

# Spinor Darboux Equations of Curves in Euclidean 3-Space

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ABSTRACT. In this paper, the spinor formulation of Darboux frame on an oriented surface is given. Also, the relation between spinor formulation of Frenet frame and Darboux frame is obtained.

## 1. INTRODUCTION

Spinors in general were discovered by Elie Cartan in 1913 [2]. Later, spinors were adopted by quantum mechanics in order to study the properties of the intrinsic angular momentum of the electron and other fermions. Today, spinors enjoy a wide range of physics applications. In mathematics, particularly in differential geometry and global analysis, spinors have since found broad applications to algebraic and differential topology, symplectic geometry, gauge theory, complex algebraic geometry, index theory [13, 11].

In the differential geometry of surfaces the Darboux frame is a natural moving frame constructed on a surface. It is the analog of the Frenet- Serret frame as applied to surface geometry. A Darboux frame exists at any non-umbilic point of a surface embedded in Euclidean space. The construction of Darboux frames on the oriented surface first considers frames moving along a curve in the surface, and then specializes when the curves move in the direction of the principal curvatures [10].

In [2, 4], the triads of mutually orthogonal unit vectors were expressed in terms of a single vector with two complex components, which is called a spinor. In the light of the existing studies, reducing the Frenet equations to a single spinor equation, equivalent to the three usual vector equations, is a consequence of the relationship between spinors and orthogonal triads of vectors. The aim of this paper is to show that the Darboux equations can be expressed with a single equation for a vector with two complex components.

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## 2. PRELIMINARIES

The Euclidean 3-space provided with standard flat metric is given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where  $(x_1, x_2, x_3)$  is a rectangular coordinate system of  $E^3$ . Recall that, the norm of an arbitrary vector  $x \in E^3$  is given by  $\|x\| = \sqrt{\langle x, x \rangle}$ . Let  $\alpha$  be a curve in Euclidean 3- space. The curve  $\alpha$  is called a unit speed curve if velocity vector  $\alpha'$  of  $\alpha$  satisfies  $\|\alpha'\| = 1$ .

Let us denote by  $T(s), N(s)$  and  $B(s)$  the unit tangent vector, unit normal vector and unit binormal vector of  $\alpha$  respectively. The Frenet Trihedron is the collection of  $T(s), N(s)$  and  $B(s)$ . Thus, the Frenet formulas are as

$$\begin{aligned} \frac{dT}{ds} &= \kappa(s)N(s) \\ \frac{dN}{ds} &= -\kappa(s)T(s) + \tau(s)B(s) \\ \frac{dB}{ds} &= -\tau(s)N(s). \end{aligned}$$

Here, the curvature is defined to be  $\kappa(s) = \|T'(s)\|$  and the torsion is the function  $\tau$  such that  $B'(s) = -\tau(s)N(s)$  [8].

The group of rotation about the origin denoted by  $SO(3)$  in  $R^3$  is homomorphic to the group of unitary complex  $2 \times 2$  matrices with unit determinant denoted by  $SU(2)$ . Thus, there exists a two-to-one homomorphism of  $SU(2)$  onto  $SO(3)$ . Whereas the elements of  $SO(3)$  act the vectors with three real components, the elements of  $SU(2)$  act on vectors with two complex components which are called spinors, [1, 5]. In this case, we can define a spinor as

$$(1) \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

by means of three vectors  $a, b, c \in R^3$  such that

$$(2) \quad a + ib = \psi^t \sigma \psi, \quad c = -\hat{\psi}^t \sigma \psi.$$

Here, the superscript  $t$  denotes transposition and  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  is a vector whose cartesian components are the complex symmetric  $2 \times 2$  matrices

$$(3) \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In addition to this,  $\hat{\psi}$  is the mate (or conjugate) of  $\psi$  and  $\bar{\psi}$  is complex conjugation of  $\psi$  [16]. Therefore,

$$(4) \quad \hat{\psi} \equiv - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \bar{\psi} = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}.$$

In that case, the vectors  $a, b$  and  $c$  are explicitly given by

$$\begin{aligned} a + ib &= \psi^t \sigma \psi = (\psi_1^2 - \psi_2^2, i(\psi_1^2 + \psi_2^2), -2\psi_1 \psi_2), \\ (5) \quad c &= -\hat{\psi}^t \sigma \psi = (\psi_1 \bar{\psi}_2 + \bar{\psi}_1 \psi_2, i(\psi_1 \bar{\psi}_2 - \bar{\psi}_1 \psi_2), |\psi_1|^2 - |\psi_2|^2). \end{aligned}$$

