Damped harmonic oscillator with time-dependent frictional coefficient and time-dependent frequency

Eun Ji Jang, Jihun Cha, Young Kyu Lee, and Won Sang Chung*

Department of Physics and Research Institute of Natural Science,

College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea

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Abstract

In this paper we extend the so-called dual or mirror image formalism and Caldirola's- Kanai's formalism for damped harmonic oscillator to the case that both frictional coefficient and time-dependent frequency depend on time explicitly. As an solvable example, we consider the case that frictional coefficient $\gamma(t) = \frac{\gamma_0}{1+qt}, (q > 0)$ and angular frequency function $w(t) = \frac{w_0}{1+qt}$. For this choice, we construct the quantum harmonic Hamiltonian and express it in terms of $su(2)$ algebra generators. Using the exact invariant for the Hamiltonian and its unitary transform, we solve the time-dependent Schrödinger equation with time-dependent frictional coefficient and time-dependent frequency.

*Electronic address: mimip4444@hanmail.net

I. INTRODUCTION

The harmonic oscillator is a system playing an important role both in classical and quantum mechanics. It appears in various physical applications running from condensed matter to semiconductors (see e.g.[1] for references to such problems). The harmonic oscillator equation with time-dependent parameters [2-6] has been discussed for a sudden frequency change using a continuous treatment based on an invariant formalism [7]. This analytic treatment requires that the time-dependent parameter be a monotonic function whose variation is short compared with the typical period of the system.

In 1931, Bateman presented [8] the so-called dual or mirror image formalism for damped oscillator. The Lagrangian for the linearly damped free particle was first obtained by Caldirola [9] and Kanai [10]. They considered the following Lagrangian

$$\mathcal{L} = e^{\lambda t}\frac{1}{2}m\dot{x}^2,$$

which gives the equation of motion for the the linearly damped free particle

$$\ddot{x} = -\lambda\dot{x}$$

During the last four decades, many techniques has been developed to incorporate classical linearly damped or dissipative systems into the framework of quantum mechanics [11]. The problem of the classical linearly damped oscillator has been studied extensively [12, 13]. Incorporating the linear damping to the quantum theory usually introduces a complex Hamiltonian with a symplectic structure [14,15].

In this paper we extend the so-called dual or mirror image formalism for damped harmonic oscillator to the case that both frictional coefficient and time-dependent frequency depend on time explicitly. We also establish the relation between Bateman's and Caldirola's-Kanai's form for the damped oscillator problem with time-dependent frictional coefficient and time-dependent frequency. As an solvable example, we consider the case that frictional coefficient $\gamma(t) = \frac{\gamma_0}{1+qt}, (q > 0)$ and angular frequency function $w(t) = \frac{w_0}{1+qt}$. For this choice, we construct the quantum harmonic Hamiltonian and express it in terms of $su(2)$ algebra generators. Using the exact invariant for the Hamiltonian and its unitary transform, we solve the time-dependent Schrödinger equation with time-dependent frictional coefficient and time-dependent frequency.

II. BATEMAN DUAL OSCILLATOR FORMALISM

In 1931, Bateman presented [8] the so-called dual or mirror image formalism for damped oscillator. According to this, the energy dissipated by the oscillator is completely absorbed at the same time by the mirror image oscillator, and thus the energy of the total system is conserved. The dual system is

$$m\ddot{x} + m\gamma\dot{x} + kx = 0 \tag{1}$$

$$m\ddot{y} - m\gamma\dot{y} + ky = 0 \tag{2}$$

where γ, k are constant. Here the second equation describes time reversal process to the first equation and together they represent so called dual damped oscillator system.

