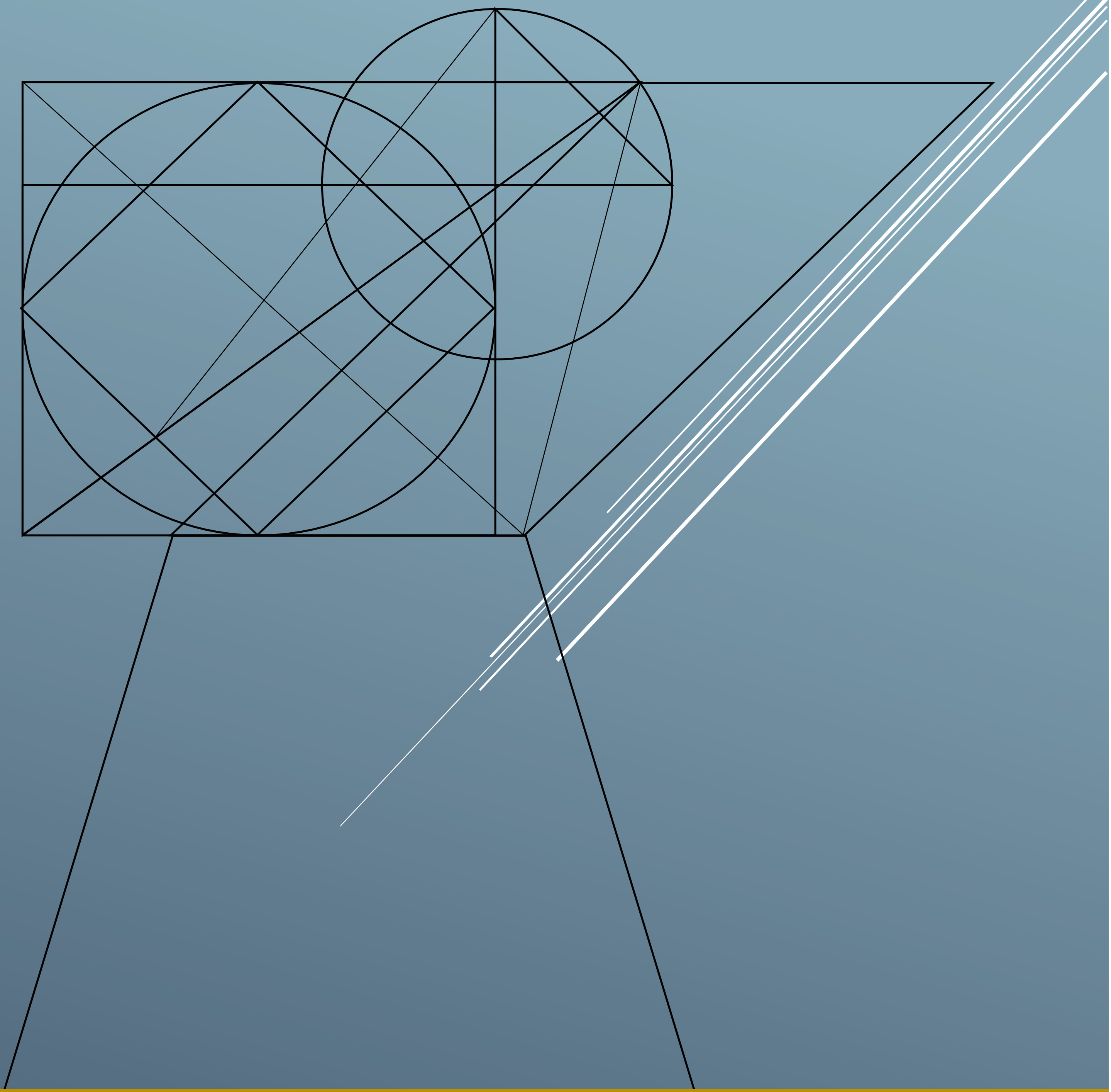

Florentin Smarandache

Compiled and Solved Problems in Geometry and Trigonometry

FLORENTIN SMARANDACHE

255 Compiled and Solved Problems in Geometry and Trigonometry

(from Romanian Textbooks)

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Explanatory Note

This book is a translation from Romanian of "Probleme Compilate şi Rezolvate de Geometrie şi Trigonometrie" (University of Kishinev Press, Kishinev, 169 p., 1998), and includes problems of 2D and 3D Euclidean geometry plus trigonometry, compiled and solved from the Romanian Textbooks for 9th and 10th grade students, in the period 1981-1988, when I was a professor of mathematics at the "Petrache Poenaru" National College in Balcesti, Valcea (Romania), Lycée Sidi El Hassan Lyoussi in Sefrou (Morroco), then at the "Nicolae Balcescu" National College in Craiova and Dragotesti General School (Romania), but also I did intensive private tutoring for students preparing their university entrance examination. After that, I have escaped in Turkey in September 1988 and lived in a political refugee camp in Istanbul and Ankara, and in March 1990 I immigrated to United States. The degree of difficulties of the problems is from easy and medium to hard. The solutions of the problems are at the end of each chapter. One can navigate back and forth from the text of the problem to its solution using bookmarks. The book is especially a didactical material for the mathematical students and instructors.

The Author

Problems in Geometry (9th grade)

1. The measure of a regular polygon's interior angle is four times bigger than the measure of its external angle. How many sides does the polygon have?

Solution to Problem 1

2. How many sides does a convex polygon have if all its external angles are obtuse?

Solution to Problem 2

3. Show that in a convex quadrilateral the bisector of two consecutive angles forms an angle whose measure is equal to half the sum of the measures of the other two angles.

Solution to Problem 3

4. Show that the surface of a convex pentagon can be decomposed into two quadrilateral surfaces.

Solution to Problem 4

5. What is the minimum number of quadrilateral surfaces in which a convex polygon with 9, 10, 11 vertices can be decomposed?

Solution to Problem 5

6. If $(\widehat{ABC}) \equiv (\widehat{A'B'C'})$, then $\exists$ bijective function $f = (\widehat{ABC}) \to (\widehat{A'B'C'})$ such that for $\forall$ 2 points $P, Q \in (\widehat{ABC})$, $\|PQ\| = \|f(P)\|, \|f(Q)\|$, and vice versa.

Solution to Problem 6

7. If $\Delta ABC \equiv \Delta A'B'C'$ then $\exists$ bijective function $f = ABC \to A'B'C'$ such that $(\forall)$ 2 points $P, Q \in ABC$, $\|PQ\| = \|f(P)\|, \|f(Q)\|$, and vice versa.

Solution to Problem 7

8. Show that if $\Delta ABC \sim \Delta A'B'C'$, then $[ABC] \sim [A'B'C']$.

Solution to Problem 8

9. Show that any two rays are congruent sets. The same property for lines.

Solution to Problem 9

10. Show that two disks with the same radius are congruent sets.

Solution to Problem 10

11. If the function $f: M \to M'$ is isometric, then the inverse function $f^{-1}: M \to M'$ is as well isometric.

Solution to Problem 11

12. If the convex polygons $L = P_1, P_2, \dots, P_n$ and $L' = P_1', P_2', \dots, P_n'$ have $|P_i, P_{i+1}| \equiv |P_i', P_{i+1}'|$ for $i = 1, 2, \dots, n-1$, and $P_i\widehat{P_{i+1}P}_{i+2} \equiv P_i'\widehat{P_{i+1}'P'}_{i+2}$, $(\forall)$ $i = 1, 2, \dots, n-2$, then $L \equiv L'$ and $[L] \equiv [L']$.

Solution to Problem 12

13. Prove that the ratio of the perimeters of two similar polygons is equal to their similarity ratio.

Solution to Problem 13

14. The parallelogram $ABCD$ has $\|AB\| = 6$, $\|AC\| = 7$ and $d(AC) = 2$. Find $d(D, AB)$.

Solution to Problem 14

15. Of triangles ABC with $||BC|| = a$ and $||CA|| = b$, a and b being given numbers, find a triangle with maximum area.

Solution to Problem 15

16. Consider a square $ABCD$ and points E, F, G, H, I, K, L, M that divide each side in three congruent segments. Show that $PQRS$ is a square and its area is equal to $\frac{2}{9}\sigma[ABCD]$.

Solution to Problem 16

17. The diagonals of the trapezoid $ABCD$ ($AB||DC$) cut at O.
    a. Show that the triangles AOD and BOC have the same area;
    b. The parallel through O to AB cuts AD and BC in M and N. Show that $||MO|| = ||ON||$.

Solution to Problem 17

18. E being the midpoint of the non-parallel side $[AD]$ of the trapezoid $ABCD$, show that $\sigma[ABCD] = 2\sigma[BCE]$.

Solution to Problem 18

19. There are given an angle $(\widehat{BAC})$ and a point D inside the angle. A line through D cuts the sides of the angle in M and N. Determine the line MN such that the area ΔAMN to be minimal.

Solution to Problem 19

20. Construct a point P inside the triangle ABC, such that the triangles PAB, PBC, PCA have equal areas.

Solution to Problem 20

21. Decompose a triangular surface in three surfaces with the same area by parallels to one side of the triangle.

Solution to Problem 21

22. Solve the analogous problem for a trapezoid.

Solution to Problem 22

23. We extend the radii drawn to the peaks of an equilateral triangle inscribed in a circle $L(O,r)$, until the intersection with the circle passing through the peaks of a square circumscribed to the circle $L(O,r)$. Show that the points thus obtained are the peaks of a triangle with the same area as the hexagon inscribed in $L(O,r)$.

Solution to Problem 23

24. Prove the leg theorem with the help of areas.

Solution to Problem 24

25. Consider an equilateral ΔABC with $\|AB\| = 2a$. The area of the shaded surface determined by circles $L(A,a), L(B,a), L(A,3a)$ is equal to the area of the circle sector determined by the minor arc $(\widehat{EF})$ of the circle $L(C,a)$.

Solution to Problem 25

26. Show that the area of the *annulus* between circles $L(O,r_2)$ and $L(O,r_2)$ is equal to the area of a disk having as diameter the tangent segment to circle $L(O,r_1)$ with endpoints on the circle $L(O,r_2)$.

Solution to Problem 26

27. Let $[OA],[OB]$ two $\perp$ radii of a circle centered at $[O]$. Take the points C and D on the minor arc $\widehat{ABF}$ such that $\widehat{AC} \equiv \widehat{BD}$ and let E, F be the projections of CD onto OB. Show that the area of the surface bounded by $[DF], \left[FE[EC]\right]$ and arc $\widehat{CD}$ is equal to the area of the sector determined by arc $\widehat{CD}$ of the circle $C(O, \|OA\|)$.

Solution to Problem 27

28. Find the area of the regular octagon inscribed in a circle of radius r.

Solution to Problem 28

29. Using areas, show that the sum of the distances of a variable point inside the equilateral triangle ABC to its sides is constant.

