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Fuzzy α-discounting method for multi-criteria decision-making

Atilla Karaman[a]* and Metin Dagdeviren[b]

[a]*Department of Operations Research, Defense Sciences Institute, Turkish Military Academy, Ankara, Turkey;* [b]*Faculty of Engineering, Department of Industrial Engineering, Gazi University, Ankara, Turkey*

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The α-Discounting Method was developed to be an alternative to and extension of the Analytical Hierarchy Process (AHP) to solve multi-criteria decision-making (MCDM) problems with non-commensurable and conflicting criteria. In contrast to the AHP, this method works not only for pairwise comparisons but also for *n*-wise comparisons if relative importance of criteria can be expressed in a system of linear homogenous equations. This method also has a comparative advantage as it can transform those MCDM problems, classified as inconsistent by the AHP, into a consistent form. This study briefly compares the two methods and then develops the Fuzzy α-Discounting Method for Multi-Criteria Decision Making (Fα-DM MCDM). Two illustrative fuzzy MCDM problems from the literature have been solved to show how the Fα-DM MCDM works.

Keywords: MCDM; α-Discounting Method; consistency in fuzzy decision problems; fuzzy set theory

1. Introduction

It is not an easy task for a decision maker to choose from among the alternatives s/he has because the decision-making process mostly includes multiple criteria with different directions instead of a mono-criterion with one direction. The conflict situation which the decision maker faces emerges while reaching a comprehensive judgment as a result of balancing the criteria that are competing with each other and which are oriented toward multiple goals (Belton and Stewart 2003). This is a fundamental problem in multi-criteria decision-making (MCDM) known as the aggregation problem (Roy 2005). As decision-making problems involve criteria that are non-commensurable (different units) and conflicting (competing with each other), the MCDM gains importance from the view point of decision makers as it provides the tools and techniques that balance those multiple criteria (Belton and Stewart 2003).

The MCDM methods proposed in the literature have both advantages and disadvantages. Analytical Hierarchy Process (AHP), as one of the most used MCDM methods, facilitates the structuring of the decision-making problems through a hierarchical approach; yet, it has received criticisms including that by Dodd, Donegan, and McMaster (1995) for the arbitrary measure of consistency proposed in the method. In the scope of this paper, the development of Fuzzy α-Discounting Method for Multi-Criteria Decision Making (Fα-DM MCDM) for pairwise comparisons using linguistic variables (variables whose underlying values are derived from verbal preference statements) has been motivated by this criticism from Dodd, Donegan, and McMaster (1995). In the case of exceeding the inconsistency threshold of 10% in the AHP, the analysts need to return back to the decision maker(s) and repeat all the procedures to get corrected preference statements of the decision maker(s) to present an acceptable final output.

Besides the burden of overcoming the inconsistency issue, the analysts may sometimes need to work with decision makers who prefer to express relationships among criteria through verbal (fuzzy) statements instead of crisp numerical values. In this respect, the main objective of this paper is to develop a novel method that ensures consistency for the preference statements of the decision makers that are reflected in decision matrices of the fuzzy MCDM problems with pairwise comparisons. To reach this objective, we have chosen to widen the scope of the α-Discounting Method (α-D) for MCDM developed by Smarandache (2010), an alternative to and extension of the AHP (Saaty 1972) to solve linear deterministic MCDM problems by assuring consistency for the decision matrices of concern, as a basis for the development of the aforementioned method for fuzzy decision matrices constructed both in single person and group decision-making settings. Another objective of this paper is to also integrate a revised version of a novel defuzzification method (Converting Fuzzy Data into Crisp Scores Method – CFCS – developed by Opricovic and Tzeng (2003)) in the aforementioned method to obtain the crisp weights that are assigned to the criteria in the end. Applicable to both the mixed set of crisp and fuzzy numbers, the defuzzification method CFCS has

*Corresponding author. Email: tllkrmn@gmail.com

advantages in treating the triangular fuzzy numbers (TFNs) with the same mean.

The Fα-DM MCDM proposed in this paper contributes to the MCDM literature by expanding Smarandache's (2010) method to include fuzzy MCDM problems and to ensure consistency for the constructed fuzzy decision matrices which was only possible for weakly inconsistent linear deterministic MCDM problems with pairwise comparisons. The Fα-DM MCDM deals with the fuzzy MCDM problems, constructed with TFNs within the scope of this paper, by first decomposing the fuzzy decision matrices into three separate (***l***), (***m***), and (***u***) matrices that constitute the respective fuzzy numbers from the lower boundary, most likely, and upper boundary fuzzy values, respectively. These fuzzy matrices are then checked for consistency by testing if the determinant (det) of the respective matrix equals 0. In the case of det = 0, the matrix is said to be a *consistent* matrix. In the contrary situation (where det ≠ 0), the matrix is found to be an *inconsistent* matrix and needs to be either discounted or increased with the α coefficient. During the formation of the fuzzy decision matrix through the linguistic variables assigned to the criteria by the decision maker, it is not expected that the decision maker assigns linguistic variables fully consistent with all pairwise comparisons. This is either due to the size of the MCDM problem, environmental conditions surrounding the decision maker that may influence her/his rationality, or her/his partial knowledge of the factors concerning the MCDM problem that may hinder her/him in assigning linguistic variables to criteria pairs consistent with others in the fuzzy decision matrix. In this respect, consistency is needed to ensure valid results from the solution of the MCDM problem of concern. By applying Fα-DM MCDM in inconsistent linear fuzzy decision matrices, there is a high likelihood of assuring consistency for the decomposed fuzzy matrices at an initial stage while solving the fuzzy MCDM problem. After the multiplication of the matrix elements above the diagonal with testing the α coefficient, the new matrix is rechecked to see if it gives det = 0. Provided that all matrices give det = 0, defuzzification is applied to have the crisp criteria weights. In the next section, we introduce, in detail, the consistency/inconsistency of an MCDM problem within the context of this paper. We briefly compare the original α-D MCDM and AHP, and then introduce the Fα-DM MCDM for pairwise comparisons in a fuzzy setting. Then, we solve two illustrative fuzzy MCDM problems from the literature to show how the Fα-DM MCDM works.

