A Short Review of the Thermal WIMP Model of Dark Matter with improved Numerical Analysis

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1 Introduction

Decoding the nature of Dark Matter is one of the most enigmatic problems in Physics currently. Recent experimental observations estimate that the dark matter content of the universe can be given by the density $\Omega_b h^2 = 0.1123 \pm 0.0035$. However no conclusive experimental detection of Dark Matter has been found yet, and it is important to build a theoretical model for Dark Matter that can produce interesting experimental insights.

This report contains a review of the Thermal WIMP model of Dark Matter, specifically highlighting the derivation of the basic equation using Boltzmann Equation and some results from Cosmology. The treatment begins with re-derivation of some elementary results and assumption of some more involved results to finally reach the relevant equation. We then proceed to the solution of the equation, first analytically under some restrictive assumptions and then to a numerical solution. Finally we go on to construct an improved numerical solution considering the time evolution of all the parameters involved, mirroring a result which has been published very recently[4].

2 The Thermal WIMP Model

2.1 Boltzmann Equation

The set of all possible positions of $\vec{q}$ and $\vec{p}$ is called the phase space of the system. We define a probability density function $f(\vec{q}, \vec{p}, t)$, such that at any instant of time t, the number of particles in a small phase space volume $d^3\vec{q}d^3\vec{q}$ is given by

$$dN \propto f(\vec{q}, \vec{p}, t) d^3\vec{q} d^3\vec{q}$$

The simplest non-trivial Hamiltonian studied in Kinetic theory is

$$\mathcal{H}(\vec{p_1}, \vec{p_2}, ..., \vec{p_N}, \vec{q_1}, \vec{q_2}, ..., \vec{q_N}) = \sum_{i=2}^{N} [\frac{\vec{p_i}^2}{2m} + U(\vec{q_i})] + \frac{1}{2} \sum_{i,j=1}^{N} \mathcal{V}(\vec{q_i} - \vec{q_j})$$

Note: Here we consider the classical kinetic energy of N particles, each of mass m, in an external potential U and a two-body interaction $\mathcal{V}$ between the particles.

Consider particles described by f, each experiencing an external force $\vec{F}$ not due to other particles. Say, at time t, dN particles were within $d^3\vec{q}d^3\vec{p}$ volume around $\vec{q}, \vec{p}$, then at time $t + \Delta t$, the new position will be $\vec{q} + \Delta\vec{q} = \vec{q} + \frac{\Delta t}{m}\vec{p}$; the new momentum will be $\vec{p} + \Delta\vec{p} = \vec{p} + \vec{F}\Delta t$.

Now,

$$q_i' = q_i + \dot{q}_i \Delta t + \mathcal{O}(\Delta t)^2 \Rightarrow dq_i' = dq_i + \frac{\partial \dot{q}_i}{\partial q_i} dq_i \Delta t + \mathcal{O}(\Delta t)^2$$

$$p_i' = p_i + \dot{p}_i \Delta t + \mathcal{O}(\Delta t)^2 \Rightarrow dp_i' = dp_i + \frac{\partial \dot{p}_i}{\partial p_i} dp_i \Delta t + \mathcal{O}(\Delta t)^2$$

Thus,

$$\begin{aligned} d^3\vec{q'} d^3\vec{p'} &= \prod_{i=1}^{3} dq_i' dp_i' \\ &= \prod_{i=1}^{3} dq_i dp_i [1 + (\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}) \Delta t + \mathcal{O}(\Delta t)^2] \end{aligned} \tag{1}$$

Again, using Hamilton's equations,

$$\frac{\partial \dot{q}_i}{\partial q_i} = \frac{\partial}{\partial q_i}(\frac{\partial \mathcal{H}}{\partial p_i}) = \frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i} = \frac{\partial}{\partial p_i}(\frac{\partial \mathcal{H}}{\partial q_i}) = -\frac{\partial \dot{p}_i}{\partial p_i}$$

Using the above result in (1), and neglecting second order terms of Δt, we get

$$d^3\vec{q'}d^3\vec{p'} = d^3\vec{q}d^3\vec{p}$$

This means that dN particles are transported to the vicinity of $\vec{q'}, \vec{p'}$ within a spread of volume which is same as the volume spread in the phase space initially. Thus the density doesn't change with time if we follow a particular set of initial points, much like an incompressible fluid. However, this was explained assuming no collision.