Solution to Problem 29

30. Consider a given triangle ABC and a variable point $M \in |BC|$. Prove that between the distances $x = d(M, AB)$ and $y = d(M, AC)$ is a relation of $kx + ly = 1$ type, where k and l are constant.

Solution to Problem 30

31. Let M and N be the midpoints of sides $[BC]$ and $[AD]$ of the convex quadrilateral $ABCD$ and $\{P\} = AM \cap BN$ and $\{Q\} = CN \cap ND$. Prove that the area of the quadrilateral $PMQN$ is equal to the sum of the areas of triangles ABP and CDQ.

Solution to Problem 31

32. Construct a triangle having the same area as a given pentagon.

Solution to Problem 32

33. Construct a line that divides a convex quadrilateral surface in two parts with equal areas.

Solution to Problem 33

34. In a square of side l, the middle of each side is connected with the ends of the opposite side. Find the area of the interior convex octagon formed in this way.

Solution to Problem 34

35. The diagonal $[BD]$ of parallelogram $ABCD$ is divided by points M, N, in 3 segments. Prove that $AMCN$ is a parallelogram and find the ratio between $\sigma[AMCN]$ and $\sigma[ABCD]$.

Solution to Problem 35

36. There are given the points A, B, C, D, such that $AB \cap CD = \{p\}$. Find the locus of point M such that $\sigma[ABM] = \sigma[CDM]$.

Solution to Problem 36

37. Analogous problem for $AB||CD$.

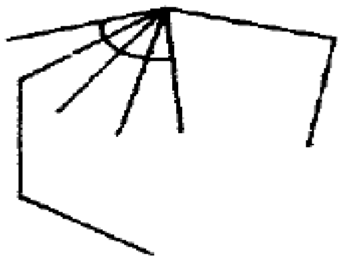
Solution to Problem 37

38. Let $ABCD$ be a convex quadrilateral. Find the locus of point x_1 inside $ABCD$ such that $\sigma[ABM] + \sigma[CDM] = k$, k – a constant. For which values of k the desired geometrical locus is not the empty set?

Solution to Problem 38

Solutions

Solution to Problem 1.



$$\frac{180\,(n-2)}{n} = 4\frac{180}{5} \Rightarrow n = 10$$

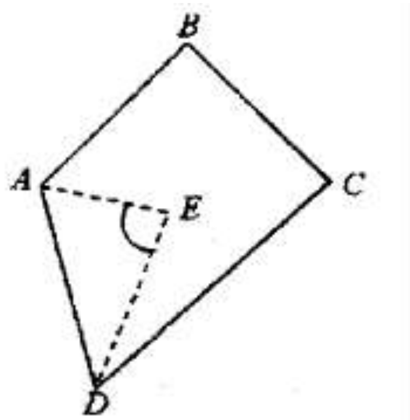
Solution to Problem 2.

$$\begin{array}{c}\text{Let } n = 3\\ x_1, x_2, x_3 \sphericalangle \text{ ext}\end{array} \Rightarrow \left.\begin{array}{l} x_1 > 90^0 \\ x_2 > 90^0 \\ x_3 > 90^0 \end{array}\right\} \Rightarrow x_1 + x_2 + x_3 > 270^0, \text{ so } n = 3 \text{ is possible.}$$

$$\begin{array}{c}\text{Let } n = 4\\ x_1, x_2, x_3, x_4 \sphericalangle \text{ ext}\end{array} \Rightarrow \left.\begin{array}{c} x_1 > 90^0 \\ \vdots \\ x_3 > 90^0 \end{array}\right\} \Rightarrow x_1 + x_2 + x_3 + x_4 > 360^0, \text{ so } n = 4 \text{ is impossible.}$$

Therefore, $n = 3$.

Solution to Problem 3.



$$m(\widehat{AEB}) = \frac{m(\hat{D}) + m(\hat{C})}{2}$$

$$m(\hat{A}) + m(\hat{B}) + m(\hat{C}) + m(\hat{D}) = 360°$$

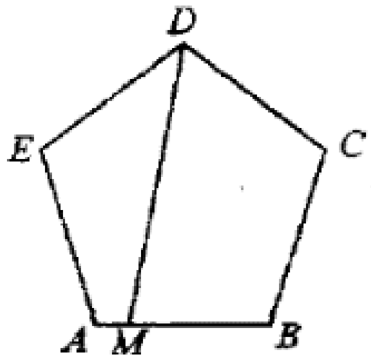
$$\frac{m(\hat{A}) + m(\hat{B})}{2} = 180° - \frac{m(\hat{C}) + m(\hat{D})}{2}$$

$$m(\widehat{AEB}) = 180° - \frac{m(\hat{A})}{2} - \frac{m(\hat{B})}{2} =$$

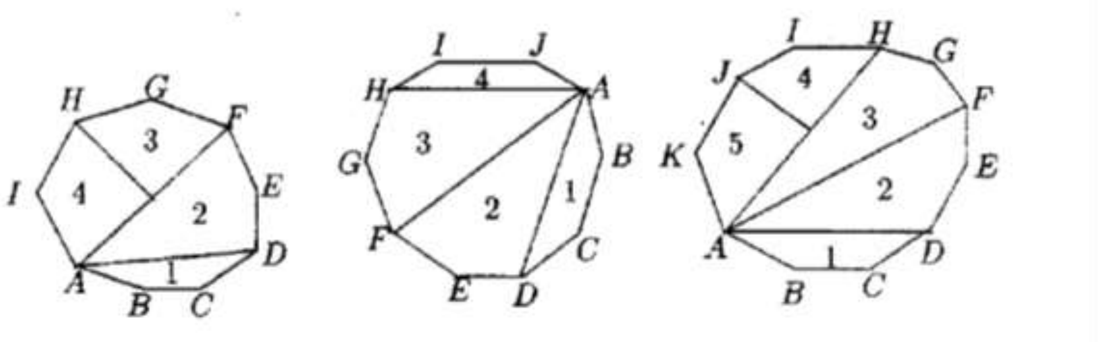
$$= 180° - 180° + \frac{m(\hat{C}) + m(\hat{D})}{2} = \frac{m(\hat{C}) + (\hat{D})}{2}$$

Solution to Problem 4.

Let $\widehat{EDC} \Rightarrow A, B \in \text{int.}\widehat{EDC}$. Let $M \in |AB| \Rightarrow M \in \text{int.}\widehat{EDC} \Rightarrow |DM \subset \text{int.}\widehat{EDC}, |EA| \cap |DM = \emptyset \Rightarrow DEAM$ quadrilateral. The same for $DCBM$.



Solution to Problem 5.



9 vertices;
4 quadrilaterals.

10 vertices;
4 quadrilaterals.

11 vertices;
5 quadrilaterals.

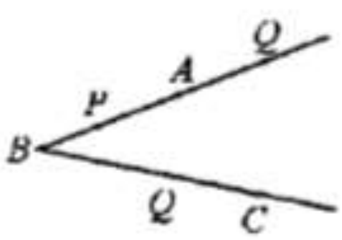
Solution to Problem 6.

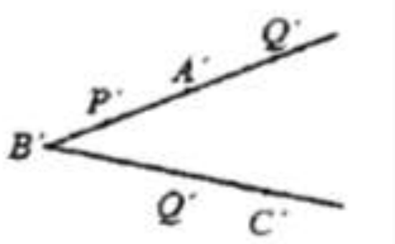
We assume that $\widehat{ABC} \equiv \widehat{A'B'C'}$. We construct a function $f : \widehat{ABC} \to \widehat{A'B'C'}$ such that

$$\begin{cases} f\,(\mathrm{B}) = \mathrm{B}' \\ \text{if } \mathrm{P} \in\ |\mathrm{BA}, \quad f(\mathrm{P}) \in \mathrm{B'A'} \end{cases}$$

$P \in |BC, f(P) \in B'C'$ such that $\|BP\| = \|B'P'\|$ where $P' = f(F)$.

The so constructed function is bijective, since for different arguments there are different corresponding values and ∀ point from $A'B'C'$ is the image of a single point from $\widehat{ABC}$ (from the axiom of segment construction).





If $P, Q\ \in$ this ray,

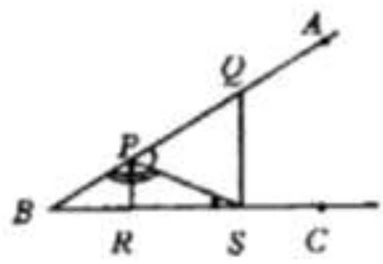
$$\left.\begin{matrix} \|\mathrm{BP}\| = \|\mathrm{B'P'}\| \\ \|\mathrm{BQ}\| = \|\mathrm{B'Q'}\| \end{matrix}\right\} \Rightarrow \|\mathrm{PQ}\| = \|\mathrm{BQ}\| - \|\mathrm{BP}\| = \|\mathrm{B'Q'}\| - \|\mathrm{B'P'}\| = \|\mathrm{P'Q'}\| = \|f(\mathrm{P}), f(\mathrm{Q})\|.$$

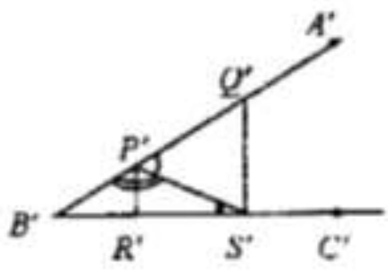
If $P, Q\ \in$ a different ray,

$$\left.\begin{matrix} \|\mathrm{BP}\| = \|\mathrm{B'P'}\| \\ \|\mathrm{BQ}\| = \|\mathrm{B'Q'}\| \\ \widehat{\mathrm{PBQ}} \equiv \widehat{\mathrm{P'B'Q'}} \end{matrix}\right\} \Rightarrow \Delta\mathrm{PBQ} = \Delta\mathrm{P'B'Q'} \Rightarrow \|\mathrm{PQ}\| = \|\mathrm{P'Q'}\| = \|f\,(\mathrm{P}), f\,(\mathrm{Q})\|.$$

Vice versa.

Let $f : \mathrm{ABC} \to \mathrm{A'B'C'}$ such that f bijective and $\|\mathrm{PQ}\| = \|f(\mathrm{P}), f(\mathrm{Q})\|$.





