Gravity in Curved Phase-Spaces : Towards Geometrization of Matter

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia. 30314, perelmanc@hotmail.com

September 2015

Abstract

After reviewing the basic ideas behind Born's Reciprocal Relativity theory, the geometry of the (co) tangent bundle of spacetime is studied via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. The curvature tensors in the (co) tangent bundle of spacetime are explicitly constructed leading to the analog of the Einstein vacuum field equations. The geometry of Hamilton Spaces associated with curved phase spaces follows. An explicit construction of a gauge theory of gravity in the $8D$ co-tangent bundle T^*M of spacetime is provided, and based on the gauge group $SO(6,2) \times_s R^8$ which acts on the tangent space to the cotangent bundle $T_{(x,p)}T^*M$ at each point $(\mathbf{x}, \mathbf{p})$. Several gravitational actions associated with the geometry of curved phase spaces are presented. We conclude with a discussion about the geometrization of matter, QFT in accelerated frames, **T**-duality, double field theory, and generalized geometry.

1 Introduction : Born's Reciprocal Relativity in Phase Space

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where

$X^i, T, E, P^i; i = 1,2,3$ are *all* boosted (rotated) into each-other, were given by [2] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$.

The $U(1,3) = SU(1,3) \otimes U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dt \wedge dp_0 + \delta_{ij} dx^i \wedge dp^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $8D$ phase-space (in natural units $\hbar = c = 1$)

$$(d\sigma)^2 \;=\; (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 + \frac{1}{b^2}\,\left((dE)^2 \;-\; (dp_x)^2 \;-\; (dp_y)^2 \;-\; (dp_z)^2\right) \tag{1.1}$$

the rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [2] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions y, z, p_y, p_z intact. There is now a subgroup $U(1,1) = SU(1,1) \otimes U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; (dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; (dP)^2}{b^2} \;=$$

$$(d\tau)^2 \left(1 \;+\; \frac{(dE/d\tau)^2 \;-\; (dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 \;-\; \frac{F^2}{F^2_{max}} \right) \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in (2.2). The proper force interval $(dE/d\tau)^2 - (dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(2.2) involves the ratios of two proper *forces*.

If (in natural units $\hbar = c = 1$) one sets the maximal proper-force to be given by $b \equiv m_P A_{max}$, where $m_P = (1/L_P)$ is the Planck mass and $A_{max} = (1/L_p)$, then $b = (1/L_P)^2$ may also be interpreted as the maximal string tension. The units of b would be of $(mass)^2$. In the most general case there are four scales of time, energy, momentum and length that can be constructed from the three constants $b, c, \hbar$ as follows

$$\lambda_t \;=\; \sqrt{\frac{\hbar}{bc}};\;\; \lambda_l \;=\; \sqrt{\frac{\hbar\ c}{b}};\;\; \lambda_p \;=\; \sqrt{\frac{\hbar\ b}{c}};\;\; \lambda_e \;=\; \sqrt{\hbar\ b\ c} \tag{1.3}$$

The gravitational constant can be written as $G = \alpha_G\ c^4/b$ where α_G is a dimensionless parameter to be determined experimentally. If $\alpha_G = 1$, then the four scales (2.3) coincide with the *Planck* time, length, momentum and energy, respectively.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (2.2) invariant are [2]

$$T' \;=\; T\ cosh\xi \;+\; (\frac{\xi_v\ X}{c^2} \;+\; \frac{\xi_a\ P}{b^2})\ \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$E' \;=\; E\; cosh\xi \;+\; (-\xi_a\; X \;+\; \xi_v P)\; \frac{sinh\xi}{\xi} \tag{1.4b}$$

$$X' \;=\; X\; cosh\xi \;+\; (\xi_v\; T \;-\; \frac{\xi_a\; E}{b^2})\; \frac{sinh\xi}{\xi} \tag{1.4c}$$

$$P' \;=\; P\; cosh\xi \;+\; (\frac{\xi_v\; E}{c^2} \;+\; \xi_a\; T)\; \frac{sinh\xi}{\xi} \tag{1.4d}$$

ξ_v is the velocity-boost rapidity parameter and the ξ_a is the force (acceleration) boost rapidity parameter of the primed-reference frame. These parameters are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) \;=\; \frac{F}{F_{max}} \tag{1.5}$$

It is straightforwad to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - c^2(dP)^2]$. Only the *combination*

$$(d\sigma)^2 \;=\; (d\tau)^2\; \left(\; 1 \;-\; \frac{F^2}{F^2_{max}}\;\right) \tag{1.6}$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = -\; dT \wedge E + dX \wedge dP$.

To show the consistency of eqs-(1.4, 1.5, 1.6) let us describe the following scenario. A massive free particle does not experience any force, thus the momentum is conserved so that $\frac{dp_a}{d\tau} = 0$ and the flat phase space interval is $(d\sigma)^2 = (d\tau)^2$. In an accelerated frame of reference the massive particle experiences a pseudo-force which implies that $\frac{dp'_a}{d\tau'} \neq 0$. Upon choosing an infinite rapidity parameter $\xi_a = \infty$ in eqs-(1.5), the value of the pseudo-force reaches its maximal proper value $F_{max} = \mathbf{b}$. Also, $(d\tau')^2 = \infty$ when the acceleration rapidity parameter is ∞, as one can verify from eqs-(1.4) by simple inspection. Since the interval in flat phase space (1.6) , in an inertial frame and accelerated frame of reference, respectively, remains invariant under the transformations (1.4) one has that $(d\sigma)^2 = (d\tau)^2 = (d\tau')^2(1 - F^2/F^2_{max}) = \infty \times 0 \neq 0$. If $(d\tau)^2$ were zero, in the inertial non-accelerated frame of reference , this would mean that the massive free particle would have followed a null geodesic, which it cannot do since only massless photons can.

We explored in [5] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The $8D$ tangent bundle of spacetime and the physics of a limiting value of the proper acceleration in spacetime [4] has been studied by Brandt [3] . Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. The purpose of this work is to analyze in further detail the geometry of the (co) tangent bundle of spacetime via the introduction of nonlinear connections associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. The procedure provided in section **2** $differs$ from the one taken by Brandt [3]. The curvature tensors in the (co) tangent bundle of spacetime are explicitly constructed in section **2** leading to the analog of the Einstein vacuum field equations.

In section **3** the geometry of Hamilton Spaces associated with curved phase spaces is analyzed within the context of the maximal proper force principle in Born's reciprocal relativity. In section **4** a gauge theory of gravity in the $8D$ cotangent bundle T^*M of spacetime is constructed. Several gravitational actions associated with the geometry of curved phase spaces are presented. It should be emphasized that our results described in section **4** are quite different than those obtained earlier by us in [12] and by [3]. We conclude with a discussion about the geometrization of matter, QFT in accelerated frames, **T**-duality, double field theory, and generalized geometry.