2. α-Discounting Method for MCDM

Developed by Smarandache (2010), the α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM) is an alternative to and extension of the AHP whose foundations were laid by Miller (1966) who formulated criteria under a hierarchy for the first time. This was then advanced by Saaty (1972) under the name AHP. In contrast to the AHP which can only deal with pairwise comparisons, α-D MCDM can handle any n-wise (with $n \geq 2$) comparisons of criteria in the form of linear homogeneous equations.

2.1. *Classification of linear decision-making problems*

In α-D MCDM, decision-making problems which can be expressed as linear homogeneous equations are divided into three groups based on their consistency levels (Smarandache 2010):

(i) *Consistent linear decision-making problems*: Any substitution of a variable x_i from an equation into another equation returns a result in agreement with all equations.

(ii) *Weakly inconsistent linear decision-making problems* (*WD*): In this type of problems, at least one substitution of a variable x_i from an equation into another equation returns a result in disagreement with at least another equation in the following ways:

$$WD(1)\left\{\begin{array}{ll} x_i = k_1 \cdot x_j, & k > 1; \\ x_i = k_2 \cdot x_j, & k_2 > 1, \quad k_2 \neq k_1 \end{array}\right\} \tag{1}$$

or

$$WD(2)\left\{\begin{array}{ll} x_i = k_1 \cdot x_j, & 0 < k_1 < 1; \\ x_i = k_2 \cdot x_j, & 0 < k_2 < 1, \quad k_2 \neq k_1 \end{array}\right\} \tag{2}$$

or

$$WD(3)\{x_i = k \cdot x_i, \quad k \neq 1\}. \tag{3}$$

(iii) *Strongly inconsistent linear decision-making problems* (*SD*): In this type of problems, at least one substitution of a variable x_i from an equation into another equation results in disagreement with at least another equation in the following way:

$$SD(4)\left\{\begin{array}{l} x_i = k_1 \cdot x_j; \\ x_i = k_2 \cdot x_j, \end{array}\right\} \text{ while } 0 < k1 < 1 < k2 \text{ or } 0 < k2 < 1 < k1. \tag{4}$$

2.2. **Procedural steps for α-D MCDM**

α-D MCDM transforms an inconsistent decision-making problem into a consistent state by discounting/increasing the linear judgment scale coefficients at/to some values. The procedural steps for α-D MCDM can be described as follows (Smarandache 2010):

Table 1. Fuzzy numbers corresponding to linguistic variables.

Intensity of fuzzy scale	Definition of linguistic variables	Fuzzy number	Value
$\tilde{1}$	Similar importance	(L, M, U)	(1, 1, 1)
$\tilde{3}$	Moderate importance	(L, M, U)	(2, 3, 4)
$\tilde{5}$	Intense importance	(L, M, U)	(4, 5, 6)
$\tilde{7}$	Demonstrated importance	(L, M, U)	(6, 7, 8)
$\tilde{9}$	Extreme importance	(L, M, U)	(8, 9, 9)
$\tilde{2}, \tilde{4}, \tilde{6}, \tilde{8}$	Intermediate values	(L, M, U)	(_, _, _)

(1) Let $\boldsymbol{C} = \{C_1, C_2, \ldots, C_n\}$, $n \geq 2$, be a set for the hierarchical components (criteria and alternatives). Get the relationships from the decision maker(s) first for the criteria and then the alternatives and form the preference statements set, $\boldsymbol{P} = \{P_1, P_2, \ldots, P_m\}$, $m \geq 1$. Each preference P_i is a linear homogeneous equation of the criteria $C_1, C_2, \ldots, C_n$: $P_i = f(C_1, C_2, \ldots, C_n)$. For the weights of the hierarchy components, let us construct a basic belief assignment (bba): m: C $\rightarrow$ [0, 1] such that $m(C_i) = x_i$, $0 < x_i < 1$ and $\sum_{i=1}^{n} m(C_i) = \sum_{i=1}^{n} x_i = 1$. Consequently, the total weight of the hierarchy components in their own group equals 1.

(2) Find all variables x_i in accordance with the set of preferences P to get an $m \times n$ linear homogeneous system of equations whose associated matrix is $\boldsymbol{A} = (a_{ij})$, $1 \leq i \leq m$ and $1 \leq j \leq n$. In order for this system to have nontrivial solutions, the rank of the matrix $\boldsymbol{A}$ should be strictly less than n.

(3) Calculate the determinant of matrix $\boldsymbol{A}$, det($\boldsymbol{A}$).

(4) If det($\boldsymbol{A}$) = 0, then the decision problem is found to be consistent as the system of equations is dependent. There is no need to parameterize the system with α. In the case of parameterization, use the Fairness Principle to set all parameters equal $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \alpha > 0$. Solve the system to find a general solution. Replace α parameters and secondary variables,[1] to reach a particular solution. Obtain the priority vector; i.e. corresponding eigenvector for the maximum eigenvalue, by normalizing the particular solution.

(5) If det($\boldsymbol{A}$) ≠ 0, then the decision problem is found to be inconsistent as the homogeneous linear system has only a trivial solution. Based on the type of inconsistency, go to either step 6 or step 7.