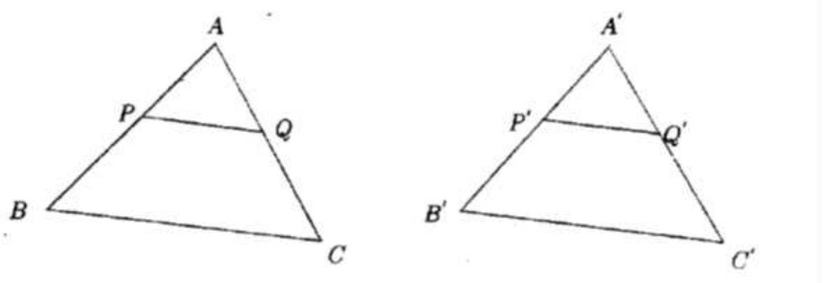
Let $P, Q \in |BA$ and $RS \in |BC$.

$$\left.\begin{array}{l}\|PQ\| = \|P'Q'\| \\ \|PS\| = \|P'S'\| \\ \|QS\| = \|Q'S'\|\end{array}\right\} \Longrightarrow \Delta PQS \equiv \Delta P'Q'S' \Longrightarrow \widehat{QPS} \equiv \widehat{Q'P'S'} \Longrightarrow \widehat{BPS} \equiv \widehat{B'P'S'} \quad (1);$$

$$\left.\begin{array}{l}\|PS\| = \|P'S'\| \\ \|RP\| = \|R'P'\| \\ \|PS\| = \|P'S'\|\end{array}\right\} \Longrightarrow \Delta PRS \equiv \Delta P'R'S'' \Longrightarrow \widehat{PSB} \equiv \widehat{P'S'B'} \quad (2).$$

From (1) and (2) $\Longrightarrow \widehat{PBC} \equiv \widehat{P'B'S'}$ (as diff. at 180°) i.e. $\widehat{ABC} \equiv \widehat{A'B'C'}$.

Solution to Problem 7.



Let $\Delta ABC \equiv \Delta A'B'C'$.

We construct a function $f : ABC \to A'B'C'$ such that $f(A) = A', f(B) = B', f(C) = C'$

and so $\quad P \in |AB| \to P' = f(P) \in |A'B'|$ such that $||AP|| = ||A'P'||$;

$P \in |BC| \to P' = f(P) \in |B'C'|$ such that $||BP|| = ||B'P'||$;

$P \in |CA| \to P' = f(P) \in |C'A'|$ such that $||CP|| = ||C'P'||$.

The so constructed function is bijective.

Let $P \in |AB|$ and $a \in |CA| \Longrightarrow P' \in |A'B'|$ and $Q' \in |C'A'|$.

$$\left.\begin{array}{l}\|AP\| = \|A'P'\| \\ \|CQ\| = \|C'Q'\| \\ \|CA\| = \|C'A'\|\end{array}\right\} \Longrightarrow \|AQ\| = \|A'Q'\|; A \equiv A' \Longrightarrow \Delta APQ \equiv \Delta A'P'Q' \Longrightarrow \|PQ\| = \|P'Q'\|.$$

Similar reasoning for (∀) point P and Q.

Vice versa.

We assume that ∃ a bijective function $f : ABC \to A'B'C'$ with the stated properties.

We denote $f(\mathrm{A}) = \mathrm{A}'', f(\mathrm{B}) = \mathrm{B}'', f(\mathrm{C}) = \mathrm{C}''$

$$\Longrightarrow \|AB\| = \|A''B''\|, \|BC\| = \|B''C''\|, \|AC\| = \|A''C''\| \Rightarrow \Delta ABC = \Delta A''B''C''.$$

Because $f(\mathrm{ABC}) = f([\mathrm{AB}] \cup [\mathrm{BC}] \cup [\mathrm{CA}]) = f([\mathrm{AB}]) \cup f([\mathrm{BC}]) \cup f([\mathrm{CA}])$

$$= [A''B''] \cup [B''C''][C''A''] = A''B''C''.$$

But by the hypothesis $f(ABC) = f(A'B'C')$, therefore

$$A''B''C'' = \Delta A'B'C' \Longrightarrow \Delta ABC \equiv \Delta A'B'C'.$$

Solution to Problem 8.

If $\Delta ABC \sim \Delta A'B'C'$ then $(\forall)$ $f: ABC \to A'B'C'$ and $k > 0$ such that:

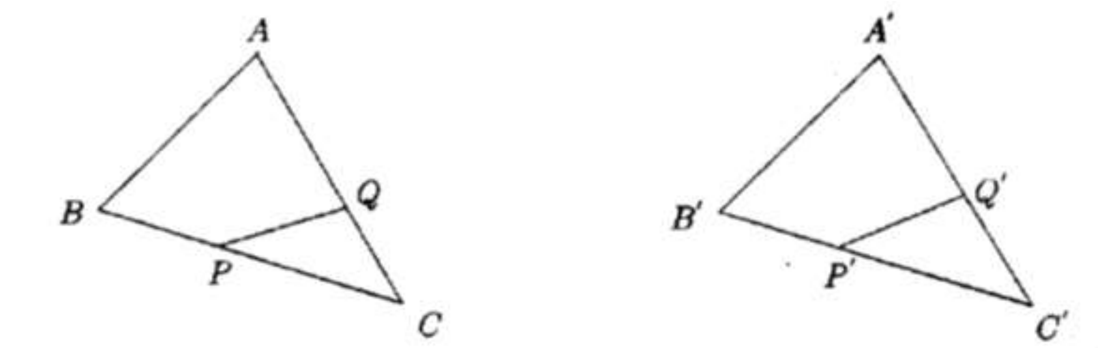
$\|PQ\| = k\, \|f(P), f(Q)\|, P, Q \in ABC$;

$$\Delta ABC \sim \Delta A'B'C' \Longrightarrow \left.\begin{array}{c} \frac{\|AB\|}{\|A'B'\|} = \frac{\|BC\|}{\|B'C'\|} = \frac{\|CA\|}{\|C'A'\|} = k \\ \hat{A} \equiv \widehat{A'}; \hat{B} \equiv \widehat{B'}; \hat{C} \equiv \widehat{C'} \end{array}\right\} \Longrightarrow \begin{array}{l} \|AB\| = k\|A'B'\| \\ \|BC\| = k\|B'C'\| \\ \|CA\| = k\|C'A'\| \end{array}.$$

We construct a function $f: ABC \to A'B'C'$ such that $f(A) = A', f(B) = B', f(C) = C'$;

if $P \in |BC| \to P \in |B'C'|$ such that $\|BP\| = k\|B'P'\|$;

if $P \in |CA| \to P \in |C'A'|$ such that $\|CP\| = k\|C'P'\|$; k – similarity constant.



Let $P, Q \in AB$ such that $P \in |BC|, Q \in |AC| \Longrightarrow P' \in |B'C'|$ and $\|BP\| = k\|B'P'\|$

$Q' \in |A'C'|$ and $\|CQ\| = k\|C'Q'\|$ (1);

As $\|BC\| = k\|B'C'\| \Longrightarrow \|PC\| = \|BC\| - \|BP\| = k\|B'C'\| - k\|B'P'\| =$

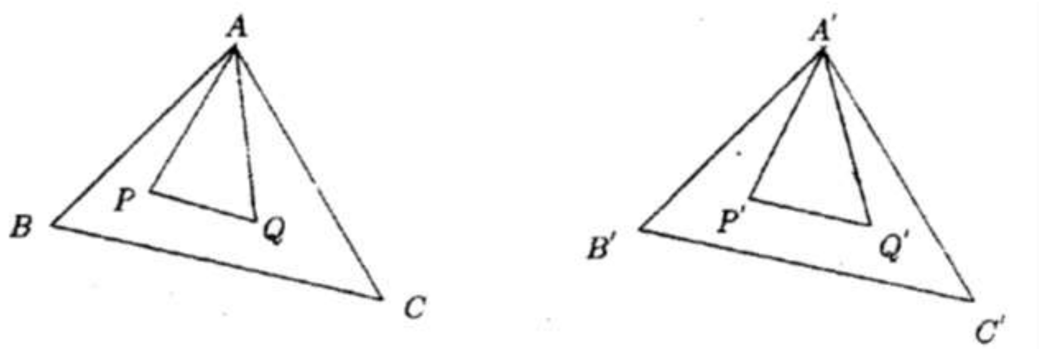
$= k(\|B'C'\| - \|B'P'\|) = k\|P'C'\|$ (2);

$\hat{C} \equiv \widehat{C'}$ (3).

From (1), (2), and (3) $\Longrightarrow \Delta PCQ \sim \Delta P'C'Q' \Longrightarrow \|PQ\| = k\|P'Q'\|$.

Similar reasoning for $P, Q \in ABC$.

We also extend the bijective function previously constructed to the interiors of the two triangles in the following way:



Let $P \in \text{int.} ABC$ and we construct $P' \in \text{int.} A'B'C'$ such that $\|AP\| = k\|A'P'\|$ (1).

Let $Q \in \text{int.} ABC \to Q' \in \text{int.} A'B'C'$ such that $\widehat{BAQ} \equiv \widehat{B'A'Q'}$ and $\|AQ\| = k\|A'Q'\|$ (2).

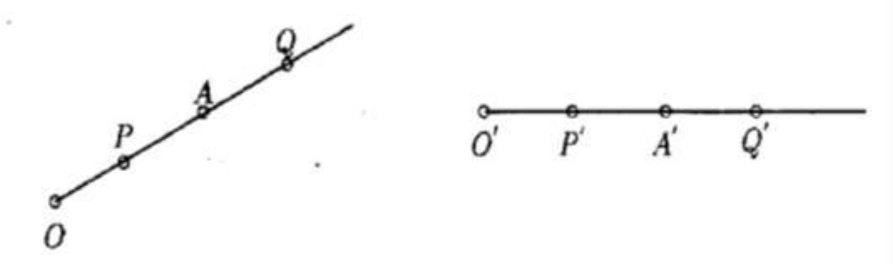
From (1) and (2),

$$\frac{\mathrm{AP}}{\mathrm{A'P'}} = \frac{\mathrm{AQ}}{\mathrm{A'Q'}} = k, \widehat{\mathrm{PAQ}} \equiv \widehat{\mathrm{P'A'Q'}} \Rightarrow \Delta\mathrm{APQ} \sim \Delta\mathrm{A'P'Q'} \Rightarrow \|\mathrm{PQ}\| = k\|\mathrm{P'Q'}\|,$$

but $P, Q \in [ABC]$, so $[ABC] \sim [A'B'C']$.

Solution to Problem 9.

a. Let $|OA$ and $|O'A'$ be two rays:



Let $f: |OA \to |O'A'$ such that $f(O) = O'$ and $f(P) = P'$ with $||OP|| = ||O'P'|$.

The so constructed point P' is unique and so if $P \neq Q \Rightarrow ||OP|| \neq ||OQ|| \Rightarrow ||O'P'|| \neq ||O'Q'|| \Rightarrow P' \neq Q'$ and $(\forall)P' \in |O'A'$ $(\exists)$ a single point $P \in |OA$ such that $||OP|| = ||O'P'||$.

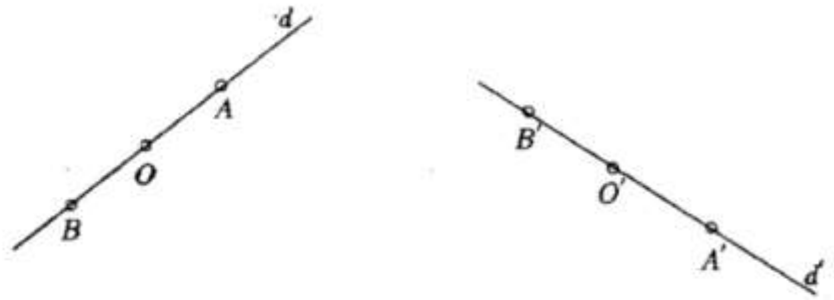
The constructed function is bijective.