2 Geometry of the (Co) Tangent Bundle of Spacetime

In this section we shall present the essentials behind the geometry of the tangent and cotangent space. We will follow closely the description by authors [9], [10], where one may study also in detail the geometry of Lagrange-Finsler and Hamilton-Cartan Spaces and their higher order generalizations. The metric associated with the tangent space TM_d can be written in the in the following block diagonal form

$$(ds)^2 \ = \ g_{ij}(x^k, y^a) \ dx^i d \ x^j \ + \ h_{ab}(x^i, y^a) \ \delta y^a \ \delta y^b \tag{2.1}$$

$(i, j, k = 1, 2, 3, ...d; \ a, b, c = 1, 2, 3, ...d)$ if instead of the standard coordinate-basis one introduces the anholonomic frames (non-coordinate basis) defined as

$$\delta_i \ = \ \partial_i - N_i^b(x, y) \ \partial_b \ = \ \partial / \partial x^i - N_i^b(x, y) \ \partial_b; \quad \partial_a \ = \ \frac{\partial}{dy^a} \tag{2.2}$$

and its dual basis is

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \ \ \delta^a = dy^a + N_k^a(x, y) \ dx^k) \tag{2.3}$$

where the N–coefficients define a nonlinear connection, N–connection structure, see details in [9], [10]. As a very particular case one recovers the ordinary linear connections if $N_i^a(x,y) = \Gamma^a_{bi}(x)\, y^b$.

The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$, with nontrivial anholonomy coefficients

$$\begin{aligned} W^k_{ij} &= 0; \;\; W^k_{aj} = 0; \;\; W^k_{ia} = 0; \;\; W^k_{ab} = 0; \;\; W^c_{ab} = 0 \\ W^a_{ij} &= -\Omega^a_{ij}; \;\; W^a_{bj} = -\partial_b N^a_j; \;\; W^b_{ia} = \partial_a N^b_j \end{aligned} \tag{2.4}$$

where

$$\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j \tag{2.5}$$

is the nonlinear connection curvature (N–curvature). This is the same object as $F^a_{\mu\nu}$ described in the previous section when $N^a_j \leftrightarrow A^a_\mu$.

A metric of type given by eq-(2.1) with arbitrary coefficients $g_{ij}(x^k, y^a)$ and $h_{ab}(x^k, y^a)$ defined with respect to a N–elongated basis is called a distinguished metric. A linear connection $D_{\delta_\gamma}\delta_\beta = \Gamma^\alpha{}_{\beta\gamma}(x,y)\delta_\alpha$ associated to an operator of covariant derivation D is compatible with a metric $g_{\alpha\beta}$ and N–connection structure on a pseudo–Riemannian spacetime if $D_\alpha g_{\beta\gamma} = 0$. The linear distinguished connection is parametrized by irreducible (horizontal, vertical) h–v–components, $\Gamma^\alpha{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc} \right)$ such that [9], [10].

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk} \right) \\ L^a{}_{bk} &= \partial_b N^a_k + \frac{1}{2} h^{ac} \left(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k \right) \\ C^i{}_{jc} &= \frac{1}{2} g^{ik}\partial_c g_{jk}; \quad C^a{}_{bc} = \frac{1}{2} h^{ad} \left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc} \right). \end{aligned} \tag{2.6}$$

This defines a canonical linear connection (distinguished by a N–connection) which is similar to the metric connection introduced by Christoffel symbols in the case of holonomic bases. The anholonomic coefficients $w^\gamma{}_{\alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^\alpha_{\beta\gamma}\delta_\alpha$. One arrives at

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta} + w^\alpha{}_{\beta\gamma}, \tag{2.7}$$

and at the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_\beta{}^\alpha{}_{\gamma\tau}\delta_\alpha$

$$R_\beta{}^\alpha{}_{\gamma\tau} = \delta_\tau \Gamma^\alpha{}_{\beta\gamma} - \delta_\gamma \Gamma^\alpha{}_{\beta\tau} + \Gamma^\sigma{}_{\beta\gamma}\Gamma^\alpha{}_{\sigma\tau} - \Gamma^\sigma{}_{\beta\tau}\Gamma^\alpha{}_{\sigma\gamma} + \Gamma^\alpha{}_{\beta\sigma} w^\sigma{}_{\gamma\tau} \tag{2.8}$$

One should note the key presence of the last term in (2.8) due to the nonvanishing anholonomic coefficients $w^\gamma{}_{\alpha\beta}$.

The torsion distinguished tensor has the following irreducible, nonvanishing, h–v–components, $T^\alpha{}_{\beta\gamma} = \left(T^i{}_{jk}, C^i{}_{ja}, S^a{}_{bc}, T^a{}_{ij}, T^a{}_{bi} \right)$ given by

$$T^i_{jk} = L^i_{jk} - L^i_{kj}; \;\; T^i_{ja} = C^i_{ja}; \;\; T^i_{aj} = - C^i_{ja}$$

$$T^i_{ja} = 0; \quad T^a_{bc} = S^a_{bc} = C^a_{bc} - C^a_{cb}$$

$$T^a_{ij} = -\Omega^a_{ij}; \quad T^a_{bi} = \partial_b N^a_i - L^a_{bi}; \quad T^a_{ib} = - T^a_{bi} \tag{2.9}$$

and where $\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j$ can be interpreted as the "field strength" associated with the nonlinear connection N^a_i.

The curvature distinguished tensor has the following irreducible, non-vanishing, h–v–components $R_\beta{}^\alpha{}_{\gamma\tau} = \left(R^i_{hjk}, R^a_{bjk}, P^i_{jka}, P^c_{bka}, S^i_{jbc}, S^a_{bcd}\right)$ given by [9], [10],

$$R^i_{hjk} = \delta_k L^i_{hj} - \delta_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{jk} \tag{2.10a}$$

$$R^a_{bjk} = \delta_k L^a_{bj} - \delta_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{jk} \tag{2.10b}$$

$$P^i_{jka} = \partial_a L^i_{jk} + C^i_{jb} T^b_{ka} - \left(\delta_k C^i_{ja} + L^i_{lk} C^l_{ja} - L^l_{jk} C^i_{la} - L^c_{ak} C^i_{jc}\right) \tag{2.10c}$$

$$P^c_{bka} = \partial_a L^c_{bk} + C^c_{bd} T^d_{ka} - \left(\delta_k C^c_{ba} + L^c_{dk} C^d_{ba} - L^d_{bk} C^c_{da} - L^d_{ak} C^c_{bd}\right) \tag{2.10d}$$

$$S^i_{jbc} = \partial_c C^i_{jb} - \partial_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb} \tag{2.10e}$$

$$S^a_{bcd} = \partial_d C^a_{bc} - \partial_c C^a_{bd} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec} \tag{2.10f}$$

Having reviewed the geometry of the tangent bundle TM we proceed with the cotangent bundle case T^*M (phase space). In the case of the cotangent space of a d-dim manifold T^*M_d the metric can be equivalently rewritten in the block diagonal form [9] as

$$(ds)^2 = g_{ij}(x^k, p_a)\, dx^i d\, x^j + h^{ab}(x^k, p_c)\, \delta p_a\, \delta p_b \tag{2.11}$$

