Logical independence inherent in Elementary Algebra seen in context of quantum randomness

Steve Faulkner

20th September 2015

Abstract As opposed to the classical logic of *true* and *false*, when elementary algebra is treated as a *formal axiomatised system*, formulae in that algebra are either *provable*, *disprovable* or otherwise, *logically independent* of axioms. This logical independence is well-known to Mathematical Logic. The intention here is to cover the subject in a way accessible to physicists. This work is part of a project researching logical independence in quantum mathematics, for the purpose of advancing a complete theory of quantum randomness.

Keywords mathematical logic, formal system, axioms, mathematical propositions, Soundness Theorem, Completeness Theorem, logical independence, mathematical undecidability, foundations of quantum theory, quantum mechanics, quantum physics, quantum indeterminacy, quantum randomness.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [7], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al offer explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in (Boolean) mathematical propositions [8,9]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

While those Boolean propositions *do* convey definitive information about quantum randomness, any insight they offer is obscure. In order to advance a full and complete theory of quantum randomness, understanding is needed of logical independence, inherent in *standard textbook quantum theory.* A likely place to start looking is in *elementary algebra* as a *formal axiomatised system* – the formal version of the very familiar algebra upon which applied mathematics and mathematical physics rest. Logical independence in this system is well-known to Mathematical Logic [12].

Any *formal system* comprises: a precise language, rules for writing formulae (propositions) and further rules of deduction. Information is designated in two 'strengths': *propositions* assert information that is questionable, and *axioms* are propositions adopted as 'true'.

In such a system, any two propositions are **either** *logically dependent* – in which case, each proves, or disproves the other – **or otherwise** they are *logically independent*, in which case, neither proves, nor disproves the other. A helpful perspective on

Steve Faulkner
Logical Independence in Physics. Information flow and self-reference in Elementary Algebra.
E-mail: StevieFaulkner@googlemail.com

this is the viewpoint of Gregory Chaitin's information-theoretic formulation [6]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms, then those axioms can neither prove nor disprove the proposition.

A good (efficient) axiom-set is a selection of propositions, all logically independent of one another. An important point to note is that there is no contradiction in a theory consisting of information whose source is some axiom-set, plus extra information whose source is a logically independent proposition. These might typically be axioms asserting the theory's set conditions, plus a proposition posing a question.

Elementary algebra is the abstraction of the familiar arithmetic used to combine numbers in the rational, real and complex number systems, through operations of *addition* and *multiplication*. These number systems are *infinite fields*. I denote this algebra – FIELDS ALGEBRA[1] – as distinct from any other algebra or arithmetic, such as Peano arithmetic or the arithmetic of integers. At a fundamental level, in some form or other, quantum theory rests on mathematical rules of FIELDS ALGEBRA.

Now, FIELDS ALGEBRA may be treated as a formal system, based on axioms listed in Table 1. These, I denote – AXIOMS of INFINITE FIELDS (or just AXIOMS). Essentially, these are the conventional *field axioms* appended with additional axioms that exclude modulo arithmetic. The point of this is that FIELDS ALGEBRA should cover only *infinite fields*.

Collectively, the AXIOMS of INFINITE FIELDS assert a definite set of information, deriving a definite set of theorems. I denote these – THEOREMS. Any proposition (in the language) is either a THEOREM or is otherwise logically independent. And so, any given formula can be regarded as a proposition in FIELDS ALGEBRA, that may prove to be a THEOREM, or may otherwise prove to be logically independent. Which of these is actually the case is decided in a process that compares information in that formula against information contained in the AXIOMS. In practice, that means deriving the formula from AXIOMS, to discover: that either it *is* a THEOREM, or otherwise, to discover, whatever extra information is needed to complete its derivation – that AXIOMS cannot provide.

2 Language

The material of this paper spans *formal arithmetic* and formulae typically seen in mathematical physics. These do not share the same language; indeed the language of the former is far smaller. For example, there is no definition for the symbol: 4 and many statements are needed, typified by: $4 = 1 + 1 + 1 + 1$. In the interest of accessibility, these low-level definitions are left to intuition.