Since the vector  $a + ib \in C^3$  is an isotropic vector, by means of an easy computation one can find that  $a, b$  and  $c$  are mutually orthogonal [2, 4, 6]. Also,  $|a| = |b| = |c| = \bar{\psi}^t \psi$  and  $\langle a \wedge b, c \rangle = \det(a, b, c) > 0$ . On the contrary, if the vectors  $a, b$  and  $c$  are mutually orthogonal vectors of same magnitudes such that  $(\det(a, b, c) > 0)$ , then there is a spinor, defined up to sign such that the equation (2) holds. Under the conditions stated above, for two arbitrary spinors  $\phi$  and  $\psi$ , there exist the following equalities

$$\begin{aligned} \overline{\phi^t \sigma \psi} &= -\hat{\phi}^t \sigma \hat{\psi}, \\ (6) \quad \widehat{a\phi + b\psi} &= \bar{a}\hat{\phi} + \bar{b}\hat{\psi}, \end{aligned}$$

and

$$\hat{\hat{\psi}} = -\psi,$$

where  $a$  and  $b$  are complex numbers. The correspondence between the spinors and the orthogonal bases given by (2) is two to one, because the spinors  $\psi$  and  $-\psi$  correspond to the same ordered orthonormal bases  $\{a, b, c\}$ , with  $|a| = |b| = |c|$ , and  $\langle a \wedge b, c \rangle$ . In addition to that, the ordered triads  $\{a, b, c\}$ ,  $\{b, c, a\}$  and  $\{c, a, b\}$  correspond to different spinors. Since the matrices  $\sigma$  (given by (3)) are symmetric, any pairs of spinors  $\phi$  and  $\psi$  satisfy  $\phi^t \sigma \psi = \psi^t \sigma \phi$ . The set  $\psi, \hat{\psi}$  is linearly independent for the spinor  $\psi \neq 0$  [4].

### 3. SPINOR DARBOUX EQUATIONS

In this section we investigate the spinor equation of the Darboux equations for a curve on the oriented surface in Euclidean 3-space  $E^3$ . Let  $M$  be an oriented surface in Euclidean 3-space and let us consider a unit speed regular curve  $\alpha(s)$  on the surface  $M$ . Since the curve  $\alpha(s)$  is also in space, there exists Frenet frame  $\{T, N, B\}$  at each point of the curve where  $T$  is unit tangent vector,  $N$  is principal normal vector and  $B$  is binormal vector, respectively. The Frenet equations of curve  $\alpha(s)$  is given by

$$\begin{aligned} \frac{dT}{ds} &= \kappa N \\ (7) \quad \frac{dN}{ds} &= -\kappa T + \tau B \\ \frac{dB}{ds} &= -\tau N \end{aligned}$$

where  $\kappa$  and  $\tau$  are curvature and torsion of the curve  $\alpha(s)$ , respectively [2]. According to the result concerned with the spinor (given by section 2) there

exists a spinor  $\psi$  such that

$$(8) \quad N + iB = \psi^t \sigma \psi, \quad T = -\hat{\psi}^t \sigma \psi$$

with  $\bar{\psi}^t \psi = 1$ . Thus, the spinor  $\psi$  represents the triad  $\{N, B, T\}$  and the variations of this triad along the curve  $\alpha(s)$  must correspond to some expression for  $\frac{d\psi}{ds}$ . That is, the Frenet equations are equivalent to the single spinor equation

$$(9) \quad \frac{d\psi}{ds} = \frac{1}{2}(-i\tau\psi + \kappa\hat{\psi})$$

where  $\kappa$  and  $\tau$  denote the torsion and curvature of the curve, respectively. The equation (9) is called spinor Frenet equation, [4].

Since the curve  $\alpha(s)$  lies on the surface  $M$ , there exists another frame of the curve  $\alpha(s)$  which is called Darboux frame and denoted by  $\{T, g, n\}$ . In this frame,  $T$  is the unit tangent of the curve,  $n$  is the unit normal of the surface  $M$  and  $g$  is a unit vector given by  $g = n \wedge T$ . Since the unit tangent  $T$  is common in both Frenet frame and Darboux frame, the vectors  $N$ ,  $B$ ,  $g$ , and  $n$  lie on the same plane. So that, the relations between these frames can be given as follows

$$(10) \quad \begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where  $\theta$  is the angle between the vectors  $g$  and  $N$ . The derivative formulae of the Darboux frame is

$$(11) \quad \begin{bmatrix} \frac{dT}{ds} \\ \frac{dg}{ds} \\ \frac{dn}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix}$$

where  $\kappa_g$  is the geodesic curvature,  $k_n$  is the normal curvature and  $\tau_g$  is the geodesic torsion of  $\alpha(s)$  [10].