$i, j, k = 1, 2, 3, ...d,\ a, b, c = 1, 2, 3, ...d$, if instead of the standard coordinate basis one introduces the following anholonomic frames (non-coordinate basis)

$$\delta_i = \delta/\delta x^i = \partial_{x^i} + N_{ia}\, \partial^a = \partial_{x^i} + N_{ia}\, \partial_{p_a}; \quad \partial^a \equiv \partial_{p_a} = \frac{\partial}{\partial p_a} \tag{2.12}$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \quad \delta p_a = dp_a - N_{ja}\, dx^j \tag{2.13}$$

where the N–coefficients define a nonlinear connection, N–connection structure. An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \ = \ H^k_{ij} \ \delta_k; \ \ D_{\delta_j}(\partial^a) \ = \ - \ H^a_{bj} \ \partial^b; \tag{2.14a}$$

$$D_{\partial^a}(\delta_i) \ = \ C^{ka}_i \ \delta_k; \ \ D_{\partial^a}(\partial^b) \ = \ - \ C^{ba}_c \ \partial^c \tag{2.14b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \ = \ \frac{1}{2} T^i_{jk} \ dx^j \wedge dx^k \ + \ C^{ia}_j \ dx^j \wedge \delta p_a \tag{2.15a}$$

$$\Omega_a \ = \ \frac{1}{2} R_{jka} \ dx^j \wedge dx^k \ + \ P^b_{aj} \ dx^j \wedge \delta p_b \ + \ \frac{1}{2} S^{bc}_a \ \delta p_b \wedge \delta p_c \tag{2.15b}$$

and the curvature 2-forms are

$$\Omega^i_j \ = \ \frac{1}{2} R^i_{jkm} \ dx^k \wedge dx^m \ + \ P^{ia}_{jk} \ dx^k \wedge \delta p_a \ + \ \frac{1}{2} S^{iab}_j \ \delta p_a \wedge \delta p_b \tag{2.16}$$

$$\Omega^a_b \ = \ \frac{1}{2} R^a_{bkm} \ dx^k \wedge dx^m \ + \ P^{ac}_{bk} \ dx^k \wedge \delta p_c \ + \ \frac{1}{2} S^{acd}_b \ \delta p_c \wedge \delta p_d \tag{2.17}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i, \ \partial^a = \partial/\partial p_a$ is given by $dx^i, \ \delta p_a = dp_a - N_{ja} dx^j$.

The distinguished torsion tensors are of the form [9]

$$T^i_{jk} \ = \ H^i_{jk} \ - \ H^i_{kj}; \ \ S^{ab}_c \ = \ C^{ab}_c \ - \ C^{ba}_c; \ \ P^a_{bj} \ = \ H^a_{bj} \ - \ \partial^a \ N_{jb}$$

$$R_{ija} \ = \ \frac{\delta N_{ja}}{\delta x^i} \ - \ \frac{\delta N_{ia}}{\delta x^j} \tag{2.18}$$

The distinguished tensors of the curvature are of the form

$$R^i_{kjh} \ = \ \delta_h H^i_{kj} \ - \ \delta_j H^i_{kh} \ + \ H^l_{kj} \ H^i_{lh} \ - \ H^l_{kh} \ H^i_{lj} \ - \ C^{ia}_k \ R_{jha} \tag{2.19}$$

$$P^{ab}_{cj} \ = \ \partial^a \ H^b_{cj} \ + \ C^{ad}_c \ P^b_{dj} \ - \ \left(\delta_j \ C^{ab}_c \ + \ H^b_{dj} \ C^{da}_c \ + \ H^a_{dj} \ C^{bd}_c \ - \ H^d_{cj} \ C^{ab}_d\right) \tag{2.20}$$

$$P^{ak}_{ij} \ = \ \partial^a \ H^k_{ij} \ + \ C^{al}_i \ T^k_{lj} \ - \ \left(\delta_j \ C^{ak}_i \ + \ H^a_{bj} \ C^{bk}_i \ + \ H^k_{lj} \ C^{al}_i \ - \ H^l_{ij} \ C^{ak}_l\right) \tag{2.21}$$

$$S^{abc}_d \ = \ \partial^c \ C^{ab}_d \ - \ \partial^b \ C^{ac}_d \ + \ C^{eb}_d \ C^{ac}_e \ - \ C^{ec}_d \ C^{ab}_e; \ \ etc.......... \tag{2.22}$$

where we have omitted the other components and once again we have for our notation $\partial^a = \partial/\partial p_a$ and $\delta/\delta x^i = \partial_{x^i} + N_{ia}\, \partial^a$. Equipped with these curvature tensors one can perform suitable contractions involving g_{ij}, h^{ij} to obtain two curvature scalars of the $\mathcal{R}, \mathcal{S}$ type

$$\mathcal{R} \ = \ \delta^j_i \ R^i_{kjl} \ g^{kl}; \quad \mathcal{S} \ = \ \delta^d_b \ S^{abc}_d \ h_{ac} \tag{2.23}$$

and construct a $2d$-dim gravitational phase space action involving a linear combination of the curvature scalars

$$S \ = \ \frac{1}{2\kappa^2} \int \ d^d x \ d^d p \ \sqrt{|det\ g|} \ \sqrt{|det\ h|} \ (\ c_1 \ \mathcal{R} \ + \ c_2 \ \mathcal{S} \ ) \tag{2.24}$$

where c_1, c_2 are real-valued numerical coefficients and κ^2 is the gravitational coupling constant. In this case, the vacuum field equations associated with the geometry of the cotangent bundle are

$$\frac{\delta S}{\delta g_{ij}} = 0, \quad \frac{\delta S}{\delta h^{ab}} = 0, \quad \frac{\delta S}{\delta N_{ia}} = 0 \tag{2.25}$$

The generalized (vacuum) field equations corresponding to gravity in the curved $2d$-dimensional (co) tangent bundle, and which are obtained from a direct variation of the tangent space/phase space actions with respect to the respective fields

$$g_{ij}(x^k, y^a), \ h_{ab}(x^k, y^a), \ N^a_i(x^k, y^a); \quad g_{ij}(x^k, p_a), \ h^{ab}(x^k, p_a), \ N_{ia}(x^k, p_a) \tag{2.26}$$

needs to be investigated further. Field equations of the form

$$\mathcal{R}_{ij} \ - \ \frac{1}{2} (\ \mathcal{R} + \mathcal{S} \ ) \ g_{ij} \ = 0 \ ; \quad \mathcal{S}_{ab} \ - \ \frac{1}{2} (\ \mathcal{R} + \mathcal{S} \ ) \ h_{ab} \ = 0 \tag{2.27}$$

$$\frac{\delta(\mathcal{R} + \mathcal{S})}{\delta N^a_i} \ = \ 0 \tag{2.28}$$

were studied by Vacaru [10].

When $i, j = 1, 2, \ldots, d$, and $a, b = 1, 2, \ldots, d$ the number of field equations is

$$\frac{1}{2} d(d+1) \ + \ \frac{1}{2} d(d+1) \ + \ d^2 \ = \ \frac{2d(2d+1)}{2} \tag{2.29}$$

which match the number of independent degrees of freedom of a metric g_{MN} in $2d$-dimensions. One should emphasize however, that a careful analysis [12] reveals that there is *no* mathematical equivalence among the above eqs-(2.25) in the (co) tangent bundle, with the ordinary Einstein vacuum field equations in a Riemannian spacetime in $2d$-dimensions

$$\mathbf{R}_{MN}(X) \ - \ \frac{1}{2} \ \mathbf{g}_{MN}(X) \ \mathbf{R}(X) \ = 0; \quad M, N = 1, 2, 3, ..., 2d \tag{2.30}$$

To finalize it is very important to remark that in section **1** we discussed Quaplectic transformations (like eqs-(1.4)) in $flat$ phase spaces which leave invariant the Born-Green line interval (1.1) and the symplectic two form $\Omega = -dt \wedge dp_0 + dx^i \wedge dp_i$. In the (co) tangent space description analyzed in this section one has covariance under a more *restricted* set of coordinate transformations of the form [9]

$$x'^i \;=\; x'^i(x^j), \quad y'^i \;=\; y^j \; \frac{\partial x'^i}{\partial x^j} \tag{2.31}$$

$$x'^i \;=\; x'^i(x^j), \quad p'_i \;=\; p_j \; \frac{\partial x^j}{\partial x'^i} \tag{2.32}$$

whereas Quaplectic transformations in flat phase space, in general, are of the form $x'^i \;=\; x'^i(x^j, p_j), p'_i(x^j, p_j)$. Thus one cannot accommodate the Quaplectic transformations to curved phase spaces (the cotangent bundle T^*M) in the manner described in eq-(2.32). The geometry of phase space was extensively studied in the lengthy monograph by [6].

