations.

Nine steps of linearisation

Let's start from Navier-Stokes equations [1, 2, 3]

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}$$

We know that in the left side is full time derivative

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \tag{1}$$

Now we could integrate both sides by time and obtain for $\mathbf{v}(\mathbf{r}, t)$

$$\mathbf{v}(\mathbf{r}, t) = -\frac{1}{\rho} \int_0^t \nabla_{\mathbf{r}} p(\mathbf{r}, \tau) d\tau + \nu \int_0^t \nabla_{\mathbf{r}}^2 \mathbf{v}(\mathbf{r}, \tau) d\tau \tag{2}$$

We obtained formal solution for velocities as follow

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{F}(\mathbf{r}, t) \tag{3}$$

Eq. (3) is valid for any $\mathbf{r}'$ in the boundary where we try to solve NS equation. We could choose $\mathbf{r}'$ so that

$$\mathbf{r}' = \mathbf{r} - \int_0^t \mathbf{v}(\mathbf{r}, \tau) d\tau \tag{4}$$

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and

$$\mathbf{v}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)=\mathbf{F}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t) \tag{5}$$

or by taking full derivative of both sides we obtain

$$\frac{D\mathbf{v}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{Dt}=\frac{D\mathbf{F}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{Dt}= \tag{6}$$

$$-\nabla p(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)+\nu\nabla^2_{(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}\mathbf{v}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t) \tag{7}$$

It is easy to prove that

$$\frac{D\mathbf{v}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{Dt}=\frac{\partial\mathbf{v}(\mathbf{r}-\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{\partial t}. \tag{8}$$

So, we obtain parabolic equations

$$\frac{\partial\mathbf{v}\,(\mathbf{r}',t)}{\partial t}=-\frac{1}{\rho}\nabla_{\mathbf{r}'}p\,(\mathbf{r}',t)+\nu\nabla^2_{\mathbf{r}'}\mathbf{v}\,(\mathbf{r}',t) \tag{9}$$

which are analytically solvable for defined boundaries using for example Green function method. When we obtain velocities $\mathbf{v}(\mathbf{r}',t)$ in the local coordinates we could go back to the global coordinates by inserting new one coordinates $\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau$ into obtained analytical solution which implies that

$$\nabla_{\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau}\mathbf{v}(\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)=\nabla^2_{\mathbf{r}}\mathbf{v}(\mathbf{r},t)$$

and

$$\frac{\partial\mathbf{v}(\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{\partial t}=\frac{\partial\mathbf{v}(\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{\partial t}+$$
$$\frac{\partial(\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)}{\partial t}\cdot\nabla_{\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau}\mathbf{v}(\mathbf{r}'+\int_0^t \mathbf{v}(\mathbf{r},\tau)d\tau,t)=$$
$$\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t}+\frac{\partial\mathbf{r}}{\partial t}\cdot\nabla_{\mathbf{r}}\mathbf{v}(\mathbf{r},t)=\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t}+\mathbf{v}(\mathbf{r},t)\cdot\nabla_{\mathbf{r}}\mathbf{v}(\mathbf{r},t).$$

References

[1] Navier, C. L. M. H. Mem acad. R. sci. paris, Vol. 6, 389-416, 1823.

[2] Cauchy, A.L. Exercises de mathematique, p.183, Paris, 1828.

[3] Stokes. G. G. trans. Camb. Phil. Soc., vol 8, 287–305, 1845.