The logical standing of unitarity in wave mechanics — in context of quantum randomness

Homogeneity of space is non-unitary

Steve Faulkner

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Abstract The homogeneity symmetry is re-examined and shown to be non-unitary. This is motivated by the prospect that logical independence in elementary algebra, entering quantum mathematics, will constitute the basis for a theory explaining quantum randomness.

Keywords foundations of quantum theory, quantum physics, quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space, unitary, non-unitary, unitarity, mathematical logic, formal system, elementary algebra, information, axioms, mathematical propositions, logical independence, quantum indeterminacy, quantum randomness.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [7], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al offer explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in (Boolean) mathematical propositions [8,9]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Tomasz Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

While those Boolean propositions *do* convey definitive information about quantum randomness, any insight they offer is obscure. In order to advance a theory for quantum randomness, understanding is needed of logical independence, inherent in *standard textbook quantum theory.* A likely place to begin is *elementary algebra as a logical system* – the formal version of the very familiar algebra upon which applied mathematics and mathematical physics rest. This is the algebra of *infinite fields.* Logical independence in this system is well-known to Mathematical Logic [12].

In a related article by this author [4], logical independence in elementary algebra is discussed. Of particular interest is *logical independence* of the imaginary scalars, seen in contrast to *logical dependence* of the rational scalars – and – the possible prospect that these two types of logical information might pass into quantum mathematics.

As it happens, the passage of that logical information is blocked. It is prevented by an alteration that quantum theory imposes on elementary algebra. Quantum

Steve Faulkner
Logical Independence in Physics. Information flow and self-reference in Elementary Algebra.
E-mail: StevieFaulkner@googlemail.com

mathematics can be regarded as an altered elementary algebra. Specifically, *unitarity* (or self-adjointness) is imposed axiomatically – *by Postulate.*

Historically, the reason for unitarity is the universal need for preserved invariance of probability amplitude. And so, interpretationally, unitarity is seen as ontologically fundamental to all symmetries of Nature [5, p109][6, p34]. This would indicate that unitarity should be a blanket condition covering the whole of quantum theory – and should be regarded as *axiomatic.* In short, unitarity is never in absence.

This paper shows that the homogeneity symmetry, generally understood to imply the Canonical Commutation Relation [5, p115][11, p44], is not itself unitary.

2 The basic symmetry of wave mechanics: homogeneity of space

The *Canonical Commutation Relation*

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of wave mechanics. The professed significance of this relation is that it represents the homogeneity of space, and *that* is accepted by quantum theorists as unitary. In this paper, I re-examine and scrutinise the Canonical Relation's derivation and establish that the homogeneity symmetry is itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains other (unitary) information also.

Imposing homogeneity on a system is identical to imposing a null physical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *form invariance.* This is the concept, from relativity, that symmetry transformations leave (physical) formulae fixed in *form*, though *values* may alter [10]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x}\,|f_{\mathsf{x}}(x)\rangle = \mathsf{x}\,|f_{\mathsf{x}}(x)\rangle\,. \tag{1}$$

The san-serif x, here, is a label for f_{x} whose eigenvalue is x. The variable x (curly) is the function domain. The use of two different variables here may seem unusual and pointless. In fact, logically they are different. x is quantified existentially but x is quantified universally.

With form held fixed, as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally (Lie group and Lie algebra). To maintain the form of (1), under translation, the basis $|f_{\mathsf{x}}\rangle$ is cleverly managed: while the translation transforms the basis from $|f_{\mathsf{x}}\rangle$ to $|f_{\mathsf{x}-\epsilon}\rangle$, a similarity transformation is also applied, chosen to revert $|f_{\mathsf{x}-\epsilon}\rangle$ back to $|f_{\mathsf{x}}\rangle$. In this way $|f_{\mathsf{x}}\rangle$ is held static. The similarity transformation is a member of the one-parameter subgroup of the general linear group GL, $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}$, with the transformation parameter ϵ coinciding with the displacement parameter. We shall see later, that similarity transforms can be found only for a certain class of functions f. The overall scheme of transformations is depicted in Figure 1.

Now, in standard theory of quantum symmetries, textbook understanding is that $\mathsf{S}(\epsilon)$ is intrinsically and necessarily unitary. It is in *that* unitarity where the Canonical Relation finds its unitary origins. The textbook reason for that unitarity, and the purpose it serves, is the preserved existence of the scalar product and invariance of probability amplitude.