At this stage we have question: Is it possible to extend Bateman dual oscillator Formalism to the case that both frictional coefficient and frequency are time-dependent. Now we will show that it is possible. Let us consider the damped oscillator with time-dependent frictional coefficient and time-dependent frequency. The equation of motion is given by

$$\ddot{x} + \gamma(t)\dot{x} + k(t)x = 0 \tag{3}$$

where we set the mass of a particle to be unity. According to Bateman dual oscillator Formalism, we can set the equation of motion for the mirror image oscillator as

$$\ddot{x} + \xi(t)\dot{x} + \kappa(t)x = 0 \tag{4}$$

The total system which consists of the system of damped harmonic oscillator and its time-reversed image is described by the Lagrangian

$$L = \dot{x}\dot{y} + F(t)(x\dot{y} - y\dot{x}) - G(t)xy \tag{5}$$

The canonical momenta for this dual system can be obtained from the eq.(5) by using the following formulas

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{y} - F(t)y \tag{6}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{x} + F(t)x \tag{7}$$

Clearly, these differ from the mechanical momenta. The equations of motion read

$$\ddot{x} + 2F(t)\dot{x} + (\dot{F} + G)x = 0 \tag{8}$$

$$\ddot{y} - 2F(t)\dot{y} + (-\dot{F} + G)y = 0 \tag{9}$$

Comparing the eq.(8) (or (9)) with the eq.(1) (or (2)), we obtain

$$F(t) = \frac{1}{2}\gamma(t), \quad \dot{F} + G = k(t), \qquad -\dot{F} + G = \kappa(t) \tag{10}$$

Then, we have

$$\xi(t) = -\gamma(t), \quad \kappa(t) = k(t) - \dot{\gamma}(t) \tag{11}$$

Thus, the dual equations of motion read

$$\ddot{x} + \gamma(t)\dot{x} + k(t)x = 0 \tag{12}$$

$$\ddot{y} - \gamma(t)\dot{y} + (k(t) - \dot{\gamma}(t))y = 0 \tag{13}$$

and the Lagrangian for the total system is given by

$$L = \dot{x}\dot{y} + \frac{1}{2}\gamma(t)(x\dot{y} - y\dot{x}) - (k(t) - \dot{\gamma}(t))xy \tag{14}$$

To reach the Batemans dual Hamiltonian we can use Legendre transformation;

$$H = \dot{x}p_x + \dot{y}p_y - L, \tag{15}$$

then we get the Bateman Hamiltonian in the following form

$$H = p_x p_y + \frac{1}{2}\gamma(t)(yp_y - xp_x) + \left(k(t) - \frac{1}{2}\dot{\gamma}(t) - \frac{1}{4}\gamma^2(t)\right)xy \tag{16}$$

Because γ, k depend on time, we have $\frac{\partial H}{\partial t} \neq 0$ which implies that H is not conserved. That is to say, the total system is not a closed one but an open one. Therefore the energy dissipated by the original oscillator is incompletely absorbed by the dual of the system. We can write the canonical equations of Hamilton as follows,

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_y - \frac{1}{2}\gamma x \tag{17}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_x + \frac{1}{2}\gamma y \tag{18}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{1}{2}\gamma p_x - \left(k - \frac{1}{2}\dot{\gamma} - \frac{1}{4}\gamma^2\right)y \tag{19}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{1}{2}\gamma p_y - \left(k - \frac{1}{2}\dot{\gamma} - \frac{1}{4}\gamma^2\right)x \tag{20}$$

III. FROM BATEMAN TO CALDIROLA-KANAI

We are going to establish the relation between Bateman's and Caldirola's- Kanai's form. We expect that proper exclusion of the y component will lead doubled Bateman Hamiltonian to the Caldirola Hamiltonian formalism. Now we apply a canonical transformation. Generating function of the transformation is $F = F(x, y, P_x, P_y)$, where

$$F = xP_x\sqrt{\Gamma(t)} + yP_y\frac{1}{\sqrt{\Gamma(t)}} \tag{21}$$

We find

$$X = \frac{\partial F}{\partial P_x} = x\sqrt{\Gamma(t)}, \qquad p_x = \frac{\partial F}{\partial x} = P_x\sqrt{\Gamma(t)} \tag{22}$$

$$Y = \frac{\partial F}{\partial P_y} = \frac{y}{\sqrt{\Gamma(t)}}, \qquad p_y = \frac{\partial F}{\partial y} = \frac{P_y}{\sqrt{\Gamma(t)}} \tag{23}$$

