Non-relativistic quantum mechanics and classical mechanics as special cases of the same theory.

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Abstract

We start by rewriting classical mechanics in a quantum mechanical fashion and point out that the only difference with quantum theory resides at one point. There is a classical analogon of the collapse of the wavefunction and an extension of the usual Born rule is proposed which might solve this problem. We work only with algebra's over the real (complex) numbers, general number fields of finite characteristic allowing for finite dimensional representations of the commutation relations are not considered given that such fields are not well ordered and do not give rise to a well defined probability interpretation. Our theory generalizes however to discrete spacetimes and finite dimensional algebra's. Looking at physics this way, spacetime itself distinguishes itself algebraically by means of well chosen commutation relations and there is further nothing special about it meaning it has also a particle interpretation just like any other dynamical variable. Likewise, there is no reason for the dynamics to be Hamiltonian and therefore we have a nonconservative formulation of quantum physics at hand. The harmonic oscillator (amongst few others) distinguishes itself because the algebra forms a finite dimensional Lie algebra; the classical and quantum (discrete) harmonic oscillator are studied in a some more generality and examples are given which are neither classical, nor quantum.

1 Introduction

Quantum gravity is a longstanding problem concering the unification of two theories, both with their own problems. In quantum physics, one has the mysterious nonlocal collapse of the wavefunction with time and space playing very distinct roles while in gravitational phyiscs there is the complete lack of local observables -even classically- if one doesn't consider point particles or more general rigid objects. This is a serious problem since relativistic quantum theory has told us that fields are the way to go and point particles should be banned from the quantum world. Logically, there seem to be only three ways out; or one knows how to do physics without coordinates at all and reinstates "local" observables *or* one changes gravity and makes coordinates dynamical, so that Einsteins theory was just an accidental consideration due to the assumption of an infinitely rigid spacetime *or* one is happy with both theories as they stand

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and one resorts to a semiclassical theory with rigid objects representing our measurement apparati. Of course, the third option neglects the simple fact that observers are also made out of atoms and therefore should be subject to the laws of quantum mechanics. Concerning the first option, the author knows of Causal Set theory and the theory of Causal Dynamical Triangulations, both of which are still plagued by the absence of local observables though. In this paper, we shall take the second option and consider events to be particles relative to conventional matter but particles which should have finite mass instead of infinite one. This is the relational viewpoint, matter is defined with respect to coordinates and coordinates are defined with respect to matter so that the distinction between them must be a very subtle one and not so grand as it is now with a rigid spacetime framework. This will make sense when the reader arrives to section seven, this means of course that general covariance in the sense of Einstein has to be given up which is not too harsh since general covariance only seems to make sense for commutative coordinates. In this paper, we try to reach this goal by writing coordinates and particles in the same language; this language must be quantum mechanical since particles obey the laws of quantum physics. Hence, we start by writing classical physics of point particles in the languange of operator algebra; first we start with first order differential equations and later on we go over to second order differential equations. The operator language is that flexible that it unifies the description in continuous time as well as discrete time and in section three we develop a consistent quantum theory in discrete time which is nonunitary but nevertheless preserves the Heisenberg commutation relations. Our framework also incoorporates a nonconservative extension of quantum theory or quantum theory without a Hamiltonian if one wants to and this opens the possibility for a time dependent Planck constant as the reader may discover in section four. In section five, we treat the quantum mechanical oscillator in discrete time which is of considerable more difficulty than the continuous case while in section six, scalar field theory on a Minkowski background is written as a covariant theory in our operator language. Section seven then contains the introduction to a "novel" view on quantum gravity. We now turn to section two which contains a formal introduction to the idea to turn physics into linear algebra.

2 An introduction of the formalism.

It is of considerable interest to release physics of its spacetime background or to write classical physics in a quantum mechanical language or to free quantum mechanics from its classical limit in its very formulation and hence get a true quantum theory. Likewise, classical physics is much broader than just Lagrangian or Hamiltonian physics and I have always wonderded why quantum theory should not be extendible to those dynamics. In this section, we will adress all these issues in a very simple way, by reformulating physics as just algebra and retrieving it by looking at the standard (Schrodinger) representations. Classical physics has a collapse of the wavefunction rule just like quantum mechanics has, it is just so that the wave function is rather trivial in the former case. To get a taste of what we are going to do, let us consider a simple one dimensional dynamical system as