If $P, Q \in |OA, P \in |OQ| \to P'Q' \in |O'A'$ such that $\|\mathrm{OP}\| = \|\mathrm{O'P'}\|; \|\mathrm{OQ}\| = \|\mathrm{O'Q'}\| \Rightarrow$

$$\|\mathrm{PQ}\| = \|\mathrm{OQ}\| - \|\mathrm{OP}\| = \|\mathrm{O'Q'}\| - \|\mathrm{O'P'}\| = \|\mathrm{P'Q'}\| (\forall)\mathrm{P}; Q \in |\mathrm{OA}$$

$\Rightarrow$ the two rays are congruent.

b. Let d and d' be two lines.



Let $O \in d$ and $O' \in d'$. We construct a function $f: d \to d'$ such that $f(O) = O'$ and $f\ (|OA) = |O'A'$ and $f\ (|OB) = |O'B'$ as at the previous point.

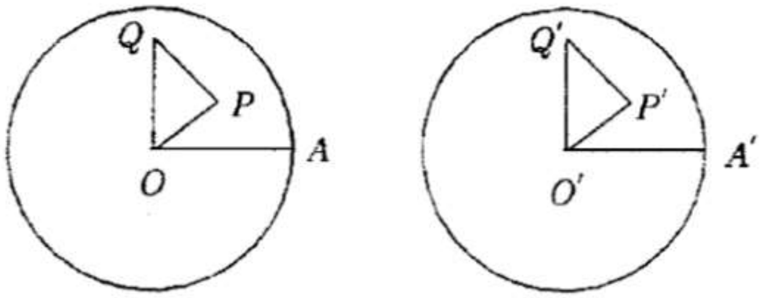
It is proved in the same way that f is bijective and that $||PQ|| = ||P'Q'||$ when P and Q belong to the same ray.

If P, Q belong to different rays:

$$\left.\begin{array}{l}\|OP\| = \|O'P'\| \\ \|OQ\| = \|O'Q'\|\end{array}\right\} \Rightarrow \|PQ\| = \|OP\| + \|OQ\| = \|O'P'\| + \|O'Q'\| = \|P'Q'\|$$

and so the two rays are congruent.

Solution to Problem 10.



We construct a function $f: D \to D'$ such that $f(O) = O', f(A) = A'$ and a point $(\forall)\ P \in D \to P' \in D'$ which are considered to be positive.

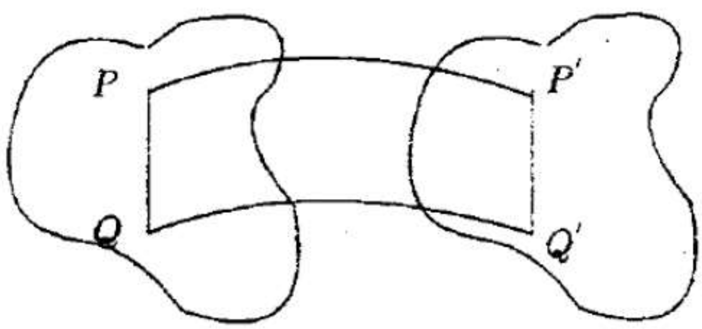
From the axiom of segment and angle construction $\Rightarrow$ that the so constructed function is bijective, establishing a biunivocal correspondence between the elements of the two sets.

Let $Q \in D \to Q' \in D'$ such that $||OQ'|| = ||OQ||$; $\widehat{AOQ} \equiv \widehat{A'O'Q'}$.

As:

$$\left.\begin{array}{l}\|OP\| = \|O'P'\| \\ \|OQ\| = \|O'Q'\| \\ \widehat{POQ} \equiv \widehat{P'O'Q'}\end{array}\right\} \Rightarrow \Delta OPQ \equiv \widehat{P'O'Q'} \Rightarrow \|PQ\| = \|P'Q'\|, (\forall)\ P, Q \in D \Rightarrow D \equiv D'.$$

Solution to Problem 11.



$f: M \to M'$ is an isometry $\Longrightarrow f$ is bijective and $(\forall)\ P, Q \in M$ we have $\|PQ\| = \|f(P), f(Q)\|$, f – bijective $\Longrightarrow f$ – invertible and f^{-1} – bijective.

$$\left.\begin{array}{c}\|P'Q'\| = \|f(P); f(Q)\| = \|PQ\| \\ \|f^{-1}(P'); f^{-1}(Q')\| = \left\|f^{-1}\big(f(P)\big), f^{-1}\big(f(Q)\big)\right\| = \|PQ\|\end{array}\right\} \Longrightarrow$$

$$\|P'Q'\| = \left\|f^{-1}\big(f(P')\big), f^{-1}\big(f(Q')\big)\right\|, (\forall)P', Q' \in M,$$

therefore $f^{-1}: M' \to M$ is an isometry.

Solution to Problem 12.

We construct a function f such that $f(P_i) = P_i'$, $i = 1, 2, \ldots, n$, and if $P \in |P_i, P_{i+1}|$.

The previously constructed function is also extended inside the polygon as follows: Let $O \in \mathrm{int}.L \to O' \in \mathrm{int}.L'$ such that $O\widehat{P_iP_{i+1}} \equiv O'\widehat{P'_iP'_{i+1}}$ and $\|OP_i\| = \|O'P'_i\|$. We connect these points with the vertices of the polygon. It can be easily proved that the triangles thus obtained are congruent.

We construct the function $g: [L] \to [L']$ such that

$$g(P) = \begin{cases} f(P), \text{if } P \in L \\ O', \text{if } P = O \\ P', \text{if } P \in [P_iOP_{i+1}] \text{ such that} \end{cases}$$

$$\widehat{P_iOP} \equiv \widehat{P_i'O'P'}\ (\forall)i = 1, 2, \ldots, n-1$$

The so constructed function is bijective $(\forall)\ P, Q \in [L]$. It can be proved by the congruence of the triangles POQ and $P'O'Q'$ that $||PQ|| = ||P'Q'||$, so $[L] = [L']$

$\Longrightarrow$ *if two convex polygons are decomposed in the same number of triangles respectively congruent, they are congruent.*

Solution to Problem 13.

$L = P_1P_2 \dots, P_n;\ L' = P_1'P_2' \dots, P_n'$

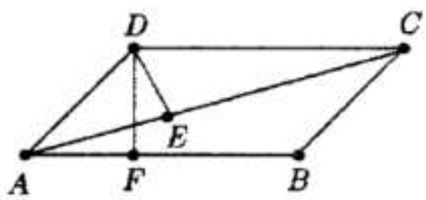
$L \sim L' \Longrightarrow (\exists)K > 0$ and $f: L \to L'$ such that $\|PQ\| = k\|f(P)f(Q)\|\ (\forall)P, Q \in L$, and $P_I' = f(P_i)$.

Taking consecutively the peaks in the role of P and Q, we obtain:

$$\left.\begin{array}{c} \|P_1P_2\| = k\|P_1'P_2'\| \Rightarrow \dfrac{\|P_1P_2\|}{\|P_1'P_2'\|} = k \\ \|P_2P_3\| = k\|P_2'P_3'\| \Rightarrow \dfrac{\|P_2P_3\|}{\|P_2'P_3'\|} = k \\ \vdots \\ \|P_{n-1}P_n\| = k\|P_{n-1}'P_n'\| \Rightarrow \dfrac{\|P_{n-1}P_n\|}{\|P_{n-1}'P_n'\|} = k \\ \|P_nP_1\| = k\|P_n'P_1'\| \Rightarrow \dfrac{\|P_nP_1\|}{\|P_n'P_1'\|} = k \end{array}\right\} \Longrightarrow$$

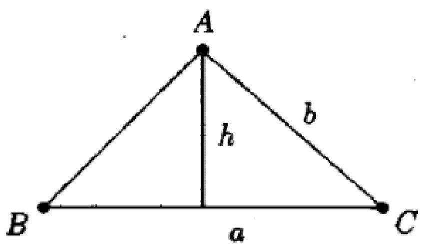
$$\Longrightarrow k = \frac{\|P_1P_2\|}{\|P_1'P_2'\|} = \frac{\|P_2P_3\|}{\|P_2'P_3'\|} = \dots = \frac{\|P_1P_2\| + \|P_2P_3\| + \dots + \|P_{n-1}P_n\| + \|P_nP_1\|}{\|P_1'P_2'\| + \|P_2'P_3'\| + \dots + \|P_{n-1}'P_n'\| + \|P_n'P_1'\|} = \frac{P}{P'}.$$

Solution to Problem 14.



$$\sigma[ADC] = \tfrac{2\cdot 7}{2} = 7;\ \sigma[ABCD] = 2 \cdot 7 = 14 = 6\|DF\| \Longrightarrow \|DF\| = \tfrac{14}{6} = \tfrac{7}{3}.$$

Solution to Problem 15.



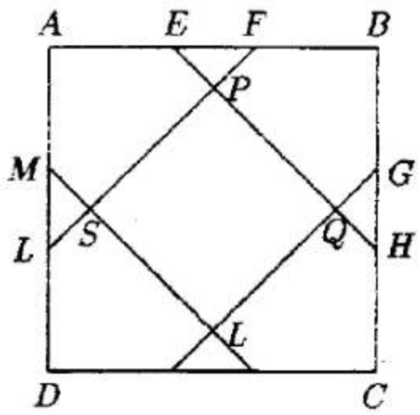
$$h = b \cdot \sin C \leq b;$$

$$\sigma[ABC] = \frac{a \cdot h}{2} \text{ is max. when } h \text{ is max.}$$

$$\max. h = b \text{ when } \sin C = 1$$

$$\Rightarrow m(C) = 90 \Rightarrow ABC \text{ has a right angle at } C.$$

Solution to Problem 16.



$\|MD\| = \|DI\| \Longrightarrow MDI$ – an isosceles triangle.

$\Longrightarrow m(\widehat{DMI}) = m(\widehat{MID}) = 45^0$;

The same way, $m(\widehat{FLA}) = m(\widehat{AFL}) = m(\widehat{BEH}) = m(\widehat{EHB})$.

$\|RK\| \Longrightarrow \|SP\| = \|PQ\| = \|QR\| = \|RS\| \Longrightarrow SRQP$ is a square.