3 Hamilton Spaces and Maximal Proper Force in the Cotangent Bundle

Having studied the geometry of the (co) tangent bundle, let us begin with the $8D$ cotangent space (phase-space) infinitesimal interval given by

$$(d\sigma)^2 \;=\; g_{ij}(x,p)\; dx^i \; dx^j \;+\; \frac{h^{ab}(x,p)}{\mathbf{b}^2} \; (dp_a \;-\; N_{ac}\; dx^c) \; (dp_b \;-\; N_{bd}\; dx^d) \tag{3.1}$$

after defining $g_{ij}(x,p) dx^i dx^j = (d\tau)^2$, the interval can be rewritten as

$$(d\tau)^2 \;+\; \frac{h^{ab}}{\mathbf{b}^2} \; (\frac{dp_a}{d\tau} \;-\; N_{ac} \; \frac{dx^c}{d\tau}) \; (\frac{dp_b}{d\tau} \;-\; N_{bd} \; \frac{dx^d}{d\tau}) \; (d\tau)^2 \tag{3.2}$$

furthermore, the interval also be recast as

$$(d\sigma)^2 \;=\; (d\tau)^2 \; \left(1 \;-\; \frac{F^2}{F^2_{max}} \right); \quad F^2_{max} \;=\; \mathbf{b}^2 \tag{3.3}$$

and given in terms of a generalized (spacelike) proper force squared F^2 defined as

$$-F^2 \;\equiv\; h^{ab} \; (\frac{dp_a}{d\tau} \;-\; N_{ac} \; \frac{dx^c}{d\tau}) \; (\frac{dp_b}{d\tau} \;-\; N_{bd} \; \frac{dx^d}{d\tau}) \tag{3.4}$$

After writing

$$\frac{\delta p_a}{d\tau} \ = \ \frac{dp_a}{d\tau} \ - \ N_{ac} \ \frac{dx^c}{d\tau} \tag{3.5}$$

eq-(3.4) becomes

$$-F^2 \ \equiv \ h^{ab} \ (\frac{\delta p_a}{d\tau}) \ (\frac{\delta p_b}{d\tau}) \tag{3.6}$$

furthermore, we may express eq-(3.6) in terms of the Hamiltonian after using the Born reciprocally invariant Hamilton's equations of motion

$$\frac{dp_a}{d\tau} \ = \ - \ \partial_a H \ = \ - \ \frac{\partial H}{\partial x^a}, \ \ \frac{dx^a}{d\tau} \ = \partial^a H \ = \ \frac{\partial H}{\partial p_a} \tag{3.7}$$

such that

$$\frac{dp_a}{d\tau} \ = \ - \ \partial_a H \ \Rightarrow$$

$$\frac{\delta p_a}{d\tau} \ = \ \frac{dp_a}{d\tau} \ - \ N_{ab} \ \frac{dx^b}{d\tau} \ = \ - \ \delta_\alpha H \ = \ - \ (\frac{\partial H}{\partial x^a} \ + \ N_{ab} \frac{\partial H}{\partial p_b}) \tag{3.8}$$

and the generalized proper force squared, finally, can be recast as

$$F^2 \ = \ F^a F_a \ = \ - \ \pi^{ab} \ (\frac{\delta p_a}{d\tau}) \ (\frac{\delta p_b}{d\tau}) \ = \ - \ \pi^{ab} \ (\delta_a H) \ (\delta_b H) \tag{3.9}$$

A particle moving along the *autoparallel* trajectories associated with the Hamilton space is described by $\frac{\delta p_a}{d\tau} = \delta_a H = 0$ leading to a zero generalized force as expected. This is consistent with the fact that the Hamilton equations of motion become the autoparallel equations associated to the (nonlinear) connection $N_{ab}(x,p)$. It was shown in [7] that *homogeneous* Hamiltonians $H(x,\lambda p) = \lambda^r H(x,p)$ lead to $\delta_a H = 0$, and consequently, to a zero generalized force. Hamiltonians which are not homogeneous furnish force-like terms that drag the particles away from the autoparallel motion [7].

Given a Hamiltonian $H(x,p)$ associated with a Hamilton space, the generalized proper force squared displayed in eqs-(3.4, 3.9) is given in terms of the metric and Nonlinear connection which are defined, respectively, as follows [9]

$${}^{(H)}g_{ab} \ = \ \frac{1}{2} \ \frac{\partial^2 H}{\partial p^a \partial p^b}, \ \ h^{ab} \ = \ {}^{(H)}g^{ab} \tag{3.10}$$

$$N_{ab}(x,p) \ = \ \frac{1}{4} \ (\ \{g_{ab}(x,p), H\} \ - \ g_{ac}\partial_b\partial^c H \ - \ g_{bc}\partial_a\partial^c H \ ) \tag{3.11}$$

given $\partial_c \equiv \frac{\partial}{\partial x^c}, \partial^c \equiv \frac{\partial}{\partial p_c}$, the Poisson bracket is

$$\{g_{ab}, \ H\} \ = \ (\partial_c g_{ab}) \ (\partial^c H) \ - \ (\partial^c g_{ab}) \ (\partial_c H) \ =$$

$$(\frac{\partial}{\partial x^c} g_{ab}) \ (\frac{\partial}{\partial p_c} H) \ - \ (\frac{\partial}{\partial p_c} g_{ab}) \ (\frac{\partial}{\partial x^c} H) \tag{3.12}$$

Therefore, in Hamilton spaces the metric and nonlinear connection are determined by the Hamiltonian $H(x,p)$. The nonlinear connection in Hamilton spaces is symmetric $N_{ab} = N_{ba}$ and has zero nonmetricity : the metric is covariantly constant $\nabla^{(H)} g_{ab} = 0$, see [7], [9] for references.

In the special case when one has a quadratic Hamiltonian given by

$$H = g_{ab}(x) \ \frac{p^a \ p^b}{m} \ + \ V(x) \tag{3.13}$$

one will have

$$h^{ab}(x) \ = \ g^{ab}(x); \quad N^a_b(x,p) \ = \ p^c \ \Gamma^a_{bc}(x);$$
$$\frac{dx^a}{d\tau} \ = \ \frac{p^a}{m}; \quad F^a \ = \ \frac{dp^a}{d\tau} \ + m^{-1} \ \Gamma^a_{bc} \ p^b \ p^c \tag{3.14}$$

therefore, in this special case the connection $N^a_b(x,p)$ is linear in the momentum and proportional to the Levi-Civita connection $\Gamma^a_{bc}(x)$. Hence, F^a is now the physical proper force experienced by a particle of mass m moving in a Riemannian spacetime, with metric $g_{ab}(x)$ and whose infinitesimal proper time interval is $g_{ab}(x)dx^a dx^b = (d\tau)^2$. For a timelike trajectory the force (acceleration) is spacelike.