$$\dot{x} = f(x).$$

This equation contains time, a time derivative and a variable x; usually we think of these as a variable, an operator and a dependent variable respectively. What I propose is that we should think of all of them in the same way, as operators acting on a (rigged) Hilbert space. That is, we promote them to T, D and X respectively and write out the commutator algebra between them. Obviously $[D,T] = i1$, $[D,X] = if(X)$ and $[X,T] = 0$ where we think of D as $D = i\frac{d}{dt}$. One could now go and look for infinite dimensional complex representations of this algebra where all generators are replaced by Hermitian operators. I would however insist that the derivative operator D is not Hermitian however because for discrete systems the right or left difference operator is not anti-Hermitian. What one does classically, is first to choose D, T and then look for X in the commutative algebra generated by $T, 1$. This formulation is already more general, one could indeed consider $T = 1 \otimes t$ and $D = 1 \otimes i\frac{d}{dt}$ acting on some tensor product Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_t$ where $\mathcal{H}_t$ equals $L^2(\mathbb{R}, dt)$ but contrary to the previous classical example $\mathcal{H}_s$ is not equal to $\mathbb{C}$ but can be a more general (possibly finite dimensional) Hilbert space and $X = \sum_i a^i \otimes f^i(t)$ where the $f^i(t)$ are differentiable functions of t which are to be interpreted as multiplication operators. As the reader immediately notices, the algebra itself is very special with the triple $D, T, 1$ forming a non semi-simple Lie subalgebra which is not invariant though by means of $[D, X]$. Likewise, X, T forms an abelian Lie subalgebra which is not invariant by means of D and finally one has that the possibly infinite dimensional subalgebra generated by D and X is seriously reducible by means of $[D,X] = if(X)$ but this last statement is not too special as one can replace $f(x)$ by $f(x,t)$. One therefore could now think about generalizing first order differential equations by changing this algebra. Notice that the function $f(x,t) = ax + bt + c$ plays a very special role in the above formalism in the sense that the whole dynamical system reduces to a Lie algebra. There is something funny about first order systems providing $f(x)$ does not have any zeroes, since then one can make a change of variables $y = g(x)$ such that the algebra for y reads $[D,Y] = i1, [Y,T] = 0$ so that there is no distinction between T and Y algebraically. For second order systems X will get its own derivative operator in quantum mechanics, namely $\frac{1}{\hbar}P$ where P is the conjugate momentum and the dynamics consists in the commutation relations between two derivations. There is a distinction between both derivations however in the sense that the time derivative does not appear in the right hand side while the momentum does. That is again a way for spacetime to appear out of algebra. Writing out the algebra for the action $I = \int f(x)(\dot{x})^2 + h(x)dt$ (a term $g(x)\dot{x}$ can be ignored since it is a total derivative) gives:

$$[D,T] = i1 \tag{1}$$

$$[D,X] = \frac{i}{2}\{F(X), P\} \tag{2}$$

$$[X,T] = 0 \tag{3}$$

$$[P,X] = i\hbar 1 \tag{4}$$

$$[P,T] = 0 \tag{5}$$

$$[D,P] = -\frac{i}{4}\{F'(X), P^2\} - iQ'(X) \tag{6}$$

where we have used the anti-commutator $\{A, B\} = AB + BA$. As the reader notices, one can introduce the so called Hamiltonian operator

$$H(X, P) = \frac{1}{4}\{F(X), P^2\} + Q(X)$$

such that in case $\hbar \neq 0$,

$$[D, X] = -\frac{1}{\hbar}[X, H], [D, P] = -\frac{1}{\hbar}[P, H]$$

which is why the function F appeared in the way it did. Notice how thight this formulation is, as the reader may easily show, the integrability condition for the commutator relation $[P, X] = i\hbar 1$

$$[D, [P, X]] = 0$$

given $[D, X] = iF(X, P)$ and $[D, P] = iG(X, P)$ is precisely equivalent to the existence of a conserved Hamiltonian operator $H(X, P)$. Therefore, if we move away from Hamiltonian systems we will have to change that commutation relation. In case $\hbar = 0$ one would have to enlarge the algebra with a derivative operator for X in order to define the Poisson bracket. This indicates the above formulation is a much better one and shows us also how to generalize away from Hamiltonian systems. As said before, P^2 appears in the commutator of $[D, P]$ but not D! This is how D distinguishes itself from P and since T commutes with everything apart from D, spacetime singles itself out algebraically in this way. Again, we have a finite dimensional Lie algebra if and only if F is a constant function and $Q(x) = ax^2 + bx + c$ which means we must study the harmonic oscillator. So far, we must conclude that the only way in which quantum mechanics differs from classical physics resides in one commutator which is the only place where $\hbar$ shows up. Indeed, as we shall see now, also the collapse of the wavefunction holds trivially in classical physics. The idea is that we look for representations on (rigged) Hilbert space of the above algebra (as unbounded operators of course), classically as well as quantum mechanically this representation would be of the tensor product type $\mathcal{H}_s \otimes \mathcal{H}_t$ and since the Heisenberg relations require a continuum of eigenvalues for the T operator[1], we identify $T = 1 \otimes t$ and $D = 1 \otimes i\frac{d}{dt}$ on $\mathcal{H}_t = L^2(\mathbb{R}, dt)$. Now, physical states are defined to be timeless which, in the continuum, we could formulate as $D\psi = 0$ however, this formulation is somewhat inconvenient in the discrete case as there the D operator is invertible. Denote therefore by $|t\rangle$ the distributional eigenstates of the t operator, then we can write down the constant state formally as $|1\rangle = \int |t\rangle dt$[2]. Therefore, we consider only states of the form $\psi \otimes |1\rangle$ where $\psi \in \mathcal{H}_s$. Hence, in classical physics where $\mathcal{H}_s = \mathbb{C}$, $\psi = e^{i\theta}$ which is trivial. Of course, one can generalize away from standard classical physics by choosing $\mathcal{H}_s$ to be nontrivial. In that case one requires also a collapse of the wavefunction. The reader notices that the standard Heisenberg rule comes from defining the operators X_t on $\mathcal{H}_s$ as follows

$$X(\psi \otimes |1\rangle) = \int dt (X_t \psi) \otimes |t\rangle.$$

[1] We shall propose modifications of the Heisenberg algebra with an invariant length later on as to incoorporate discrete dynamical systems.