$\|AB\| = a, \|AE\| = \frac{2a}{3}, \|MI\| = \sqrt{\frac{4a^2}{9} + \frac{4a^2}{9}} = \frac{2a\sqrt{2}}{3}$;

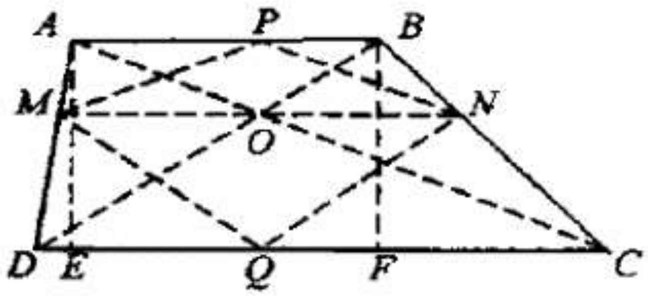
$2\|RI\|^2 = \frac{a^2}{9} \Longrightarrow \|RI\|^2 = \frac{a^2}{18} \Longrightarrow \|RI\| = \frac{a}{3\sqrt{2}} = \frac{a\sqrt{2}}{6}$;

$\|SR\| = \frac{2a\sqrt{2}}{3} - 2\frac{a\sqrt{2}}{6} = \frac{a\sqrt{2}}{3}$;

$\sigma[SRQP] = \frac{2a^2}{9} = \frac{2}{9}\sigma[ABCD]$.

Solution to Problem 17.

$$\left.\begin{aligned} \sigma[ACD] &= \frac{\|DC\| \cdot \|AE\|}{2} \\ \sigma[BCD] &= \frac{\|DC\| \cdot \|BF\|}{2} \\ \|AE\| &= \|BF\| \end{aligned}\right\} \Longrightarrow \sigma[ACD] = \sigma[BCD]$$



$$\left.\begin{aligned}\sigma[AOD] &= \sigma[AMO] + \sigma[MOD] \\ \sigma[AMO] = \sigma[MPO] &= \frac{\|MO\| \cdot \|OP\|}{2} \\ \sigma[MOD] = \sigma[MOQ] &= \frac{\|OM\| \cdot \|OQ\|}{2}\end{aligned}\right\} \Rightarrow \sigma[AOD] = \frac{\|OM\|(\|OP\| + \|OQ\|)}{2} = \frac{\|OM\| \cdot h}{2}$$

The same way,

$$\sigma[BOC] = \frac{\|ON\| \cdot h}{2}.$$

Therefore,

$$\sigma[AOD] = \sigma[BOC] \Rightarrow \frac{\|OM\| \cdot h}{2} = \frac{\|ON\| \cdot h}{2} \Rightarrow \|OM\| = \|ON\|.$$

Solution to Problem 18.

$\|AE\| = \|ED\|$;

We draw $MN \perp AB; DC$;

$\|EN\| = \|EM\| = \frac{h}{2'}$;

$$\sigma[BEC] = \frac{(\|AB\| + \|DC\|) \cdot h}{2} - \frac{\|AB\| \cdot h}{4} - \frac{\|DC\| \cdot h}{4} = \frac{(\|AB\| + \|DC\|) \cdot h}{4} = \frac{1}{2}\sigma[ABCD];$$

Therefore, $[ABCD] = 2\sigma[BEC]$.

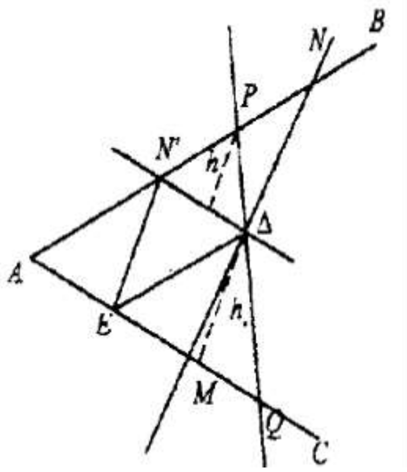
Solution to Problem 19.

$\sigma[AEDN']$ is ct. because A, E, D, N' are fixed points.

Let a line through D, and we draw $\|$ to sides ND and DE.

No matter how we draw a line through D, $\sigma[QPA]$ is formed of: $\sigma[AEDN] + \sigma[NPO] + \sigma[DEQ]$.

We have $\sigma[AEDN]$ constant in all triangles PAQ.

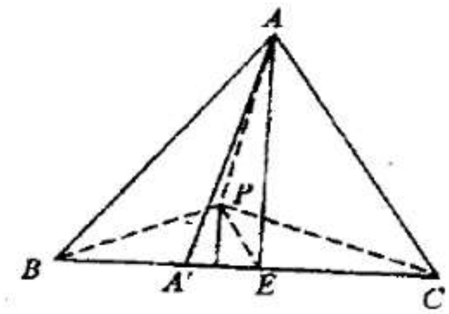


Let's analyse:

$$\sigma[PN'D]+\sigma[DEQ]=\frac{\|N'D\|\cdot h_1}{2}+\frac{\|EQ'\|\cdot h_2}{2}=\frac{\|N'D\|}{2}\left(h_1+\frac{\|EQ\|}{\|ND\|}\cdot h_2\right)$$
$$=\frac{\|N'D\|}{2}\cdot\left(h_1+\frac{h_2}{h_1}\cdot h_2\right)=\frac{\|N'D\|}{2h_1}[(h_1-h_2)^2+2h_1h_2].$$

ΔAMN is minimal when $h_1=h_2 \Longrightarrow D$ is in the middle of $|PQ|$. The construction is thus: ΔANM where $NM \,||\, EN'$. In this case we have $|ND|\equiv|DM|$.

Solution to Problem 20.



$$\sigma[ABC]=\frac{\|BC\|\cdot\|AA'\|}{2}$$

Let the median be $|AE|$, and P be the centroid of the triangle.

Let $PD\perp BC$. $\sigma[BPC]=\frac{\|BC\|\cdot\|PD\|}{2}$.

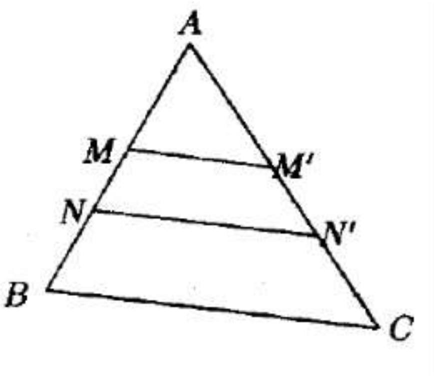
$$\left.\begin{matrix}AA'\perp BC\\ PD\perp BC\end{matrix}\right\}\Rightarrow AA'\parallel PD\Rightarrow \Delta PDE\sim\Delta AA'E\Rightarrow\frac{\|PD\|}{\|AA'\|}=\frac{\|PE\|}{\|AE\|}=\frac{1}{3}\Rightarrow\|PD\|=\frac{\|AA'\|}{3}\Rightarrow\sigma[BPC]$$
$$=\frac{\|BC\|\cdot\frac{\|AA'\|}{3}}{2}=\frac{1}{3}\frac{\|BC\|\cdot\|AA'\|}{2}=\frac{1}{3}\sigma[ABC].$$

We prove in the same way that $\sigma[PAC]=\sigma[PAB]=\frac{1}{3}\sigma[ABC]$, so the specific point is the centroid.

Solution to Problem 21.

Let $M, N \in AB$ such that $M \in |AN|$.

We take $MM' \parallel BC$, $MN' \parallel BC$.

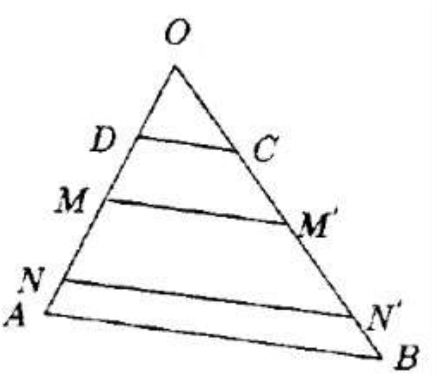


$$\Delta AMM' \sim \Delta ABC \Rightarrow \frac{\sigma[AMM']}{\sigma[ABC]} = \left(\frac{AM}{AB}\right)$$

$$\sigma[AMM'] = \frac{1}{3}\sigma, \left(\frac{\|AM\|}{\|AB\|}\right)^2 = \frac{1}{3}, \|AM\| = \frac{\|AB\|}{\sqrt{3}};\ \Delta ANN' \sim \Delta ABC \Longrightarrow$$

$$\left.\begin{array}{l} \dfrac{\sigma[ANN']}{\sigma[ABC]} = \left(\dfrac{\|AN\|}{\|AB\|}\right)^2 \\ \sigma[ANN'] = \dfrac{2}{3}\sigma[ABC] \end{array}\right\} \Longrightarrow \left(\frac{\|AN\|}{\|AB\|}\right)^2 = \frac{2}{3} \Longrightarrow \|AN\| = \sqrt{\frac{2}{3}\|AB\|}.$$

Solution to Problem 22.



$\|OD\| = a, \|OA\| = b$;

$\sigma[\Delta CM'M] = \sigma[MM'N'N] = \sigma[NN'BA] = \frac{1}{3}\sigma[ABCD]$;

$$\Delta ODC \sim \Delta OAB \Longrightarrow \frac{\sigma[ODC]}{\sigma[OAB]} = \frac{\|OD\|^2}{\|OA\|^2} = \frac{a}{b} \Longrightarrow \frac{\sigma[ODC]}{\sigma[OAB] - \sigma[ODC]} = \frac{a^2}{b^2 - a^2} \Longrightarrow \frac{\sigma[ODC]}{\sigma[ABCD]}$$

$$= \frac{a^2}{b^2 - a^2} \quad (1)$$

$$\Delta ODC \sim \Delta OMM' \Rightarrow \frac{\sigma[ODC]}{\sigma[OMM']} = \left(\frac{\|OD\|}{\|OM\|}\right)^2 \Rightarrow \frac{\sigma[ODC]}{\sigma[OMM'] - \sigma[ODC]} = \frac{a^2}{\|OM\|^2}$$

$$\Rightarrow \frac{\sigma[ODC]}{\sigma[DCMM']} = \frac{a^2}{\|OM\|^2 - a^2} \Rightarrow \frac{\sigma[ODC]}{\frac{1}{3}\sigma[ABCD]} = \frac{a^2}{\|OM\|^2 - a^2} \quad (2)$$

$$\Delta ONN' \sim \Delta ODC \Rightarrow \frac{\sigma[ODC]}{\sigma[DNN']} = \frac{\|OD\|}{\|ON\|} = \frac{a^2}{\|ON\|} \Rightarrow \frac{\sigma[IDC]}{\sigma[ONN'] - \sigma[ODC]} = \frac{a^2}{\|ON\|^2}$$

$$\Rightarrow \frac{\sigma[ODC]}{\sigma[DCNN']} = \frac{a^2}{\|ON\|^2 - a^2} \Rightarrow \frac{\sigma[ODC]}{\frac{2}{3}\sigma[ABCD]} = \frac{a^2}{\|ON\|^2 - a^2} \quad (3)$$

We divide (1) by (3):

$$\frac{2}{3} = \frac{\|ON\|^2 - a^2}{b^2 - a^2} \Rightarrow 3\|ON\|^2 - 3a^2 = 2b^2 - 2a^2 \Rightarrow \|ON\|^2 = \frac{a^2 + 2b^2}{3}.$$

Solution to Problem 23.