Logical connectives used are: *not* ($\neg$), *and* ($\wedge$) , or ($\vee$), *implies* ($\Rightarrow$) and *if-and-only-if* ($\Leftrightarrow$). Turnstile symbols are used: *derives* ($\vdash$) and *models* ($\models$). Also used are the quantifiers: *there-exists* ($\exists$) and *for-all* ($\forall$).

Use of Quantified formulae is crucial in the conveyance of full information. For instance, quantifiers eliminate ambiguities suffered by ordinary equations. To illustrate: the equation $y = x^2$ doesn't express whether $\forall y \exists x \left(y = x^2\right)$ or $\forall x \exists y \left(y = x^2\right)$ is intended. Yet, logically, these two are very different.

3 Examples of logic in FIELDS ALGEBRA

The propositions (1) – (5) are five examples illustrating the three distinct logical values possible under FIELDS AXIOMS. Notice that these formulae do not assert equality; they assert *existence*. Each is a proposition asserting existence for some instance of a variable α, complying with an equality, specifying a particular numerical value.

$$\exists \alpha \mid \alpha = 3 \tag{1}$$
$$\exists \alpha \mid \alpha^2 = 4 \tag{2}$$
$$\exists \alpha \mid \alpha^2 = 2 \tag{3}$$
$$\exists \alpha \mid \alpha^2 = -1 \tag{4}$$
$$\exists \alpha \mid \alpha^{-1} = 0 \tag{5}$$

[1] FIELDS should not be confused with the field concept commonly used in physics.

AXIOMS of INFINITE FIELDS

	ADDITIVE GROUP	
A0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	CLOSURE
A1	$\exists 0\forall\alpha \mid \alpha + 0 = \alpha$	IDENTITY 0
A2	$\forall\alpha\exists\beta \mid \alpha + \beta = 0$	INVERSE
A3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	ASSOCIATIVITY
A4	$\forall\alpha\forall\beta \mid \alpha + \beta = \beta + \alpha$	COMMUTATIVITY
	MULTIPLICATIVE GROUP	
M0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta \times \gamma$	CLOSURE
M1	$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha$	IDENTITY 1
M2	$\forall\beta\exists\alpha \mid \alpha \times \beta = 1 \wedge \beta \neq 0$	INVERSE
M3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$	ASSOCIATIVITY
M4	$\forall\alpha\forall\beta \mid \alpha \times \beta = \beta \times \alpha$	COMMUTATIVITY
D	$\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$	DISTRIBUTIVITY
	$0 \neq 1; \quad 0 \neq 1 + 1; \quad \cdots\cdots \quad 0 \neq 1 + \cdots + 1$	NO MODULO ARITH

Table 1 AXIOMS of INFINITE FILEDS. These are written as sentences in *first-order logic*. They comprise the *standard field axioms* with added axioms that exclude modulo arithmetic. Variables: $\alpha, \beta, \gamma, 0, 1$ represent objects the axiom-set acts upon. Semantic interpretations of objects complying with AXIOMS are known as *scalars*. The fact FIELDS ALGEBRA is intrinsically existential is clearly seen in the general use of the 'there exists' quantifier: $\exists$.

Of the five examples, AXIOMS prove only (1) and (2). Proofs are given below in this section. Also, AXIOMS prove the negation of (5); in point of fact, (5) contradicts, and is inconsistent with AXIOM M2. The remaining two, (4) and (3), are neither proved nor negated, and are logically independent of AXIOMS.