[2] $D|1\rangle = 0$ in the continuum.

This is how one retrieves the standard picture, for representations different from a tensor product however, one should find out a more generalized Born rule and this is left for future work.

The remainder of this paper consists in making "discrete modifications" to the above algebra and studying the harmonic oscillator in more generality as well classically as quantum mechanically, discrete and continuous. Let us begin by making a remark for first order dynamical systems with nontrivial $\mathcal{H}_s$, the most easy system being $\dot{x} = -\alpha x$ there it is easy to see that the only solution is of the form $X(t) = M \otimes e^{-\alpha t}$ with M any hermitian matrix. It can be easily shown, quite generally, that the most general situation for *any* first order differential equation $\dot{x} = f(x)$ is one that reduces to a solution of the form $X(t) = UD(t)U^{\dagger}$ where $D(t)$ is a diagonal matrix with real solutions on the diagonal and U is a constant unitary matrix on $\mathcal{H}_s$. Therefore, to get quantum effects, one needs to go to second order and study if classically anything interesting might happen. We proceed now with defining a discrete operator algebra and let ourselves be guided by the prototypical examples of a multiplication operator and a derivative operator. For example

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and the right(or forward) derivative operator

$$D = i \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

which is invertible and satisfies the commutation relation

$$[D, T] = i1 + D.$$

Introducing a fundamental time $a > 0$, this generalizes to

$$[D, T] = i1 + aD.$$

One easily remarks for first order systems that $[D, X] = if(X)$ cannot be sought for in the commutative algebra generated by $T, 1$; so either $[X, T] = 0$ cannot hold or the equations of motion need to be generalized. It is sufficient though for X to belong to the algebra generated by $D, T, 1$ where every element can be written in the finite form $\sum_j f_j(T)(D + \frac{i}{a}1)^j$ due to the commutation relations (and finite because of Cayley Hamilton). However D cannot be Hermitian and therefore X is not Hermitian too (it would just be upper triangular in our above example). This doesn't give any interpretational difficulties as long as we would agree that that we are only interested in $f_0(T)$ which is a sensible viewpoint[3]. Note also upfront that not any f is allowed for, for example $f(x) = 1$ has no solutions which is a pretty bad situation. Therefore, I propose that the correct axiomatic system has to leave the commutation relations partially open in the following sense; in order not to depart too much from the continuum situation

[3]Notice that $f_0(T)$ is uniquely determined for the representation above, the $f_i(T)$ with $i \geq 1$ contains free parameters though.

we must uphold that $[X,T]=0$ and substitute $[D,X]=if(X)+g(X,T)D$ for *some* function $g(X,T)$ which must be interpreted as pure gauge. That is, we do not need to specify g upfront, the algebra is fully specified by a and f, it is just so that g will show up. The other viewpoint which has not the usual continuum limit (but is a general differential operator) would be to consider the commutation relation between $[X,T]=g(X,T,D)$ as pure gauge and insist upon $[D,X]=if(X)$ with the drawback that not all f are allowed for. Actually, it is possible to give an explicit formula for the gauge function $g(X,T)$, it is just $af(X)$. This follows from the beautiful formula

$$[D,X(T,1)]=\frac{i}{a}\left(e^{a\frac{d}{dT}}-1\right)X(T,1)+\left(e^{a\frac{d}{dT}}-1\right)X(T,1)D$$

as the reader may prove by induction on the monomials; $\frac{d}{dT}$ is just the standard derivative defined on a commutative variable T by $\frac{d}{dT}T=1$ and the Leibniz rule. To summarize, our proposal therefore is

$$\begin{aligned}[D,T] &= i1+aD &(7)\\ [D,X] &= if(X)+af(X)D &(8)\\ [X,T] &= 0 &(9)\end{aligned}$$

Just to avoid trivial cases such as $D=-\frac{i}{a}1$ let us agree that if T is of rank n in finite dimensional representations, then $[D,T]$ must be of rank $n-1$. Let us illustrate this algebra by means of the generalized exponential function defined by $[D,X]=iX+aXD$ and $[X,T]=0$ assuming that T has no degenerate eigenvalues. The latter just equals the ordinary exponential function $X(T,1)=\beta e^{\alpha T}$ since

$$\frac{1}{a}\left(e^{a\frac{d}{dT}}-1\right)e^{\alpha T}=\frac{1}{a}\left(e^{a\alpha}-1\right)e^{\alpha T}=e^{\alpha T}$$

and the last equality holds if and only if

$$\alpha=\frac{\ln(a+1)}{a}$$

which could be seen as the discrete correction to the unity. We must prove uniqueness and existence of solutions to the above equations for X in the commutative algebra spanned by $1,T$; that is, there exists only a one parameter family of solutions. This is most easily proven when T has no degenerate eigenvalues and D is the standard right derivative, a more general proof is lacking at this moment. On the geometrical side, one should notice that vectorfields of the form $f(T)D$ do not close anymore to a Lie algebra since the commutator gets D^2 corrections. This is the algebraic price to pay for the introduction of a fundamental time scale and the abandonment of locality in time. We now proceed to the formulation of discrete second order systems and shall notice that more broad possibilities arise here.