$\|OA\| = r \to \|DE\| = 2r$; $\sigma_{\text{hexagon}} = \frac{3r^2\sqrt{2}}{2}$ (1)

$DEFO$ a square inscribed in the circle with radius $R \Rightarrow$

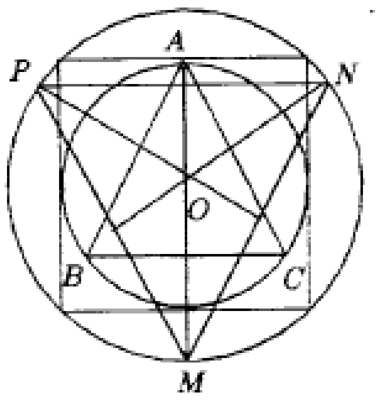
$\Rightarrow l_4 = R\sqrt{2} = \|DE\| \Rightarrow P\sqrt{2} = 2r \Rightarrow R = r\sqrt{2}$

$\|OM\| = R = r\sqrt{2}$

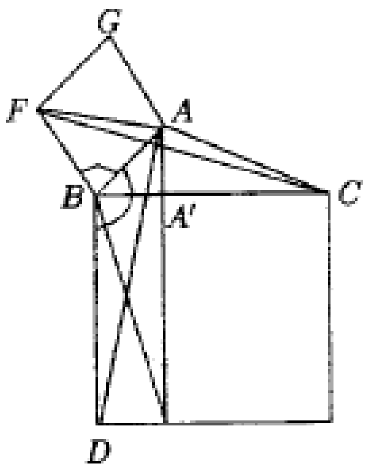
$$\sigma[OMN] = \frac{\|OM\| \cdot \|ON\| \sin 120}{2} = \frac{r\sqrt{2} \cdot r\sqrt{2} \cdot \frac{\sqrt{3}}{2}}{2} = \frac{r^2\sqrt{3}}{2}$$

$$\sigma[MNP] = 3\sigma[OMN] = 3\frac{r^2\sqrt{3}}{2} = \frac{3r^2\sqrt{2}}{2} \quad (2)$$

From (1) and (2) $\Rightarrow \sigma[MNP] = \sigma_{\text{hexagon}}$.



Solution to Problem 24.



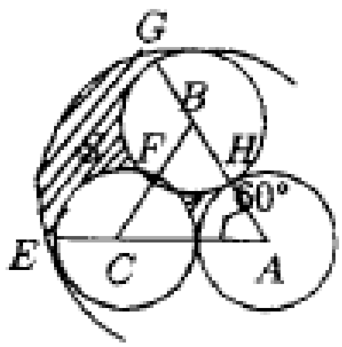
$\|AB\|^2 = \|BC\| \cdot \|BA'\|$

We construct the squares $BCED$ on the hypotenuse and $ABFG$ on the leg.

We draw $AA' \perp BC$.

$\sigma[ABFG] = \|AB\|^2$
$\sigma[A'BDH] = \|BD\| \cdot \|BA'\| = \|BC\| \cdot \|BA'\| \ldots$

Solution to Problem 25.



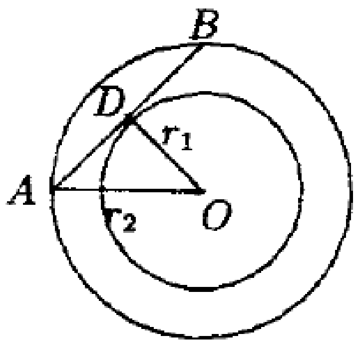
$$\sigma(s_1) = \sigma[ABC] - 3\sigma[\text{sect.}\,ADH]$$
$$\sigma[ABC] = \frac{l^2\sqrt{3}}{4} = \frac{(2a)^2\sqrt{3}}{4} = a^2\sqrt{3}$$
$$\sigma[\text{sect.}\,ADH] = \frac{r^2}{2} m(\widehat{DH})$$
$$m(\widehat{DH}) = \frac{\pi}{180} m(\widehat{DH}) = \frac{\pi}{180} \cdot 60^0 = \frac{\pi}{3}$$
$$\sigma[\text{sect.}\,ADH] = \frac{a^2}{2} \cdot \frac{\pi}{3} = \frac{\pi a^2}{2}$$
$$\sigma(s_1) = a^2\sqrt{3} - 3\frac{\pi a^2}{6} = a^2\sqrt{3} - \frac{\pi a^2}{2} \quad (1)$$

$$\begin{aligned}\sigma(s_2) = \sigma[\text{sect.}\,AEG] - \sigma[\text{sect.}\,ABC] - \sigma[\text{sect.}\,ECF] - \sigma[\text{sect.}\,GBF]\\ = \frac{(3a)^2}{2}\cdot\frac{\pi}{180}\cdot 60 - a^2\sqrt{3} - \frac{a^2}{2}\cdot\frac{\pi}{180}\cdot 120 = \frac{9a^2}{2}\cdot\frac{\pi}{3} - a^2\sqrt{3} - \frac{\pi a^2}{3} - \frac{\pi a^2}{3}\\ = \frac{3\pi a^2}{2} - \frac{2\pi a^2}{3} - a^2\sqrt{3} \quad (2)\end{aligned}$$

From (1) and (2)

$$\Longrightarrow\ \sigma(s_1) + \sigma(s_2) = \frac{3\pi a^2}{2} - \frac{2\pi a^2}{3} - \frac{\pi a^2}{2} = \frac{2\pi a^2}{6} = \frac{\pi a^2}{3}.$$

Solution to Problem 26.



$$\|AD\|^2 = r_2^2 - r_1^2$$

$$\sigma[L(O,r_1)] = \pi r_1^2$$

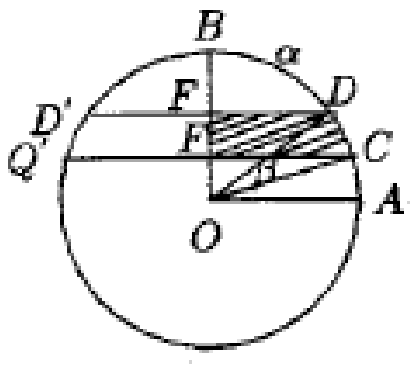
$$\sigma[L(O,r_2)] = \pi r_2^2$$

$$\sigma[\text{annulus}] = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2) \quad (1)$$

$$\sigma[\text{disk diameter}\|AB\|] = \pi\|AD\|^2 = \pi(r_2^2 - r_1^2) \quad (2)$$

From (1) and (2) $\Longrightarrow\ \sigma[\text{annulus}] = \sigma[\text{disk diameter}]$.

Solution to Problem 27.



$$\sigma[CDEF] = \frac{\sigma[CDD'C']}{2}$$
$$\sigma[CDD'C'] = \sigma_{\text{seg}}[CDBD'C'] - \sigma_{\text{seg}}[DBD']$$

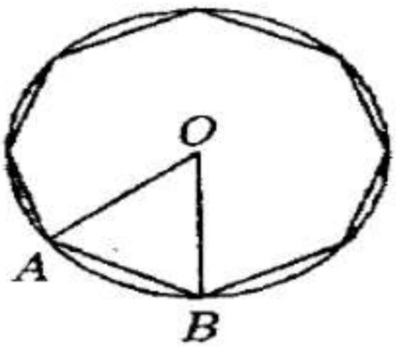
We denote $m(\widehat{AC}) = m(\widehat{BD}) = a \Rightarrow m(\widehat{CD}) = \frac{\pi}{2} - 2a$

$$\sigma_{\text{sect.}} = \frac{r^2}{2}(\alpha - \sin\alpha)$$
$$\sigma[CDBD'C'] = \frac{r^2}{2}[\pi - 2\alpha - \sin(\pi - 2\alpha)] = q\frac{r^2}{2}[\pi - 2\alpha - \sin(\pi - 2\alpha)]$$
$$\sigma_A = \frac{r^2}{2}(2\alpha - \sin 2\alpha)$$
$$\sigma[CDD'C'] = \sigma[CDBD'C'] - \sigma[DBD'] = \frac{r^2}{2}(\pi - 2\alpha - \sin 2\alpha) = \frac{r^2}{2}(2\alpha - \sin 2\alpha)$$
$$= \frac{r^2}{2}(\pi - 2\alpha - \sin 2\alpha - 2\alpha + \sin 2\alpha) = \frac{r^2}{2}(\pi - 4\alpha)$$
$$\Rightarrow \sigma[CDEF] = \frac{\sigma[CDD'C']}{2} = \frac{r^2}{4}(\pi - 4\alpha) = \frac{r^2}{2}\left(\frac{\pi}{2} - 2\alpha\right) \quad (1)$$
$$\sigma[\text{sect.}\,COD] = \frac{r^2}{2}m(\widehat{CD}) = \frac{r^2}{2}\left(\frac{\pi}{2} - 2\alpha\right) \quad (2)$$

From (1) and (2) $\Rightarrow \sigma[CDEF] = \sigma[\text{sect.}\,COD]$.

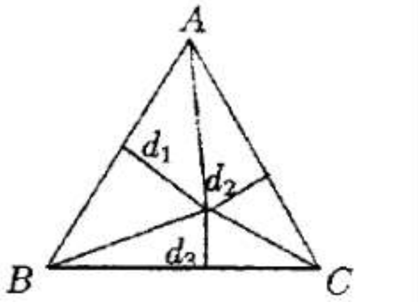
$$\|O_1F\| = \|OE\|$$
$$\sigma[\text{square}] = \|DE\|^2 = \|OA\|^2 = \frac{bc}{2} = V[ABC]$$

Solution to Problem 28.



$$\mu(\widehat{AOB}) = \frac{\pi}{4}$$
$$\sigma[AOB] = \frac{r^2\sin\frac{\pi}{4}}{2} = \frac{r^2\frac{\sqrt{2}}{2}}{2} = \frac{r^2\sqrt{2}}{4}$$
$$\sigma[\text{orthogon}] = 8\cdot\frac{r^2\sqrt{2}}{4} = 2\sqrt{2r^2}$$

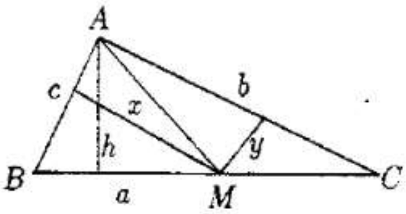
Solution to Problem 29.



$$\sigma[ABC] = \sigma[AMB] + \sigma[AMC] + \sigma[MBC]$$
$$\Longrightarrow ah_a = ad_3 + ad_2 + ad_1$$
$d_1 + d_2 + d_3 = h_a$ (a is the side of equilateral triangle)

$\Longrightarrow d_1 + d_2 + d_3 = \frac{a\sqrt{3}}{2}$ (because $h_a = \frac{a\sqrt{3}}{2}$).

