18th International Conference on Information Fusion
Washington, DC - July 6-9, 2015

On the Quality Estimation of Optimal Multiple Criteria Data Association Solutions

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***Abstract*—In this paper, we present a method to estimate the quality (trustfulness) of the solutions of the classical optimal data association (DA) problem associated with a given source of information (also called a criterion). We also present a method to solve the multi-criteria DA problem and to estimate the quality of its solution. Our approach is new and mixes classical algorithms (typically Murty's approach coupled with Auction) for the search of the best and the second best DA solutions, and belief functions (BF) with PCR6 (Proportional Conflict Redistribution rule # 6) combination rule drawn from DSmT (Dezert-Smarandache Theory) to establish the quality matrix of the global optimal DA solution. In order to take into account the importances of criteria in the fusion process, we use weighting factors which can be derived by different manners (ad-hoc choice, quality of each local DA solution, or inspired by Saaty's Analytic Hierarchy Process (AHP)). A simple complete example is provided to show how our method works and for helping the reader to verify by him or herself the validity of our results.**

***Index Terms*—Data association, Multi-criteria analysis, belief functions, PCR6, DSmT.**

I. Introduction

Efficient algorithms for modern multisensor-multitarget tracking (MS-MTT) systems [1], [2] require to estimate and predict the states (position, velocity, etc) of the targets evolving in the surveillance area covered by a set of sensors. These estimation and prediction are based on sensors measurements and dynamical models assumptions. In the monosensor context, MTT requires classicallyto solve the data association (DA) problem to associate the available measurements at a given time with the predicted states of the targets to update their tracks using filtering techniques (Kalman filter, Particle filter, etc). In the multisensor MTT context, we need to solve more difficult multi-dimensional assignment problems under constraints. Fortunately, efficient algorithms have been developed in the operational research and tracking communities for formalizing and solving these optimal assignments problems (see the related references detailed in the sequel).

Before going further, it is necessary to recall briefly the basis of DA problem and the methods to solve it. This problem can be formulated as follows: We have $m > 1$ targets T_i $(i = 1, \dots, m)$, and $n > 1$ measurements[1] z_j $(j = 1, \dots, n)$ at a given time k, and a $m \times n$ rewards (gain/payoff) matrix $\mathbf{\Omega} = [\omega(i, j)]$ whose elements $\omega(i, j) \geq 0$ represent the payoff (usually homogeneous to the likelihood) of the association of target T_i with measurement z_j, denoted (T_i, z_j). The data association problem consists in finding the global optimal assignment of the targets with some measurements by maximizing[2] the overall gain in such a way that no more than one target is assigned to a measurement, and reciprocally.

Without loss of generality, we can assume $\omega(i, j) \geq 0$ because if some elements $\omega(i, j)$ of $\mathbf{\Omega}$ were negative, we can always add the constant value[3] to all elements of $\mathbf{\Omega}$ to work with a new payoff matrix $\mathbf{\Omega}' = [\omega'(i, j)]$ having all elements $\omega'(i, j) \geq 0$, and we get same optimal assignment solution with $\mathbf{\Omega}$ and with $\mathbf{\Omega}'$. Moreover, we can also assume without loss of generality $m \leq n$ because otherwise we can always swap the roles of targets and measurements in the mathematical problem definition by working directly with $\mathbf{\Omega}^t$ instead, where the superscript t denotes the transposition of the matrix. The optimal assignment problem consists of finding the $m \times n$ binary association matrix $\mathbf{A} = [a(i, j)]$ which Maximize the global rewards

$$R(\mathbf{\Omega}, \mathbf{A}) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} \omega(i, j) a(i, j) \tag{1}$$

$$\text{Subject to} \quad \begin{cases} \sum_{j=1}^{n} a(i, j) = 1 & (i = 1, \dots, m) \\ \sum_{i=1}^{m} a(i, j) \leq 1 & (j = 1, \dots, n) \\ a(i, j) \in \{0, 1\} \end{cases} \tag{2}$$

The association indicator value $a(i, j) = 1$ means that the corresponding target T_i and measurement z_j are associated, and $a(i, j) = 0$ means that they are not associated $(i = 1, \dots, m$ and $j = 1, \dots, n)$.

[1] In a multi-sensor context targets can be replaced by tracks provided by a given tracker associated with a type of sensor, and measurements can be replaced by another tracks set. In different contexts, possible equivalents are assigning personnel to jobs or assigning delivery trucks to locations.

[2] In some problems, the matrix $\mathbf{\Omega} = [\omega(i, j)]$ represents a cost matrix whose elements are the negative log-likelihood of association hypotheses. In this case, the data association problems consists in finding the best assignment that minimizes the overall cost.

[3] equals to the absolute value of the minimum of $\mathbf{\Omega}$.

The solution of the optimal assignment problem stated in (1)–(2) is well reported in the literature and several efficient methods have been developed in the operational research and tracking communities to solve it. The most well-known algorithms are Kuhn-Munkres (or Hungarian) algorithm [3], [4] and its extension to rectangular matrices proposed by Bourgeois and Lassalle in [5], Jonker-Volgenant method[6], and Auction [7]. More sophisticated methods using Murty's method [8], and some variants [9], [10], [11], [12], [13], [14], [15], are also able to provide not only the best assignment, but also the m-best assignments. We will not present in details all these classical methods because they have been already well reported in the literature [16], [17].