Substituting new variables into the Hamiltonian at the eq. (16), we get

$$H = P_xP_y + \left(k - \frac{1}{2}\dot{\gamma} - \frac{1}{4}\gamma^2\right)XY \tag{24}$$

New hamiltonian, lets call it $H' = H + \frac{\partial F}{\partial t}$, is

$$H' = P_xP_y + \Omega^2(t)XY \tag{25}$$

where

$$\Omega^2(t) = k - \frac{1}{2}\dot{\gamma} - \frac{1}{4}\gamma^2 \tag{26}$$

This Hamiltonian describes undamped oscillations at the time-dependent frequency $\Omega(t)$.

Now we want to extend the real coordinates and momenta into the complex plane and to introduce a canonical transformation from X, Y, Px, Py to $Q, \bar{Q}, P, \bar{P}$:

$$X = \frac{1}{2}(Q + \bar{Q}) + \frac{i}{2\Omega}(P - \bar{P}) \tag{27}$$

$$Y = \frac{1}{2}(Q + \bar{Q}) - \frac{i}{2\Omega}(P - \bar{P}) \tag{28}$$

$$P_x = \frac{1}{2}(P + \bar{P}) + \frac{i\Omega}{2}(Q - \bar{Q}) \tag{29}$$

$$P_y = \frac{1}{2}(P + \bar{P}) - \frac{i\Omega}{2}(Q - \bar{Q}) \tag{30}$$

Writing the eq. (24) in terms of the new coordinates we get

$$H' = \left(\frac{P^2}{2} + \frac{1}{2}\Omega^2Q^2\right) + \left(\frac{\bar{P}^2}{2} + \frac{1}{2}\Omega^2\bar{Q}^2\right) \tag{31}$$

This Hamiltonian represents the energy of two independent identical oscillators, because of this we will only concentrate on one oscillator, the $Q-P$ system. The canonical generator of the system is,

$$F_1 = \Pi Q + \frac{1}{4}\gamma(t)Q^2 \tag{32}$$

Then transforms Q, P into Z, Π, we find

$$P = \frac{\partial F_1}{\partial Q} = \Pi + \frac{\gamma}{2}Q \tag{33}$$

$$Z = \frac{\partial F_1}{\partial \Pi} = Q \tag{34}$$

We can write H' in terms of new variables,

$$H_1 = H' + \frac{\partial F_1}{\partial t} = \frac{1}{2}\Pi^2 + \frac{\gamma}{2}\Pi Z + \frac{1}{2}k(t)Z^2 \tag{35}$$

We will make one more transformation from Z, Π to z, π generated by

$$F_2 = \frac{\pi Z}{\sqrt{\Gamma(t)}}, \tag{36}$$

where

$$\Pi = \frac{\partial F_2}{\partial Z} = \frac{\pi}{\sqrt{\Gamma(t)}} \tag{37}$$

$$z = \frac{\partial F_2}{\partial \pi} = \frac{Z}{\sqrt{\Gamma(t)}} \tag{38}$$

Since we can write H_2 as

$$H_2 = H_1 + \frac{\partial F_2}{\partial t} = \frac{1}{2\Gamma(t)}\pi^2 + \frac{1}{2}k(t)\Gamma(t)z^2 \tag{39}$$

This Hamiltonian is the Caldirola-Kanai-type Hamiltonian.