In the absence of collision, the phase space density function must satisfy the condition that

$$f(\vec{q'}, \vec{p'}, t + \Delta t)d^3\vec{q}d^3\vec{p} = f(\vec{q}, \vec{p}, t)d^3\vec{q}d^3\vec{p}$$

However, now if we include the effect of collision, the collection of phase points that we were following will manifest a change in the particle density in the phase space volume $d^3\vec{q}d^3\vec{p}$. The change in the number of particles in the vicinity of a particular point in the 6N-dimensional phase space is given by

$$\begin{aligned} dN_{coll} &= (\frac{\partial f}{\partial t})_{coll}\Delta t d^3\vec{q}\vec{p} \\ &= f(\vec{q'}, \vec{p'}, t + \Delta t)d^3\vec{q}d^3\vec{p} - f(\vec{q}, \vec{p}, t)d^3\vec{q}d^3\vec{p} \\ &= \Delta f d^3\vec{q}d^3\vec{p} \end{aligned} \tag{2}$$

Dividing both sides of eq(1) by $d^3\vec{q}d^3\vec{p}\Delta t$ and taking the infinitesemal limits, we get,

$$\frac{df}{dt} = (\frac{\partial f}{\partial t})_{coll}$$

In Hamiltonian mechanics the above result can be succinctly captured in the operator notation,

$$\mathbf{L}[f] = \mathbf{C}[f] \tag{3}$$

where $\mathbf{L}$ is the Liouville operator which determines how the phase space density function evolves and $\mathbf{C}$ is the collision operator which is captures the effect of collision in the evolution of the phase space density.

The generalization of $\mathbf{L}$ to general relativity is given by

$$\mathbf{L} = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol. The metric used in standard Cosmology is the so called FRW metric, in which the distribution has the property $f(\vec{q}, \vec{p}, t) = f(|\vec{p}|, t)$. It can be shown that the Liouville operator then reduces to the form

$$\mathbf{L} = E\frac{\partial}{\partial t} - Hp^2\frac{\partial}{\partial E} \tag{4}$$

where H is the Hubble constant ($H = \frac{\dot{a}}{a}$, a being the scale factor in cosmology). We will return to the form of the Collision operator after a short discussion on the evolution of the early universe.

2.2 Evolution of the early universe

We discuss in short about the thermal history of the universe in its first few minutes. The discussion will be largely qualitative and we will just list out many results without proof.

The two important parameters that should be kept in mind while discussing the thermal history of the universe are the rate of interactions (Γ) and the rate of expansion of the universe (Hubble's constant or H). When $\Gamma \gg H$, the time scale of the system allows it to go interact with each other and reach equilibrium without much effect of the expansion of the universe. However, as the universe cools, Γ decreases faster than H, and hence at one point of time, the particles can decouple from the thermal bath which is in equilibrium, as the particles may have moved so far apart that they can't interact with each other. This is the basic premise of the thermal WIMP model of Dark Matter. In this model, these DM particles are thought to be weakly interacting massive particles which decoupled from the thermal bath early during the evolution of the universe; what we estimate to be the DM density is nothing but the 'cold' relic of the once 'hot' WIMPs.

For the purpose of our calculation we need the following results. Let T be the temperature of the photon gas. The total radiation density is the sum over the energy densities of the *relativistic species* only

$$\rho_r = \sum_i = \frac{\pi^2}{30} g_{eff}(T) T^4 \tag{5}$$

The sum over particles may receive tow contributions:

• Relativistic species in thermal equilibrium with the photons, $T_i = T \gg m_i$,

$$g_{eff}^{th}(T) = \sum_{bosons} g_i + \frac{7}{8} \sum_{fermions} g_i$$

When the tempertaure drops below the mass m_i of the particle species it becomes non-relativistic and is removed from the sum above. Away from mass thresholds, the thermal contributions is independent of the temperature.

• Relativistic species which are not in thermal equilibrium with the photons; for the decoupled species having a different temperature T_i, we have

$$g_{eff}^{dec}(T) = \sum_{bosons} g_i (\frac{T_i}{T})^4 + \frac{7}{8} \sum_{fermions} g_i (\frac{T_i}{T})^4$$

A similar expression can be written for the entropy density as well, for which we get

$$s = \frac{2\pi^2}{45} h_{eff}(T) T^3 \tag{6}$$

Now as before we can define $h_{eff} = h_{eff}^{th} + h_{eff}^{dec}$, with $h_{eff}^{th} = g_{eff}^{th}$; and for the contribution from decoupled species we get

$$h_{eff}^{dec}(T) = \sum_{bosons} g_i (\frac{T_i}{T})^3 + \frac{7}{8} \sum_{fermions} g_i (\frac{T_i}{T})^3 \neq g_{eff}^{dec}(T)$$

As can be seen, for finding out both the energy density and the entropy density of the universe, it is important to find out the contributions from all the particles available. In Table 1, we list out the particles and their degrees of freedom in the standard model of particle physics.

For latter analysis, we require the exact form of how the g_{eff} varies with temperature. For this I have taken data points from M. Laine's paper[2]. Figure 1 shows the variation of h_{eff} with temperature.