The purpose of this paper is to propose a solution for two important problems related with the aforementioned Data Association issue:

• **Problem 1 (mono-criterion):** Suppose that the DA reward $\mathbf{\Omega}_1$ has been established based on a unique criterion C_1 then we want to evaluate the quality[4] of each association (pairing) provided in the optimal solution by one of the aforementioned algorithms. The choice of the algorithm does not matter as soon as they are able to provide the optimal DA solution represented by a binary matrix $\mathbf{A}_1$ (assumed to be unique here for convenience). So based on $\mathbf{\Omega}_1$ and $\mathbf{A}_1$, we want to estimate the quality matrix $\mathbf{Q}_1$ of the optimal pairing solutions given in $\mathbf{A}_1$. This quality matrix will be useful to select optimal association pairings that have sufficient quality to be used to update the tracking filters, and not to use the optimal data associations that have a poor quality, which will save computational time and avoid to potentially degrade tracking performances.

• **Problem 2 (multi-criteria):** We assume that we have different Rewards matrices $\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_K$ $(K > 1)$, established from different criteria from which we can draw optimal DA solutions $\mathbf{A}_1, \ldots, \mathbf{A}_K$ with their corresponding quality matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ (obtained by the method used for solving Problem 1). We assume that each criterion C_k, $k = 1, \ldots, K$ has its own importance with respect to the others which is expressed either by a given relative importance $K \times K$ matrix $\mathbf{M}$, or directly by a weighting $M \times 1$ vector $\mathbf{w}$. The problem 2 consists in finding the optimal (i.e. the one generating the best global quality) DA solution based on all information drawn from the independent multiple criteria we have, that is from $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ and $\mathbf{M}$ (or $\mathbf{w}$) in a well-justified and comprehensive manner.

This paper is organized as follows: in section 2 we present a method for solving problem 1 which uses both 1st-best and 2nd-best DA solutions provided by Murty's algorithm. Our method is based on Belief Functions (BF), the Proportional Conflict Redistribution fusion rule #6 (PCR6) developed in Dezert-Smarandache Theory (DSmT) framework [19], and the pignistic probability transform. Section 3, proposes a solution for Problem 2 exploiting Saaty's AHP method, BF and also Murty's algorithm. Section 4 presents a full simple detailed example to show how the method works for readers who want to check by themselves our results. Section 5 will conclude this paper with perspectives.

[4]In this paper, the quality of a pairing of the optimal DA solution refers to a confidence score which corresponds to a degree of trustfulness one grants to this pairing for taking the decision to use it, or not.

II. Solution of problem 1 (mono-criterion)

This solution has already been addressed in details in [21] and we will just briefly present here the main ideas for making this paper self containing. In problem 1, we want to establish a confidence level (i.e. a quality indicator) of the pairings of the optimal data association solution. More precisely, we are searching for an answer to the question: how to measure the quality of the pairings $a(i, j) = 1$ provided in the optimal assignment solution $\mathbf{A}$? The necessity to establish a quality indicator is motivated by the following three main practical reasons:

1) In some practical tracking environment with the presence of clutter, some association decisions $(a(i, j) = 1)$ are doubtful. For these unreliable associations, it is better to wait for new information (measurements) instead of applying the hard data association decision, and making potentially serious association mistakes.

2) In some multisensor systems, it can be also important to save energy consumption for preserving a high autonomy of the system. For this goal, only the most trustful specific associations provided in the optimal assignment have to be used instead of all of them.

3) The best optimal assignment solution is not necessarily unique. In such situation, the establishment of quality indicators may help in selecting one particular optimal assignment solution among multiple possible choices.

It is worth noting that the 1st-best, as well as the 2nd-best, optimal assignment solutions are unfortunately not necessarily unique. Therefore, we need to take into account the possible multiplicity of assignments in the analysis of the problem. The multiplicity index of the best optimal assignment solution is denoted $\beta_1 \geq 1$, and the multiplicity index of the 2nd-best optimal assignment solution is denoted $\beta_2 \geq 1$, and we will denote the sets of corresponding assignment matrices by $\mathcal{A}_1 = \{\mathbf{A}_1^{(k_1)}, k_1 = 1 \ldots, \beta_1\}$ and by $\mathcal{A}_2 = \{\mathbf{A}_2^{(k_2)}, k_2 = 1 \ldots, \beta_2\}$. Here are three simple examples with different multiplicities in solutions:

Example 1: If we take $\mathbf{\Omega} = \begin{bmatrix} 8 & 1 & 2 \\ 5 & 3 & 3 \end{bmatrix}$, then $\beta_1 = 2$ and $\beta_2 = 1$ because the 1st best and 2nd best DA solutions are

$$\mathbf{A}_1^{k_1=1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{A}_1^{k_1=2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example 2: If we take $\mathbf{\Omega} = \begin{bmatrix} 6 & 3 & 9 \\ 1 & 4 & 1 \end{bmatrix}$, then $\beta_1 = 1$ and $\beta_2 = 2$ because the 1st best and 2nd best DA solutions are

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{A}_2^{k_2=1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{A}_2^{k_2=2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Example 3: If we take $\mathbf{\Omega} = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$, then $\beta_1 = 2$ and $\beta_2 = 2$ because the 1st best and 2nd best DA solutions are

$$\mathbf{A}_1^{k_1=1} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad \mathbf{A}_1^{k_1=2} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix},$$
$$\mathbf{A}_2^{k_2=1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad \mathbf{A}_2^{k_2=2} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}.$$