Accordingly, instances of α, in (1) and (2), are numbers consistent with AXIOMS and accepted as scalars, proved to *necessarily* exist; the instance of α in (5) is inconsistent with AXIOMS and rejected as *necessarily* non-existent; and instances of α in (4) and (3) are numbers consistent with AXIOMS and accepted as scalars whose existences are not provable, and not necessary, but *possible.*

In the cases of propositions (1) and (2), logical dependence, on AXIOMS, is established by the fact that these propositions (syntactically) derive, directly from AXIOMS. Likewise for the negation of (5). In contrast, however, logical independence of (4) and (3) is not provable by direct derivation because AXIOMS do not assert such information. In essence, that is the whole point of the discussion. What *does* confirm logical independence is a proposition's *truth-table*, viewed from the context of the Soundness Theorem and its converse, the Completeness Theorem. Briefly, Soundness says: if a formula is provable, it will be true, irrespective of whether variables are understood as rational, real or complex (or any other field). Completeness says: if a formula is true, irrespective of how variables are understood, then it will be provable. Hence, if there is disagreement in a truth-table, jointly, Soundness and Completeness except an *excluded middle* whose formulae are neither provable nor disprovable. This is the predicament of Proposition (4). Sections 4 and 7 explain the detail.

Proof of (1): that $\exists\alpha \mid \alpha = 3$

$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	AXIOM A0	(6)
$\forall\beta\exists\alpha \mid \alpha = \beta + \beta$	$\gamma \curvearrowright \beta\,(6)$	(7)
$\forall\gamma\exists\alpha \mid \gamma = \beta + \beta$	$\alpha \curvearrowright \gamma\,(7)$	(8)
$\forall\beta\exists\alpha \mid \alpha = \beta + \beta + \beta$	Subst. (8), (6)	(9)
$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha$	AXIOM M1	(10)
$\exists\beta \mid \beta = 1$	by (10)	(11)
$\exists\alpha \mid \alpha = 1 + 1 + 1$	Subst. (11), (9))	(12)

Substitution involving quantifiers

$$\forall\beta\underline{\forall\gamma}\exists\alpha \mid \ \alpha = \beta + \gamma$$
$$\forall\beta\underline{\exists\gamma} \mid \ \gamma = \beta + \beta$$
$$\Rightarrow \ \forall\beta\exists\alpha \mid \ \alpha = \beta + \beta + \beta$$

In the example above, an existential quantifier of one proposition must be matched with a universal quantifier of the other. These are highlighted by underlining.

Notation
$\gamma \curvearrowright \beta$ indicates swapping to different bound variable. This is always allowed under the quantifier, so long as all instances are swapped

Proof of (2): that $\exists\alpha \mid \alpha^2 = 4$

$$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma \qquad \text{AXIOM A0} \qquad (13)$$
$$\forall\beta\exists\alpha \mid \alpha = \beta + \beta \qquad \gamma \curvearrowright \beta\,(13) \qquad (14)$$
$$\forall\alpha \mid \alpha \times \alpha = \alpha \times \alpha \qquad \text{identity rule} \qquad (15)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = (\beta + \beta) \times (\beta + \beta) \qquad \text{Subst.}\,(14),(15) \qquad (16)$$
$$\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma) \qquad \text{AXIOM D} \qquad (17)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = \beta \times (\beta + \beta) + \beta \times (\beta + \beta) \qquad \text{by}\,(17),(16) \qquad (18)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = (\beta \times \beta) + (\beta \times \beta) + (\beta \times \beta) + (\beta \times \beta) \qquad \text{by}\,(17),(18) \qquad (19)$$
$$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha \qquad \text{AXIOM M1} \qquad (20)$$
$$\exists\beta \mid \beta = 1 \qquad \text{by}\,(20) \qquad (21)$$
$$\exists\alpha \mid \alpha \times \alpha = (1 \times 1) + (1 \times 1) + (1 \times 1) + (1 \times 1) \qquad \text{Subst.}\,(21),(19) \qquad (22)$$
$$\exists\alpha \mid \alpha \times \alpha = 1 + 1 + 1 + 1 \qquad \text{by}\,(20),(22) \qquad (23)$$

4 Soundness and Completeness

Model theory is a branch of Mathematical Logic applying to all first-order theories, and hence to FIELDS ALGEBRA [4,5]. Our interest is in two standard theorems: the Soundness Theorem and its converse, the Completeness Theorem, and theorems that follow from them. These theorems formalise the link connecting the truth (semantic information) of a formula and its provability (syntactic information). Together, their combined action identifies an *excluded middle*, comprising the set of all non-provable, non-negatable propositions — those that are logically independent of AXIOMS.