3 Discrete quantum mechanics.

It is far from trivial to formulate an algebra for discrete quantum mechanics which closes. Let me notice from the outset that it is impossible to have $[P,X]=$

$i\hbar 1$ and a dynamics which can be derived from a conserved Hamiltonian $H(X,P)$ without allowing for X,P to depend upon D ! This is most easily understood by the fact that $[D, Z(T,1)] = iF(T,1) + aF(T,1)D$. Hence, one must abandon upon $[X,T] = [P,T] = 0$ for such scheme to work out. Hence, since D is not Hermitian, neither are X,P but as discussed before, it is covenient to expand $X = \sum f_j(T)(D + \frac{i}{a}1)^j$ and consider only $f_0(T)$. As mentioned before, such line of reasoning produces the wrong continuum limit and it is better to apply the following trick of doublage; that is define

$$\mathcal{H}(X,P,D) = H(X,P) - aiH(X,P)D$$

then the equation

$$[D,X] = -\frac{1}{\hbar}[X,\mathcal{H}]$$

is equivalent to

$$\left(1 + \frac{ia}{\hbar}H(X,P)\right)[D,X] = -\frac{1}{\hbar}[X,H] + \frac{ai}{\hbar}[X,H]\,D$$

and the right hand side is of the form $iF(T) + aF(T)D$ if X,P are functions of $T,1$ only. The same remark holds for $[D,P]$ so both these commutators are defined *implicitely* in the discrete case. Moreover, we have that X,P will become slightly non Hermitian (depending on a) as $\left(1 + \frac{ia}{\hbar}H(X,P)\right)$ is non Hermitian. Since our system is Hamiltonian, the integrability of $[P,X] = i\hbar 1$ is assured as can be seen as follows

$$[D,[P,X]] = \frac{1}{\hbar}[\mathcal{H},[P,X]] = \frac{1}{\hbar}[H(X,P) - iaH(X,P)D,[P,X]]$$

and the last expression is since $[H(X,P),[P,X]] = 0$ equivalent to

$$-\frac{ia}{\hbar}\left(H(X,P)D\,[P,X] - [P,X]\,H(X,P)D\right) = -\frac{ia}{\hbar}\left(H(X,P)\,[D,[P,X]] + H(X,P)\,[P,X]\,D - i\hbar H(X,P)D\right)$$

which results in

$$\left(1 + \frac{ia}{\hbar}H(X,P)\right)[D,[P,X]] = 0.$$

This last equation implies in general that

$$[D,[P,X]] = 0$$

which is what we needed to prove. Therefore, our proposal for a quantum mechanics which is discrete in time reads

$$[D,T] = i1 + aD \tag{10}$$
$$[D,X] = -\frac{1}{\hbar}[X,\mathcal{H}] \tag{11}$$
$$[P,T] = 0 \tag{12}$$
$$[D,P] = -\frac{1}{\hbar}[P,\mathcal{H}] \tag{13}$$
$$[X,T] = 0 \tag{14}$$
$$[P,X] = i\hbar 1 \tag{15}$$

where, as before, $\mathcal{H}(X,P,D) = H(X,P) - aiH(X,P)D$. Let us calculate discrete corrections to the free particle $H(X,P) = \frac{P^2}{2m}$, the harmonic oscillator is postponed to section five. The reader may easily verify that

$$X = \frac{M}{m + \frac{iaM^2}{2\hbar}} \otimes T + N \otimes 1$$

and

$$P = M \otimes 1$$

with M, N selfadjoint and satisfying

$$[M, N] = i\hbar 1.$$

What concerns a suitable probability interpretation, one might simply put that only $\frac{X+X^\dagger}{2}$ is observed; also, this teaches us that the evolution is not unitary even if the system is of Hamiltonian nature. One can propose a further generalization by making X discrete as well, that is by putting $[P, X] = i\hbar 1 + bP$ with b a fundamental length scale. We leave this open for future work.