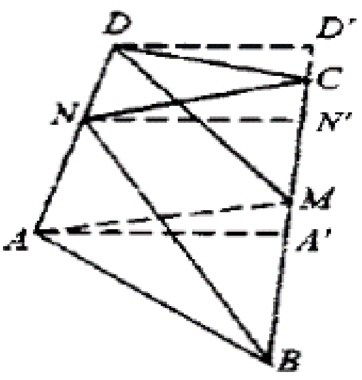
Solution to Problem 30.



$$ABC - \text{given } \Delta \Longrightarrow a, b, c, h - \text{constant}$$
$$\sigma[ABC] = \frac{ah}{2}$$
$$\sigma[ABC] = \sigma[AMB] + \sigma[AMC]$$
$$\Longrightarrow \frac{ah}{2} = \frac{cx}{2} + \frac{by}{2} \Longrightarrow cx + by = ab \Longrightarrow \frac{c}{ah}x + \frac{b}{ah}y = 1 \Longrightarrow kx + ly = 1,$$

where $k = \frac{c}{ah}$ and $l = \frac{b}{ah}$.

Solution to Problem 31.



We draw $AA' \perp BC; NN' \perp BC; DD' \perp BC \Rightarrow \left.\begin{matrix} AA' \parallel NN' \parallel DD' \\ \|AN\| = \|ND\| \end{matrix}\right\} \Rightarrow MN'$median line in the trapezoid $AA'D'D \Longrightarrow \|NN'\| = \frac{\|AA'\|+\|DD'\|}{2}$, $\sigma[BCN] = \frac{\|BC\|+\|NN'\|}{2}$.

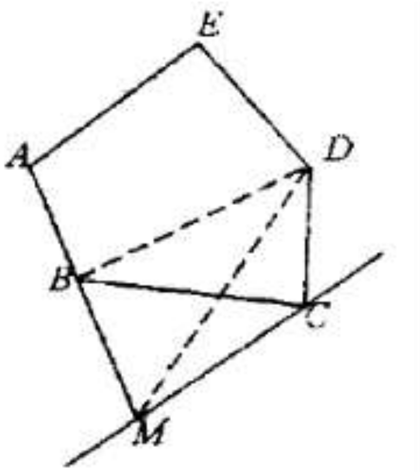
$$\sigma[BAM] + \sigma[MDC] = \frac{\|BM\| \cdot \|AA'\|}{2} + \frac{\|MC\| \cdot \|OD'\|}{2} \Rightarrow \|BM\| = \|MC\| = \frac{\|BC\|}{2} \Rightarrow$$

$$\Rightarrow \sigma[BAM] + \sigma[MDC] = \frac{\|BC\|}{2}\left(\frac{\|AA'\| + \|DD'\|}{2}\right) = \frac{\|BC\| \cdot \|NN'\|}{2} = \sigma[BCN]$$

$$\left.\begin{matrix} \sigma[BCN] = \sigma[PMQN] + \sigma[BPM] + \sigma[MQC] \\ \sigma[BCN] + \sigma[MDC] = \sigma[BAP] + \sigma[BPM] + \sigma[MQC] + \sigma[CDQ] \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \sigma[PMQN] = \sigma[BPA] + \sigma[CDQ].$$

Solution to Problem 32.



First, we construct a quadrilateral with the same area as the given pentagon. We draw through C a parallel to BD and extend |AB| until it intersects the parallel at M.

$\sigma[ABCDE] = \sigma[ABDE] + \sigma[BCD]$,

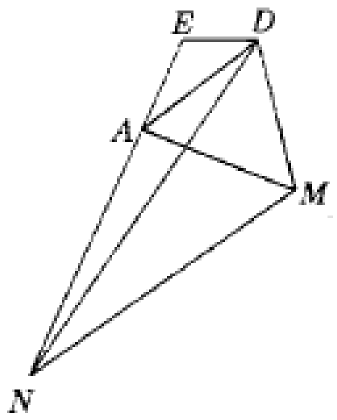
$\sigma[BCD] = \sigma[BDM]$ (have the vertices on a parallel at the base).

Therefore, $\sigma[ABCDE] = \sigma[AMDE]$.

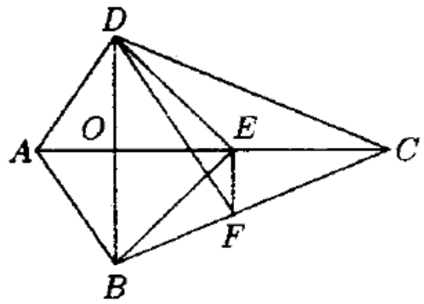
Then, we consider a triangle with the same area as the quadrilateral $AMDE$.

We draw a parallel to AD, N is an element of the intersection with the same parallel.

$\sigma[AMDE] = \sigma[ADE] + \sigma[ADE] = \sigma[ADE] + \sigma[ADN] = \sigma[EDN]$.



Solution to Problem 33.



$$\|AE\| = \|EC\|$$

$$\|EF\|BD \Longrightarrow \sigma[BDF] = \sigma[BDE]$$

$$\sigma[ABFD] = \sigma[ABED] \text{ (1)}$$

$\sigma[ADE] = \sigma[DEC]$ equal bases and the same height;

$$\sigma[ADE] = \sigma[DEC]$$

$$\sigma[ABED] = \sigma[BEDC] \text{ (2)}$$

$\sigma[DEF] = \sigma[BEF]$ the same base and the vertices on parallel lines at the base;

$$\sigma[DCF] \neq \sigma[DEC]+\sigma[ECF]+\sigma[DEF] \neq \sigma[DEC]+\sigma[ECF]+\sigma[BEF] = \sigma[BEDC] \quad (3)$$

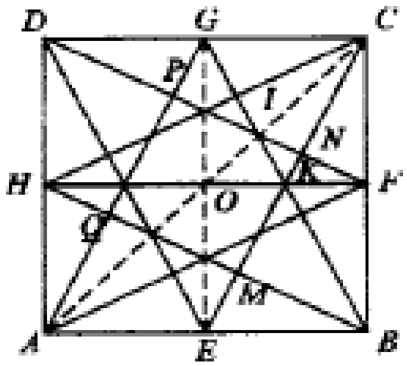
$$(1),(2),(3) \Rightarrow \sigma[ABFD] = \sigma[DCF]$$

Solution to Problem 34.

$$\left.\begin{array}{l}\Delta DCF \equiv \Delta CBE \Rightarrow \widehat{CFD} \equiv \widehat{CEB} \\ m(\widehat{CEB}) + m(\widehat{ECB}) = 90^\circ\end{array}\right\} \Rightarrow m(\widehat{CFN}) + m(\widehat{ECB}) = 90^\circ \Rightarrow$$

$$\Rightarrow m(\widehat{CNF}) = 90^\circ \Rightarrow CE \perp DF$$

$$\left.\begin{array}{l}|CF| \equiv |EB| \\ m(\widehat{MBE}) = m(\widehat{NCF}) \\ m(\widehat{NEB}) + m(\widehat{CFN})\end{array}\right\} \Rightarrow \Delta DCF \equiv \Delta BME \Rightarrow \begin{array}{l}|CN| \equiv |MB| \\ |NF| \equiv |ME|\end{array}$$



It is proved in the same way that:

$$\left.\begin{array}{l}|CN| \equiv |NB| \equiv |AQ| \equiv |DP| \\ |NF| \equiv |ME| \equiv |NQ| \equiv |PE| \\ |CE| \equiv |NB| \equiv |AG| \equiv |DF|\end{array}\right\} \Rightarrow$$

$$\Rightarrow |PN| \equiv |NM| \equiv |QM| \equiv |PQ|$$

$\Longrightarrow MNPQ$ rhombus with right angle$\Longrightarrow MNPQ$ is a square.

$$\left.\begin{array}{l}|DG| \equiv |FB| \\ \widehat{DIG} \equiv \widehat{FIB} \\ \widehat{GDI} \equiv \widehat{FBI}\end{array}\right\} \Rightarrow \Delta DGI \equiv \Delta DFI \Rightarrow \left.\begin{array}{l}|GI| \equiv |IF| \\ |GC| \equiv |CF| \\ |IC| \equiv |IC| \text{ com.}\end{array}\right\} \Rightarrow \Delta GIC \equiv \Delta FIC \Rightarrow$$

$$\Rightarrow \widehat{GCI} \equiv \widehat{FCI} \Rightarrow I \in |AC|$$

It is proved in the same way that all the peaks of the octagon are elements of the axis of symmetry of the square, thus the octagon is regular.

$$\|CF\| = \frac{1}{2},\ \|RF\| = \frac{1}{4},\ \|CR\| = \sqrt{\frac{1}{4} + \frac{1}{16}} = \frac{\sqrt{5}}{4}$$

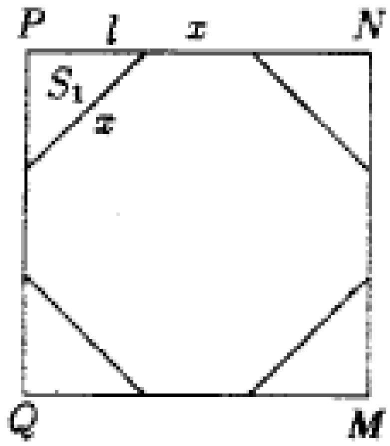
$$\|NF\| \cdot \frac{\sqrt{5}}{4} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \Rightarrow \|NF\| = \frac{1}{8} \cdot \frac{4}{\sqrt{5}} = \frac{1}{2\sqrt{5}}$$

$$\|EC\| = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$$

$$\|BM\| \cdot \frac{\sqrt{5}}{2} = 1 \cdot \frac{1}{2} \Rightarrow \|BM\| = \frac{1}{\sqrt{5}}$$

$$\|MN\| = \frac{\sqrt{5}}{2} - \frac{1}{2\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{5 - 1 - 2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$$

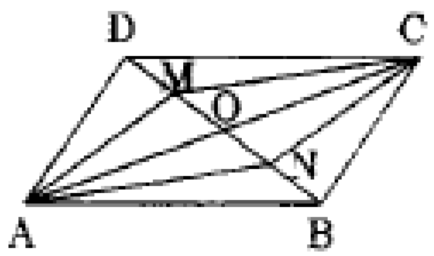
Consider the square separately.



$$2l^2 = 2x^2$$
$$x = l\sqrt{2}$$
$$2l + l\sqrt{2} = \frac{1}{\sqrt{5}}$$
$$l(2+\sqrt{2}) = \frac{1}{\sqrt{5}} \Rightarrow l = \frac{1}{\sqrt{5}(2+\sqrt{2})}$$
$$\sigma[QMNP] = \frac{1}{5}$$

$$S_1 = \frac{x^2}{2}$$
$$S = \sigma[QMNP] - 4S_1 = \frac{1}{5} - 2l^2 = \frac{1}{5} - 2\frac{1}{5(6+4\sqrt{2})} = \frac{1}{5} - \frac{1}{5(3+2\sqrt{2})} =$$
$$= \frac{1}{5}\cdot\frac{3+2\sqrt{2}-1}{3+2\sqrt{2}} = \frac{1}{5}\cdot\frac{2(1+\sqrt{2})}{3+2\sqrt{2}} = \frac{2}{5} - \frac{(1+\sqrt{2})(3-2\sqrt{2})}{9-8} = \frac{2}{5}(3-2\sqrt{2}+3\sqrt{2}-4) =$$
$$= \frac{2}{5}(\sqrt{2}-1).$$

Solution to Problem 35.



$$\|OM\| = \|NM\| = \|NB\|$$
$$\|DC\|? \Longrightarrow \Delta MOC = \Delta NBA \Longrightarrow \|MC\| = \|AN\|$$

It is proved in the same way that $\Delta DAM = \Delta BCN \Longrightarrow \|MC\| = \|NC\|$.