(6) If the linear decision-making problem is weakly inconsistent, then parameterize the right-hand side coefficients, and denote the system matrix $\boldsymbol{A}(\alpha)$. Compute the α values which makes det($\boldsymbol{A}(\alpha)$) equal 0 in order to get the parametric equation (If the Fairness Principle is used, set all parameters equal to each other and solve for $\alpha > 0$). Replace α in $\boldsymbol{A}(\alpha)$, and solve the resulting dependent homogeneous linear system. Similarly as in step 4, replace each secondary variable by 1, and normalize the particular solution to get the priority vector.

(7) If the linear decision-making problem is strongly inconsistent, get more information from the decision-maker(s) to remove inconsistencies found in the decision-making problem. In strong inconsistency situation, the Fairness Principle may not work properly. Another principle should be considered to deal with the inconsistencies.

In weak inconsistent decision-making problems, where α-D MCDM/Fairness Principle is used, the degree of consistency and inconsistency of the decision-making problem at the initial stage is calculated based on the α value obtained as follows (Smarandache 2010):

- If $0 < \alpha < 1$, then α is the degree of consistency and $\beta = 1 - \alpha$ is the degree of inconsistency of the decision-making problem.
- If $\alpha > 1$, then $1/\alpha$ is the degree of consistency and $\beta = 1 - (1/\alpha)$ is the degree of inconsistency of the decision-making problem.

2.3. *A comparison of AHP and α-D MCDM*

The following have been stated for the comparison of AHP and α-D MCDM by Smarandache (2010):

(1) For consistent decision-making problems, AHP and α-D MCDM/Fairness Principle give the same result.

(2) The general solution of α-D MCDM includes all particular solutions, that of AHP as well.

(3) α-D MCDM is not restricted to pairwise comparisons, it can also use n-wise comparisons between criteria.

(4) α-D MCDM also works for preferences that can be transformed into homogeneous nonlinear equations and/or inequalities.

Table 2. Fuzzy pairwise comparison of each criterion.

Quality criteria	Durability (D)	Aesthetics (A)	Reliability ($R1$)	Reputation ($R2$)
Durability (D)	1, 1, 1	2, 3, 4	1/6, 1/5, 1/4	6, 7, 8
Aesthetics (A)	1/4, 1/3, 1/2	1, 1, 1	1/8, 1/7, 1/6	1, 2, 3
Reliability ($R1$)	4, 5, 6	6, 7, 8	1, 1, 1	8, 9, 9
Reputation ($R2$)	1/8, 1/7, 1/6	1/3, 1/2, 1	1/9, 1/9, 1/8	1, 1, 1

Despite its widespread usage, AHP has received many criticisms, including but not limited to (i) AHP not being a valid methodology as additive and multiplicative operations are not applicable on AHP scale values which are also arbitrarily determined (Barzilai 2010); (ii) the problem of rank reversal that may arise during decomposition and aggregation of hierarchic structures (Belton and Gear 1983); and (iii) an arbitrary measure of consistency (Dodd, Donegan, and McMaster 1995). Although the first and second criticism may also be valid for the α-D MCDM, this novel method does provide a new approach to assure consistency for the whole pairwise comparison process.

3. Fuzzy α-Discounting Method for MCDM

In this section, the Fuzzy α-Discounting Method for Multi-Criteria Decision Making (Fα-DM MCDM) is introduced after background information on the fuzzy set theory is given. Then, illustrative examples from the literature are given to show how Fα-DM MCDM is used. Fuzzy AHP (FAHP) had been used to reach a solution in the original problems found in illustrative examples. Unlike the crisp version of AHP, various researchers have proposed different FAHP methods which systematically approached the selection and justification problem of alternatives using fuzzy set theory and hierarchical structure analysis, and yet it is still not clear which one of these FAHP methods is preferable to others (Demirel, Demirel, and Kahraman 2008).

3.1. *Fuzzy set theory*

A fuzzy set is characterized by a membership function which assigns to each object in the set its grade of membership within the real unit interval [0, 1] (Zadeh 1965). A value of 0 indicates the object is not a member of the given set, while a value of 1 shows that the object fully belongs to the given set. Values between 0 and 1 imply graded degrees of membership.

Another feature that distinguishes the fuzzy set theory from the classical set theory is the use of linguistic variables that derive their value not from numbers, but from words or sentences in a natural or artificial language. A linguistic variable is formally characterized by a five-tuple consisting (x, $\boldsymbol{T}(x)$, $\boldsymbol{U}$, $\boldsymbol{G}$, $\boldsymbol{M}$) where x is the name of variable; $\boldsymbol{T}(x)$ is the term set of x (collection of linguistic values of x); $\boldsymbol{U}$ is the universe of discourse; $\boldsymbol{G}$ is a *syntactic rule* generating the terms in $\boldsymbol{T}(x)$; and $\boldsymbol{M}$ is a *semantic rule* associating with each linguistic value x its meaning (Zadeh 1975).

Representing the extension of the confidence intervals concept, fuzzy numbers are fuzzy subsets of the set of real numbers $\mathbb{R}$. According to the definition, a fuzzy number $\tilde{A}$ belonging to a fuzzy set and whose membership function is represented with x is denoted as follows (Dubois and Prade 1978):

$$\mu_{\tilde{A}}(x) : \boldsymbol{R} \rightarrow [0, 1]\ (0 \leqq \mu_{\tilde{A}}(x) \leqq 1, x \in \boldsymbol{X}) \tag{5}$$

and has the following properties:

(1) $\mu_{\tilde{A}}(x)$ is a piecewise continuous function from the set of real numbers to the [0, 1] closed interval.
(2) $\mu_{\tilde{A}}(x)$ is a convex fuzzy subset. In this case, the membership degree for the increasing values of x is either monotonously increasing/decreasing or first monotonously increasing then monotonously decreasing.
(3) $\mu_{\tilde{A}}(x)$ is the normalized state of a fuzzy subset. In this case, it is sufficient that the membership degree is 1 for at least one element.