Briefly: any given (first-order) axiom-set is *modelled* by particular mathematical structures. That is to say, there are certain structures, consistent with each individual axiom of that axiom-set. In the case of FIELDS ALGEBRA, these modelling structures are the *infinite fields.* These are closed structures consisting of numbers known as *scalars.* In practical terms, if a proposition is logically independent of AXIOMS, this independence may be diagnosed by demonstrating disagreement on whether the proposition is true – between any two *models.* Of relevance to quantum theory is Proposition (4); this is true in the complex plane, but false in the real line.

Theorem 1 *The Soundness Theorem:*

$$\Sigma \vdash \mathcal{S} \Rightarrow \forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S})\,. \qquad (24)$$

If structure $\mathcal{M}$ models axiom-set Σ and Σ derives sentence $\mathcal{S}$, then every structure $\mathcal{M}$ models $\mathcal{S}$.
Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

A *sentence* is a formula where there is no occurrence of any variable not *bound* by a quantifier. For example: $\forall\alpha\forall\beta\,(\alpha + \beta = \beta + \alpha)\,.$

Theorem 2 *The Completeness Theorem:*

$$\Sigma \vdash \mathcal{S} \Leftarrow \forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S})\,. \qquad (25)$$

If structure $\mathcal{M}$ models axiom-set Σ and every structure $\mathcal{M}$ models sentence $\mathcal{S}$, then *Σ derives sentence $\mathcal{S}$.*
Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

5 Logically Dependent $\mathcal{S}$

Jointly, Theorems 1 and 2 imply the 2-way implication which is Theorem 3:

Theorem 3 *Soundness And Completeness:*

$$\Sigma \vdash \mathcal{S} \Leftrightarrow \forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S})\,. \qquad (26)$$

If structure $\mathcal{M}$ models axiom-set Σ, then axiom-set Σ derives sentence $\mathcal{S}$, if-and-only-if, all structures $\mathcal{M}$ model sentence $\mathcal{S}$.
Alternatively**:** *A sentence is provable under an axiom-set, if-and-only-if, that sentence is true for all models of that axiom-set.*

In addition, supplementary to Theorem 3, for every provable sentence $\mathcal{S}$ there is a corresponding disprovable negation $\neg\,\mathcal{S}$, also subject to Theorems 1 and 2, resulting in Theorem 4, a second, but complimentary 2-way implication:

Theorem 4 *Soundness And Completeness covering Negations:*

$$\Sigma \vdash \neg\,\mathcal{S} \Leftrightarrow \forall\mathcal{M}\,(\mathcal{M} \models \neg\,\mathcal{S})\,. \tag{27}$$

If structure $\mathcal{M}$ models axiom-set Σ, then axiom-set Σ derives the negation of sentence $\mathcal{S}$, if-and-only-if, all structures $\mathcal{M}$ model the negation of $\mathcal{S}$.
Alternatively**:** *A sentence is disprovable under an axiom-set, if-and-only-if, that sentence is false for all models of that axiom-set.*

6 Logically Independent $\mathcal{S}$

And so, while Theorem 3 covers all provable sentences under axiom–set Σ, Theorem 4 covers the set of disprovable negations under axiom–set Σ. Of special interest is the remaining set. This is an *excluded middle,* not covered, either by Theorem 3 or by Theorem 4, comprising sentences that are *neither provable, nor disprovable.* Happily, whereas there is no suggestion of any excluded middle in the left hand sides of (26) and (27), the right hand sides jointly define one. This excluded middle is the set of sentences $\mathcal{S}$ excluded by the right hand sides of both (26) and (27), thus:

$$\neg\forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S}) \wedge \neg\forall\mathcal{M}\,(\mathcal{M} \models \neg\,\mathcal{S})\,. \tag{28}$$