IV. DAMPED HARMONIC OSCILLATOR WITH TIME-DEPENDENT FRICTIONAL COEFFICIENT AND TIME-DEPENDENT FREQUENCY

Let us consider the one dimensional system with a linear frictional force with time-dependent frictional coefficient and the harmonic force with time-dependent frequency. For a particle with unit mass, we have the following equation of motion:

$$\frac{dv}{dt} = -\gamma(t)v - w^2(t)x \tag{40}$$

or

$$\ddot{x} + \gamma(t)\dot{x} + w^2(t)x = 0 \tag{41}$$

Let us assume that the Lagrangian takes the form

$$\mathcal{L} = \Gamma(t)G(v) + K(t)x^2, \tag{42}$$

where $v = \dot{x}$. The momentum is then given by

$$p = \Gamma(t)G'(v) \tag{43}$$

The equation of motion for the Lagrangian (42) reads

$$\frac{dv}{dt} = -\frac{\dot{\Gamma}}{\Gamma}\frac{G'}{G''} + \frac{2K(t)}{\Gamma G''}x \tag{44}$$

Comparing the eq.(44) with the eq.(40), we have

$$\frac{\dot{\Gamma}}{\Gamma} = \gamma(t), \quad \frac{G'}{G''} = v, \quad \frac{2K(t)}{\Gamma G''} = -w^2(t) \tag{45}$$

Solving the eq.(5), we get

$$\Gamma(t) = e^{\int_0^t \gamma(s)ds}, \quad G = \frac{v^2}{2}, \quad K(t) = -\frac{1}{2}w^2(t)\Gamma(t) \tag{46}$$

The Lagrangian is then given by

$$\mathcal{L} = \frac{1}{2}e^{\int_0^t \gamma(s)ds}[\dot{x}^2 - w^2(t)x^2] \tag{47}$$

When $\gamma(t) = const, w(t) = const$, the eq.(47) reduces to the Bateman Lagrangian [1]. Using the Legendre transform, we have the corresponding Hamiltonian:

$$H = \frac{1}{2\Gamma(t)}p^2 + \frac{1}{2}\Gamma(t)w^2(t)x^2 \tag{48}$$

V. QUANTUM HARMONIC OSCILLATOR WITH THE TIME-DEPENDENT ANGULAR FREQUENCY

The classical equation of motion for the damped harmonic oscillator with general frictional coefficient $\gamma(t)$ and time-dependent frequency $w(t)$ is given by

$$\ddot{x} + \gamma(t)\dot{x} + w^2(t)x = 0, \tag{49}$$

where we set a mass of a particle to be unity. We only consider a special case that frictional coefficient and the frequency of the system decreases rationally. Then, $\gamma(t)$ and $w(t)$ in the above equation becomes

$$\gamma(t) = \frac{\gamma_0}{1+qt}, \quad w(t) = \frac{w_0}{1+qt}, \tag{50}$$

where $q > 0$ is assumed and γ_0, w_0 are positive constants.

The corresponding quantum Hamiltonian for the eq.(49) is

$$H = \frac{p^2}{2\Gamma(t)} + \frac{1}{2}\Gamma(t)w^2(t)x^2, \tag{51}$$

where $[x,p] = i$ (we set $\hbar = 1$) and

$$\Gamma(t) = e^{\int_0^t \gamma(s)ds} = (1+qt)^{\gamma_0/q} \tag{52}$$

The Hamiltonian (51) can be written as

$$H = \frac{1}{\Gamma(t)}J_- + \Gamma(t)w^2(t)J_+, \tag{53}$$

where

$$J_+ = \frac{1}{2}x^2, \quad J_- = \frac{1}{2}p^2 \tag{54}$$

If we introduce the operator

$$J_0 = \frac{i}{4}(px + xp), \tag{55}$$

we have $su(2)$ algebra defined by

$$[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm, \tag{56}$$

where we have $J_\pm^\dagger = J_\pm, J_0^\dagger = -J_0$. Here we note that the Hamiltonian giving the eq.(49) is not unique. If we set $H = H_-(t)J_- - iH_0(t)J_0 + H_+(t)J_+$, we obtain the eq.(49) if the following relations hold:

$$\dot{H}_- = -\gamma(t)H_-, \quad H_-H_+ - \frac{1}{4}H_0^2 - \frac{1}{2}\dot{H}_0 - \frac{1}{2}\gamma(t)H_0 = w^2(t) \tag{57}$$

For simplicity, throughout this paper we set $H_0 = 0$ and then we have the Hamiltonian (53).