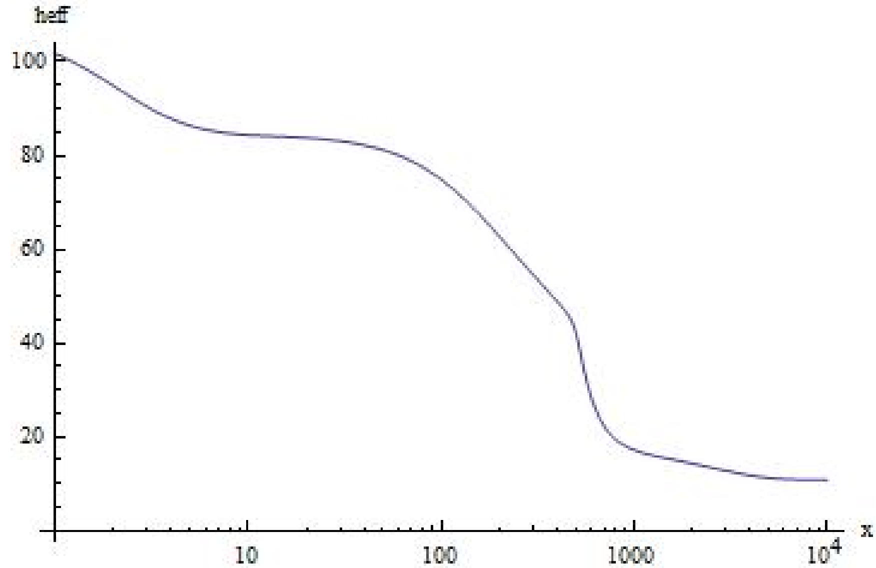
2.3 From equilibrium to decoupling

The parameter that we are concerned about is the time evolution of the number density of a particle species, in this case the candidate Dark Matter particle. We need an equation that governs the evolution of the number density of the DM particle which was initially in equilibrium with the thermal bath but which decoupled at some point during the evolution of the universe. More so, the final relic density of the remaining Dark matter must satisfy the observational results.

Table 1: The Standard Model Zoo[1]

Type		**Mass**	**Spin**	**g**
quarks	$t, \overline{t}$	173 GeV	0.50	2.2.3=12
	$b, \overline{b}$	4 GeV		
	$c, \overline{c}$	1 GeV		
	$s, \overline{s}$	100 MeV		
	$d, \overline{d}$	5 MeV		
	$u, \overline{u}$	2 MeV		
gluons	g	0	1.00	8.2=16
leptons	$\tau^{\pm}$	1777 MeV	0.50	2.2 + 2.1=6
	$\mu^{\pm}$	106 MeV		
	$e^{\pm}$	511 KeV		
	$\nu_\tau, \overline{\nu_\tau}$	< 0.6 eV		
	$\nu_\mu, \overline{\nu_\mu}$	< 0.6 eV		
	$\nu_e, \overline{\nu_e}$	< 0.6 eV		
gauge bosons	W^+	80 GeV	1.00	3
	W^-	80 GeV		
	Z^0	91 GeV		
	γ	0		2
Higgs boson	H^0	125 GeV	0.00	1

Figure 1: The evolution of h_{eff} with x, where x=m/T, m=100 GeV



The number density is given by

$$n = \int f(p,t) \frac{g d^3 p}{(2\pi)^3}$$

In order to get the evolution equation for n, eq (3) is to be integrated over the particle momenta and summed over the spin degrees of freedom.

For the simple Thermal WIMP model we consider the computation for the case of annihilation of the DM particle and its antiparticle to produce standard model particles which are immediately gets into equilibrium with the thermal bath. Hence we are basically considering the interaction

$$\chi + \overline{\chi} \leftrightarrow s + \overline{s}$$

For the sake of generality, we present the computations for the case of annihilations of two particles, 1 and 2, into two others, 3 and 4. At the end, we can infact sum over all possible final results (channels). By integrating the Boltzmann equation by parts (3) over the particle momenta, it can be rewritten as

$$g_1 \int \mathbf{C}[f] \frac{d^3 p_1}{(2\pi)^3 E_1} = \frac{1}{a^3} \frac{d}{dt}(a^3 n_1) = \dot{n_1} + 3Hn_1 \tag{7}$$

Note, in the above calculation we are following the evolution of particle 1. Also, it is important to note that we are implicitly using the fact that $\dot{n_1} = \int \frac{df(E_1,t)}{dt} \frac{g_1 d^3 p}{(2\pi)^3} = \int \frac{\partial f(E_1,t)}{\partial t} \frac{g_1 d^3 p}{(2\pi)^3}$. This is so because $\frac{\partial E_1}{\partial t} = 0$ as we are considering a system where there is no external force.