Now, by writing the negations of (26) and (27):

$$\neg\,(\Sigma \vdash \mathcal{S}) \Leftrightarrow \neg\forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S})\,; \tag{29}$$

$$\neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \neg\forall\mathcal{M}\,(\mathcal{M} \models \neg\,\mathcal{S})\,; \tag{30}$$

we may then match (28) with its corresponding left side, so as to construct:

$$\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \neg\forall\mathcal{M}\,(\mathcal{M} \models \mathcal{S}) \wedge \neg\forall\mathcal{M}\,(\mathcal{M} \models \neg\,\mathcal{S})\,. \tag{31}$$

This includes all sentences excluded by (26) and (27). On the left, it limits all sentences that are neither provable nor negatable, to those on the right, that are neither true nor false, across all structures that model the axiom-set. For theories whose axiom-set is modelled by more than one single structure – where $\mathcal{M}_1$ and $\mathcal{M}_2$ are distinct, we deduce:

Theorem 5 *The logically independent, excluded middle:*

$$\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \Leftrightarrow \exists\mathcal{M}_1\,(\mathcal{M}_1 \models \mathcal{S}) \wedge \exists\mathcal{M}_2\,(\mathcal{M}_2 \models \neg\,\mathcal{S})\,. \tag{32}$$

Axiom-set Σ derives neither sentence $\mathcal{S}$ nor its negation, if-and-only-if, there exist structures $\mathcal{M}_1$ and *$\mathcal{M}_2$ which each model axiom-set Σ, such that $\mathcal{M}_1$ models $\mathcal{S}$, and $\mathcal{M}_2$ models the negation of $\mathcal{S}$.*
Alternatively**:** *A sentence is true for some but not all models of an axiom-set, if-and-only-if, that sentence is logically independent of that axiom-set.*

A good reference is the section on logical independence, written by Edward Stabler, in his 1948 book. [12].

7 The action of Soundness and Completeness on fields algebra

Section 7 discusses the employment of Theorems 1 and 5, specifically applied to FIELDS ALGEBRA. These result in Theorems 6 and 7. And these two new theorems provide us with two practical tests – performed *by inspection* – telling us about a proposition's provability.

Propositions under test are *existential propositions* – those asserting existence of variables. Tests are applied by examining the proposition's truth-table. To illustrate, Table 2 lists the five truth-tables for propositions (1) to (5). The T and F entries are answers to the question: is the proposition adjacent, in this row, True or False for the interpretation, assigned, above the column?

Proposition	Interpretations		
	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha \mid \alpha = 3$	T	T	T
$\exists\alpha \mid \alpha \times \alpha = 4$	T	T	T
$\exists\alpha \mid \alpha \times \alpha = 2$	T	T	F
$\exists\alpha \mid \alpha \times \alpha = -1$	T	F	F
$\exists\alpha \mid \alpha^{-1} = 0$	F	F	F

Table 2 Truth-tables for some existential propositions. T and F denote true and false. The T and F entries are answers to the question: is the proposition to the left, true, for the interpretation above? Disagreement along a row confirms independence for that proposition. True in $\mathbb{Q}$ confirms dependence due to Theorem 6.

In advance of stating Theorems 6 and 7, we may say: *by inspection* of Table 2, the first two, and the last propositions, corresponding to (1), (2) and (5), are logically dependent; and the third and fourth propositions, corresponding to (3) and (4), are logically independent.

Theorem 6 ***Logical Dependence*** *is demonstrated if a proposition is* True *while interpreting its variables as rational.*

Proof Proof is in the following steps: 1 – 5. *by inspection*

A problem, here, must be recognised. An infinite number of AXIOMS is posed to exclude all modulo arithmetic.