To establish the quality of the specific associations (pairings) (i, j) satisfying $a_1(i, j) = 1$ belonging to the optimal assignment matrix $\mathbf{A}_1$, we propose to use both $\mathbf{A}_1$ and 2nd-best assignment solution $\mathbf{A}_2$. The basic idea is to use the values $a_1(i, j) = 1$ in the best, and $a_2(i, j)$ in the 2nd-best assignments to identify the change (if any) of the optimal pairing (i, j). In fact, we assume[5] that higher quality of an entry in a quality matrix suggests that its association in an optimal solution is more stable across those good solutions. The connection between the stability of an association across the good solutions and the stability over an error in measurement is done through the components of the reward matrices (the inputs of our method) which must take into account the measurement uncertainties. Based on this assumption, our quality indicator will be defined using both the stability of the pairing and its relative impact in the global reward. This proposed method works also when the 2nd-best assignment solution $\mathbf{A}_2$ is not unique (as shown in examples 2 and 3). Our method helps to select the best (most trustful) optimal assignment in case of multiplicity of $\mathbf{A}_1$ matrices. We do not claim that the definition of the quality matrix proposed in this work is the best proposal. However, we propose a new comprehensive way of solving this problem from a practical standpoint.

To take into account efficiently the reward values of each specific association given in the best assignment $\mathbf{A}_1$ and in the 2nd-best assignment $\mathbf{A}_2^{k_2}$ for estimating the quality of DA solutions, we propose to use the following construction of quality indicators depending on the type of matching:

• When $a_1(i, j) = a_2^{k_2}(i, j) = 0$, one has full agreement on "non-association" (T_i, z_j) in $\mathbf{A}_1$ and in $\mathbf{A}_2^{k_2}$ and this non-association (T_i, z_j) has no impact on the global rewards $R_1(\Omega, \mathbf{A}_1)$ and $R_2(\Omega, \mathbf{A}_2^{k_2})$, and it will be useless. Therefore, we can set its quality arbitrarily to any arbitrary value, typically we take $q^{k_2}(i, j) = 0$ because these values are not useful at all for the application (i.e. tracking) standpoint.

• When $a_1(i, j) = a_2^{k_2}(i, j) = 1$, one has a full agreement on the association (T_i, z_j) in $\mathbf{A}_1$ and in $\mathbf{A}_2^{k_2}$. his association (T_i, z_j) has however different impacts in the global rewards values $R_1(\Omega, \mathbf{A}_1)$ and $R_2(\Omega, \mathbf{A}_2^{k_2})$. To qualify the quality of this association (T_i, z_j), we define the two basic belief assignments (BBA's) on $X \triangleq (T_i, z_j)$ and $X \cup \neg X$ (the ignorance), for $s = 1, 2$ as follows:

$$\begin{cases} m_s(X) = a_s(i, j) \cdot \omega(i, j)/R_s(\Omega, \mathbf{A}_s) \\ m_s(X \cup \neg X) = 1 - m_s(X) \end{cases} \tag{3}$$

[5]This assumption has however not been proven formally yet and its validity is a challenging open-question left for future research works.

Applying the conjunctive fusion rule (here one has no conflicting mass), we get

$$\begin{cases} m(X) = m_1(X)m_2(X) + m_1(X)m_2(X \cup \neg X) \\ \qquad\quad + m_1(X \cup \neg X)m_2(X) \\ m(X \cup \neg X) = m_1(X \cup \neg X)m_2(X \cup \neg X) \end{cases} \tag{4}$$

Applying the pignistic transformation[6] [20], we get finally $BetP(X) = m(X) + \frac{1}{2} \cdot m(X \cup \neg X)$ and $BetP(\neg X) = \frac{1}{2} \cdot m(X \cup \neg X)$. Therefore, we choose as quality indicator for the association (T_i, z_j) the value $q^{k_2}(i, j) \triangleq BetP(X) = m(X) + \frac{1}{2} \cdot m(X \cup \neg X)$.

• When $a_1(i, j) = 1$ and $a_2^{k_2}(i, j) = 0$, one has a disagreement (conflict) on the association (T_i, z_j) in $\mathbf{A}_1$ and in (T_i, z_{j_2}) in $\mathbf{A}_2^{k_2}$, where j_2 is the measurement index such that $a_2(i, j_2) = 1$. To qualify the quality of this non-matching association (T_i, z_j), we define the two following basic belief assignments (BBA's) of the propositions $X \triangleq (T_i, z_j)$ and $Y \triangleq (T_i, z_{j_2})$

$$\begin{cases} m_1(X) = a_1(i, j) \cdot \frac{\omega(i,j)}{R_1(\Omega, \mathbf{A}_1)} \\ m_1(X \cup Y) = 1 - m_1(X) \end{cases} \tag{5}$$

and

$$\begin{cases} m_2(Y) = a_2(i, j_2) \cdot \frac{\omega(i,j_2)}{R_2(\Omega, \mathbf{A}_2^{k_2})} \\ m_2(X \cup Y) = 1 - m_2(Y) \end{cases} \tag{6}$$

Applying the conjunctive fusion rule, we get $m(X \cap Y = \emptyset) = m_1(X)m_2(Y)$ and

$$\begin{cases} m(X) = m_1(X)m_2(X \cup Y) \\ m(Y) = m_1(X \cup Y)m_2(Y) \\ m(X \cup Y) = m_1(X \cup Y)m_2(X \cup Y) \end{cases} \tag{7}$$