We need to understand the collision term for further analysis. We provide a heuristic explanation of first the collision operator followed by a discussion on the form of the LHS of eq(5). The collision term is given by:

$$\begin{aligned} C[f(p_1)] = \frac{1}{2} \sum_{spins} \iiint (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\ \times [W(p_1, p_2 \to p_3, p_4) f(p_1) f(p_2)(1 \pm f(p_3))(1 \pm f(p_4)) \\ -W(p_3, p_4 \to p_1, p_2) f(p_3) f(p_4)(1 \pm f(p_1))(1 \pm f(p_2))] \times d\Pi_2 d\Pi_3 d\Pi_4 \end{aligned} \tag{8}$$

A few things to note here: the 4-fold δ function ensures conservation of the 4-momentum in the collision process. Secondly, we are integrating over the relativistic phase space volume where the volume element is given by $d\Pi = g \frac{1}{(2\pi)^3} \frac{d^3 p}{2E}$. Also, W here refers to the collision probability which is given by the squared matrix element of the Transition Matrix. The $\pm$ sign refers to the effect of the nature of the particles (bosons or fermions) on the collision. Heuristically, this makes sense as in the equation we are taking care of the effect of 'in' and 'out' of the particles in the phase space.

We can now express eq(5) as

$$\begin{aligned} \dot{n_1} + 3Hn_1 = \sum_{spins} \iiiint (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\ \times [W(p_1, p_2 \to p_3, p_4) f(p_1) f(p_2)(1 \pm f(p_3))(1 \pm f(p_4)) \\ -W(p_3, p_4 \to p_1, p_2) f(p_3) f(p_4)(1 \pm f(p_1))(1 \pm f(p_2))] \times d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 \end{aligned} \tag{9}$$

We now employ some assumptions to simplify the integral that we have to solve in eq(5).

1. We neglect statistical factors. For massive particles decoupling early during the evolution of the universe, we can be fairly certain that Bose condensation or Fermi degeneracy doesn't arise we can replace the Fermi-Dirac and Bose-Einstein statistics by Maxwell Boltzmann and neglect the blocking and stimulated emission factors.

2. By CP-invariance, it can be argued that $W(p_1, p_2 \to p_3, p_4) = W(p_3, p_4 \to p_1, p_2)$. We assume that the interaction $\chi + \overline{\chi} \leftrightarrow s + \overline{s}$ there is no CP violation. A more useful relation is the unitarity

of the transition matrix:

$$\sum_{spins} \iiiint (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) W(p_1, p_2 \to p_3, p_4) d\Pi_3 d\Pi_4 = \\ \sum_{spins} \iiiint (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) W(p_3, p_4 \to p_1, p_2) d\Pi_3 d\Pi_4 \tag{10}$$

We now define the cross section times velocity for the relevant interaction as:

$$\sum_{spins} \iiiint (2\pi)^4 \delta^4 (p_1 + p_2 - p_3 - p_4) W(p_1, p_2 \to p_3, p_4) d\Pi_3 d\Pi_4 = \\ 4E_1 E_2 g_1 g_2 v_m \sigma_{1,2\to 3,4} \tag{11}$$

where $v_m = [(p_1.p_2)^2 - m_1^2 m_2^2]^{1/2}$, i.e. the Moller velocity is defined in a way to make the quantity $v_m n_1 n_2$ Lorentz invariant. The factor $g_1 g_2$ arises because of the averaging of all the spins. For all processes $1 + 2 \to$ *anything* we can replace $\sigma_{1,2\to 3,4}$ by σ.

3. We assume that the standard model particles thus formed are in kinetic and chemical equilibrium, which means that as soon as they are produced, they go into equilibrium with other particles; hence their distributions can be replaced by their equilibrium (MB) distribution : $f_3 f_4 = f_3^{eq} f_4^{eq}$

4. In equilibrium, if we assume MB distribution and impose energy conservation, we get $f_1^{eq} f_2^{eq} = f_3^{eq} f_4^{eq}$

Now using all the above results in eq(5), we eq(7) reduces to the form:

$$\dot{n_1} + 3Hn_1 = -\iint \sigma v_m (dn_1 dn_2 - dn_1^{eq} dn_2^{eq}) \tag{12}$$

We now use the following argument to simplify the expression. We assume that 1,2 remain in kinetic equilibrium even when it deviates from chemical equilibrium. In that case it can be deduced from symmetry arguments that the distribution functions in kinetic equilibrium is proportional to the distribution function in chemical equilibrium with a proportional factor which is independent of momentum; hence the factor $\frac{f_1 f_2}{f_1^{eq} f_2^{eq}}$ can be taken out of the integral, so that we will get the following equation

$$\dot{n_1} + 3Hn_1 = - < \sigma v_m > (n_1 n_2 - n_1^{eq} n_2^{eq}) \tag{13}$$

where

$$< \sigma v_m >= \frac{\iint \sigma v_m dn_1^{eq} dn_2^{eq}}{\iint dn_1^{eq} dn_2^{eq}}$$

Our basic aim is to solve eq (12) for the interaction $\chi + \overline{\chi} \to s + \overline{s}$ to find out the evolution of DM particles and find the relic density.