4 An example of a nonconservative quantum theory.

Let us write down an algebra which closes and such that no Hamiltonian $H(X, P, T)$ exists:

$$\begin{aligned}
[D,T] &= i1 && (16)\\
[D,X] &= i(P+\alpha X) && (17)\\
[P,T] &= 0 && (18)\\
[D,P] &= i\beta X && (19)\\
[X,T] &= 0 && (20)\\
[P,X] &= i\hbar e^{\alpha T}. && (21)
\end{aligned}$$

As the reader may easily verify the last commutation relation is mandatory since

$$[D, [P, X]] = i\alpha\, [P, X]$$

and $i\hbar e^{\alpha T}$ is the only solution. Hence, this is a theory with a time dependent quantum constant; if $\alpha < 0$ the universe will become more classical as $t \to \infty$ and was more quantum in the past. We shall now integrate these equations and construct explicit solutions. In the standard Schroedinger representation, one can derive that

$$\partial_t^2 X - \beta X - \alpha \partial_t X = 0$$

which can be solved in full generality to

$$X = M \otimes e^{\kappa_+ t} + N \otimes e^{\kappa_- t}$$

with M, N general Hermitian operators and $\kappa_\pm = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2}$ where we assume that $\beta > -\frac{\alpha^2}{4}$. Hence

$$P = -\kappa_- M \otimes e^{\kappa_+ t} - \kappa_+ N \otimes e^{\kappa_- t}$$

and therefore the commutator equals

$$[P,X] = (\kappa_+ - \kappa_-)\,[M,N] \otimes e^{\alpha t}$$

which imposes that

$$[M,N] = \frac{i\hbar}{\sqrt{\alpha^2+4\beta}}1$$

which is a standard quantum mechanical commutation relation with a new Planck constant $\frac{\hbar}{\sqrt{\alpha^2+4\beta}}$. The evolution of the operators is not unitary as the commutation relation $[P,X] = i\hbar e^{\alpha T}$ reveals. Hence, we have a nontrivial superposition principle and this example is therefore as quantum mechanical as standard Hamiltonian quantum mechanics. Note that in case $\beta < -\frac{\alpha^2}{4}$ we get out a modified harmonic oscillator where

$$X = M \otimes e^{\kappa_+ t} + M^\dagger \otimes e^{\kappa_- t}$$

with $\kappa_\pm = \frac{\alpha \pm i\sqrt{-4\beta-\alpha^2}}{2}$ and M satisfies

$$\left[M, M^\dagger\right] = \frac{\hbar}{\sqrt{-4\beta-\alpha^2}}1$$

which is the algebra of bosonic creation and annihilation operators. In the critical case $\beta = -\frac{\alpha^2}{4}$,

$$X = M \otimes e^{\frac{\alpha t}{2}} + N \otimes t e^{\frac{\alpha t}{2}}$$

with N, M Hermitian and satisfying

$$[N,M] = i\hbar 1.$$

The reader may notice that our algebra remains otherwise the same if we replace $[D,P] = i\beta X$ by $[D,P] = iQ(X)$ for any function Q and therefore we can incorporate a wide class of nonlinear systems. These can of course not be explicitely integrated, which is the advantage of time independent Hamiltonian systems. The clever reader must notice that a change of variables $P' = Pe^{-\alpha T}$ transforms our system into one with a time dependent Hamiltonian, therefore the X, P' satisfy quantum theory and a undergo a unitary evolution which cannot be explicitely integrated since the Hamiltonian is time dependent (however a path integral formulation for X, P' is possible). This is well known for one dimensional classical dynamical systems where a change of variables which explicitely depends upon time can turn a system without a (conserved) energy into one with a conserved enery. For example, consider $\ddot{x} - \alpha\dot{x} + \beta x = 0$ and make the transformation of variable to $y = xe^{-\frac{\alpha t}{2}}$ then y has a conserved energy but x doesn't. The same comment holds of course if we would perform the change of variables $X' = Xe^{-\frac{\alpha}{2}T}$ and $P' = Pe^{-\frac{\alpha}{2}T}$ so that it are now X', P' which have a conserved Hamiltonian

$$H = \frac{P'^2}{2} + \frac{\alpha}{4}\{X', P'\} - \frac{\beta X'2}{2}$$

and therefore undergo standard unitary evolution. As the reader may explicity check for our solutions with $\beta > -\frac{\alpha^2}{4}$ the Hamiltonian is

$$H = -\frac{\alpha^2+4\beta}{4}\{M,N\}$$

and with $U(t) = e^{-i\frac{Ht}{\hbar}}$ one can check that

$$X(t) = e^{\frac{\alpha t}{2}} U(t) \, (M \otimes 1 + N \otimes 1) \, U(t)^{\dagger}$$