Because we need to work with a normalized combined BBA, we can choose different rules of combination (say either Dempster-Shafer's rule, Dubois-Prade's rule, Yager's rule [19], etc). In this work, we propose to use the Proportional Conflict Redistribution rule no. 6 (PCR6) proposed originally in DSmT framework [19] because it has been proved very efficient in practice [28], [29]. Hence with PCR6, we get:

$$\begin{cases} m(X) = m_1(X)m_2(X \cup Y) + m_1(X) \cdot \frac{m_1(X)m_2(Y)}{m_1(X)+m_2(Y)} \\ m(Y) = m_1(X \cup Y)m_2(Y) + m_2(X) \cdot \frac{m_1(X)m_2(Y)}{m_1(X)+m_2(Y)} \\ m(X \cup Y) = m_1(X \cup Y)m_2(X \cup Y) \end{cases} \tag{8}$$

Applying the pignistic probability transformation, we get finally $BetP(X) = m(X) + \frac{1}{2} \cdot m(X \cup Y)$ and $BetP(Y) = m(Y) + \frac{1}{2} \cdot m(X \cup Y)$. Therefore, we choose the quality indicators as follows: $q^{k_2}(i, j) = BetP(X)$, and $q^{k_2}(i, j_2) = BetP(Y)$.

[6]We have chosen here BetP for its simplicity and because it is widely known, but DSmP could be used instead for expecting better performances [19].

The absolute quality factor $Q_{abs}(\mathbf{A}_1)$ of the optimal assignment given in $\mathbf{A}_1$ conditioned by $\mathbf{A}_2^{k_2}$, for any $k_2 \in \{1, 2, \ldots, \beta_2\}$ is defined as

$$Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2}) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} a_1(i,j) q^{k_2}(i,j) \tag{9}$$

The absolute average quality factor $Q_{aver}(\mathbf{A}_1)$ per association of the optimal assignment given in $\mathbf{A}_1$ conditioned by $\mathbf{A}_2^{k_2}$, for any $k_2 \in \{1, 2, \ldots, \beta_2\}$ is defined by

$$Q_{aver}(\mathbf{A}_1, \mathbf{A}_2^{k_2}) = \frac{1}{m} Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2}) \tag{10}$$

where m is the number of "1" in the optimal DA matrix $\mathbf{A}_1$ (i.e. the number of targets).

To take into account the eventual multiplicities (when $\beta_2 > 1$) of the 2nd-best assignment solutions $\mathbf{A}_2^{k_2}$, $k_2 = 1, 2, \ldots, \beta_2$, we need to combine the $\mathbf{Q}_I(\mathbf{A}_1, \mathbf{A}_2^{k_2})$ values. Several methods can be used for this, in particular we can use either:

– A weighted averaging approach: The quality indicator components $q(i,j)$ of the quality matrix $\mathbf{Q}$ are then obtained by averaging the qualities obtained from each comparison of $\mathbf{A}_1$ with $\mathbf{A}_2^{k_2}$. More precisely, one will take

$$q(i,j) \triangleq \sum_{k_2=1}^{\beta_2} w(\mathbf{A}_2^{k_2}) q^{k_2}(i,j) \tag{11}$$

where $w(\mathbf{A}_2^{k_2})$ is a weighting factor in $[0,1]$, such that $\sum_{k_2=1}^{\beta_2} w(\mathbf{A}_2^{k_2}) = 1$. Since all assignments $\mathbf{A}_2^{k_2}$ have the same global reward value R_2, then we suggest to take $w(\mathbf{A}_2^{k_2}) = 1/\beta_2$. A more elaborate method would consist of using the quality indicator of $\mathbf{A}_2^{k_2}$ based on the 3rd-best solution, which can be itself computed from the quality of the 3rd assignment solution based on the 4th-best solution, and so on by a similar mechanism.

– A belief-based approach: (see [18] for basics on belief functions): A second method would express the quality by a belief interval $[q^{\min}(i,j), q^{\max}(i,j)]$ in $[0,1]$ instead of single real number $q(i,j)$ in $[0,1]$. More precisely, one can compute the belief and plausibility bounds of the quality by taking $q^{\min}(i,j) \equiv Bel(a_1(i,j)) = \min_{k_2} q^{k_2}(i,j)$ and $q^{\max}(i,j) \equiv Pl(a_1(i,j)) = \max_{k_2} q^{k_2}(i,j)$. Hence for each possible pair (i,j), one can define a basic belief assignment (BBA) $m_{ij}(.)$ on the frame of discernment $\Theta \triangleq \{T = \text{trustful}, \neg T = \text{not trustful}\}$, which characterizes the quality of the pairing (i,j) in the optimal assignment solution $\mathbf{A}_1$, as follows

$$\begin{cases} m_{ij}(T) = q^{\min}(i,j) \\ m_{ij}(\neg T) = 1 - q^{\max}(i,j) \\ m_{ij}(T \cup \neg T) = q^{\max}(i,j) - q^{\min}(i,j) \end{cases} \tag{12}$$

Because only the optimal associations[7] (i,j) such that $a_1(i,j) = 1$ are useful in tracking algorithms to update the tracks, we do not need to pay attention (compute and store) the qualities of components (i,j) such that $a_1(i,j) = 0$. In fact all components (i,j) such that $a_1(i,j) = 0$ should be set to zero by default in $\mathbf{Q}$ matrix.