3 Solution of the number density evolution equation

3.1 Approximate analytic solution

We first recast eq(12) by change of variables. We assume similar DM particles (χ is the same as $\overline{\chi}$) colliding to give SM particles. We first define $Y = \frac{n}{s}$ and $x = \frac{m}{T}$. Then Y is a proxy variable for the number density and x is the proxy variable for time or consequently the temperature of the universe (inverse). We then get from eq(12)

$$\frac{dY}{dx} = \frac{1}{3H} \frac{ds}{dx} < \sigma v_m > (Y^2 - Y^{eq2})$$

Here

$$Y_{eq} = \frac{n_{eq}}{s} = \frac{g}{(2\pi)^{3/2}} x^{3/2} e^{-x}$$

In the FRW universe, we know

$$H = (\frac{8}{3}\pi G\rho)^{1/2}$$

With above result, (5) and (6), we arrive at the following form of equation for the evolution of Y,

$$\frac{dY}{dx} = -(\frac{45G}{\pi})^{-1/2} \frac{g_*^{1/2}}{x^2} < \sigma v_m > (Y^2 - Y^{eq2}) \tag{14}$$

Here we define

$$g_*^{1/2} = \frac{h_{eff}}{g_{eff}^{1/2}} (1 + \frac{1}{3} \frac{T}{h_{eff}} \frac{dh_{eff}}{dT})$$

We intend to find Y_∞; for that we may assume that $Y \gg y_{eq}$ for $fx \gg x_f$. Hence we get

$$\frac{1}{y_\infty} \approx \frac{1}{y(x_f)} + \sqrt{\frac{\pi}{45G}} m \int_{x_f}^{\infty} dx \frac{< \sigma v_m >}{x^2} \sqrt{g_*(x)}$$

Neglecting $\frac{1}{y(x_f)}$ and assuming $< \sigma v_m >$ to be constant we get the result

$$Y_\infty \approx \sqrt{\frac{45G}{\pi g_*(x_f)}} \frac{x_f}{m} \frac{1}{< \sigma v_m >}$$

The relic density is given by

$$\Omega h^2 = \frac{s_0 Y_\infty m h^2}{\rho_{crit}}$$

In the above results, if we put the present entropy density to be $2890cm^2$, $g_*(x_f) \approx 100$, $\rho_{crit} = 1.05 \times 10^-5 h^2 GeV cm^{-3}$, $x_f \approx 20$ (the reason behind taking x_f in this way will be clear in the next section) and take the mass of each particle to be $\approx 100GeV$, then we can match the relic density $\Omega h^2 = 0.1126$ with the interaction strength of $< \sigma v_m > \approx 2 \times 10^{-26} cm^3/s$. The interaction strength is approximately the same as the interaction cross section that we expect from Weak interaction.

This is sometimes referred to as the 'WIMP Miracle' as the relic density is exactly matched by massive particles interacting with a cross section similar to that of Weak Interaction.

3.2 Preliminary Numerical Solution

Here we develop a numerical method to find the evolution of the WIMPs. We start by the equation

$$\frac{dY}{dx} = -\frac{s(m)}{H(m)} \frac{<\sigma v_m>}{x^2} (Y^2 - Y_{eq}^2) \tag{15}$$