Considering the equations (1), (2) and (8) there exists a spinor  $\phi$ , defined up to sign, such that

$$(12) \quad g + in = \phi^t \sigma \phi, \quad T = -\hat{\phi}^t \sigma \phi$$

and

$$\bar{\phi}^t \phi = 1.$$

That is, the spinor  $\phi$  represents the triad  $\{g, n, T\}$  of the curve  $\alpha(s)$  on the surface  $M$ . The variations of the triad  $\{g, n, T\}$  along curve must correspond to some expression for  $\frac{d\phi}{ds}$ . Since  $\{\phi, \hat{\phi}\}$  is a basis for the two component spinors ( $\phi \neq 0$ ), there are two functions  $f$  and  $h$ , such that

$$(13) \quad \frac{d\phi}{ds} = f\phi + h\hat{\phi}$$

where the functions  $f$  and  $g$  are possibly complex-valued functions. Differentiating the first equation in (12) and using the equation (13) we have

$$\begin{aligned} \frac{dg}{ds} + i \frac{dn}{ds} &= \left( \frac{d\phi}{ds} \right)^t \sigma\phi + \phi^t \sigma \left( \frac{d\phi}{ds} \right) \\ &= 2f(\phi^t \sigma\phi) + 2h(\hat{\phi}^t \sigma\phi). \end{aligned}$$

Substituting the equations (11) and (12) into the last equation and after simplifying, one finds

$$(14) \quad -i\tau_g(g + in) + (-\kappa_g - i\kappa_n)T = 2f(g + in) - 2hT.$$

From the last equation we get

$$(15) \quad f = -i \frac{\tau_g}{2}, \quad h = \frac{i\kappa_n + \kappa_g}{2}$$

Thus, we have proved the following theorem.

**Theorem 3.1.** *Let the two-component spinor  $\phi$  represents the triad  $\{g, n, T\}$  of a curve parametrized by arc length on the surface  $M$ . Then, the Darboux equations are equivalent to the single spinor equation*

$$(16) \quad \frac{d\phi}{ds} = \left( -i \frac{\tau_g}{2} \right) \phi + \left( \frac{i\kappa_n + \kappa_g}{2} \right) \hat{\phi}$$

where  $\kappa_n$  and  $\kappa_g$  are normal and geodesic curvature, respectively and  $\tau_g$  is the geodesic torsion of  $\alpha(s)$ .

The equation (16) is called spinor Darboux equation. Now, we investigate the relation between the spinors  $\psi$  and  $\phi$  represent the triad  $\{N, B, T\}$  (Frenet frame) and  $\{g, n, T\}$  (Darboux Frame), respectively.

Considering the equation (10) we get

$$(17) \quad N + iB = (g + in)(\cos \theta + i \sin \theta)$$

Thus, from the equations (8), (12) and (17) we can give the following theorem.

**Theorem 3.2.** *Let the spinors  $\psi$  and  $\phi$  represent the triads  $\{N, B, T\}$  and  $\{g, n, T\}$  of a curve parametrized with arc length, respectively. Then, the relation between the spinor formulation of Frenet frame and Darboux frame is as follows*

$$\begin{aligned} \psi^t \sigma \psi &= e^{i\theta} (\phi^t \sigma \phi) \\ T &= T \end{aligned}$$

#### 4. CONCLUSIONS

In this study, we have established the spinor Darboux equations of curves on an oriented surface in  $E^3$  and then we have given the relation between spinor formulations of Frenet frame and Darboux frame. Also, we have shown that the Darboux equations are equivalent to a single spinor equation, which is a consequence of the relationship between spinors and orthogonal triads of vectors, and of using complex quantities.

There are so many applications of Darboux frame in differential geometry. So far, different curves such as Bertrand curves[7], biharmonic  $D$ -helices [15], Smarandache curves [12] have been studied according to Darboux frame. Furthermore, other studies like inextensible flows of these curves [14], tubes [3], dual Darboux frames of ruled surfaces have been given [9].

These kinds of applications can be included in future investigations by means of spinor Darboux equations of curves on surfaces. Moreover, the spinor formulations of curves can be given not only in Euclidean 3-space but also in other three dimensional spaces like Minkowski 3-space or Galilean 3-space, etc.

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