Example 4: Let's consider the rewards matrix

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 11 & 45 & 30 \\ 17 & 8 & 38 & 27 \\ 10 & 14 & 35 & 20 \end{bmatrix}$$

We get one 1st best ($\beta_1 = 1$) and four 2nd best ($\beta_2 = 4$) DA solutions with their respective qualities as follows:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow R_1(\mathbf{\Omega}, \mathbf{A}_1) = 86$$

$$\mathbf{A}_2^{k_2=1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow R_2(\mathbf{\Omega}, \mathbf{A}_2^{k_2=1}) = 82$$

$$\mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2=1}) \approx \begin{bmatrix} 0 & 0 & 0.59 & 0 \\ 0 & 0 & 0 & 0.41 \\ 0 & 0.65 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_2^{k_2=2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_2(\mathbf{\Omega}, \mathbf{A}_2^{k_2=3}) = 82$$

$$\mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2=2}) \approx \begin{bmatrix} 0 & 0 & 0.89 & 0 \\ 0 & 0 & 0 & 0.56 \\ 0 & 0.45 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_2^{k_2=3} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_2(\mathbf{\Omega}, \mathbf{A}_2^{k_2=3}) = 82$$

$$\mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2=3}) \approx \begin{bmatrix} 0 & 0 & 0.89 & 0 \\ 0 & 0 & 0 & 0.76 \\ 0 & 0.52 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_2^{k_2=4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow R_2(\mathbf{\Omega}, \mathbf{A}_2^{k_2=4}) = 82$$

$$\mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2=4}) \approx \begin{bmatrix} 0 & 0 & 0.59 & 0 \\ 0 & 0 & 0 & 0.56 \\ 0 & 0.35 & 0 & 0 \end{bmatrix}$$

Note that the absolute quality factors are :

$$Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=1}) \approx 1.66, \qquad Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=2}) \approx 1.91$$
$$Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=3}) \approx 2.19, \qquad Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=4}) \approx 1.51$$

Therefore, we can see that

$$Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=3}) > Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=2}) > Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=1}) > Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2=4})$$

which makes perfectly sense because $\mathbf{A}_1$ has more matching pairings with $\mathbf{A}_2^{k_2=3}$ than with others 2nd-best assignments $\mathbf{A}_2^{k_2}$ ($k_2 \neq 3$). These pairings have also the strongest impact in the global reward value. Therefore, the quality matrix $\mathbf{Q}$

[7] given in the optimal solution found using Murty's algorithm.

differentiates the quality of each pairing in the optimal assignment $\mathbf{A}_1$ as expected. This method provides an effective and comprehensive solution to estimate the quality of each specific association provided in the optimal assignment solution $\mathbf{A}_1$. The averaged qualities per association are:

$$Q_{aver}(\mathbf{A}_1, \mathbf{A}_2^{k_2=1}) \approx 0.55, \qquad Q_{aver}(\mathbf{A}_1, \mathbf{A}_2^{k_2=2}) \approx 0.63$$
$$Q_{aver}(\mathbf{A}_1, \mathbf{A}_2^{k_2=3}) \approx 0.73, \qquad Q_{aver}(\mathbf{A}_1, \mathbf{A}_2^{k_2=4}) \approx 0.50$$

The global quality matrix is then given by (using the averaging approach)

$$\mathbf{Q}(\mathbf{A}_1, \mathcal{A}_2) = \frac{1}{\beta_2} \sum_{k_2=1}^{\beta_2} \mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2}) \approx \begin{bmatrix} 0 & 0 & 0.74 & 0 \\ 0 & 0 & 0 & 0.57 \\ 0 & 0.49 & 0 & 0 \end{bmatrix}$$

The global quality indexes $Q_{abs}(\mathbf{A}_1, \mathcal{A}_2)$ and $Q_{aver}(\mathbf{A}_1, \mathcal{A}_2)$ are then approximately equal to 1.8 and 0.6 respectively.

One can also improve the estimation of the quality matrix by using the absolute quality factor of each solution $\mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2})$, for $k_2 = 1, \dots \beta_2$ to define the normalized weighting factors as follows:

$$\mathbf{w} = [w_{k_2}, k_2 = 1, \dots \beta_2]'$$

with $w_{k_2} \triangleq \frac{Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2})}{K}$, and where the normalization factor K is given by $K = \sum_{k_2=1}^{\beta_2} Q_{abs}(\mathbf{A}_1, \mathbf{A}_2^{k_2})$. In this example, we get the weights

$$\mathbf{w} = [w_1 \ w_2 \ w_3 \ w_4]' \approx [\frac{1.66}{7.27} \ \frac{1.91}{7.27} \ \frac{2.19}{7.27} \ \frac{1.51}{7.27}]' = [0.2283 \ 0.2627 \ 0.3012 \ 0.2077]'$$

The global quality matrix is then given by (using the averaging approach)

$$\mathbf{Q}(\mathbf{A}_1, \mathcal{A}_2) = \sum_{k_2=1}^{\beta_2} w_{k_2} \mathbf{Q}(\mathbf{A}_1, \mathbf{A}_2^{k_2}) \approx \begin{bmatrix} 0 & 0 & 0.76 & 0 \\ 0 & 0 & 0 & 0.58 \\ 0 & 0.49 & 0 & 0 \end{bmatrix}$$

If we prefer to use the Belief Interval Measure (BIM) instead of the previous averaging approach, we will get in this example the following imprecise qualities values:

Optimal. assignments	BIM
$(1, 3)$	$\approx [0.59, 0.89]$
$(2, 4)$	$\approx [0.41, 0.76]$
$(3, 2)$	$\approx [0.35, 0.65]$

Based on the comparisons of (pessimistic) lower bounds, or (optimistic) upper bounds of BIM, we observe that we get a consistent ordering of the qualities of the optimal solutions (same ordering as with the averaging method).