To finalize this section we shall discuss the consistency of the vacuum field equations (2.27,2.28) in the case of Hamiltonian spaces. Given a solution $g_{kl}(x,p)$ to the vacuum field equations (2.27), one can always associate a Hamiltonian function given by

$$H(x,p) \ = \ \left(2 \int \ g_{ab}(x,p) \ dp^a \ dp^b \right) \ + \ A_a(x) \ p^a \ + \ V(x) \tag{3.15}$$

where $A_a(x), V(x)$ are arbitrary functions. The nonlinear connection coefficients are provided in eq-(3.11). The key question remains whether or not the nonlinear connection coefficients described by the expression in eq-(3.11), in terms of the above Hamiltonian (3.15) and the solutions $g_{kl}(x,p)$ to the metric vacuum field equations (2.27), are in fact also solutions to the nonlinear connection vacuum field equations (2.28).

Setting $A_a(x) = 0, V(x) = 0$, a class of solutions for the Hamiltonian can be chosen to be

$$H(x,p) \ = \ \sum_{n=1}^{N} \ G_{a_1 a_2 .....a_n}(x) \ p^{a_1} \ p^{a_2} \ ..... \ p^{a_n} \tag{3.16}$$

leading to a metric

$${}^{(H)}g_{ab}(x,p) \ = \ \sum_{n=1}^{N} \ G_{a_1 a_2 ....a_n}(x) \ \delta^{a_1}_a \ \delta^{a_2}_b \ p^{a_3} \ p^{a_4} ..... \ p^{a_n} \ + \ .... \tag{3.17}$$

and to a nonlinear connection N_{ij} given in eq-(3.11). One must verify whether or not the functions $G_{a_1 a_2 .....a_n}(x), n = 1, 2, \ldots, N$ allows us to construct nontrivial solutions to the vacuum field equations (2.27,2.28) for the metric $g_{ab}, h^{ab} = g^{ab}$ and nonlinear connection N_{ab}. This is a very difficult consistency problem that needs to be investigated further.

4 Gauge Theories of Gravity in the Cotangent Bundle

In this section we will construct a gauge theory of gravity in the $8D$ cotangent bundle T^*M based on the gauge group given by the semidirect product $SO(6,2) \times_s R^8$. Let us begin with a Lie group $\mathcal{G}$; its associated Lie algebra is spanned by the generators $\mathcal{L}_A, A = 1, 2, \ldots,$ dim $\mathcal{G}$, and whose structure constants are f^C_{AB}. The Lie algebra commutator is $[\mathcal{L}_A, \mathcal{L}_B] = f^C_{AB}\mathcal{L}_C$. The components of the gauge field strength in the $8D$ cotangent bundle T^*M, and corresponding to the Lie-algebra valued gauge fields $\mathcal{A}^A_i \mathcal{L}_A, \mathcal{A}^A_a \mathcal{L}_A$, are

$$\mathcal{F}^A_{ij} \ = \ \delta_i \mathcal{A}^A_j \ - \ \delta_j \mathcal{A}^A_i \ + \ [\ \mathcal{A}_i, \ \mathcal{A}_j \ ]^A =$$
$$(\frac{\partial}{\partial x^i} \ + \ \mathbf{b} \ N_{ib} \ \frac{\partial}{\partial p_b}) \ \mathcal{A}^A_j \ - \ (\frac{\partial}{\partial x^j} \ + \ \mathbf{b} \ N_{jb} \ \frac{\partial}{\partial p_b}) \ \mathcal{A}^A_i \ +$$
$$\mathcal{A}^B_i \ \mathcal{A}^C_j \ f^A_{BC} \tag{4.1}$$

$$\mathcal{F}^A_{ab} \ = \ \frac{\partial}{\partial p^a}\mathcal{A}^A_b \ - \ \frac{\partial}{\partial p^b}\mathcal{A}^A_a \ + \ \mathcal{A}^B_a \ \mathcal{A}^C_b \ f^A_{BC} \tag{4.2}$$

$$\mathcal{F}^A_{ia} \ = \ \delta_i \mathcal{A}^A_a \ - \ \partial_a \mathcal{A}^A_i \ + \ \mathcal{A}^B_i \ \mathcal{A}^C_a \ f^A_{BC} \tag{4.3}$$
$$\mathcal{F}^A_{ai} \ = \ \partial_a \mathcal{A}^A_i \ - \ \delta_i \mathcal{A}^A_a \ + \ \mathcal{A}^B_a \ \mathcal{A}^C_i \ f^A_{BC} \tag{4.4}$$

there is anti-symmetry in the indices $\mathcal{F}^A_{ia} = -\mathcal{F}^A_{ai}$ and the particular Lie-algebra-valued two-form field strength is $\mathcal{F}^A_{ia} dx^i \wedge \delta p^a$ where $dx^i \wedge \delta p^a \ = - \ \delta p^a \wedge dx^i$.

We shall choose the gauge group to be the semidirect product $SO(6,2) \times_s R^8$ which is the extension of the $4D$ Poincare group $SO(3,1) \times_s R^4$ given by the semidirect product of the Lorentz group with the translations. The flat metric in the tangent space to the cotangent bundle $T_{(x,p)}T^*M$, at the point (x,p), is $\eta_{AB} = diag\ (-,+,+,+,-,+,+,+)$. There are two timelike directions corresponding to the temporal coordinate x^0 and the energy p^0.

The $SO(6,2)$ Lie algebra generators $\mathcal{L}_{AB}$ obey the commutation relations

$$[\mathcal{L}_{AB}, \ \mathcal{L}_{CD}] \ = \ (\eta_{BC}\mathcal{L}_{AD} - \eta_{AC}\mathcal{L}_{BD} - \eta_{BD}\mathcal{L}_{AC} + \eta_{AD}\mathcal{L}_{BC}). \tag{4.5}$$

The other commutators associated with the translation generators $\mathcal{P}_A$ are

$$[\mathcal{L}_{AB},\ \mathcal{P}_C] \ = \ (\ \eta_{BC}\ \mathcal{P}_A \ - \ \eta_{AC}\ \mathcal{P}_B\ ); \qquad [\mathcal{P}_A,\ \mathcal{P}_B] \ = \ 0 \tag{4.6}$$

The metric G_{MN} in the $8D$ cotangent bundle T^*M is given by

$$G_{MN} \ = \ G_{MN}(x,p) \ =$$

$$\begin{pmatrix} g_{ij}(x,p) \ + \ h_{ab}(x,p)\ N_i^a(x,p)\ N_j^b(x,p) & - \ N_i^a(x,p)\ h_{ab}(x,p) \\ - \ N_j^b(x,p)\ h_{ab}(x,p) & h_{ab}(x,p) \end{pmatrix} \tag{4.7}$$

One could also have complex (Hermitian) metrics of the form $G_{MN} = G_{(MN)} + iG_{[MN]}$ with an antisymmetric piece $G_{[MN]}$. We refer to [11] for a study of gauge theories of Born Reciprocal Gravity based on the Quaplectic group [2] given by the semidirect product of the (pseudo) unitary group with the Weyl-Heisenberg group.