In MCDM problems, TFNs are frequently used as they are handled easily in computations. TFN is characterized with three value judgments (l, m, u) where l represents the lower bound, m represents the mod (or most likely value), and u represents the upper bound. The membership function of a TFN is expressed as follows:

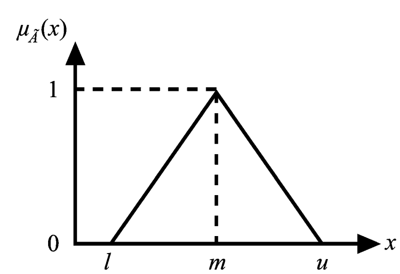

Figure 1. Membership function for TFNs.

Table 3. Matrix (***l***) constructed from the fuzzy pairwise comparison.

(***l***)	*D*	*A*	*R*1	*R*2
D	1	2	1/6	6
A	1/2	1	1/8	1
*R*1	6	8	1	8
*R*2	1/6	1	1/8	1

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-l)/(m-l) & \text{if } l \leq x \leq m, \\ (u-x)/(u-m) & \text{if } m \leq x \leq u, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

A graphical representation of the membership function for TFNs is displayed in Figure 1.

Based on the properties and extended definitions of fuzzy numbers, the algebraic operations for two TFNs, $\tilde{\boldsymbol{A}}_{\mathbf{1}} = (l_1, m_1, u_1)$ and $\tilde{\boldsymbol{A}}_{\mathbf{2}} = (l_2, m_2, u_2)$, and a positive real number r are done in the following way:

- Summation (⊕): $\tilde{\boldsymbol{A}}_{\mathbf{1}} \oplus \tilde{\boldsymbol{A}}_{\mathbf{2}} = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$
- Subtraction (⊖): $\tilde{\boldsymbol{A}}_{\mathbf{1}} \ominus \tilde{\boldsymbol{A}}_{\mathbf{2}} = (l_1 - u_2, m_1 - m_2, u_1 - l_2)$ $l_i, m_i, u_i > 0$
- Multiplication (⊗): $\tilde{\boldsymbol{A}}_{\mathbf{1}} \otimes \tilde{\boldsymbol{A}}_{\mathbf{2}} = (l_1 l_2, m_1 m_2, u_1 u_2)$
- Multiplication of a real number r with a TFN: $r \otimes \tilde{\boldsymbol{A}}_{\mathbf{1}} = (rl_1, rm_1, ru_1)$ $r, l_i, m_i, u_i > 0$
- Division (⊘): $\tilde{\boldsymbol{A}}_{\mathbf{1}} \oslash \tilde{\boldsymbol{A}}_{\mathbf{2}} = (l_1/u_2, m_1/m_2, u_1/l_2)$
- Inversion: $\tilde{\boldsymbol{A}}_{\mathbf{1}}^{\mathbf{1}} = (1/u_1, 1/m_1, 1/l_1)$ $l_i, m_i, u_i > 0$

3.2. *Single person decision-making with Fα-DM MCDM*

In this section, we introduce the Fα-DM MCDM algorithm for decision-making problems with one single decision maker. Within the concept of single person decision-making with Fα-DM MCDM, the preference statements of a single decision maker is obtained through a questionnaire that includes questions for pairwise comparisons among the criteria of concern. As the number of criteria/alternatives increases, there is an inclination for an inconsistent outcome (Olson et al. 1995) based on the increasing complexity of the decision-making problem. In this respect, the algorithms proposed in this paper are both for the single person and group decision-making settings. The probability of returning to the decision maker(s) to give preference statements as a result of an inconsistent outcome is highly reduced thanks to the α coefficient(s) calculated within the procedure of the algorithms. In the case of ending up with more than one α coefficient, the proposed algorithms also do not differentiate between them and take an average of the weights that are obtained after the multiplication of each α coefficient with the respective fuzzy decision matrix. After multiplication with α coefficients, the elements of the fuzzy decision matrices should have values between 1/9 and 9 to be consistent with the intensity scale.

The procedural steps for the Fα-DM MCDM are as follows:

(1) Structure the criteria of the decision-making problem under a hierarchy.
(2) Given the linguistic variables with determined scales, and their corresponding fuzzy importance grades; collect the subjective preference statements of the decision maker among the criteria and construct the pairwise fuzzy decision matrix.
(3) Assuming that the fuzzy importance grades are TFN, group the pairwise fuzzy decision matrix in three matrices; i.e. (***l***) lower bound, (***m***) most likely, and (***u***) upper bound value matrix, respectively.
(4) Reorder the elements of these (***l***), (***m***), and (***u***) matrices such that the above cells of the diagonal contain the lower bound, most likely, and upper bound values in l_{ij}, m_{ij}, and u_{ij}, respectively of the respective TFN.
(5) Calculate determinants for (***l***), (***m***), and (***u***) matrices. If obtained determinant values equal 0, then the respective matrices are found to be consistent. Calculate the weights of the criteria by first normalizing the column values and then taking averages of the row column values of the respective normalized matrices.
(6) If det ≠ 0 in any of the matrices created in step 5, then the respective matrix/matrices is/are inconsistent. Find the $\alpha > 0$ values resulting in det = 0 in inconsistent matrix/matrices based on the Fairness Principle.
(7) If the solution set for the α coefficient obtained in step 6 has:

   (a) Only one element, multiply the upper part of the matrix diagonal with α and bottom with 1/α. Test if the new values result in det = 0. If the matrix is consistent, then calculate the weights of the criteria.
   (b) More than one element, choose the α value based on the following:

      - For 0 < α < 1 values, select the max α value as the degree of consistency for the original MCDM problem (α) is the highest.
      - For α > 1, select the min α value as the degree of consistency for the original MCDM problem (1/α) is the highest.