III. Solution of the 2nd problem (multi-criteria)

In this section, we evaluate the global DA association solution, with estimation of its quality, based on the knowledge of the qualities of multiple optimal DA solutions established separately based on distinct association criteria C_k, $k = 1, \dots, K$. More precisely, given the set of quality matrices $\mathbf{Q}k$ ($k = 1, \dots, K$) defined by the components $q^k(i, j)$ according to Eq.(11), how to establish the global optimal DA solution with its overall quality matrix $\mathbf{Q}$? Moreover, we want to take into account the importance of each criteria (when defined) in the establishment of the solution.

In fact this 2nd problem is linked to the previous one and the method developed for solving our first problem will also help to solve this second problem as it will be shown in the following. Our solution is based on four distinct steps:

• **Step 1**: Estimation of the normalized weighting vector $\mathbf{w}$ of the criteria: Two simple approaches are proposed to establish the normalized criteria ranking (weighting) vector.

1) **Direct method**: The weightings factors can be directly established either by an external source of information, or by the system designer. If these weightings factors are not available, we propose to compute them from the qualities indicators derived by the method used to solve the 1st problem (see the previous section). For example, if we consider K criteria providing quality factors $Q_{abs}^k(\mathbf{A}_1(C_k), \mathcal{A}_2(C_k))$, $k = 1, 2, \dots, K$, then we compute the normalized $K \times 1$ weighting vector $\mathbf{w} = [w_1 w_2 ... w_K]'$ with the k-th component given by

$$w_k \triangleq \frac{Q_{abs}^k(\mathbf{A}_1(C_k), \mathcal{A}_2(C_k))}{\sum_{j=1}^K Q_{abs}^j(\mathbf{A}_1(C_j), \mathcal{A}_2(C_j))} \qquad (13)$$

where $Q_{abs}^k(\mathbf{A}_1(C_k), \mathcal{A}_2(C_k))$ is the absolute quality factor obtained from the quality matrix $\mathbf{Q}^k(\mathbf{A}_1, \mathcal{A}_2)$ of the optimal DA for the criteria C_k.

2) **Saaty's method**: This method is part of Saaty's AHP method widely used for multi-criteria decision analysis in operational research [22], [23], [24], and it has been connected with information fusion and belief functions in [25], [26], [27]. The relative importance of one criterion over another must be expressed by the system designer using a pairwise $K \times K$ comparison matrix (also called knowledge matrix) $\mathbf{M} = [m_{pq}]$ where the element m_{pq} of the matrix defines the importance of criteria C_p with respect to the criteria C_q, with $p, q \in \{1, 2, \dots, K\}$. For example, see [25] for details, let's consider only $K = 3$ criteria, if the comparison matrix is given by

$$\mathbf{M} = \begin{bmatrix} (1/1) & (1/3) & (4/1) \\ (3/1) & (1/1) & (5/1) \\ (1/4) & (1/5) & (1/1) \end{bmatrix}$$

it means that the element $m_{13} = 4/1$ indicates that the criteria C_1 is four times as important as the criteria C_3

for the system designer (or decision-maker), etc. From this pairwise matrix, Saaty demonstrated that the ranking of the priorities of the criteria can be obtained from the normalized eigenvector, denoted $\mathbf{w}$, associated with the principal/max eigenvalue of the matrix $\mathbf{M}$, denoted λ. In our example, one gets $\lambda = 3.0857$ and and $\mathbf{w} = [0.2797\, 0.6267\, 0.0936]'$ which shows that C_2 criterion is the most important criterion with the weight 0.6267, then the criterion C_1 is the second most important criterion with weight 0.2797, and finally C_3 criterion is the least important criterion with weight 0.0936.

• **Step 2**: Combined estimation of the qualities of each target association

Once the normalized weighting vector $\mathbf{w}$ of the criteria has been obtained, we need at first to compute the combined/weighted estimation of the qualities of each target association with the n available measurements. This is done by building the following $n \times K$ matrix

$$\mathbf{Q}_i \triangleq [\mathbf{q}_i(C_1) \dots \mathbf{q}_i(C_K)] \tag{14}$$

where each column $\mathbf{q}_i(C_k)$ of the matrix $\mathbf{Q}_i$ corresponds to the transpose of the i-th row of the quality matrix $\mathbf{Q}^k(\mathbf{A}_1, \mathcal{A}_2)$.

Then following AHP approach, we multiply this $n \times K$ matrix $\mathbf{Q}_i$ by the normalized criteria ranking $K \times 1$ vector $\mathbf{w}$ (obtained either from the direct method of Saaty's one) to get the combined estimation of the qualities of each target association. More precisely, for the i-th target, we obtain the following $n \times 1$ vector

$$\mathbf{q}_i \triangleq \mathbf{Q}_i \mathbf{w} \tag{15}$$

• **Step 3**: Search for the optimal global assignment based on combined qualities derived from the criteria.