We then shift the x-dependent prefactor by defining a variable

$$y = \frac{s(m)}{H(m)} <\sigma v_m> Y = \lambda Y$$

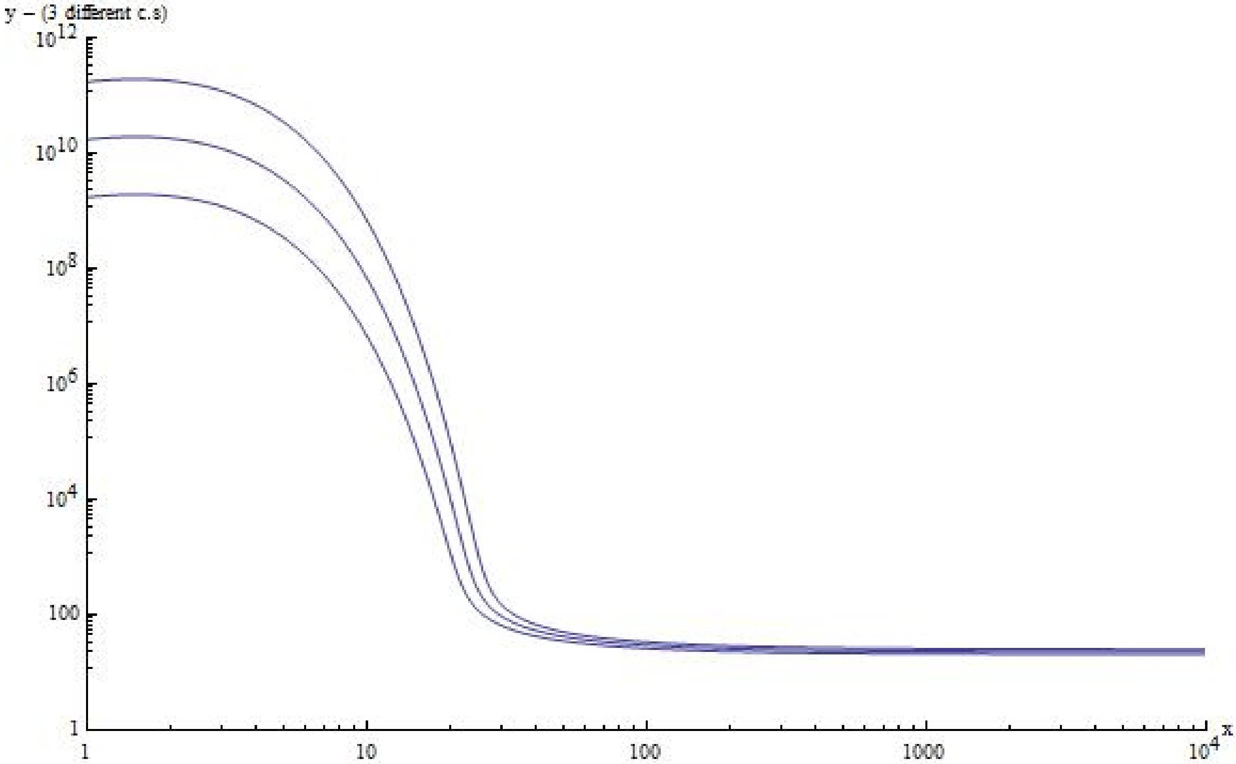
Then the above equation reduces to

$$\frac{dy}{dx} = -\frac{1}{x^2}(y^2 - y_{eq}^2) \tag{16}$$

with

$$y_{eq} = 0.290 \frac{m}{\sqrt{g_*}\sqrt{8\pi G}} < \sigma v_m > x^{3/2} e^{-x}$$

Figure 2: The evolution of y with $x = m/T$ for m = 100 GeV, for three different cross sections of interaction



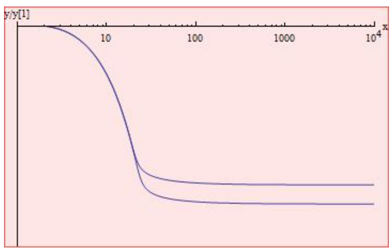
We are assuming the DM particles to be Majorana fermions with 2 degrees of freedom.

We do the numerical analysis in two parts so that it doesn't blow up [the Mathematica code is attached in appendix A]. The analysis is done for 100 GeV WIMPs for three different cross sections 10^{-9},10^{-10},10^{-11} GeV^{-2}. [viz. Fig 2]

Here it is evident that at $x_f \approx 20$, the number density falls to its relic value, without much dependence on the cross section at all.

A more suggestive way to plot the graph that shows how the relic density decreases with increasing cross section is given below:

Figure 3: The evolution of y with x = m/T for m = 100 GeV, for two different cross sections of interaction



3.3 Improved numerical analysis

We now present an improved numerical analysis mirroring a result that has been published very recently[4]. Here we take into account the evolution of g and h with time and that leads to some interesting results. We recast the well known evolution equation by means of another substitution $W = lnY$, to get

$$\frac{dW}{dx} = \frac{\lambda}{x^2}[1 + \frac{1}{3}\frac{d(lnh)}{d(lnT)}]\frac{h}{g^{1/2}}(e^{(2W_{eq}-W)} - e^W) \tag{17}$$

Here $\lambda = 2.76 \times 10^{35} m < \sigma v_m >$ and $W_{eq} = lnY_{eq}$, with $Y_{eq} = 0.290\frac{x^{3/2}e^{-x}}{h}$ (m in GeV, and $< \sigma v_m >$ in $cm^3 s^{-1}$). We now estimate, given the mass of each DM particle what should be the cross section to match the relic density of DM.

The mathematica code is attached in Appendix B. The basic idea of the numerical approach is to first reduce the stiffness of the original equation by necessary variable changes, then import the result of variation of g and h with T from the text file generated by M. Laine et al[3] and finally solve the equation with the degrees of freedom as dynamic parameters. This analysis is done for a given mass for a given range of cross sections, and then the cross section that matches the final relic density (here it is taken to be $\Omega_b h^2 = 0.11$) is chosen. This process is iterated over a range of masses of DM particles. Finally we plot the required cross section for a given mass of DM particle in Fig. 4.