Table 4. Matrix (***m***) constructed from the fuzzy pairwise comparison.

(***m***)	*D*	*A*	*R*1	*R*2
D	1	3	1/5	7
A	1/3	1	1/7	2
*R*1	5	7	1	9
*R*2	1/7	½	1/9	1

- For the above cases, the selected α value should not make the matrix elements exceed the 1–9 scale after the multiplication process. If such a case occurs, select other α values, which will yield a new matrix not exceeding the 1–9 scale or the minimum number of matrix elements exceeding the 1–9 scale after multiplication.

Test if the new values result in det = 0. If the matrix is consistent, then calculate the weights of the criteria.

(8) Apply a defuzzification method on the fuzzy weights obtained from the new (***l***), (***m***), and (***u***) matrices to get the final weights for each criterion.

In this paper, we used a revised version of the CFCS (Converting Fuzzy Data into Crisp Scores) Method developed by Opricovic and Tzeng (2003). CFCS is developed in a manner similar to the work of Chen and Hwang (1992). CFCS can produce defuzzified numbers corresponding to the fuzzy weights using a similar approach that finds left and right scores through fuzzy min and fuzzy max and the total score is determined as a weighted average according to the membership functions. CFCS is also applicable to the mixed set of crisp and fuzzy criteria. Revision in the form of weight ranking for criteria in an ascending order is required because the weights in the form of $\boldsymbol{l}_{ij} < \boldsymbol{m}_{ij} < \boldsymbol{u}_{ij}$ are not always available. The procedural steps of the revised version of CFCS are as follows:

(1) For each criterion, sort the fuzzy (***l***), (***m***), and (***u***) weights in an ascending order.

(2) Normalization:

$$i^{\max} = \max_{ij},\ i^{\min} = \min_{ij}\ \Delta^{\max}_{\min} = \max_{ij} - \min_{ij}$$

compute for all alternatives a_j, $j = 1, \ldots, J$

$$x_{lj} = (\boldsymbol{l}_{ij} - i^{\min})/\Delta^{\max}_{\min}\ x_{mj} = (\boldsymbol{m}_{ij} - i^{\min})/\Delta^{\max}_{\min}\ x_{rj} = (r_{ij} - i^{\min})/\Delta^{\max}_{\min}$$

(3) Compute left (*ls*) and right (*rs*) normalized values, for $j = 1, \ldots, J$

$$x^{ls}_{j} = x_{mj}/(1 + x_{mj} - x_{lj})\ x^{rs}_{j} = x_{rj}/(1 + x_{rj} - x_{mj})$$

(4) Compute total normalized crisp value:

$$\boldsymbol{x}^{\text{crisp}}_{j} = [x^{ls}_{j}(1 - x^{ls}_{j}) + x^{rs}_{j}x^{rs}_{j}]/[1 - x^{ls}_{j} + x^{rs}_{j}] \text{ for } j = 1, \ldots, J$$

(5) Compute crisp weight values, for $j = 1, \ldots, J$

$$f_{ij} = i^{\min} + \Delta^{\max}_{\min}\ w_{ij} = f_{ij}/\sum f_{ij}$$

3.2.1. *Illustrative example*

As an illustrative example, we took the same example found in the work of Tzeng and Huang (2011) that was solved using fuzzy AHP. To compute the α values, we used SymPy (2012), an open-source Python library for symbolic mathematics. In brief, the problem mentions a decision maker in a company who wants to determine criteria weights for an optimal product quality based on appropriate budget allocation. The criteria to evaluate are durability, esthetics, reliability, and reputation. The fuzzy pairwise comparison matrix is shown in Tables 1 and 2:

The proposed Fα-DM MCDM is applied beginning from step 4 as follows:

- Construct the (***l***), (***m***), and (***u***) matrices in an orderly way as shown in Tables 3–5:
- Compute det(***l***), det(***m***), and det(***u***):

det(***l***) = 0 Consistent matrix. Normalized criteria weights = [0.1974, 0.0748, 0.6638, and 0.0640].

det(***m***) = 0.078609221 and det(***u***) = –0.399305556. Apply α-D MCDM to transform (***m***) and (***u***) matrices into consistent ones.

- Apply α-D MCDM on (***m***) and (***u***):

We ran the following SymPy pseudocode to find α values that give det(***m***) = 0

>>> import matrix library

>>> equalize x as an unknown variable

>>> construct the matrix "M" with elements from (m) that are multiplied by x for those found in the upper diagonal and by 1/x for the lower diagonal

>>> solve for the α coefficients making det(M) = 0

The solution set for α is found as [14/9, 35/9, 7/6, and 7/15]. Among them 7/6 is the α value after whose multiplication with matrix elements (matrix elements over the diagonal are multiplied with 7/6 and below the diagonal with 6/7) yields only one matrix element to exceed the 1–9 scale and the criteria weights = [0.2558, 0.0898, 0.6119, and 0.0425]. After the application of the same procedure for (***u***), the solution set for α is found as [2, 3/8, 32/9, and 2/3] among which 2/3 is chosen as its multiplication with matrix elements does not cause them to exceed the 1–9 scale. The criteria weights = [0.2056, 0.0855, 0.6470, and 0.0620].