From the set of m vectors $\mathbf{q}_i$ (i=1,2,…,m) we need to solve now a new optimal DA association problem with the (global) $m \times n$ rewards matrix defined by

$$\mathbf{\Omega}_G \triangleq [\mathbf{q}_1 \ \mathbf{q}_2 \ \dots \mathbf{q}_m]' \tag{16}$$

Murty's algorithm is then used again here to get the optimal DA solution(s) providing the best global reward, and to generate also all the 2nd-best solutions that are necessary to estimate its quality in Step 4.

• **Step 4**: Estimation of the quality of the optimal DA solution.

We use the method described in Section 2 for solving the problem 1 to estimate the quality of the optimal DA solution. If several 1st-best DA solutions occur, we choose the solution generating the highest Q_{abs} quality index.

IV. A SIMPLE ILLUSTRATIVE EXAMPLE

For the sake of simplicity, let's consider the following example with $m = 3$ targets, $n = 5$ measurements, and 3 criteria C_1, C_2 and C_3 associated with the (randomly chosen) rewards matrices:

$$\mathbf{\Omega}(C_1) = \begin{bmatrix} 100 & 20 & 33 & 5 & 27 \\ 11 & 80 & 25 & 37 & 62 \\ 38 & 2 & 24 & 78 & 46 \end{bmatrix}$$

$$\mathbf{\Omega}(C_2) = \begin{bmatrix} 87 & 35 & 43 & 20 & 95 \\ 28 & 83 & 25 & 10 & 29 \\ 10 & 7 & 72 & 41 & 29 \end{bmatrix}$$

$$\mathbf{\Omega}(C_3) = \begin{bmatrix} 25 & 78 & 49 & 60 & 9 \\ 30 & 26 & 79 & 20 & 49 \\ 20 & 20 & 3 & 47 & 81 \end{bmatrix}$$

A. Qualities of optimal data associations

Applying the method described in section 1, we easily obtain the following quality matrices of optimal DA solutions:

- For criterion C_1, one gets $\beta_1 = 1$ and $\beta_2 = 1$, and the following 1st best and 2nd best DA solutions

$$\mathbf{\Omega}(C_1) \Rightarrow \begin{cases} \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{cases}$$

providing the 1st and 2nd best global rewards $R(\mathbf{\Omega}(C_1), \mathbf{A}_1) = 258$ and $R(\mathbf{\Omega}(C_1), \mathbf{A}_2) = 240$. Applying the method described in Section 2, we obtain the following quality matrix related with the optimal DA based on criterion C_1:

$$\mathbf{Q}^1 \approx \begin{bmatrix} 0.82 & 0 & 0 & 0 & 0 \\ 0 & 0.52 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.76 & 0 \end{bmatrix}$$

- For criterion C_2, one gets $\beta_1 = 1$ and $\beta_2 = 1$, and the following 1st best and 2nd best DA solutions

$$\mathbf{\Omega}(C_2) \Rightarrow \begin{cases} \mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{cases}$$

providing the 1st and 2nd best global rewards $R(\mathbf{\Omega}(C_2), \mathbf{A}_1) = 250$ and $R(\mathbf{\Omega}(C_2), \mathbf{A}_2) = 242$. Applying the method described in Section 2, we obtain the following quality matrix related with the optimal DA based on criterion C_2:

$$\mathbf{Q}^2 \approx \begin{bmatrix} 0 & 0 & 0 & 0 & 0.51 \\ 0 & 0.78 & 0 & 0 & 0 \\ 0 & 0 & 0.74 & 0 & 0 \end{bmatrix}$$

- For criterion C_3, one gets $\beta_1 = 1$ and $\beta_2 = 1$, and the following 1st best and 2nd best DA solutions

$$\mathbf{\Omega}(C_3) \Rightarrow \begin{cases} \mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{cases}$$

providing the 1st and 2nd best global rewards $R(\mathbf{\Omega}(C_3), \mathbf{A}_1) = 238$ and $R(\mathbf{\Omega}(C_3), \mathbf{A}_2) = 220$. Applying the method described in Section 2, we obtain the following quality matrix related with the optimal DA based on criterion C_3:

$$\mathbf{Q}^3 \approx \begin{bmatrix} 0 & 0.53 & 0 & 0 & 0 \\ 0 & 0 & 0.78 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.79 \end{bmatrix}$$

B. Multicriteria-based DA solution with its quality

- Case 1: If we assume that all criteria have the same weights in the search of optimal DA solution, then we take the normalized weighting vector as $\mathbf{w} = [1/3\ 1/3\ 1/3]'$. Therefore, the weighted average $\mathbf{\Omega}_G = \sum_{k=1}^{K=3} w_k \mathbf{Q}^k$ of the quality matrices $\mathbf{Q}^1$, $\mathbf{Q}^2$ and $\mathbf{Q}^3$ gives us the following rewards matrix

$$\mathbf{\Omega}_G \approx \begin{bmatrix} 0.27 & 0.17 & 0 & 0 & 0.17 \\ 0 & 0.43 & 0.26 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.26 \end{bmatrix}$$

Now we solve the DA association problem to maximize the global quality reward using Murty's algorithm and we get the following 1st best and 2nd best DA solutions:

$$\mathbf{\Omega}_G \Rightarrow \begin{cases} \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{cases}$$

with the 1st and 2nd best global rewards $R(\mathbf{\Omega}_G, \mathbf{A}_1) \approx 0.97$ and $R(\mathbf{\Omega}_G, \mathbf{A}_2) \approx 0.96$. Applying the method described in Section II to estimate the quality of this optimal DA solution, we obtain the following quality matrix:

$$\mathbf{Q} \approx \begin{bmatrix} 0.74 & 0 & 0 & 0 & 0 \\ 0 & 0.84 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 \end{bmatrix}$$