The frame E_M^A fields are introduced such that

$$G_{MN} \ = \ E_M^A\ E_N^B\ \eta_{AB} \tag{4.8}$$

where $A, B = 1, 2, \ldots, 8$ are the indices of the tangent space to the $8D$ cotangent bundle $T_{(x,p)}T^*M$, at each point (x,p). $M, N = 1, 2, \ldots, 8$ are the indices of the cotangent bundle T^*M of the $4D$ spacetime manifold M.

The Lie-algebra valued gauge field is

$$\mathbf{A}_M \ = \ \Omega_M^{AB}\ \mathcal{L}_{AB} \ + \ E_M^A\ \mathcal{P}_A \tag{4.9}$$

where Ω_M^{AB} (analog of the spin connection) is the field that gauges the $SO(6,2)$ symmetry. E_M^A gauges the (Abelian) translations in $T_{(x,p)}T^*M$. Defining the derivative operators as

$$\hat{\partial}_M \ \equiv \ (\delta_i,\ \partial_a) \ = \ (\frac{\partial}{\partial x^i} \ + \ N_{ib}\frac{\partial}{\partial p_b},\ \frac{\partial}{\partial p_a}) \tag{4.10}$$

the Lie-algebra valued field strength is given by

$$\mathbf{F}_{MN} \ = \ \hat{\partial}_M \mathbf{A}_N \ - \ \hat{\partial}_N \mathbf{A}_M \ + \ [\mathbf{A}_M,\ \mathbf{A}_N] \tag{4.11}$$

The curvature two-form associated with the spin connection $\Omega_M^{AB} \ = \ - \ \Omega_M^{BA}$ is

$$\mathcal{R}_{MN}^{AB} \ \equiv \ \mathcal{F}_{MN}^{AB} \ = \ \hat{\partial}_M \Omega_N^{AB} \ - \ \hat{\partial}_N \Omega_M^{AB} \ + \ \Omega_{[M}^{AC}\ \Omega_{N]}^{CB} \tag{4.12}$$

and whose explicit components are

$$\mathcal{R}_{ij}^{AB} \ \equiv \ \mathcal{F}_{ij}^{AB} \ = \ (\frac{\partial}{\partial x^i} \ + \ \mathbf{b}\ N_{ib}\ \frac{\partial}{\partial p_b})\ \Omega_j^{AB} \ - \ (\frac{\partial}{\partial x^j} \ + \ \mathbf{b}\ N_{jb}\ \frac{\partial}{\partial p_b})\ \Omega_i^{AB} \ +$$

$$\Omega_{[i}^{AC}\ \Omega_{j]}^{CB} \tag{4.13}$$

$$\mathcal{R}^{AB}_{ab} \ \equiv \ \mathcal{F}^{AB}_{ab} \ = \ \frac{\partial}{\partial p^a} \ \Omega^{AB}_b \ - \ \frac{\partial}{\partial p^b} \ \Omega^{AB}_a \ + \ \Omega^{AC}_{[a} \ \Omega^{CB}_{b]} \tag{4.14}$$

$$\mathcal{R}^{AB}_{ia} \ \equiv \ \mathcal{F}^{AB}_{ia} \ = (\frac{\partial}{\partial x^i} \ + \ \mathbf{b} \ N_{ib} \ \frac{\partial}{\partial p_b}) \ \Omega^{AB}_a \ - \ \frac{\partial}{\partial p^a} \ \Omega^{AB}_i \ + \ \Omega^{AC}_{[i} \ \Omega^{CB}_{a]} \tag{4.15}$$

and $\mathcal{F}^{AB}_{ai} \ = \ -\mathcal{F}^{AB}_{ia}$. A summation over the repeated indices is implied and $[MN]$ denotes the anti-symmetrization of indices with weight one.

The explicit components of the torsion two-form defined as

$$\mathcal{T}^{A}_{MN} \ \equiv \ \mathcal{F}^{A}_{MN} \ = \ \hat{\partial}_M E^A_N \ - \ \hat{\partial}_N E^A_M \ + \ \Omega^{AC}_{[M} \ E^C_{N]} \tag{4.16}$$

are

$$\mathcal{T}^{A}_{ij} \ \equiv \ \mathcal{F}^{A}_{ij} \ = \ (\frac{\partial}{\partial x^i} \ + \ \mathbf{b} \ N_{ib} \ \frac{\partial}{\partial p_b}) \ E^A_j \ - \ (\frac{\partial}{\partial x^j} \ + \ \mathbf{b} \ N_{jb} \ \frac{\partial}{\partial p_b}) \ E^A_i \ + \ \Omega^{AC}_{[i} \ E^C_{j]} \tag{4.17}$$

$$\mathcal{T}^{A}_{ab} \ \equiv \ \mathcal{F}^{A}_{ab} \ = \ \frac{\partial}{\partial p^a} \ E^A_b \ - \ \frac{\partial}{\partial p^b} \ E^A_a \ + \ \Omega^{AC}_{[a} \ E^C_{b]} \tag{4.18}$$

$$\mathcal{T}^{A}_{ia} \ \equiv \ \mathcal{F}^{A}_{ia} \ = (\frac{\partial}{\partial x^i} \ + \ \mathbf{b} \ N_{ib} \ \frac{\partial}{\partial p_b}) \ E^A_a \ - \ \frac{\partial}{\partial p^a} \ E^A_i \ + \ \Omega^{AC}_{[i} \ E^C_{a]} \tag{4.19}$$

and $\mathcal{F}^A_{ai} = -\mathcal{F}^A_{ia}$.

The frame fields allow us to construct the curvature tensor on the cotangent bundle T^*M as follows

$$\mathcal{R}^{Q}_{MNP} \ \equiv \ \mathcal{R}^{AB}_{MN} \ \ E^Q_A \ E_{BP} \ = \ \mathcal{F}^{AB}_{MN} \ \ E^Q_A \ E_{BP} \tag{4.20}$$

where the explicit components $\mathcal{F}^{AB}_{MN}$ are obtained in eqs- (4.13-4.15) . E^M_A is the inverse frame field such that $E^M_A \ E^B_M \ = \ \delta^B_A$ and $E_{AM} \ E^M_B \ = \ \eta_{AB}$. The contraction of indices yields the Ricci-like tensors.

$$\mathcal{R}_{MP} \ = \ \delta^N_Q \ \mathcal{R}^Q_{MNP} \tag{4.21a}$$

A further contraction yields the generalized Ricci scalar

$$\mathcal{R} \ = \ G^{MP} \ \mathcal{R}_{MP} \tag{4.21b}$$

The Torsion tensors are

$$\mathcal{T}_{MNQ} \ = \ \mathcal{F}^A_{MN} \ E_{AQ}, \ \ \mathcal{T}^Q_{MN} \ = \ \mathcal{F}^A_{MN} \ E^Q_A, \ \ \mathcal{T}_M \ = \ \delta^N_Q \ \mathcal{T}^Q_{MN} \tag{4.22}$$