Table 5. Matrix ($\boldsymbol{u}$) constructed from the fuzzy pairwise comparison.

($\boldsymbol{u}$)	D	A	$R1$	$R2$
D	1	4	1/4	8
A	1/4	1	1/6	3
$R1$	4	6	1	9
$R2$	1/8	1/3	1/9	1

- Defuzzify the obtained weights for each criterion with the revised CFCS (RCFCS) method:

(1) After sorting in an ascending order, normalize the fuzzy criteria weights given in Table 6:
$i^{\max}$ =0.66380 $i^{\min}$ = 0.0425 $\Delta_{\min}^{\max}$ = 0.6213 (Table 7)
(2) Compute left and right scores (ls, rs): Table 8
(3) Compute total normalized crisp value:
$\boldsymbol{x}_j^{\text{crisp}}$= [0.2767, 0.0687, 0.9697, and 0.0306]
(4) Compute crisp weight values:
$\boldsymbol{f}_{ij}$ = [0.2144, 0.0851, 0.6450, and 0.0615] and $\boldsymbol{w}_{ij}$ = [0.2131, 0.0846, 0.6412, and 0.0611]

In the original problem, Tzeng and Huang used the geometric mean method to find the fuzzy weight vector and the center of area method to defuzzify the weight vector into crisp values. They computed the weights assigned to each criterion as [0.2278, 0.0898, 0.6572, and 0.0498] amounting to a total of 1.0246. Although the weights computed with Fα-DM MCDM, [0.2131, 0.0846, 0.6412, and 0.0611], result in the same ranking ($R1 > D > A > R2$) as FAHP did, the weights are lower for D, A, and $R1$, and only higher for $R2$ which is an indication of different methods yielding different results. However, it should be noted that due to the relatively small size of the illustrative problem covered in this section, the criteria rankings happened to be in the same order in both methods. Also, the weights obtained as a result of Fα-DM MCDM do not exceed the maximum value of 1.

3.3. *Group decision-making with Fα-DM MCDM*

Fα-DM MCDM for a group decision-making situation is applied in the same way as single person decision-making with Fα-DM MCDM, with an additional aggregation procedure for pairwise comparisons between criteria from a group of decision makers. It should be noted that the illustrative example covered in this paper contains full information from the decision makers; however, the Fα-DM MCDM for group decision-making has the potential to deal with missing information.

The procedural steps for Fα-DM MCDM in a group decision setting are identical to the procedural steps in single person decision setting with the following addition in Step 2:

- Given the linguistic variables with determined scales, and their corresponding fuzzy importance grades; collect the subjective preference statements from a group of decision makers among the criteria to construct the pairwise fuzzy decision matrix. Then, aggregate these matrices into a fuzzy synthetic pairwise comparison matrix using the geometric mean method suggested by Buckley (1985):

$$\tilde{a}_{ij} = \left(\tilde{a}_{ij}^{1} \otimes \cdots \otimes \tilde{a}_{ij}^{n}\right)^{1/n}. \quad (7)$$

Table 6. Fuzzy criteria weights.

	l	m	$u = r$
D	0.1974	0.2056	0.2558
A	0.0748	0.0855	0.0898
$R1$	0.6119	0.6470	0.6638
$R2$	0.0425	0.0620	0.0640

Table 7. Normalized fuzzy criteria weights.

Normalized matrix	l	m	r
D	0.2494	0.2625	0.3433
A	0.0521	0.0693	0.0762
$R1$	0.9166	0.9729	1.0000
$R2$	0.0000	0.0314	0.0346

Table 8. Left and right scores.

	ls	rs
D	0.2591	0.3177
A	0.0681	0.0757
$R1$	0.9210	0.9736
$R2$	0.0304	0.0345

3.3.1. *Illustrative example*

To illustrate the application of Fα-DM MCDM in a group decision-making situation, we took another example found in the work of Tzeng and Huang (2011) that was

Table 9. Linguistic variables and corresponding fuzzy numbers.

Linguistic variables	LAb	LVs	LEs	LWk	LEq	Eq	Wk	Es	Vs	Ab
Fuzzy numbers	$\tilde{9}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{7}$	$\tilde{9}$
TFN values	$\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{7}\right)$	$\left(\frac{1}{9}, \frac{1}{7}, \frac{1}{5}\right)$	$\left(\frac{1}{7}, \frac{1}{5}, \frac{1}{3}\right)$	$\left(\frac{1}{5}, \frac{1}{3}, 1\right)$	$\left(\frac{1}{3}, 1, 1\right)$	(1, 1, 3)	(1, 3, 5)	(3, 5, 7)	(5, 7, 9)	(7, 9, 9)

Table 10. Linguistic variable assignment to criteria for each firm.

