Metric operator equations for quantum gravity

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Abstract. The search for a consistent theory of quantum gravity has motivated the development of radically different approaches. This seeks consists of constructing a mathematical apparatus that encapsulates both concepts of quantum theory and general relativity. However, none approach has been definitive and the problem remains open. As the quantization of the metric is an alternative, this paper shows how a metric operator may be explicitly obtained by introducing a temporal operator, defining an induced metric and invoking some spacetime symmetries. This makes it possible to relate the effective acoustic metric to the model proposed here. The metric operator equations are expressed in terms of a hamiltonian operator describing the degrees of freedom of quantum vaccum whose dynamics gives rise to the metric field. These findings may help understand and study the quantum vacuum at Planck scale, consisting of one more tool for the community working on quantization of gravity.

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1. Introduction

A number of approaches have aimed to develop a consistent formulation of quantum theory of gravity [1, 2]. This formulation consists of constructing a mathematical apparatus that encapsulates both concepts of quantum theory and general relativity. Quantum theory is a framework for all fundamental interactions and gravitation is very well described by the metric field in general relativity. Both theories have gone through several experimental tests making the unification fairly attractive. However, the satisfactory unified formulation is still an open problem. The older approach is to directly quantize the theory of general relativity resulting in canonical quantization methods such as *Loop Quantum Gravity* [3]. Another approach is *String Theory* that incorporates the ideia that a theory of quantum gravity could emerge in an unified description with all elementar interactions and matter fields [4]. Less orthodox ideas inspired from analogies with condensed matter also state a scenario where gravity/spacetime and other fields are emergent phenomena [5, 6, 7, 8]. The major motivation for the construction of a quantum theory of gravity is to understand fundamental issues such as the origin of the universe, the evaporation of black holes and the structure of space and time.

The incompatibility between quantum theory and general relativity can already be realized in their foundations through the well-known problem of time [8, 9]. In general relativity, spatial and temporal variables are treated as dynamical ones. In standard quantum theory, spatial variables are dynamical ones but the time is just an absolute parameter. In quantum field theory, all variables also receive democratic treatment, but now spatial and temporal variables are treated as parameters. Fields are the dynamical variables in Minkowiski spacetime. For completeness, this article shows how the democratic treatment among all dynamic and parametric variables contributes to establish a procedure for quantization of the metric field. In particular, a metric operator can be obtained by introducing a temporal operator, defining an induced metric and invoking some spacetime symmetries. It also shows how the analogy with the effective acoustic metric leads to an analogue model to the one proposed here. This analogy suggests that the metric operator equations are expressed in terms of a hamiltonian operator describing the degrees of freedom of quantum vaccum whose dynamics gives rise to the metric field.

2. Formulation of the metric operator

In order to construct the metric operator one must admit the coexistence of all dynamic and parametric variables. In particular, with regards to standard quantum mechanics, one introduces a temporal operator T besides the three position operators $X^i, (i = 1, 2, 3)$. One also considers three spatial parameters x^i in addition to the temporal parameter t. The dynamic variables become operators, $X^\mu = (T, X^1, X^2, X^3)$, in the Hilbert space and the parametric variables, $\xi^\alpha = (t, x^1, x^2, x^3)$, take their values in the Euclidean space. Throughout this article one uses natural units, $\hbar = c = 1$, unless stated otherwise. The connection between the two sorts of variables can be made through an induced metric operator. In fact, the simplest way of defining this metric operator is in analogy to an induced metric. It can be written as ‡

$$\hat{g}_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is a diagonal matrix introducing the Lorentzian signature into the metric, so that $X_\mu = \eta_{\mu\nu} X^\nu$.

To calculate (1), one must use spacetime symmetry transformations and their representations in the Hilbert space [10, 11]. Let $|\xi^\alpha>$ and $P^\alpha = (H, P^1, P^2, P^3)$ be eigenvectors and translation generators of the corresponding operators X^μ, respectively. As a spacetime displacement, $\xi \to \xi' = \xi + \lambda$, implies a displacement of the eigenvectors $|\xi>$, one can write the state eigenvectors transformed as

$$|\xi + \lambda >= \mathrm{e}^{\mathrm{i}\lambda^\alpha P_\alpha} |\xi >, \tag{2}$$

where λ are infinitesimal displacements. On the other hand, since the corresponding operators X^μ bear a similar relationship one can expand them in the form

$$\begin{aligned} X^\mu(\xi + \lambda) &= \mathrm{e}^{\mathrm{i}\lambda^\alpha P_\alpha} X^\mu(\xi) \mathrm{e}^{-\mathrm{i}\lambda^\alpha P_\alpha} \\ &= X^\mu(\xi) + \mathrm{i}\lambda^\alpha [P_\alpha, X^\mu(\xi)] + \mathcal{O}(|\lambda|^2). \end{aligned} \tag{3}$$