And so, because its presence is thought *intrinsically necessary*, unitarity is imposed axiomatically on the theory, *by Postulate.* The upshot is that standard theory *imposes* Hilbert space on vectors $|f_{\mathsf{x}}\rangle$. This imposed unitarity is added information, extra to the information of homogeneity. In consequence, in standard theory, the symmetry for wave mechanics is a *resultant* – unitary subgroup of homogeneity.

As an experiment, we proceed, in this paper, by treating unitarity as a purely separate issue from homogeneity and allowing $\mathsf{S}(\epsilon)$ it's widest generality, so that homogeneity is faithfully and genuinely conveyed through the theory. The experiment begins with the eigenvalue equation for position (1) being rewritten, as the eigenformula in the quantified proposition (2). From here on, all informal assumptions are to be shed and the Dirac notation is dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator $\mathbf{x}$, eigenfunctions f_{x} and eigenvalues x, seen from the reference frame O_{x}:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x) \tag{2}$$

Translation: Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through ϵ. See Figure 2. The *principle of relativity* guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_{x} in (2), thus:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_{\mathsf{x}} \mid \quad \mathbf{x} f'_{\mathsf{x}}(x') = \mathsf{x}' f'_{\mathsf{x}}(x') \tag{3}$$

A relation for $\mathbf{x}$ is to be evaluated, so $\mathbf{x}$ is held static for all reference frames. The translation transforms position, thus:

$$\forall \epsilon \forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{4}$$

and transforms the function, thus:

$$\forall \epsilon \forall x' \forall f'_{\mathsf{x}} \exists f_{\mathsf{x}} \exists x \mid \quad f_{\mathsf{x}}(x) \mapsto f'_{\mathsf{x}}(x') = f_{\mathsf{x}-\epsilon}(x - \epsilon) \tag{5}$$

Substituting (4) and (5) into (3) gives the translated formula:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}-\epsilon}(x - \epsilon) = (\mathsf{x} + \epsilon) f_{\mathsf{x}-\epsilon}(x - \epsilon) \,. \tag{6}$$

Similarity: Now applying the similarity transformation. This involves the (one parameter) linear operator $\mathsf{S}_{(\epsilon)}$. Such an $\mathsf{S}_{(\epsilon)}$ exists only if there exists a space of functions ψ_{x}, that is complete, normalisable, not restricted to separable[1] functions, and is a subset of the (translatable) functions f_{x}. Logically, the act of assuming such an $\mathsf{S}_{(\epsilon)}$ hypothesises that such a class of functions does indeed exist. No such function space is guaranteed. Accordingly, the assertion of proposition (7) is newly assumed information entering the system.

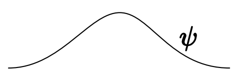

Figure 3 The linear transformations S exist only for bounded ψ_{x}.

$$\forall x \forall \epsilon \forall \psi_{\mathsf{x}-\epsilon} \exists \mathsf{S} \exists \psi_{\mathsf{x}} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} \psi_{\mathsf{x}}(x) = \psi_{\mathsf{x}-\epsilon}(x - \epsilon) \,. \tag{7}$$

In standard theory, $\mathsf{S}_{(\epsilon)}$ is set unitary by the mathematician. Doing that restricts the space of functions ψ_{x} to the Hilbert space L^2. Here, $\mathsf{S}_{(\epsilon)}$ is a member of the one

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

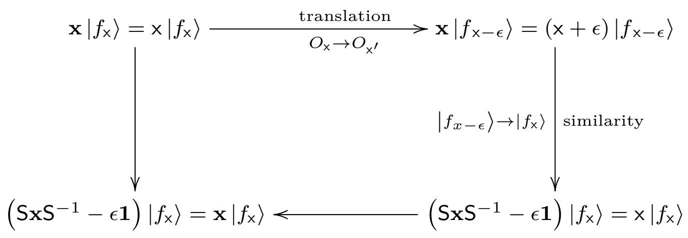

Figure 1 Scheme of transformations. The bottom left hand formula is the resulting group relation.