1. Collectively invoke the AXIOMS of Table 1, to derive each and every proposition, $\mathcal{S}$, asserting existence of all the individual rational numbers. Steps (a) – (e).
   (a) The first step in this process is to derive THEOREMS that assert existence of every positive integer, thus:

$$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma \qquad \text{AXIOM A0} \qquad (33)$$
$$\forall\beta\exists\alpha \mid \alpha = \beta + \beta \qquad \text{by (33)}; \gamma \curvearrowright \beta \qquad (34)$$
$$\forall\gamma\exists\alpha \mid \gamma = \beta + \beta \qquad \text{by (34)}; \alpha \curvearrowright \gamma \qquad (35)$$
$$\forall\beta\exists\alpha \mid \alpha = \beta + \beta + \beta \qquad \text{Subst. (35), (33)}$$
$$\forall\beta\exists\alpha \mid \alpha = \beta + \beta + \beta + \cdots \qquad \cdots \qquad (36)$$
$$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha \qquad \text{AXIOM M1} \qquad (37)$$
$$\exists\beta \mid \beta = 1 \qquad \text{by (37)} \qquad (38)$$
$$\exists\alpha \mid \alpha = 1 + 1 + 1 + \cdots \qquad \text{Subst. (38), (36)} \qquad (39)$$

   Writing (39) in more concise language:

$$\exists\alpha \mid \alpha = n \qquad \text{for } n = 1, 2, \ldots \qquad (40)$$

   (b) Now apply AXIOM A2 to (40), to derive existence of every negative integer.
   (c) Next, apply AXIOM M2 to derive existence of the reciprocal of every non-zero integer.
   (d) Then invoke AXIOM M0 to derive existence of every rational number.
   (e) Finally add existence of zero to the system, by invoking AXIOM A1.
2. By Theorem 1, The Soundness Theorem, each of these propositions $\mathcal{S}$ is true in every model of AXIOMS.
3. Hence, all numbers, whose existence is asserted by these propositions, form a set $\mathbb{Q}$, subsumed by every infinite-field.
4. This set is closed and so forms a structure modelling AXIOMS.
5. By Theorem 2, The Completeness Theorem, every existential proposition $\mathcal{S}$, asserting existence in $\mathbb{Q}$, is provable.

Theorem 7 ***Logical Independence*** *is demonstrated if a proposition is* True *while its variables are interpreted as members of one infinite-field, but* False *when interpreted as members of a different infinite-field.*

Proof Sructures modelling AXIOMS are the infinite-fields. Hence, by Theorem 5, disagreement between infinite-fields implies logical independence.