By comparing (3) with the Taylor expansion:

$$X^\mu(\xi + \lambda) = X^\mu(\xi) + \lambda^\alpha \partial_\alpha X^\mu(\xi) + \mathcal{O}(|\lambda|^2), \tag{4}$$

one gets the following equation:

$$\mathrm{i}\partial_\alpha X^\mu = [X^\mu, P_\alpha]. \tag{5}$$

By inserting (5) into (1), one has

$$\hat{g}_{\alpha\beta} = -[X^\mu, P_\alpha][X_\mu, P_\beta]. \tag{6}$$

Interestingly, we have employed only translation generators, P^μ, in order to get (6). We do not need the generators of the Lorentz algebra that incorporate boosts and rotational symmetries. In addition, the dependence of the operators X^μ with the parametric variables, i.e. $\xi \to X^\mu(\xi)$, turns them into field operators. Another point is what the hamiltonian H describes since it does not have an explicit form yet. Next we shall see that it is fairly evident to interpret it as describing the degrees of freedom of the quantum vacuum whose dynamics gives rise to the metric field.

3. Analogy with effective metric

Symmetries, conservation laws and analogies are very useful to probe theories. In particular, some aspects of the physics of condensed matter give rise to a programme known as 'analogue gravity', which allows to investigate aspects of the physics of curved

‡ This is also a 3-brane in 3+1 dimensions (spacetime filling 3-brane) if you prefer.

spacetime. According to this vision gravity/spacetime is an emergent phenomenon and quantum hydrodynamics can help the development of a quantum theory of gravity [5, 6]. For example, the physics of sound waves in a moving fluid is the simplest analogue model that mimics the physics of light waves in a curved spacetime. These models invoke a mathematical artifact named effective acoustic metric, which is an analogy for the metric of curved spacetime.

A fluid propagating relative to the laboratory can produce sound waves which in turns propagates relative to the fluid. In relation to the laboratory, the velocity of a sound vector ray propagating along the direction of the vector $\vec{n}$ ($\vec{n}^2 = 1$) is given by [7]

$$\frac{d\vec{x}}{dt} = c_{\rm s}\vec{n} + \vec{v}, \tag{7}$$

where $c_{\rm s}$ is the speed of sound relative to the fluid and $\vec{v} = \vec{v}(t, \vec{r})$ is the speed of the fluid relative to the laboratory at the instant t and at the point $\vec{r}$. By handling (7), one gets

$$-c_{\rm s}^2 dt^2 + (d\vec{x} - \vec{v}dt)^2 = 0. \tag{8}$$

It is easy to see that (8) defines a null line element ($ds^2 = 0$) in the way:

$$ds^2 = -(1 - \frac{v^2}{c_{\rm s}^2})(c_{\rm s}dt)^2 - 2\frac{\vec{v}}{c_{\rm s}} \cdot d\vec{x}(c_{\rm s}dt) - d\vec{x} \cdot d\vec{x}. \tag{9}$$

From (9) it is also easy to see that an effective metric may be defined for the propagation of sound in the fluid as

$$g_{00} = -1 + \frac{v^i v_i}{c_{\rm s}^2}, \qquad g_{0j} = -\frac{v_j}{c_{\rm s}}, \qquad g_{ij} = \delta_{ij}. \tag{10}$$

A null line element admits the omission of a multiplicative prefactor. Thus no conformal factor has been explicit in (10).

In order to show the relationship of the effective metric with the metric operator, the following commutation relations must be postulated:

$$[X^i, P^j] = {\rm i}\delta^{ij}, \quad [T, P^\mu] = {\rm i}\delta^{0\mu}. \tag{11}$$

The commutation relations (11) are sufficient for what follows. The derivation of the commutation relations with the other operators of the standard quantum mechanics will be detailed in future work. Here the relations between T and P^μ are motivated by the same reasons as the usual relations between X^i and P^j. From (6) and (11), the metric operator is simplified as

$$\begin{aligned} \hat{g}_{00} &= -[X^\mu, P_0][X_\mu, P_0] = -I - [X^i, H][X_i, H], \\ \hat{g}_{0j} &= -[X^\mu, P_0][X_\mu, P_j] = {\rm i}[X_j, H], \\ \hat{g}_{ij} &= -[X^\mu, P_i][X_\mu, P_j] = \delta_{ij}I. \end{aligned} \tag{12}$$

In addition to the commutation relations (11), one must be noted that a velocity operator is generally defined in terms of a hamiltonian operator as ${\rm i}V^i = [X^i, H]$ [10]. Also let

$\bar{v}^i(\xi)$ be the mean value of this velocity operator in a state $|\psi\rangle$, i.e., $\bar{v}^i = \langle\psi|V^i|\psi\rangle$. By inserting these last two remarks into (12), one gets

$$g_{00} = -1 + \frac{\bar{v}^i\bar{v}_i}{c^2}, \qquad g_{0j} = -\frac{\bar{v}_j}{c}, \qquad g_{ij} = \delta_{ij}. \tag{13}$$

where the velocity of light (c) has been reinserted.