A Lagrangian, linear in the curvature scalar and quadratic in torsion, can be chosen to be

$$\mathcal{L} \ = \ c_1 \ \mathcal{R} \ + \ c_2 \ \mathcal{T}_{MNQ} \ \mathcal{T}^{MNQ} \ + \ c_3 \ \mathcal{T}_M \ \mathcal{T}^M. \tag{4.23}$$

where c_1, c_2, c_3 are numerical coefficients. The action is

$$S \ = \ \frac{1}{2\kappa^2} \int_{\Omega_8} d^8Y \ \sqrt{| \ det \ G_{MN} \ |} \ \mathcal{L} \tag{4.24}$$

where κ^2 is the analog of the gravitational coupling constant and the $8D$ measure of integration is defined by

$$d^8Y \ \equiv \ dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge \delta p_1 \wedge \delta p_2 \wedge \delta p_3 \wedge \delta p_4 \ =$$

$$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dp_1 \wedge dp_2 \wedge dp_3 \wedge dp_4 \tag{4.25}$$

with

$$\delta p_a \ = \ dp_a \ - \ N_{ai} \ dx^i \tag{4.26}$$

The curvature (4.13-4.15) depends on the geometric quantities g_{ij}, h_{ab}, N_{ia} that describe the metric (4.7) and Ω_M^{AB}. The number of degrees of freedom $d(2d+1)$ (found in eq-(2.29)) associated with g_{ij}, h_{ab}, N_{ia} is the same as the number of degrees of freedom of a metric G_{MN} in $2d$ dimensions. Therefore, the net number of degrees of freedom correspond to those of G_{MN} and Ω_M^{AB} as it occurs in Poincare gauge theories of gravity. Furthermore, if the torsion (4.16) is set to zero one can solve Ω_M^{AB} in terms of E_M^A. To sum up, in the absence of torsion, the action (4.24) represents a Poincare-like gauge theory of gravity in $(2d-2, 2)$ dimensions, written in a *nonholonomic* coordinate basis, and where the gauge group is $SO(6,2) \times_s R^8$.

Bars [13] has proposed a gauge symmetry in phase space. One of the consequences of this gauge symmetry is a new formulation of physics in spacetime. Instead of one time there must be *two* times, while phenomena described by one-time physics in $3+1$ dimensions appear as various shadows of the same phenomena that occur in $4+2$ dimensions with one extra space and one extra time dimensions (more generally, $d+2$). Problems of ghosts and causality are resolved automatically by the $Sp(2,R)$ gauge symmetry in phase space.

The ordinary $4D$ Einstein-Hilbert action can be written in terms of the vielbeins e_i^a and spin connection ω_i^{ab} as

$$S \ = \ \frac{1}{16\pi G} \int \ e_i^a \wedge e_j^b \wedge R_{kl}^{cd}(\omega_i^{ab}, e_i^a) \ \epsilon_{abcd} \ \epsilon^{ijkl} \tag{4.27}$$

The natural extension of (4.27) to the $8D$ cotangent bundle T^*M is

$$\frac{1}{2\kappa^2} \int \ E_{M_1}^{A_1} \wedge E_{M_2}^{A_2} \wedge E_{M_3}^{A_3} \wedge E_{M_4}^{A_4} \wedge E_{M_5}^{A_5} \wedge E_{M_6}^{A_6} \wedge \mathcal{R}_{M_7 M_8}^{A_7 A_8} \ \epsilon_{A_1 A_2 \ldots A_8} \ \epsilon^{M_1 M_2 \ldots M_8} \tag{4.28}$$

A pending project (when the Torsion is constrained to zero) is to compare the field equations (2.24, 2.25) with the field equations obtained from a variation of the action (4.24).

One could also introduce Lanczos-Lovelock-like Lagrangians in D-dimensions, written in terms of the generalized Kronecker deltas,

$$\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} = \frac{1}{n!}\,\delta^{\mu_1\nu_1}_{[\alpha_1\beta_1}\,\delta^{\mu_2\nu_2}_{\alpha_2\beta_2}\,\ldots\,\delta^{\mu_n\nu_n}_{\alpha_n\beta_n]} \tag{4.29}$$

as

$$\mathcal{L} = \sum_{n=0}^{|D/2|} a_n\,\mathcal{R}^{(n)}, \quad \mathcal{R}^{(n)} = \frac{1}{2^n}\,\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\,\prod\,\mathcal{R}^{\alpha_r\beta_r}_{\mu_r\nu_r} \tag{4.30}$$

where $|D/2|$ is the integer part of $D/2$; a_n are coupling constants of dimensions $(length)^{2n-D}$. In the $8D$ cotangent bundle case T^*M the range of indices is $\alpha,\beta = 1,2,\ldots,8$; $\mu,\nu,\ldots,8$. The first four indices correspond to the four-dim spacetime, and the last four indices to the momentum space. Despite the product of curvatures, the advantage of Lanczos-Lovelock Lagrangians is that they lead to field equations containing only derivatives of the metric up to *second* order, and in arbitrary number of dimensions.

The introduction of matter sources in the right hand side of the field equations (2.24, 2.25) has been discussed by Vacaru [10]. In particular, he has introduced Clifford/spinor structures in Lagrange-Finsler and Hamilton-Cartan spaces. A study of gauge gravity over spinor bundles can be found in the monograph [10]. An analysis of gauge gravity and conservation laws in higher order anisotropic spaces was investigated in [8].

A discussion of Mach's principle within the context of Born Reciprocal Gravity in Phase Spaces was described in [17]. The Machian postulate states that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. It is also consistent with equating the maximal proper force $m_{Planck}(c^2/L_{Planck})$ to $M_{Universe}(c^2/R_{Hubble})$ and reflecting a maximal/minimal acceleration duality. By invoking Born's reciprocity between coordinates and momenta, a minimal Planck scale should correspond to a minimum momentum, and consequently to an upper scale given by the Hubble radius. Further details can be found in [17].

5 Conclusions : Towards the Geometrization of Matter and T-Duality

The results of this work leads us to believe that a *geometrization* of matter is of paramount importance in the quantization program of gravity based on the geometry of cotangent spaces (phase spaces). For instance, in $4D$ Riemannian spacetimes, one finds that Einstein's field equations, in units of $8\pi G = c = 1$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \tag{5.1}$$

exhibit a geometry/matter reciprocity symmetry, because after replacing

$$R_{\mu\nu} \leftrightarrow T_{\mu\nu}, \quad R = g^{\mu\nu} R_{\mu\nu} \leftrightarrow T = g^{\mu\nu} T_{\mu\nu} \tag{5.2}$$

in eq-(5.1) it yields

$$T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = R_{\mu\nu} \tag{5.3}$$

one can notice that the last eq-(5.3) is indeed *equivalent* to eq-(5.1) after simply taking the trace of eq-(5.1) in $D = 4$ and leading to $T = -R$. In this respect four dimensions is singled out.

In other dimensions than $D = 4$ one can look back at eqs-(2.27)

$$\mathcal{R}_{ij} - \frac{1}{2} (\mathcal{R} + \mathcal{S}) g_{ij} = 0 \Rightarrow$$

$$\mathcal{R}_{ij} - \frac{1}{2} \mathcal{R} = \frac{1}{2} g_{ij} \mathcal{S} = T_{ij} \tag{5.4}$$

such that when all the quantities in eq-(5.4) solely depend on the coordinates x^i (and not on the momenta p_a) one finds that the scalar *curvature* $\mathcal{S}$ in *momentum* space (times $g_{ij}/2$) plays the role of an effective stress energy tensor T_{ij} in the horizontal spacetime M. Hence, matter sources (mass in particular) can be effectively *geometrized* (mimicked) by the *momentum* space *curvature*.