	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$
Firm1						
$C1$	1	Wk	Es	Wk	Eq	Eq
$C2$		1	Wk	Eq	Eq	Wk
$C3$			1	Eq	LEs	LEs
$C4$				1	LWk	LWk
$C5$					1	Eq
$C6$						1
Firm2						
$C1$	1	LWk	Wk	Eq	Eq	LWk
$C2$		1	Wk	Wk	Wk	LWk
$C3$			1	LVs	LVs	LVs
$C4$				1	LWk	LWk
$C5$					1	LWk
$C6$						1
Firm3						
$C1$	1	LEs	Eq	Wk	Eq	LVs
$C2$		1	Wk	Ab	Vs	LEs
$C3$			1	Es	Wk	LVs
$C4$				1	Eq	LAb
$C5$					1	LVs
$C6$						1
Firm4						
$C1$	1	LWk	LEs	Wk	LVs	Es
$C2$		1	Es	Vs	LAb	Vs
$C3$			1	Wk	LAb	Wk
$C4$				1	LAb	Wk
$C5$					1	Ab
$C6$						1
Firm5						
$C1$	1	Wk	Es	LWk	LVs	LEq
$C2$		1	Wk	LEs	LVs	LWk
$C3$			1	LVs	LAb	LEs
$C4$				1	LWk	Wk
$C5$					1	Es
$C6$						1

solved using fuzzy AHP. To compute the α values, we used MATLAB (2012), a numerical computing environment and fourth-generation programming language. In brief, the problem is about the selection of a firm, after a tender process, for the provision of engineering services for the construction of a branch station of the Taipei City Police Bureau. Five firms submitted proposals which will be evaluated against six different criteria by three decision-making groups composed of five persons each. The decision makers want to find the weights to be assigned to these criteria. The linguistic variables used by the decision makers are given in Table 9 and the fuzzy pairwise comparison matrices and corresponding triangular fuzzy numbers for the five firms against six criteria are given in Tables 10 and 11.

- To construct the fuzzy synthetic pairwise comparison matrix from the matrices given above, the geometric mean method is used: $\tilde{a}_{ij} = \left(\tilde{a}^1_{ij} \otimes \tilde{a}^2_{ij} \otimes \tilde{a}^3_{ij} \otimes \tilde{a}^4_{ij} \otimes \tilde{a}^5_{ij}\right)^{1/5}$. As an example, $\tilde{a}_{12}$ in the fuzzy synthetic pairwise comparison matrix is calculated as follows:

Table 11. Corresponding fuzzy number assignment to each firm.

	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$
Firm1						
$C1$	1	$\tilde{3}$	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$
$C2$		1	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$
$C3$			1	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$
$C4$				1	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$
$C5$					1	$\tilde{1}$
$C6$						1
Firm2						
$C1$	1	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$
$C2$		1	$\tilde{3}$	$\tilde{3}$	$\tilde{3}$	$\tilde{3}^{-1}$
$C3$			1	$\tilde{7}^{-1}$	$\tilde{7}^{-1}$	$\tilde{7}^{-1}$
$C4$				1	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$
$C5$					1	$\tilde{3}^{-1}$
$C6$						1
Firm3						
$C1$	1	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	7^{-1}
$C2$		1	$\tilde{3}$	$\tilde{9}$	$\tilde{7}$	$\tilde{5}^{-1}$
$C3$			1	$\tilde{5}$	$\tilde{3}$	$\tilde{7}^{-1}$
$C4$				1	$\tilde{1}$	$\tilde{9}^{-1}$
$C5$					1	$\tilde{7}^{-1}$
$C6$						1
Firm4						
$C1$	1	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{5}$
$C2$		1	$\tilde{5}$	$\tilde{7}$	$\tilde{9}^{-1}$	$\tilde{7}$
$C3$			1	$\tilde{3}$	$\tilde{9}^{-1}$	$\tilde{3}$
$C4$				1	$\tilde{9}^{-1}$	$\tilde{3}$
$C5$					1	$\tilde{9}$
$C6$						1
Firm5						
$C1$	1	$\tilde{3}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$
$C2$		1	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$
$C3$			1	$\tilde{7}^{-1}$	$\tilde{9}^{-1}$	$\tilde{5}^{-1}$
$C4$				1	$\tilde{3}^{-1}$	$\tilde{3}$
$C5$					1	$\tilde{5}$
$C6$						1

$$\begin{aligned}\tilde{a}_{12} &= \left((1,3,5)\otimes\left(\tfrac{1}{5},\tfrac{1}{3},1\right)\otimes\left(\tfrac{1}{7},\tfrac{1}{5},\tfrac{1}{3}\right)\otimes\left(\tfrac{1}{5},\tfrac{1}{3},1\right)\otimes(1,3,5)\right)^{1/5}\\ &= \left(\left(1\times\tfrac{1}{5}\times\tfrac{1}{7}\times\tfrac{1}{5}\times1\right)^{1/5},\left(3\times\tfrac{1}{3}\times\tfrac{1}{5}\times\tfrac{1}{3}\times3\right)^{1/5},\left(5\times1\times\tfrac{1}{3}\times1\times5\right)^{1/5}\right)\\ &= (0.356,0.725,1.528)\end{aligned} \tag{8}$$

The fuzzy synthetic pairwise comparison matrix is shown in Table 12.

Table 12. Fuzzy synthetic pairwise comparison matrix.

SM	C1			C2			C3			C4			C5			C6		
C1	1	1	1	0.356	0.725	1.528	1.052	1.719	3.005	0.725	1.552	3.272	0.415	0.459	1.016	0.467	0.750	1.332
C2				1	1	1	1.246	3.323	5.348	1.380	2.068	3.323	0.573	0.803	1.310	0.491	0.859	1.719
C3							1	1	1	0.517	0.789	1.332	0.181	0.254	0.369	0.191	0.300	0.467
C4										1	1	1	0.245	0.333	0.844	0.339	0.644	1.290
C5													1	1	1	0.859	1.165	2.068
C6																1	1	1

Table 13. Fuzzy criteria weights computed with Fα-D MCDM.

	l	m	$u = r$
C1	0.1407	0.1608	0.1991
C2	0.2052	0.2256	0.2267
C3	0.0705	0.0726	0.0845
C4	0.1004	0.1031	0.1076
C5	0.2299	0.2351	0.2745
C6	0.1641	0.1908	0.2086

Table 14. Crisp criteria weights according to Fα-D MCDM and FAHP.