Let us consider the effective geometry (10) as being an analogue model for the mean value of the quantum geometry (13), with $c_{\rm s}$ and $v^i(t,\vec{r})$ playing the role of c and $\bar{v}^i(\xi)$, respectively. It is well known that the effective geometry is emergent from dynamics of fluid. Here, by forcing the identification $c \to c_{\rm s}$ and $\bar{v}^i(\xi) \to v^i(t,\vec{r})$, the effective geometry is also emergent from the quantum geometry. It is interesting to note that some relation to quantization of effective metric has already been done in [12]. In this case, it is reasonable to interpret the hamiltonian as describing the degrees of freedom which constitute the quantum fluid. On the other hand, because (6) could be derived by only considerations of spacetime symmetries, its meaning is more fundamental. That is, the hamiltonian might describe the degrees of freedom of the quantum vacuum whose dynamics gives rise to the metric field.

4. Application to free hamiltonian

The metric operator equations have been expressed in terms of a hamiltonian operator in (12). As it has already been explored, let us consider the hamiltonian as describing the degrees of freedom of the quantum vacuum whose dynamics gives rise to the dynamic metric field. One can consider the simplest case of a free hamiltonian in the form [10]

$$H = l_0 P^2/2 + E_0, \tag{14}$$

where E_0 is an internal contribuition to the energy which commutes with operators X^i and P^i. The characteristic length l_0 has been used to override the usual mass. The commutation relations (11), between T and P^μ, imply $[T, E_0] = {\rm i}$ leading to uncertainty relations: $\Delta T \Delta E_0 \geq 1/2$. It is expected that quantum gravitational effects are meaningful at the Planck scale. By setting $\Delta T = \sqrt{G}$ as minimum uncertainty for T, it follows that $\Delta E_0 \geq 1/2\sqrt{G}$ (where G is the gravitational constant) [13]. Since E_0 must be larger than its uncertainty then $E_0 > 5 \times 10^{18}$ GeV. Although this model is simplified, it is useful to illustrate an application of the equations leading to more complex cases.

The commutation relations (11), between X^i and P^j, yield now

$$[X^i, H] = {\rm i} l_0 P^i. \tag{15}$$

By inserting (15) into (12), the components of the metric operator take the form

$$\hat{g}_{00} = -I + l_0^2 P^i P_i, \qquad \hat{g}_{0j} = -l_0 P_j, \qquad \hat{g}_{ij} = \delta_{ij} I. \tag{16}$$

From (16) we see that $\hat{g}_{\alpha\beta}$ and P^i commute, i.e. $[\hat{g}_{\alpha\beta}, P^i] = 0$. Thus they have the same set of eigenvectors. If $P^i|K^i >= K^i|K^i >$, where K^i and $|K^i >$ are the eigenvectors and eigenvalues of the total momentum operator P^i, then $\hat{g}_{\alpha\beta}|K^i >= g_{\alpha\beta}|K^i >$, where $g_{\alpha\beta}$ are the eigenvalues of the metric operator given by

$$g_{00} = -1 + l_0^2 K^i K_i, \qquad g_{0j} = -l_0 K_j, \qquad g_{ij} = \delta_{ij}. \tag{17}$$

Note that for $\|l_0 K_j\| << 1$, the metric components approach of the components of the Minkowiski metric. Moreover, their contravariant components are given by

$$g^{00} = -1, \qquad g^{0i} = -l_0 K^i, \qquad g^{ij} = \delta^{ij} - l_0^2 K^i K^j. \tag{18}$$

The model given by (14) bears a ground state of "heavy constituents" with energy of the order of Planck scale. "Light quanta" (subplanckian scale) may arise as low-energy collective excitations propagating over this vacuum. The energy spectrum of theses quanta is given by [6]

$$g^{\mu\nu} p_\mu p_\nu = 0, \tag{19}$$

where $p_\mu = (-E, p_1, p_2, p_3)$ is the 4-momentum of the quanta. This equation also shows that the spectrum is independent of a multiplicative prefactor (conformal factor). From (18) and (19) one gets