Figure 4: The variation of cross section with DM particle mass

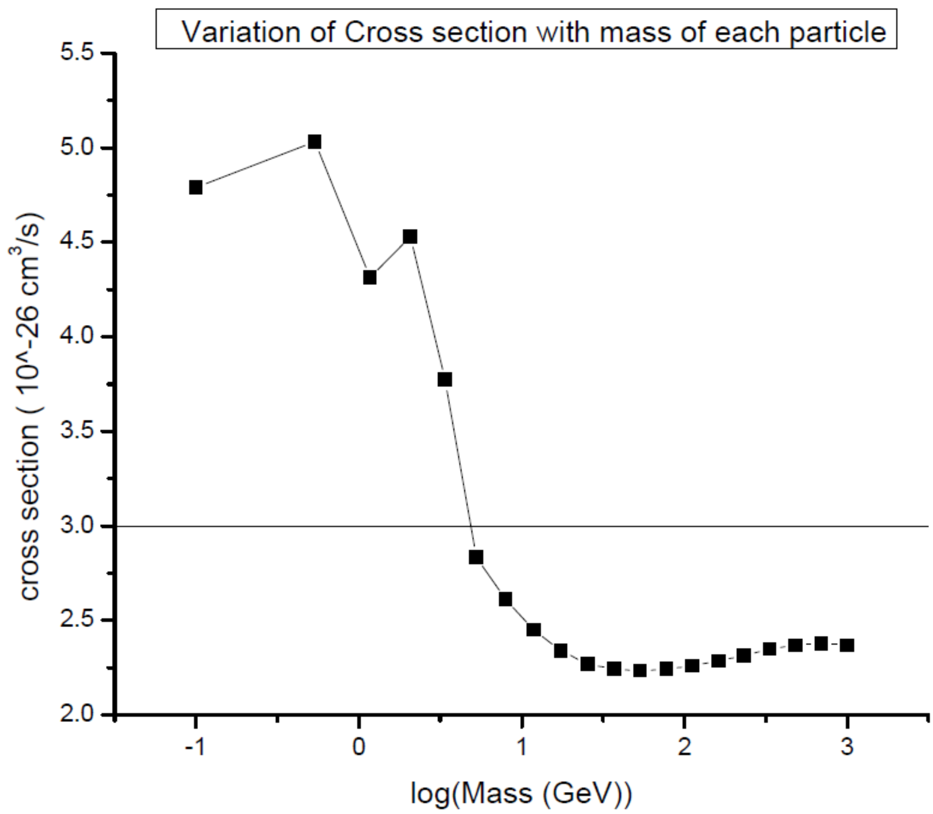


Figure 4 shows deviation for the cross section of interaction from the canonical result that is often quoted 3×10^{-26}, for light DM particles. We survey masses in the range ($10^{-1} - 10^3 GeV$). For lighter particles, we find a cross section as high as $5 \times 10^{-26} cm^3/s$.

4 Further scope

The above result opens up new scopes regarding the study of WIMP relics. In experiments, for example, detecting a light DM particle will involve different cross sections. Furthermore, the deviation in the cross section arises purely due to the fact that the variation of g with temperature (1.e. evolution of Universe) has been taken care of. But the current analysis depends on the standard model of particles. Since there definitely are particles beyond the standard model, the cross sections might change further with addition of more particles in the analysis.

5 Acknowledgement

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A Mathematica code for 3.2

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Relic Density
Calculations

Basic Calculations (*Reference Tanedo, Murayama*)

m = 1000(*GeV*);(*this equation follows from Tanedo's numerical \
calculation following Murayama's HW*)
gstarhalf = 10;
Mpl = 2.44*10^18 (*GeV*);
n = 0(*s channel*);
s = 10^-10 (*cross section in Gev^-2*);
a = 0.192*gstarhalf*Mpl*m*s;
soln1 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
y[1] == a*E^-1},
y,
{x, 1, 20}];
w = Evaluate[y[20] /. soln1];
soln2 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
y[20] == w},
y,
{x, 20, 10000}];
g1 = LogLogPlot[Evaluate[y[x] /. soln1], {x, 1, 20},
PlotRange -> {{1, 10000}, {1, 10^12}}];
g2 = LogLogPlot[Evaluate[y[x] /. soln2], {x, 20, 10000},
PlotRange -> {{1, 10000}, {1, 10^12}}];