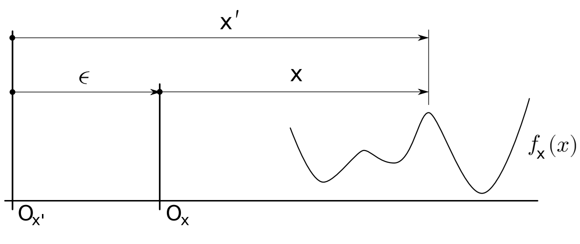

Figure 2 Passive translation of a function Two reference systems, O_{x} and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, individually act as reference systems for position of a function f_{x}. If the x-space is homogeneous, then regardless of the value of ϵ, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; f_{x} is non-observable and O_{x} and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

parameter subgroup of the infinite dimensional, (non-unitary) general linear group $\mathsf{GL}(\infty)$. This restricts ψ_{x} not to the Hilbert space L^2 but to the Banach space L^1.

The similarity transformation is formed, thus:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} \psi_{\mathsf{x}}(x) = (\mathsf{x} + \epsilon)\, \psi_{\mathsf{x}}(x) \,.$$

Introducing the trivial eigenformula: $\forall \psi_{\mathsf{x}} \forall x \forall \epsilon \mid \epsilon \mathbf{1} \psi_{\mathsf{x}}(x) = \epsilon \psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbf{1} \right) \psi_{\mathsf{x}}(x) = \mathsf{x} \psi_{\mathsf{x}}(x) \,. \tag{8}$$

Now comparing the original position eigenformula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathbf{x} \psi_{\mathsf{x}}(x) = \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbf{1} \right) \psi_{\mathsf{x}}(x) \,. \tag{9}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(\infty)$, there exists a unique linear operator $\mathbf{g}$ for real parameters ϵ, such that:

$$\forall \mathsf{S} \exists \mathbf{g} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \tag{10}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extracted, thus: $\forall \mathbf{g} \forall \eta \exists \mathbf{k} : \ \mathbf{g} = \eta \mathbf{k}$, implying:

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta \epsilon \mathbf{k}} \tag{11}$$

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)}^{-1} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta \epsilon \mathbf{k}} \tag{12}$$

Substitution of (11) and (12) into (9) gives:

$$\begin{aligned}
& \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \exp(+\eta \epsilon \mathbf{k})\, \mathbf{x} \exp(-\eta \epsilon \mathbf{k})\, \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{1} + \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \mathbf{x} \left[\mathbf{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} + \mathcal{O}\left(\epsilon^2\right)\right] \left[\mathbf{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} - \eta \epsilon \mathbf{x} \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ & \forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && [\mathbf{k} \mathbf{x} - \mathbf{x} \mathbf{k}]\, \psi_{\mathsf{x}}(x) = \left[\eta^{-1} \mathbf{1} - \mathcal{O}(\epsilon)\right] \psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{k}, \mathbf{x}]\, \psi_{\mathsf{x}}(x) = \eta^{-1} \mathbf{1} \psi_{\mathsf{x}}(x) \tag{13}$$

And by a similar proof, conditional on the existence of eigenfunctions $\chi(k)$, of $\mathbf{k}$:

$$\forall k \forall \eta \exists \mathsf{k} \exists \chi_{\mathsf{k}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{x}, \mathbf{k}]\, \chi_{\mathsf{k}}(k) = \eta^{-1} \mathbf{1} \chi_{\mathsf{k}}(k) \,. \tag{14}$$

Importantly, we see (13) and (14) is $\forall \eta$, rather than the particular case of $\eta^{-1} = -i$ that we see in the unitary subalgebra we know as the Canonical Commutation Relation:

$$[\mathbf{k}, \mathbf{x}] = -i \mathbf{1} \quad \text{or} \quad [\mathbf{p}, \mathbf{x}] = -i \hbar \mathbf{1} \tag{15}$$

Conclusion

The above establishes that the homogeneity symmetry is not a source of unitary information in wave mechanics. And therefore, if the reason given is that symmetries in Nature are ontologically unitary, for postulating that quantum theory should be unitary or self-adjoint, then either, a different reason must be found, or the postulate must be withdrawn.

This opens up the possibility of a logical modification to quantum theory, where quantum theory remains a unitary theory, but, in which that unitarity (or self-adjointness) is not axiomatically imposed *by Postulate*. And as a result, that modified quantum theory would allow the logical independence of the imaginary scalars, and the logical dependence of the rational scalars, originating in elementary algebra, to enter quantum mathematics.

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