In classical mechanics, inertial mass is that property of matter which opposes acceleration. The Quaplectic group transformations in flat phase spaces, implementing Born's Reciprocal Relativity principle [1], implies the physical equivalence of accelerated frames of reference [2]. Likewise, Special Relativity is based on the physical equivalence of inertial frames in flat Minkowski spacetime via Lorentz transformations. One of the most salient features of the Quaplectic group transformations is the *mixing* of spacetime coordinates with the energy-momentum coordinates as described in section **1**. Also, Quaplectic group transformations can change the spin of particles, which does not occur in Lorentz transformations. Born's Reciprocal Relativity, in essence, is an attempt to unify space-time-matter.

This picture of the equivalence of accelerated frames in flat phase space differs considerably from the one in ordinary Quantum Field Theory (QFT). The physics behind accelerated frames in Minkowski space is essential in the Fulling-Davies-Unruh effect, where an accelerating observer will observe blackbody radiation where an inertial observer would observe none. From the viewpoint of the accelerating observer, the vacuum of the inertial observer will look like a state containing many particles in thermal equilibrium (a warm gas of photons). The Unruh temperature [18] is the effective temperature experienced by a uniformly accelerating detector in a vacuum field. It is given by $T = \frac{\hbar a}{2\pi c k_B}$, where a is the local acceleration, and k_B is the Boltzmann constant. The Unruh temperature has the same form as the Hawking temperature after replacing a for the surface gravity at the black hole horizon.

Recently, Dasgupta [19] re-investigated the Bogoliubov transformations which relate the Minkowski inertial vacuum to the vacuum of an accelerated observer. He implemented the transformation using a non-unitary operator used in formulations of irreversible systems by Prigogine [20]. An attempt was discussed to generalize Quantum Field Theory (QFT) for accelerated frames using this new connection to Prigogine transformations. It is warranted to build a generalized QFT in accelerated frames which is compatible with the Quaplectic group transformations in Born's Reciprocal Relativity [1]. This may shed some light into the resolution of the black hole information paradox by recurring to *novel* physical principles and which are beyond the many current proposals based on standard QFT in curved Riemannian spacetimes.

Finally we add that in [17] we argued how Born Reciprocal Relativity could provide a physical mechanism to understand T-duality in string theory. Nowadays it is pursued via Double Field Theory (FDT) [15]. The idea behind DFT is to introduce a doubled space with coordinates $X^M = (x^i, \tilde{x}^i), M = 1, \ldots, 2D$, on which $O(D, D)$ acts naturally in the fundamental representation [15], [16]. One has doubled the number of all spacetime coordinates. This idea is actually well motivated by string theory on toroidal backgrounds, where these coordinates are dual both to momentum and winding modes [16]. An extension of DFT to exceptional groups, now commonly referred to as exceptional field theory, allows us to settle open problems in Kaluza-Klein truncations of supergravity that, although of conventional nature, were impossible to solve with standard techniques [16]. We have not addressed in this work how to accommodate DFT to Born Reciprocal Relativity and the geometry of (co) tangent bundles. It is becoming more clear that generalized geometries (like those involving metrics of the form in eq-(4.7)) warrant further investigation.

Acknowledgements

We thank M. Bowers for very kind assistance, to Sergiu Vacaru for many discussions and explanations about Finsler geometry, and to Igor Kanatchikov for reference [6].

References

[1] M. Born, Proc. Royal Society **A 165** (1938) 291.

M. Born, Rev. Mod. Physics **21** (1949) 463.

[2] S. Low, Jour. Phys **A** Math. Gen **35** (2002) 5711.

S. Low, Il Nuovo Cimento **B 108** (1993) 841;

S. Low, Found. Phys. **36** (2007) 1036.

S. Low, J. Math. Phys **38** (1997) 2197.

[3] H. Brandt, Contemporary Mathematics **196** (1996) 273.

H. Brandt, Found Phys. Letts **4** (1991) 523.

H. Brandt, Chaos, Solitons and Fractals **10** (2-3) (1999) 267.

[4] E. Caianiello, Lett. Nuovo Cimento **32** (1981) 65.

[5] C. Castro, " Some consequences of Born's Reciprocal Relativity in Phase Spaces" Foundations of Physics **35**, no.6 (2005) 971.

[6] Jan J. Slawianowski, *Geometry of Phase Spaces*.

[7] L. Barcaroli, L. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer "Hamilton geometry: Phase space geometry from modified dispersion relations" arXiv:1507.00922.

[8] S. Vacaru and Heinz Dehnen, "Locally Anisotropic Nonlinear Connections in Einstein and Gauge Gravity" [arXiv:gr-qc/0009039].

S. Vacaru and Y. Goncharenko, "Yang-Mills and Gauge Gravity on Generalized Lagrange and Finsler Spaces", Int. J. Theor. Phys. **34** (1995) 1955.

S. Vacaru, " Gauge Gravity and Conservation Laws in Higher Order Anisotropic Spaces" hep-th/9810229.

[9] R. Miron, D. Hrimiuc, H. Shimada and S. Sabau, *The Geometry of Hamilton and Lagrange Spaces* (Kluwer Academic Publishers, Dordrecht, Boston, 2001).

R. Miron, *Lagrangian and Hamiltonian geometries. Applications to Analytical Mechanics*, arXiv: 1203.4101 [math.DG].

[10] S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, *"Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity"* (Geometry Balkan Press, 693 pages).

S. Vacaru, "Finsler-Lagrange Geometries and Standard Theories in Physics: New Methods in Einstein and String Gravity" [arXiv : hep-th/0707.1524].

[11] C. Castro, Phys. Letts **B 668** (2008) 442.

[12] C. Castro, "Gravity in Curved Phase-Spaces, Finsler Geometry and Two Times Physics" Int. J. Mod.Phys. **A27** (2012) 1250069

C. Castro, "Born's Reciprocal Gravity in Curved Phase-Spaces and the Cosmological Constant " Found. Phys. **42**, issue 8 (2012) 1031-1055

[13] I. Bars, Int. J. Mod. Phys. **A 25** (2010). 5235. Phys. Rev. **D 54** (1996) 5203, Phys. Rev. **D 55** (1997) 2373.

[14] D. Lovelock, "The Einstein tensor and its generalizations". Journal of Mathematical Physics **12** no. (3) (1971) 498.

[15] C. Hull and B. Zwiebach, "Double Field Theory" JHEP 1204 (2012) 020.

M. Duff, J. Lu, R. Percacci, C. Pope, H. Samtleben and E. Sezgin, " Membrane Duality Revisited" arXiv:1509.02915.

[16] L. Motls, http://motls.blogspot.com/2015/07/introduction-to-double-field-theory-12.html

[17] C. Castro, "On Dual Phase Space Relativity, the Machian Princiople and Modfied Newtonain Dynamics" Prog. in Phys. **1** (April 2005) 20.

[18] W. Unruh , "Notes on black-hole evaporation". Physical Review **D 14**, no. 4 (1976) 870.

http://en.wikipedia.org/wiki/Unruh_effect

[19] A. Dasgupta, "Quantum Field Theory in Accelerated Frames" arXiv: 1508.01940.

[20] I. Prigogine, *From Being to Becoming : Time and Complexity in the Physical Sciences* (WH. Freeman and Co. 1980).