m = 100(*GeV*);
gstarhalf = 10;
Mpl = 2.44*10^18 (*GeV*);
n = 0(*s channel*);
s = 10^-10 (*cross section in Gev^-2*);
a = 0.192*gstarhalf*Mpl*m*s;
soln1 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
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y[1] == a*E^-1},
y,
{x, 1, 20}];
w = Evaluate[y[20] /. soln1];
soln2 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
y[20] == w},
y,
{x, 20, 10000}];
g3 = LogLogPlot[Evaluate[y[x] /. soln1], {x, 1, 20},
PlotRange -> {{1, 10000}, {1, 10^12}}];
g4 = LogLogPlot[Evaluate[y[x] /. soln2], {x, 20, 10000},
PlotRange -> {{1, 10000}, {1, 10^12}}];

m = 10(*GeV*);
gstarhalf = 10;
Mpl = 2.44*10^18 (*GeV*);
n = 0(*s channel*);
s = 10^-10 (*cross section in Gev^-2*);
a = 0.192*gstarhalf*Mpl*m*s;
soln1 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
y[1] == a*E^-1},
y,
{x, 1, 20}];
w = Evaluate[y[20] /. soln1];
soln2 = NDSolve[
{y'[x] == -(x^(-2 - n)) ((y[x])^2 - (a*x^(1.5)*E^(-x))^2),
y[20] == w},
y,
{x, 20, 10000}];
g5 = LogLogPlot[Evaluate[y[x] /. soln1], {x, 1, 20},
PlotRange -> {{1, 10000}, {1, 10^12}}];
g6 = LogLogPlot[Evaluate[y[x] /. soln2], {x, 20, 10000},
PlotRange -> {{1, 10000}, {1, 10^12}}];
Show[g1, g2, g3, g4, g5, g6,
AxesLabel -> {"x",
"y - (3 different c.s)"}](*This graph shows the y dependence on \
mass/cross section *)
```

B Mathematica code for 3.3

```
logspace[increments_, start_, end_] :=
Module[{a}, (a = Range[0, increments];
Exp[a/increments*Log[(end - start) + 1]] - 1 + start)];
mass = N[logspace[19, 0.1, 1000]]; (*GeV*)
sigma = Range[0, 1000]*9*10^-29 + 10^-26;(*cm^3/s*)

T = ReadList[
"G:\\Dark Matter Project\\Numerical Solution\\some data \
files\\TGeV.txt", Number];
g =
ReadList[
```

```
"G:\\Dark Matter Project\\Numerical Solution\\some data \
files\\g.txt", Number];
geff =
ReadList [
"G:\\Dark Matter Project\\Numerical Solution\\some data \
files\\effG.txt", Number];
z = Riffle [T, g];
z1 = Partition[z, 2];
g = Interpolation[z1, InterpolationOrder -> 1];
z2 = Riffle[T, geff];
z3 = Partition[z2, 2];
geff = Interpolation[z3, InterpolationOrder -> 1];

m = Part[mass, 1];
For[z5 = 0, z5 < 1001, z5++,
{s = Part[sigma, z5 + 1];
Yeq[X_] := 0.290*X^(1.5)*E^(-X)/g[m/X];
Weq[X_] := Log[Yeq[X]];
sol = NDSolve[
{U'[u] == (2.76*10^35*m*s/u^2)*
geff[m/u]*(E^(2*Weq[u] - U[u]) - E^(U[u])),
U[1] == Weq[1]},
U,
{u, 1, 1000}];
Wf = Evaluate[U[1000] /. sol];
Yf = E^Wf;
sigmah2 = Part[Yf*m*2.752*10^8, 1];
Clear[Yeq, Weq, X, sol, U, u, Wf, Yf];
If[sigmah2 < 0.11, sigma3 = ScientificForm[s]; Break[];];
Clear[sigmah2, s];
}
];
output = {{m, sigma3}};
Clear[m, z5, sigma3];

For[z6 = 2, z6 < 21, z6++,
{m = Part[mass, z6];
For[z5 = 0, z5 < 1001, z5++,
{s = Part[sigma, z5 + 1];
Yeq[X_] := 0.290*X^(1.5)*E^(-X)/g[m/X];
Weq[X_] := Log[Yeq[X]];
sol = NDSolve[
{U'[u] == (2.76*10^35*m*s/u^2)*
geff[m/u]*(E^(2*Weq[u] - U[u]) - E^(U[u])),
U[1] == Weq[1]},
U,
{u, 1, 1000}];
Wf = Evaluate[U[1000] /. sol];
Yf = E^Wf;
sigmah2 = Part[Yf*m*2.752*10^8, 1];
Clear[Yeq, Weq, X, sol, U, u, Wf, Yf];
If[sigmah2 < 0.11, sigma3 = ScientificForm[s]; Break[];];
Clear[sigmah2, s];
}
```

```
];
output = Append[output, {m, sigma3}];
Clear[m, z5, sigma3];
}
];
Export["G:\\Dark Matter Project\\Numerical Solution\\some data \
files\\output_1.txt", output, "Table"]
```

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