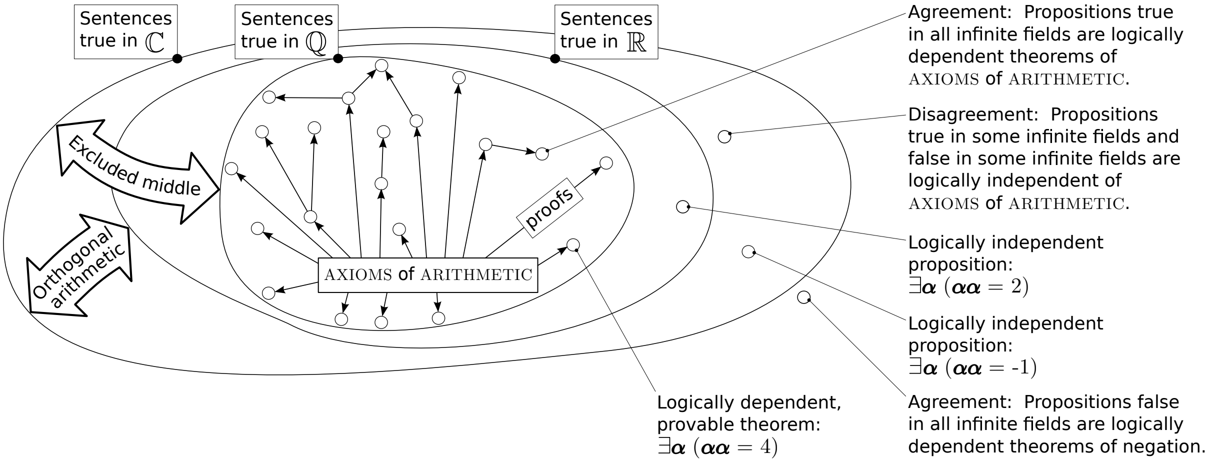

Figure 1 Truth-space for propositions (small circles) asserting existence of particular numbers. The innermost nesting is the set of all propositions, true (consistent with AXIOMS) in all infinite fields. The Completeness Theorem guarantees these are *logically dependent* theorems. The set to the exterior comprises propositions false (inconsistent with AXIOMS) in all infinite fields; these are the only propositions inconsistent with AXIOMS. The Completeness Theorem guarantees these are *logically dependent* negations. Soundness plus Completeness Theorems guarantee the excluded middle consists of *logically independent*, mathematically undecidable propositions.

8 Other examples of truth-tables.

PROPOSITION	INTERPRETATIONS		
	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha \mid \alpha = \mathsf{a}^{\mathbb{Q}}$	T	T	T
$\exists\alpha \mid \alpha = \mathsf{a}^{\mathbb{R}}$	T	T	F
$\exists\alpha \mid \alpha = \mathsf{a}^{\mathbb{C}}$	T	F	F

Table 3 Some more general examples: logical dependence of a particular, rational scalar $\mathsf{a}^{\mathbb{Q}}$, logical independence of the real scalar $\mathsf{a}^{\mathbb{R}}$ and logical independence of the complex scalar $\mathsf{a}^{\mathbb{C}}$.

PROPOSITION	INTERPRETATIONS		
	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \mid y = x^2$	T	T	T
$\forall y \exists x \mid y = x^2$	T	F	F

Table 4 Truth-tables concerning existence of x and y in the function $y = x^2$.

Table 4 demonstrates logical ambiguity present in ordinary mathematical formulae. Corresponding to the equation $y = x^2$, there are different possibilities in first-order logic: $\forall x \exists y \,|\, y = x^2$ and $\forall y \exists x \,|\, y = x^2$. The former, quantified by $\forall x \exists y$, is true for the rational field and therefore, exists by THEOREM. The latter, quantified by $\forall y \exists x$, has a disagreeing truth-table and therefore, is independent of AXIOMS.

In Table 5 we see validity of the *finite* polynomial

$$p\,(\mathsf{x}) = \mathsf{a} + \mathsf{b}\mathsf{x} + \mathsf{c}\mathsf{x}^2$$

compared with that of *infinite* series:

$$\exp\,(\mathsf{x}) \equiv \lim_{n\to\infty}\left[1 + \mathsf{x} + \frac{(\mathsf{x})^2}{2} + \cdots + \frac{(\mathsf{x})^n}{n!}\right].$$

Particular, rational scalars: a, b, c, x.

In both cases, all input variables are rational. Even so, in the case of the exponential, any rational input maps to an infinite sum that is never rational. So $\exp(\mathsf{x})$ is never logically dependent. In Table 6, the rational x is replace by the quantified $\forall x$.

Proposition	Interpretations		
	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y \mid y = p(\mathsf{x}) = \mathsf{a} + \mathsf{b}\mathsf{x} + \mathsf{c}\mathsf{x}^2$	T	T	T
$\exists y \mid y = \exp(\mathsf{x})$	T	T	F

Table 5 Truth-tables showing the logical dependence of a finite polynomial, versus, logical independence of an infinite series – where arguments are rational.

Proposition	Interpretations		
	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y \mid y = p(x) = \mathsf{a} + \mathsf{b}x + \mathsf{c}x^2$	T	T	T
$\forall x \exists y \mid y = \exp(x)$	T	T	F

Table 6 Truth-tables showing the logical dependence of a finite polynomial, versus, logical independence of an infinite series – where arguments are universally quantified.