and likewise for $P(t)$. So, the question we must ask here is which are the correct physical variables[4]? What we do of course is to make our measure stick time dependent when making such change of variables. Obviously, both give different experimental predictions and are therefore physically inequivalent. Therefore we must stress that physics is not just in the algebra, we also need to specify a basic set of physical observables, things we can really measure. In our formalism we can measure T and X, but TX is forbidden, hence only the chosen set of generators can be measured and no composite operators which implies the Hamiltonian is not an observable. Indeed, we would just not know which is the correct one if we allow for observables to mix. T and X are expressed with respect to classical measure sticks and the standard idea behind quantum gravity is to turn the measure sticks into observables so that one can, as to speak, measure the measure stick (at least if one takes the vielbein formulation of gravity) which is a rather nonsensical idea. The problem is that one doesn't write physics anymore in terms of observables but in terms of equivalence classes of hidden variables. On a Minkowski background spacetime one can get away with this since fields have a definite meaning with respect to a spacetime measure stick (or class of inertial observers) and Lorentz transformations simply transform one measure stick into another and the fields transform covariantly too. However, in general relativity, the measure stick is subject to its own dynamics depending on quantities which are measured relatively to it[5]. Classically, this doesn't pose a problem yet as measurements are trivial but quantum mechanically it does. One sees a tower of relationships here, points and derivative operators just are, the measure stick is defined relationally to points and and Dirac fields are defined with respect to the measure stick and the points, scalar fields and gauge fields do refer directly to the points and derivative operator. Diffeomorphism invariance of the dynamics makes one speak in terms of equivalence classes of all these objects so that their local meaning gets lost. Still, one cannot formulate physics based upon the equivalence classes, since the dynamics identifies certain representants and not the equivalence classes themselves. What I propose in the above is that these relationships are implicit in terms of algebra and not explicit. I hoped to present a system which cannot be transformed to a Hamiltonian system in this way but it seems we need more variables for this.

5 The discrete harmonic oscillator.

In this section we study the quantum system given by the Hamiltonian $H(X, P) = \frac{P^2}{2m} + \omega \frac{X^2}{2}$ in discrete time. Unfortunately, this cannot be expliciteley integrated as the inverse of $1 + \frac{ai}{\hbar} H$ cannot be written in perturbation series as the Hamiltonian H is generally unbounded. Therefore, we resort immediately to a representation of the algebra in which we can take the inverse and we make use of

[4]There is no contradiction here with the previous statement that X undergoes unitary evolution with a time dependent Hamiltonian since the spectrum of X and aX are the same. This can easily be shown for the multiplication operator x on $L^2(\mathbb{R}, dx)$ which is unitarily equivalent to ax for any $a > 0$.

[5]For example, a Dirac field.

the convenient formula

$$\left[F^{-1}, G\right] = -F^{-1}\left[F, G\right]F^{-1}$$

when evaluating commutators. Therefore, take the rescaled three time timestep T and D of page four, that is

$$T = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$D = \frac{i}{a}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

and write out $X = x \otimes P_1 + A(x, \partial_x) \otimes P_2 + B(x, \partial_x) \otimes P_3$ and $P = i\hbar\partial_x \otimes P_1 + C(x, \partial_x) \otimes P_2 + D(x, \partial_x) \otimes P_3$ where P_i is the projection operator on the vector with a 1 on the i'th place and zeroes otherwise. We need to solve for A, B, C, D and check whether the commutation relations

$$[C, A] = [D, B] = i\hbar 1$$

hold. The operator $1 + \frac{ai}{\hbar}H$ is given by

$$1+\frac{ai}{\hbar}H = \left(1 - \frac{ia\hbar}{2m}\partial_x^2 + \frac{ia\omega}{2\hbar}x^2\right)\otimes P_1 + \left(1 + \frac{ia}{2m\hbar}C^2 + \frac{ia\omega}{2\hbar}A^2\right)\otimes P_2 + \left(1 + \frac{ia}{2m\hbar}D^2 + \frac{ia\omega}{2\hbar}B^2\right)\otimes P_3$$

and we will use the shorthand $Z_i \otimes P_i$ for this. The equations of motion are given by

$$\begin{aligned} Z_1(A - x) &= \frac{ai\hbar}{m}\partial_x && (22) \\ Z_2(B - A) &= \frac{aC}{m} && (23) \\ Z_1(C - i\hbar\partial_x) &= -a\omega x && (24) \\ Z_2(D - C) &= -a\omega A && (25) \end{aligned}$$

as the reader may verify for himself. These equations can be readily solved to

$$\begin{aligned} A &= x + \frac{ai\hbar}{m}Z_1^{-1}\partial_x && (26) \\ C &= i\hbar\partial_x - Z_1^{-1}a\omega x && (27) \\ B &= A + \frac{a}{m}Z_2^{-1}C && (28) \\ D &= C - Z_2^{-1}a\omega A && (29) \end{aligned}$$

and it is obvious that B, D satisfy the commutation relations if A, C do. These are most easily verified by using that

$$\begin{aligned} \left[\partial_x, Z_1^{-1}\right] &= -Z_1^{-1}\frac{i\omega a}{\hbar}xZ_1^{-1} && (30) \\ \left[x, Z_1^{-1}\right] &= -Z_1^{-1}\frac{i\hbar a}{m}\partial_x Z_1^{-1} && (31) \end{aligned}$$

using the formula above. Hence, our formalism is fully consistent and can be useful for more complex systems. It is obvious how to generalize this framework to further time steps but this needs to be done on a computer and not by hand. The attentive reader notices that the operators X_i become totally nonlocal in x space making it very hard to compute eigenvalues and eigenspaces; this is of course also the case for interacting theories in the continuum and smart approximation methods need to be found.