$$-E^2 + 2 l_0 K^i p_i E + p^2 - l_0^2 (K^i p_i)^2 = 0, \tag{20}$$

where $p^2 = \|p\|^2 = \delta^{ij} p_i p_j$. By handling (20), we have

$$E = l_0 K^i p_i \pm p. \tag{21}$$

This energy spectrum is different from the standard spectrum due to an additional term containing the characteristic length (l_0). By imposing $l_0 = 0$, one obtains the standard spectrum for massless particles, $E = \|p\|$. In particular, let us consider $p = p_1$ so that $E = (l_0 K^1 \pm 1) p_1$. If $l_o \neq 0$, there are two different regimes. For $\|l_0 K^1\| << 1$, one gets $E = \|p_1\|$ again. Otherwise, for $\|l_0 K^1\| >> 1$, the "light quanta" are more coupled to the "heavy constituents", i.e., $E = l_0 K^1 p_1$. By dimensional analysis, one can estimate the energy associated to K^1 to be of the order of thermal energy, i.e. $l_0 (K^1)^2 \sim \kappa T$ or $\|l_0 K^1\| \sim \sqrt{l_0 \kappa T}$. Let us assume $l_0^{-1} \sim 10^{19}$ GeV. Then, for the current temperature of the Universe, $\kappa T \sim 10^{-4}$ eV, one has $\|l_0 K^1\| \sim 10^{-16} (<< 1)$. Whereas in the primordial Universe, $\kappa T \sim 10^{19}$ GeV, we obtain $\|l_0 K^1\| \sim 1$. Finally, the case $\|l_0 K^1\| >> 1$ is related to the transplanckian regime.

5. Connection with Einstein equations

The analogy with condensed matter suggests that the hamiltonian H should be interpreted as describing the degrees of freedom of the microscopic system (quantum vacuum) whose dynamics gives rise to the dynamic metric field. For the case explored here, note that H can never vanish because otherwise it would violate the noncommutativity with T. The contribuition of E_0 in the hamiltonian (14) suggests that these degrees of freedom come from quantum fluctuations of vacuum which affect the field metric. In order to connect with the Einstein equations, note that (6) can be expressed as

$$\hat{g}_{\alpha\beta} = -\hat{T}_{\alpha\beta}, \tag{22}$$

where $\hat{T}_{\alpha\beta} = [X^\mu, P_\alpha][X_\mu, P_\beta]$. On the other hand, the Einstein equations:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = -8\pi G T_{\alpha\beta}, \tag{23}$$

are simplified as

$$g_{\alpha\beta} = -T^{(\mathrm{vac})}_{\alpha\beta}, \tag{24}$$

apart from a conformal factor, $\rho_{\mathrm{vac}} = \Lambda/8\pi G$, and when the Einstein tensor is null, i.e. $G_{\alpha\beta} = 0$.

Now (22) and (24) can be compared. We can clearly see the equivalence of physical content. The left hand of theses equations involves metric quantities, while the right hand involves quantities of momentum-energy. For correspondence, supose that there is a state $|\Psi>$ such that we can get the following expected values:

$$<\Psi|\hat{g}_{\alpha\beta}|\Psi>= g_{\alpha\beta} \quad \text{and} \quad <\Psi|\hat{T}_{\alpha\beta}|\Psi>= T^{(\mathrm{vac})}_{\alpha\beta}. \tag{25}$$

Therefore, apart from a conformal factor and in the absence of curvature, an aspect of Einstein equations (24) can be reproduced by applying the state $|\Psi>$ in (22).

6. Discussion

The formulation of the metric operator requires the introduction of a temporal operator which is not common in standard quantum mechanics. The relationship and coexistence between an absolute and dynamic time have already been explored in [9]. One may extend those discussions to the absolute and dynamic spatial variables. That is, we never measure the absolute variables but only some parameter of physical objects such as rulers and clock-hands, from where we extract the measures of time and space. The metric operator equations could be obtained in terms of theses variables and operators of momentum and energy, as shown in (6). In order to solve these equations, it is useful to invoke the analogy with condensed matter. This allows to relate the two systems: quantum vacuum and quantum liquid.

This quantum vacuum composed of "heavy constituents" gives rise to a kind of condensed matter system. The colletive excitations and emergent symmetries of such a Planck condensed matter could describe the physics of quantum vacuum of a very similar way to description of condensed matter in real laboratory [6, 8]. As possible search for experimental evidence, the standard energy spectrum for massless particles is modified in (21) due to the appearance of an additional term. Once again, this indicates possible violation of Lorentz invariance at high energies [14]. This system of "heavy constituents" might contribute to the degrees of freedom which gives rise to the entropy of black holes [15, 16]. It can also be investigated whether there is some connection with dark matter. On the other hand, the full classical limit given by (23) for any magnitude of the Einstein tensor has not been obtained yet.

These findings arise from a new way of quantizing the metric and may be interesting for community working on quantization of gravity.

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