In order to derive quantum-mechanical solutions of the time-dependent Hamiltonian system, it is convenient to use the invariant operator I. To investigate the system quantum mechanically, we introduce the trial invariant operator as

$$I = h_1(t)J_+ + ih_2(t)J_0 + h_3(t)J_-, \tag{58}$$

where h_1, h_2, h_3 are real functions and I is Hermitian. The invariant operator I obeys

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + i[H, I] = 0 \tag{59}$$

Inserting the eq.(58) into the eq.(59), we have

$$\dot{h}_1 = -\Gamma w^2 h_2 \tag{60}$$

$$\dot{h}_2 = \frac{2}{\Gamma} h_1 - 2\Gamma w^2 h_3 \tag{61}$$

$$\dot{h}_3 = \frac{1}{\Gamma} h_2 \tag{62}$$

If we assume

$$h_1(t) = \alpha(1+qt)^A, \quad h_2(t) = \beta(1+qt)^B, \quad h_3(t) = \gamma(1+qt)^C, \tag{63}$$

we have

$$\begin{gathered} \alpha q A + \beta w_0^2 = 0, \quad \beta - \gamma q C = 0, \quad q\beta B = 2\alpha - 2w_0^2\gamma \\ A = \frac{\gamma_0}{q} + B - 1, \qquad C = B + 1 - \frac{\gamma_0}{q} \end{gathered} \tag{64}$$

Let us transform the eq.(58) with an appropriate unitary operator U like

$$I' = UIU^\dagger \tag{65}$$

We choose U as

$$U = U_2 U_1 = e^{k_2(t)J_0} e^{ik_1(t)J_+} \tag{66}$$

If we take

$$k_1 = -\frac{h_2}{2h_3}, \quad e^{-k_2(t)} = \frac{1}{\gamma}(1+qt)^{-C}, \tag{67}$$

we obtain

$$I' = \frac{p^2}{2} + \frac{1}{2}\left(\alpha - \frac{\beta^2}{4\gamma}\right)\gamma(1+qt)^{A+C} x^2 \tag{68}$$

From the fact that both I and I' are time-invariant, we know $A + C = 0$. Thus, if we choose $\gamma = 1$, we have

$$h_1 = w_0^2(1+qt)^{-1+\gamma_0/q}, \quad h_2 = q - \gamma_0, \quad h_3 = (1+qt)^{1-\gamma_0/q} \tag{69}$$

$$U = (1+qt)^{(1-\gamma_0/q)J_0}(1+qt)^{-\frac{i}{2}(q-\gamma_0)(-1+\gamma_0/q)J_+} \tag{70}$$

and

$$I' = \frac{p^2}{2} + \frac{1}{2} w_1^2 x^2, \tag{71}$$

where

$$w_1^2 = w_0^2 - \frac{1}{4}(q-\gamma_0)^2 \tag{72}$$

The eigenvalue equation for I' can be written as

$$I'|n,t\rangle' = w_1\left(n+\frac{1}{2}\right)|n,t\rangle', \quad n=0,1,2,\cdots \tag{73}$$

