Special Relativity for Beginners Part IV

(Derivation of the Relativistic Momentum as a Function of the "Contracted" Length)

The purpose of this paper is to derive the formula for the momentum of a body (or particle) as a function of its "contracted" length. The paper also shows that the Fitzgerald-Lorentz contraction is a real effect, and therefore, not an illusion.

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1. The Formula for the Momentum of a body as a Function of the Length Ratio, l_o / l

I shall begin the derivation from Einstein's formula of equivalence between mass and energy:

$$E = m c^2 \tag{1.1}$$

I shall also use equation (1.4) from paper [1]

$$m\, l = m_0 l_0 \tag{1.2}$$

Multiplying both sides of equation (1.2) by c^2 yields

$$m\, l\, c^2 = m_0 l_0 c^2 \tag{1.3}$$

In virtue of equation (1.1), the first side of equation (1.3) may be written as follows

$$E\, l = m_0 l_0 c^2 \tag{1.4}$$

Now let us consider Einstein's total relativistic energy formula

$$E^2 = p^2 c^2 + \left(m_0 c^2\right)^2 \tag{1.5}$$

We take the square root on both sides of equation (1.5) to eliminate the square on the first side

$$E = \sqrt{p^2 c^2 + \left(m_0 c^2\right)^2} \tag{1.6}$$

Multiplying by the "contracted" length, *l*, we get

$$E\,l = l\,\sqrt{p^2 c^2 + \left(m_0 c^2\right)^2} \tag{1.7}$$

Because the first side of equations (1.4) and (1.7) are identical, the second sides must also be identical. Therefore we write

$$l\,\sqrt{p^2 c^2 + \left(m_0 c^2\right)^2} = m_0\, l_0\, c^2 \tag{1.8}$$

Raising both sides to the power of two we get

$$l^2 \left(p^2 c^2 + m_0^2 c^4\right) = m_0^2\, l_0^2\, c^4 \tag{1.9}$$

Solving this equation for the momentum, *p*, of the body, yields

$$p = \pm m_0 c\, \sqrt{\left(\frac{l_0}{l}\right)^2 - 1} \tag{1.10}$$

or, put it into words

> The relativistic momentum, *p*, of a body (or particle) is proportional to the square root of: $\left(\frac{l_0}{l}\right)^2 - 1$.

2. Is the Fitzgerald-Lorentz Length Contraction an Illusion?

Equation (1.10/1.12) suggests that the Fitzgerald-Lorentz length contraction formula (which coincides with Einstein's Special Relativity's length contraction formula)

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \tag{2.1}$$

is not an illusion but a real effect. How do we know that the length contraction formula produces a real length contraction on a body whose length is measured by an observer in uniform relative motion with respect to the body? If the effect of the formula were an illusion, then, formula (1.10) should also be an illusion. However, formula (1.10) seems to describe the momentum of a body perfectly well. We may easily show that equation (1.10) is correct by rewriting it as follows

$$p = \pm m_0 c \sqrt{\left(\frac{l_0}{l_0 \sqrt{1 - \frac{v^2}{c^2}}} \right)^2 - 1} \tag{2.2}$$

After simplification we get

$$p = \pm m_0 c \sqrt{\left(\frac{1}{1 - \frac{v^2}{c^2}} \right) - 1} \tag{2.3}$$

Next, we operate algebraically as follows

$$p = \pm m_0 c \sqrt{\frac{1 - 1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = \pm m_0 c \sqrt{\frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = \pm m_0 c \frac{v}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.4}$$

Which finally yields

$$p = \pm \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.5}$$

Where we have kept the minus sign for generality reasons (e.g. if you were interested in antiparticles). This formula is usually written as

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.6}$$

or, equivalently

$$p = m v \tag{2.7}$$

Therefore we draw the conclusion that the Fitzgerald-Lorentz length contraction formula is not an illusion.

2. Conclusions

In summary, the Fitzgerald-Lorentz length contraction formula is a real effect measured by an observer in uniform relative motion with respect to a given body or particle. Therefore the relativistic length contraction is not an illusion.

Appendix 1
Nomenclature

The following are the symbols used in this paper

$c =$ speed of light in vacuum
$v =$ speed of a body or particle of mass m
$m_0 =$ rest mass of a body or particle
$m =$ relativistic mass of a body or particle
$l_0 =$ proper length of a body or particle
$l =$ "contracted" (or relativistic) length of a body or particle
$p =$ momentum of a body or particle
$E =$ total relativistic energy (or simply relativistic energy) of a body or particle

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FURTHER READING

(1) A. Einstein, *On the Electrodynamics of Moving Bodies*, Annalen der Physik, (1905).
(2) R. A. Frino, *Special Relativity for Beginners – Part I*, vixra.org: vixra 1506.0161, (2015).
(3) R. A. Frino, *Special Relativity for Beginners – Part II*, vixra.org: vixra 1507.0107, (2015).
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REFERENCES

[1] R. A. Frino, *Special Relativity for Beginners – Part III*, vixra.org: vixra 1509.xxxx, (2015).