6 Minkowski field theory in manifestly covariant form.

The intention of this section is to show to the reader that ordinary field theory on Minkowski is generally covariant if one works in the vielbein formulation and the quantum mechanical distinction between space and time doesn't pose any problems. There is no issue of background independence here since spacetime is written in the same algebraic language as the fields are. Actually, spacetime totally dissapears from this algebraic formulation as the reader will understand. This opens the door for making distinct covariant theories of gravity the algebraic way. Specifically consider the differential operators $D_a = e^{\mu}_{a}\partial_{\mu}$ where e^{μ}_{a} is the Minkowski vielbein. The Minkowski vielbein distinguishes itself from all other vielbeins in the sense that one can find solutions to the system $D_a f_b = \delta_{ab}$ where the f_b are functions on spacetime which we call the vielbein coordinates. They are all uniquely determined up to a constant and hence define a physical coordinate system with respect to which our fields will be defined. In our algebraic language, one has

$$\left[D_a, F^b\right] = \delta^b_a,\ \left[F^a, F^b\right] = 0,\ [D_a, D_b] = 0$$

and those relationships single out flat spacetime (the flat metric is of course input). Changing them, brings you into the realm of distinct curved spacetimes. The Klein Gordon field Ψ then satisfies the equation

$$\eta^{ab}\,[D_a, \Psi_b] - m^2\Psi = 0$$

where classically $[\Psi, F^a] = 0$, $[D_b, \Psi] = \Psi_b$ and $[\Psi_b, F^a] = 0$ and we will now come to the commutation relations. These are *not* in the algebra, but in the *representation* of the operators F^a. That is, suppose again that we separate the F^a, D_b from the rest and go over to a tensor product construction $\mathcal{H}_s \otimes \mathcal{H}_{st}$ where $\mathcal{H}_s$ is the system Hilbert space and $\mathcal{H}_{st}$ is the spacetime Hilbert space, then we can simultaniously diagonalize all F^a and write eigenvectors $|x^b\rangle$ spanning all of $\mathcal{H}_{st}$ and satisfying $F^a|x^b\rangle = x^a|x^b\rangle$. Then, as was the case for time in nonrelativistic physics, we can define the unit vector in $\mathcal{H}_{st}$ by $|1\rangle = \int d^4x|x^b\rangle$ and the set of operators Ψ_{x^a} by

$$\Psi\phi \otimes |1\rangle = \int d^4x(\Psi_x\phi) \otimes |x^a\rangle.$$

Likewise, one has that $[D_b, \Psi]\,\phi \otimes |1\rangle$ equals

$$((\Psi_b)_x\phi) \otimes |1\rangle$$

with $(\Psi_b)_x = \partial_b \Psi_x$. Hence, the causality statement becomes that

$$\left[\Psi_{(t,\vec{x})}, \Psi_{(t,\vec{y})}\right] = 0$$

and

$$\left[\Psi_{(t,\vec{x})}, \partial_t \Psi_{(t,\vec{y})}\right] = -i\hbar \delta^3(\vec{x} - \vec{y}).$$

So somehow, these commutation relations form a substitute for the Lie brackets $[\Psi, \Psi_a]$ which are not directly specified in our theory. For such formulation to be consistent, one must of course show that nontrivial solutions satisfying all these requirements exist and that Lorentz covariance is well implemented. One notices that partial differential equations leave the algebra partially open in the sense that we do not specify the $[D_a, \Psi_b]$ separately, this results of course in the choice of the initial values. As mentioned, this formulation is fully covariant and spacetime is fully gone from these equations (we do not make reference anymore to the spacetime dependence of the derivative operators and physical coordinates). As I said, this may serve as a guideline for making the operators D_a also "dynamical", ie. to give them nontrivial commutation relations so as to recover curved spacetime. Again, the reader notices that the only distinction between classical physics and quantum physics resides in the single appearance of $\hbar$ in the last commutation relation. One can now go and think about finding generalizations of this framework just as we did for nonrelativistic physics.

7 A "novel" view on quantum gravity.

As is well known, the diffeomorphism invariance of relativity makes it hard, if not imposssible, to define local observables and this is even so classically if we ignore the existence of point particles and speak into the language of fields. Also, it makes the Hamiltonian into a constraint and therefore jeopardizes time evolution and the quantum notion of causality. In our formalism, there is a way out by further blurring the algebraic distinction between spacetime and matter: that is, by making spacetime noncommutative and writing out further commutators. Another way to say this would be that spacetime needs to consist out of particles of finite mass. To make sense of this statement, lets rewrite the algebra of a free particle:

$$[D, T] = i1,\ [X, T] = [P, T] = 0,\ [P, X] = i\hbar 1,\ [D, X] = iP,\ [D, P] = 0.$$

