Generalized Sophomore's Dream Identity

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(July 3, 2015)

The identity of Johann Bernoulli, nowadays known as "Sophomore's Dream", states that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = \int\limits_0^1 \frac{1}{x^x} dx.$$

This paper represents the proof of generalized fact that

$$\sum_{n=k+1}^{\infty} \frac{n^k}{n^n} = \int\limits_0^1 \frac{x^k}{x^x} dx$$

for every natural k, and moreover,

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} = \int\limits_{-c}^{a-c} \frac{a^t}{(t+c)^t} dt \quad \left(= a \int\limits_0^1 t^{c-at} dt \right) \qquad (*)$$

for every appropriate a and c.

Proof:

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} = \sum_{n=1}^{\infty} a^n \int\limits_0^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-t(n+c)} dt = a \int\limits_0^{\infty} e^{-t(c+1)+ate^{-t}} dt = a \int\limits_0^1 t^{c-at} dt.$$

Even more,

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} \frac{\Gamma(n+b)}{\Gamma(n)(n+c)^b} = \int\limits_{-c}^{a-c} \left(\ln \frac{a}{t+c} \right)^b \frac{a^t}{(t+c)^t} dt.$$

Generally:

$$\sum_{n=0}^{\infty} \frac{a^n \Gamma(n+b)}{(n+c)^{n+b}} \frac{f^{(n)}(0)}{n!} = \int_0^{\infty} t^{b-1} e^{-ct} f(ate^{-t}) dt$$

One can use it, for example, to show that

$$\sum_{n=0}^{\infty} \frac{a^n \Gamma(n+b)}{(n+c)^{n+b}} \frac{\Gamma(n+d)}{n!\Gamma(d)} = \int_0^{\infty} \frac{x^{b-1} e^{-x(c-d)}}{(e^x - ax)^d} dx.$$

That's why, for example,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} = \int_0^1 \frac{dx}{(1 + x \ln x)^2}$$