Acting $\langle x|$ to both side of the above equation from the left, we can easily obtain the solution as

$$\langle x|n,t\rangle' = \left(\frac{w_1}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\sqrt{w_1}x)e^{-\frac{1}{2}w_1x^2}, \tag{74}$$

where $H_n(x)$ is n-th order Hermite polynomial. Letting $|n,t\rangle = U^\dagger|n,t\rangle'$ and using $e^{aJ_0}f(x) = f(xe^{a/2})$, we obtain

$$I|n,t\rangle = w_1\left(n+\frac{1}{2}\right)|n,t\rangle, \quad n=0,1,2,\cdots \tag{75}$$

and

$$\langle x|n,t\rangle = \left(\frac{w_1w_0^2}{\pi h_1(t)}\right)^{1/4} H_n\left(\sqrt{\frac{w_1w_0^2}{h_1(t)}}x\right)\exp\left[-\frac{w_0^2}{2h_1(t)}\left(w_1 - \frac{i(\gamma_0-q)}{2w_0^2}h_1(t)\right)x^2\right] \tag{76}$$

The step operators are then given by

$$a(t) = \sqrt{\frac{w_1h_1}{2w_0^2}}\left(1-\frac{ih_2}{2w_1}\right)x + i\sqrt{\frac{w_0^2}{2w_1h_1}}p \tag{77}$$

$$a^\dagger(t) = \sqrt{\frac{w_1h_1}{2w_0^2}}\left(1+\frac{ih_2}{2w_1}\right)x - i\sqrt{\frac{w_0^2}{2w_1h_1}}p \tag{78}$$

and the invariant is expressed as

$$I(t) = \frac{w_1}{2}\left[a^\dagger(t)a(t)+\frac{1}{2}\right] \tag{79}$$

The Fock space representation of the step operators are

$$a(t)|n,t\rangle = \sqrt{n}|n-1,t\rangle \tag{80}$$

$$a^\dagger(t)|n,t\rangle = \sqrt{n+1}|n+1,t\rangle \tag{81}$$

Expressing the position and momentum operators in terms of the step operators, we get

$$x = \sqrt{\frac{w_0^2}{2w_1h_1}}[a(t)+a^\dagger(t)] \tag{82}$$

$$p = i\sqrt{\frac{h_1}{2w_0^2 w_1}}\left[\left(w_1 + \frac{ih_2}{2}\right)a(t) - \left(w_1 - \frac{ih_2}{2}\right)a^\dagger(t)\right] \tag{83}$$

The time-dependent wave function for the Hamiltonian (51) then takes the form

$$|\psi_n(t)\rangle = |n,t\rangle e^{i\gamma_n(t)} \tag{84}$$

Inserting the eq.(84) into the the time-dependent Schrödinger equation for the Hamiltonian (51), we have

$$\gamma_n(t) = -\frac{w_0^2}{qw_1}(n+1/2)\ln(1+qt) \tag{85}$$

Thus, the time-dependent wave function for the Hamiltonian (51) reads

$$\langle x|\psi_n(t)\rangle = \left(\frac{w_1 w_0^2}{\pi h_1(t)}\right)^{1/4} H_n\left(\sqrt{\frac{w_1 w_0^2}{h_1(t)}}x\right)\exp\left[-\frac{w_0^2}{2h_1(t)}\left(w_1 - \frac{i(\gamma_0 - q)}{2w_0^2}h_1(t)\right)x^2\right][e_q^t]^{-i\frac{w_0^2}{w_1}(n+1/2)} \tag{86}$$

VI. CONCLUSION

In this paper we extended the so-called dual or mirror image formalism for damped harmonic oscillator to the case that both frictional coefficient and time-dependent frequency depend on time explicitly. We also established the relation between Bateman's and Caldirola's-Kanai's form for the damped oscillator problem with time-dependent frictional coefficient and time-dependent frequency. As an solvable example, we considered the case that frictional coefficient $\gamma(t) = \frac{\gamma_0}{1+qt}, (q > 0)$ and angular frequency function $w(t) = \frac{w_0}{1+qt}$ which is decreasing with time. For this choice, we constructed the quantum harmonic Hamiltonian and expressed it in terms of $su(2)$ algebra generators. Using the exact invariant for the Hamiltonian and its unitary transform, we constructed the Fock representation for the invariant. Finally, we obtained the time-dependent wave function for the Hamiltonian with time-dependent frictional coefficient and time-dependent frequency.

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