Picking out T, D as $t, i\partial_t$ leads to the fact that X, P describe a free particle of mass one. Choosing however X, P as $t, i\hbar\partial_t$ leads to the conclusion that T, D describe a particle of infinite mass with unit Planck constant where $T = x \otimes 1$ and $D = i\partial_x \otimes 1 - \frac{\hbar}{2} 1 \otimes \partial_t^2$. The momentum D is therefore somewhat unconventional in the sense that it depends upon its derivative operator $P = i\hbar\partial_t$ but as the reader may want to figure out, this has no observational consequences (define D_t as before, that is $D\psi \otimes |1\rangle = \int dt (D_t \psi) \otimes |t\rangle$ and one sees that $D_t = i\partial_x$ as anticipated). Commutative spacetime is, if one wants to, a spectator which only feels derivative operators while the fields "feel" (time) derivatives of fields. Such blurring needs however a more general interpretative framework, for a proposal in this direction see [3]. One might in general expect spacetime quantization to *break* the general covariance of Einstein's laws; in fact, I do not know of any nice definition of a covariant tensor in a general noncommutative theory, the very existence of this seems to be tied to very special algebraic properties of the number system such as is the case for superspaces and ordinary commutative manifolds. Therefore, general covariance would dynamically emerge on large scales in such theory but would not be a fundamental property of physcis. It would be, as said in the introduction, an accident due to the assumption of an infinitely rigid spacetime and therefore quantum gravity could restore local observables to spacetime physics. Another, more conventional, way out of this problem would be to do physics without coordinates, but then what we understand about dynamics and quantum theory needs to find a very different formulation. For more on this, see some papers in the reference list.

8 Conclusions

In this paper, we proposed a new way of looking on classical and quantum physics which resulted in the construction of novel quantum theories, either with a time dependent Planck constant or in discrete time or probably both. This novel viewpoint also enabled us to change the role of the dynamical variables with time (and hopefully spacetime) itself: conventional time can be defined as a particle of infinite mass with respect to a conventional particle. Hence, what this extension of quantum theory tells us is that particles are *relative* and not absolute as in Einstein's theory of relativity and this goes appearantly in both ways. This opens the door for a whole new class of physical theories which may reluctantly be regarded upon by the relativity community but on the other hand Einsteins theory just doesn't make any sense when restricted to fields (one could try complicated constructions to characterize an event by studying coordinate invariants and construct timelike vectors of motion as well as localized (not local) rest spaces, but this does not work in general and the definitions are highly non-unique). There might be a way out by resorting to the definition of a particle as microsocopic black hole which is a quasi local object having a physical meaning (one could enumerate them and say that $BH1$ is turning around $BH2$ from the perspective of $BH3$) but such avenues are highly speculative and not worked out to any detail whatsoever though. It remains of course to show that those "novel" theories can indeed give the correct predictions which Einstein's theory does for *point like* objects. An avenue which I did not present is to add hidden variables to quantum field theory and restore point particles once again in spacetime, this would be a terribly nonlocal theory (for those particles) and I am not aware yet that such framework even exists. Certainly Bohmian mechanics is not up to that point yet and I wonder how one will treat states with an indefinite particle number. This would be a way to rescue relativity and conventional quantum mechanics at the same time while keeping a rather conservative viewpoint on spacetime. Since I do not know how such viewpoint can be merged with particle creation and annihilation, I did not mention it in the introduction. Certainly, the laws governing such framework cannot be defined in terms of deterministic differential equations but a stochastic dynamics might do; this would also require a stochastic reformulation of relativity. At any rate, this paper offers novel possibilities for quantum mechanics and relativity to be formulated and I felt I should report this. It would be interesting to generalize this framework to higher derivative theories and see if one gets something sensible out. Classically, such theories are known to produce nonsensical results since the acceleration is an intial value and therefore such universes are completely anti-Newtonian. The standard argument that this will only be visible for particles with high momenta in the quantum world is irrelevant, its classical limit has solutions with anti-Newtonian behavior at low *velocities* but possibly high momenta; this is due to a convoluted definition of the canonical momentum in higher derivative theories (which involves higher time derivatives of the velocities). This has an impact on the asymptotic safety scenario of Weinberg who generously adds an infinite number of higher derivative Lagrangians to the path integral and considers them as an interaction. This is a *generalization* of standard quantum theory and while a Hamiltonian framework certainly exists for such Lagrangian theories, there is no standard relationship between the time derivatives of fields and their first momenta jeopardizing the usual Heisenberg commutation relations. It is reasonable to expect that such theories will be unitary, although this would need to be verified explicitely by starting from the Hamiltonian and integrating out the momenta (usually for theories with a finite number of derivatives, one momentum integration cannot be generally eliminated, but this single integration might not matter in the limit for the number of derivatives towards infinity). Moreover, possibly, such theory might have to be supplemented with *constraints* (apart from the diffeomorphism constraints) since the Lagrangian is singular which might involve adding ghosts to the action. In general, it is fair to say that the resulting theory, if it exists, is a *limit* of path integrals on the lattice based upon Lagrangians with a finite number of derivatives but cannot itself be written as a standard path integral since an action with an infinite number of terms doesn't make sense mathematically. A similar comment applies to the standard model since the bare constants have to be infinite. This hints that the fundamental theory needs to be (very?) different and this approach certainly has no local observables (it is a scattering matrix approach for the entire universe) so I wonder how one is going to tell that the earth is turning around the sun.

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