Conclusions & Discussion

The premise of this paper is that mathematical physics rests on a foundation of elementary algebra, and in doing so, inherits the information it contains or conveys. The approach taken, is to treat elementary algebra as a formal axiomatised system, in order to expose logical information that might be passed into quantum mathematics.

As a formal axiomatised system, elementary algebra becomes a theory of existence — existence of scalars, that is. In that theory, two modes of existence occur. There is existence, provable under the system axioms; this is *logically dependent* existence. Then there is existence that is neither provable nor negatable, under those axioms; This is *logically independent* existence. Rational scalars are shown to exist logically dependent on axioms, while imaginary scalars are shown logically independent. These findings suggest a description of quantum measureable observables that is inherently logically dependent, in contrast to, quantum probability amplitudes, characterised by logical independence.

Together with *non-existence* of entities such as infinity, denied by axioms, the two modes of existence form an existential system, constituting a 3-valued logic. This seems most probably the missing mathematical foundation for the 3-valued logic of Hans Reichenbach, which he showed resolves 'causal anomalies' of quantum mechanics [10,11].

Ongoing research

Standard quantum theory has a further axiom, on top of Elementary Algebra, which imposes unitarity (or self-adjointness) – *by Postulate.* This postulate is conflicting information that blocks the logical independence of imaginary scalars, so destroying the 3-valued logic.

Nevertheless, if unitary information can be shown to emerge naturally out of *quantum mathematics,* without being imposed as a Physical Principle, without the need for it being imposed *by Postulate,* thus rendering redundant the unitarity *by Postulate*, then logical independence from elementary algebra would freely enter quantum mathematics. And if that were to be possible — via the logical independence in Boolean propositions, used by Tomasz Paterek et al [8,9] — the prospect would open up of finding a theoretical link, directly connecting logical independence in Elementary Algebra, with quantum randomness.

References

1. Alain Aspect, Jean Dalibard, and Gérard Roger, *Experimental test of Bell's inequalities using time- varying analyzers*, Physical Revue Letters **49** (1982), no. 25, 1804–1807.
2. Alain Aspect, Philippe Grangier, and Gérard Roger, *Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of Bell's inequalities*, Physical Review Letters **49** (1982), no. 2, 91–94.
3. John Bell, *On the Einstein Podolsky Rosen paradox*, Physics **1** (1964), 195–200.
4. Andrew Brown, Graham Flegg, Derek Goldrei, Alison Hughes, Ros Porter, Alan Slomson, Alex Wilkie, and Hossein Zand, *M335 Studies in pure mathematics*, vol. Logic Units 5-8, The Open University, Milton Keynes UK, 1981.
5. Peter J Cameron, *Sets, logic and categories*, Springer-Verlag London Limited, 1999.
6. Gregory J Chaitin, *Gödel's theorem and information*, International Journal of Physics **21** (1982), 941–954.
7. S Kochen and E P Specker, *The problem of hidden variables in quantum mechanics*, Journal of Mathematics and Mechanics **17** (1967), 59–87.
8. Tomasz Paterek, Johannes Kofler, Robert Prevedel, Peter Klimek, Markus Aspelmeyer, Anton Zeilinger, and Caslav Brukner, *Logical independence and quantum randomness*, New Journal of Physics **12** (2010), no. 013019, 1367–2630.
9. ———, *Logical independence and quantum randomness — with experimental data*, arxiv.org/pdf/0811.4542v2.pdf (2010).
10. Hilary Putnam, *'three-valued logic'*, Philosophical Studies **8** (1957), no. 5, 73–80.
11. Hans Reichenbach, *Philosophic foundations of quantum mechanics*, University of California Press, Berkeley and Los Angeles California, 1944.
12. Edward Russell Stabler, *An introduction to mathematical thought*, Addison-Wesley Publishing Company Inc., Reading Massachusetts USA, 1948.