- Case 2: If we use the prior information given by absolute quality indicators to build the normalized weighting vector, we get

$$Q^1_{abs} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{Q}^1(i,j) \approx 2.11$$

$$Q^2_{abs} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{Q}^2(i,j) \approx 2.04$$

$$Q^3_{abs} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{Q}^3(i,j) \approx 2.10$$

and we have $Q^1_{abs} + Q^2_{abs} + Q^3_{abs} = 6.2672$. So that, the normalized weights are given by

$$\mathbf{w} = [w_1\ w_2\ w_3]' = [\frac{2.1154}{6.2672}\ \frac{2.0426}{6.2672}\ \frac{2.1091}{6.2672}]' \approx [0.3375\ 0.3260\ 0.3365]'$$

The weighted average $\mathbf{\Omega}_G = \sum_{k=1}^{K=3} w_k \mathbf{Q}^k$ of the quality matrices $\mathbf{Q}^1$, $\mathbf{Q}^2$ and $\mathbf{Q}^3$ give us now the following rewards matrix

$$\mathbf{\Omega}_G \approx \begin{bmatrix} 0.27 & 0.17 & 0 & 0 & 0.16 \\ 0 & 0.43 & 0.26 & 0 & 0 \\ 0 & 0 & 0.24 & 0.25 & 0.26 \end{bmatrix}$$

Now we solve the DA association problem to maximize the global quality reward and we get the following 1st best and 2nd best DA solutions:

$$\mathbf{\Omega}_G \Rightarrow \begin{cases} \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{cases}$$

with the 1st and 2nd best global rewards $R(\mathbf{\Omega}_G, \mathbf{A}_1) \approx 0.97$ and $R(\mathbf{\Omega}_G, \mathbf{A}_2) \approx 0.96$. Applying the method described in Section 2 to estimate the quality of this optimal DA solution, we obtain the following quality matrix:

$$\mathbf{Q} \approx \begin{bmatrix} 0.74 & 0 & 0 & 0 & 0 \\ 0 & 0.84 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 \end{bmatrix}$$

Because the normalized weights based on the absolute quality indicators, in this example, are all close to 1/3, we obtain the result of the multicriteria-based optimal DA and its quality close to what we get when assuming equi-importance of the criteria in the fusion process, which is normal.

To qualify qualitatively the quality of the pairings in the optimal DA solution, we split the quality range [0;1] into three subintervals as follows[8]

Low quality :	if $q(i,j) \in [0; 1/3]$
Medium quality :	if $q(i,j) \in [1/3; 2/3]$
High quality :	if $q(i,j) \in [2/3; 1]$

[8] Of course, other repartitions could be used instead depending on the what would prefer the system designer.

Based on this qualitative scale, we finally get for our example the final multicriteria-based DA solution

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

with the qualitative quality matrix

$$\mathbf{Q}^{qualitative} = \begin{bmatrix} High & - & - & - & - \\ - & High & - & - & - \\ - & - & - & - & Medium \end{bmatrix}$$

where the notation "$-$" means "that the quality evaluation does not apply", or is interpreted (by default) as "the worst quality".

Remark: It is worth to note that this approach provides in general not the same results as if one would combine (and weight) directly the original reward matrices of each criterion. In this example, the weighted global reward matrix $\mathbf{\Omega}_{direct} = \sum_{k=1}^{K} w_k \mathbf{\Omega}(C_k)$ would be equal to

$$\mathbf{\Omega}_{direct} \approx \begin{bmatrix} 69.07 & 45.39 & 41.75 & 29.31 & 40.85 \\ 22.93 & 61.43 & 44.46 & 22.79 & 47.44 \\ 23.13 & 9.98 & 30.78 & 55.75 & 53.53 \end{bmatrix}$$

corresponding to the quality matrix of optimal DA solution

$$\mathbf{Q}_{direct} \approx \begin{bmatrix} 0.73 & 0 & 0 & 0 & 0 \\ 0 & 0.84 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.47 & 0 \end{bmatrix}$$

One sees that these high quality solutions are fully consistent with the high quality solutions of our method. However, the medium quality solution (we get (3,4) pairing from the direct optimal assignment versus (3,5) assignment obtained by our method) mismatch. This reflects an ambiguity in the choice of the assignment of target T_3. Therefore, such assignment is unreliable because of its low quality, and should not be used to update the track of this target.

V. Conclusion

In this paper, we have proposed two methods based on belief functions for establishing: 1) the quality of pairings given by optimal data association (or assignment) solution using a chosen algorithm (typically Murty's algorithm coupled with Auction algorithm) with respect to a given criterion, and 2) the quality of the multicriteria-based optimal data association solution. Our methods are independent of the choice of the algorithm used in finding the optimal assignment solution, and, in case of multiple optimal solutions, they provide also a way to select the best optimal assignment solution (the one having the highest absolute quality factor). The methods developed in this paper are general in the sense that they can be applied to different types of association problems corresponding to different sets of constraints. This method can be extended to SD-assignment problems as well. As perspectives, we would like to extend our approach to the n-D assignment context, and then evaluate its performances in a realistic multi-target tracking scenario.

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