	Fα-D MCDM	FAHP
C1	0.1664	0.2000
C2	0.2237	0.2720
C3	0.0739	0.0940
C4	0.1043	0.1380
C5	0.2415	0.3220
C6	0.1902	0.2630

- Apply steps from 5 to 7 as described above and defuzzify the computed fuzzy criteria weights in Table 13 by applying the RCFCS.

A comparison for the crisp criteria weights computed through Fα-DM MCDM and FAHP is given in Table 14.

As in the ranking result in the single person decision-making case, the ranking result in group decision-making for Fα-DM MCDM and FAHP has been observed to be the same. Both the FAHP and Fα-DM MCDM rank the criteria as $D5 > D2 > D6 > D1 > D4 > D3$. As to the weights assigned to criteria by FAHP, it has been observed that they are higher than the weights assigned by Fα-DM MCDM. The reason for having such a difference in weight assignment is based on the different approaches in processing of fuzzy numbers and application of different defuzzification methods. It is a fact that Tzeng and Huang used best non-fuzzy performance as a defuzzification method as a result of which the assigned weights amounted to 1.289, exceeding the total maximum value of 1.

4. Conclusion and further research

Beyond the perimeters of academic circles, the AHP has been widely used by non-academicians due to its structuring the decision-making problems in simple, hierarchical application, usage of verbal preference statements to denote the numerical scale of 1–9, and setting a consistency level in MCDM problems. However, there are the criticisms of Barzilai (2010), Belton and Gear (1983), and Dodd, Donegan, and McMaster (1995) against AHP which we previously mentioned. Furthermore, AHP can only handle pairwise comparisons and the number of these pairwise comparisons increases enormously as the number of criteria/alternatives increases. α-D MCDM not only responds to this shortcoming by getting n-wise comparisons from the decision-maker but also handles weak inconsistent decision-making problems (even above the AHP consistency ratio (CR) of 10%). This study developed the Fα-DM MCDM both for single person and group decision-making situations, which groups the TFNs of the corresponding linguistic variables in ($\boldsymbol{l}$), ($\boldsymbol{m}$), and ($\boldsymbol{u}$) matrices in an ascending order by considering the values above the matrix diagonal. Although it might seem unlikely in the first place, this emerges as a necessary condition to ensure consistent allocation of the TFNs from corresponding linguistic variables in relevant matrices. Fα-DM MCDM then ensures the consistency of the relevant matrix by applying the aforementioned steps to reach a matrix determinant value of 0. As a last step, RCFCS defuzzification method is applied to obtain crisp values from the ($\boldsymbol{l}$), ($\boldsymbol{m}$), and ($\boldsymbol{u}$) matrices. Additionally, for the group decision-making situation in Fα-DM MCDM, the geometric mean method has been used to produce an aggregate synthetic fuzzy decision matrix from the decision matrices obtained through decision makers. Two MCDM problems previously solved with FAHP taken from the literature have been resolved with Fα-DM MCDM. Considering the solutions, Fα-DM MCDM and FAHP have given the same ranking but with different criteria weights in both single and group decision-making settings. The reason for different criteria weights is due to the different handling of fuzzy numbers and defuzzification methods used in both methods.

As a further research venue, the Fα-DM MCDM is capable of dealing with fuzzy MCDM problems with missing information, and even in a group decision-making setting, this novel method can include the preference statements in a mixed state both with perfect information (i.e. all pairwise comparisons have been provided by the decision makers), and with missing information before aggregating them into a fuzzy synthetic decision matrix. We also consider the MCDM problems with interval numbers to be included within the scope of further research.

Nomenclature

AHP	Analytical Hierarchy Process
CFCS	Converting Fuzzy Data into Crisp Scores Method
det	determinant
FAHP	Fuzzy Analytical Hierarchy Process
Fα-DM MCDM	Fuzzy α-Discounting Method for Multi-Criteria Decision Making
(***l***)	matrix for lower boundary fuzzy values
ls	left normalized value/score
(***m***)	matrix for most likely fuzzy values
MCDM	Multi-Criteria Decision Making
RCFCS	Revised Converting Fuzzy Data into Crisp Scores Method
rs	right normalized value/score
SD	strongly inconsistent linear decision-making problems
TFN	triangular fuzzy number
(***u***)	matrix for upper boundary fuzzy values
WD	weakly inconsistent linear decision-making problems
α-D MCDM	α-Discounting Method for Multi-Criteria Decision Making

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Note

1. Secondary variables are used to indicate the values of the main variables in a decision-making problem with linear preference statements. Assume that in a linear decision-making problem with three criteria – x, y, and z – the degrees of importance assigned to criteria x and y, as main variables, can be indicated through z, a secondary variable, based on the transitivity rule. An example from Smarandache (2010) shows this as follows: Suppose that C1 (x) is four times as important as C2 (y); C2 (y) is three times as important as C3 (z); and C3 (z) is one-twelfth as important as C1 (x). The linear homogeneous system associated to this decision-making problem is: $x = 4y$; $y = 3z$; $z = x/12$. Using z as secondary variable, the weights to be assigned to main variables – x and y – can be obtained easily. Solving this homogeneous linear system, we get its general solution that we set as a vector [$12z$ $3z$ z], where z can be any real number (z is considered a secondary variable, while $x = 12z$ and $y = 3z$ are main variables).

ORCID

Atilla Karaman http://orcid.org/0000-0002-8328-4103
Metin Dagdeviren http://orcid.org/0000-0003-2121-5978

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