Thus $ANCM$ is a parallelogram.

$$\sigma[AOB] = \frac{\|OA\| \cdot \|OB\| \sin\alpha}{2}$$

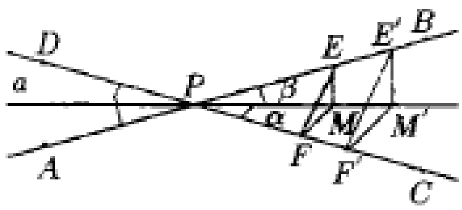
$$\sigma[AOD] = \frac{\|OA\| \cdot \|OD\| \sin(\pi - \alpha)}{2}$$

$$\sigma[ABCD] = \|OA\| \cdot \|OB\| \sin\alpha + \|OA\| \cdot \|OD\| \sin\alpha = \|OA\| \cdot \sin\alpha(\|OA\| + \|OD\|) =$$

$$= \|OA\| \cdot \|DB\| \cdot \sin\alpha$$

$$\sigma[ANCM] = \|OA\| \cdot \|MN\| \sin\alpha = \|OA\| \cdot \frac{\|DB\|}{3} \sin\alpha = \frac{\sigma[ABCD]}{3} \Rightarrow \frac{\sigma[AMCN]}{\sigma[ABCD]} = \frac{1}{3}$$

Solution to Problem 36.



To determine the angle α:

$$\sin\alpha = \frac{\|EM\|}{\|MP\|},\ \sin\beta = \frac{\|MF\|}{PM}$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{\|EM\|}{\|MF\|} = k \Rightarrow \sin\alpha = k\sin(a - \alpha) \Rightarrow$$

$$\Rightarrow \sin\alpha = k\sin a\cos\alpha + k\cos a\sin\alpha$$

We write

$$t = \mathrm{tg}\frac{\alpha}{2} \Rightarrow \frac{2t}{1+t^2} = k\sin a \cdot \frac{1-t^2}{1+t^2} + k\frac{2t}{1+t^2} \cdot \cos a \Rightarrow kt^2\cos a \cdot +2(1 - k\cos a) \cdot t -$$

$$-k\sin a = 0 \Rightarrow t_{1,2} = \frac{k\cos a - 1 \pm \sqrt{(k-1)^2 + 2k(1-\cos a)}}{k\cos a}$$

thus we have established the positions of the lines of the locus.

$$\sigma[ABM] = \frac{\|AB\| \cdot \|ME\|}{2}$$

$$\sigma[CDM] = \frac{\|CD\| \cdot \|MF\|}{2}$$

$$\sigma[ABM] = \sigma[CDM] \Rightarrow \|AB\| \cdot \|ME\| = \|CD\| \cdot \|MF\| \Rightarrow \frac{\|ME\|}{\|MF\|} = \frac{\|CD\|}{\|AB\|}$$

constant for A, B, C, D – fixed points.

We must find the geometrical locus of points M such that the ratio of the distances from this point to two concurrent lines to be constant.

$\frac{\|ME\|}{\|MF\|} = k$. Let M' be another point with the same property, namely $\frac{\|M'E'\|}{\|M'F'\|} = k$.

$$\left.\begin{array}{l}\left.\begin{array}{l} ME \perp AB \\ M'E' \perp AB \end{array}\right\} \Rightarrow ME \parallel M'E' \\ \left.\begin{array}{l} MF \perp CD \\ M'F' \perp CD \end{array}\right\} \Rightarrow MF \parallel M'F' \end{array}\right\} \Rightarrow \left.\begin{array}{l} \widehat{EMF} \equiv \widehat{E'M'F'} \\ \frac{\|ME\|}{\|MF\|} = \frac{\|M'E'\|}{\|M'F'\|} \end{array}\right\} \Rightarrow \triangle MEF \sim \triangle M'E'F' \Rightarrow$$

$$\Rightarrow \widehat{FEM} \equiv \widehat{F'E'M'} \Rightarrow \widehat{PEF} \equiv \widehat{PE'F'}$$

$$\Rightarrow \left.\begin{array}{l} \frac{\|PE\|}{\|PE'\|} = \frac{\|EF\|}{\|E'F'\|} \\ \frac{\|EF\|}{\|E'F'\|} = \frac{\|EM\|}{\|E'M'\|} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \frac{\|PE\|}{\|PE'\|} = \frac{\|EM\|}{\|E'M'\|} \\ \widehat{PEM} \equiv \widehat{PE'M'} \end{array}\right\} \Rightarrow \triangle PEM \sim \triangle PE'M' \Rightarrow \widehat{EPM} \equiv \widehat{E'PM'}$$

P, M, M' collinear $\Longrightarrow$ the locus is a line that passes through P.

When the points are in $\sphericalangle CPB$ we obtain one more line that passes through P. Thus the locus is formed by two concurrent lines through P, from which we eliminate point P, because the distances from P to both lines are 0 and their ratio is indefinite.

Vice versa, if points N and N' are on the same line passing through P, the ratio of their distances to lines AB and CD is constant.

$$\left.\begin{array}{l} ME \parallel M'E' \Rightarrow \triangle PME \sim \triangle PM'E' \Rightarrow \frac{\|EM\|}{\|E'M'\|} = \frac{\|PM\|}{\|PM'\|} \\ MF \parallel M'F' \Rightarrow \triangle PMF \sim \triangle PM'F' \Rightarrow \frac{\|FM\|}{\|F'M'\|} = \frac{\|PM\|}{\|PM'\|} \end{array}\right\} \Rightarrow \frac{\|EM\|}{\|E'M'\|} = \frac{\|FM\|}{\|F'M'\|} \Rightarrow$$

$$\Rightarrow \frac{\|EM\|}{\|FM\|} = \frac{\|E'M'\|}{\|F'M'\|} = k$$

Solution to Problem 37.

We show in the same way as in the previous problem that:

$$\frac{\|ME\|}{\|MF\|} = k \Longrightarrow \frac{\|ME\|}{\|MF\| + \|ME\|} = \frac{k}{1-k} \Longrightarrow \|ME\| = \frac{kd}{1+k},$$

and the locus of the points which are located at a constant distance from a given line is a parallel to the respective line, located between the two parallels.

If $\|AB\| > \|CD\| \Longrightarrow d(MAE) < d(MCD)$.

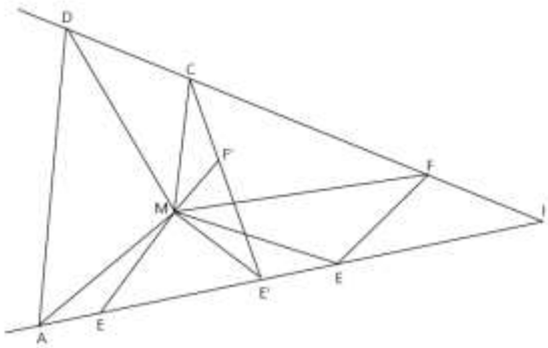
Then, if

$$\frac{ME}{MF} = k \Longrightarrow \frac{ME}{MF - ME} = \frac{k}{1-k} \Longrightarrow \frac{ME}{d} = \frac{k}{1-k} \Longrightarrow ME = \frac{kd}{1-k},$$

thus we obtain one more parallel to AB.

Solution to Problem 38.

Solution no. 1



We suppose that ABCD is not a parallelogram. Let $\{I\} = AB \cap CD$. We build $E \in (IA$ such that $IE = AB$ and $F \in (IC$ such that $IF = CD$. If M a point that verifies $\sigma[ABM] + \sigma[CDM] = 1$ (1), then, because $\sigma[ABM] = \sigma[MIE]$ and $\sigma[CDM] = \sigma[MIF]$, it results that $\sigma[MIE] + \sigma[MIF] = k$ (2).

We obtain that $\sigma[MEIF] = k$.

On the other hand, the points E, F are fixed, therefore $\sigma[IEF] = k' = \text{const}$. That is, $\sigma[MEF] = k - k' = \text{const}$.

Because $EF = \text{const.}$, we have $d(M, EF) = \frac{2(k-k')}{EF} = \text{const.}$, which shows that M belongs to a line that is parallel to EF, taken at the distance $\frac{2(k-k')}{EF}$.

Therefore, the locus points are those on the line parallel to EF, located inside the quadrilateral $ABCD$. They belong to the segment $[E'F']$ in Fig. 1.

Reciprocally, if $M \in [E'F']$, then $\sigma[MAB] + \sigma[MCD] = \sigma[MIE] + \sigma[MIF] = \sigma[MEIF] = \sigma[IEF] + \sigma[MEP] = k' + \frac{EF \cdot 2(k-k')}{2 \cdot EF} = k$.

In conclusion, the locus of points M inside the quadrilateral $ABCD$ which occurs for relation (1) where k is a positive constant smaller than $S = \sigma[ABCD]$ is a line segment.

If $ABCD$ is a trapeze having AB and CD as bases, then we reconstruct the reasoning as $AD \cap BC = \{I\}$ and $\sigma[MAD] + \sigma[MBC] = s - k = \text{const.}$

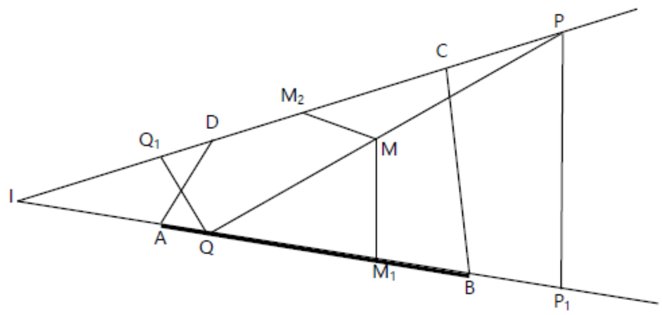
If $ABCD$ is a parallelogram, one shows without difficulty that the locus is a segment parallel to AB.

Solution no. 2 (Ion Patrascu)

We prove that the locus of points M which verify the relationship $\sigma[MAB] + \sigma[MCD] = k$ (1) from inside the convex quadrilateral $ABCD$ of area s ($k \subset s$) is a line segment.

Let's suppose that $AB \cap CD = \{I\}$, see *Fig. 2*. There is a point P of the locus which belongs to the line CD. Therefore, we have $(P; AB) = \frac{2k}{AB}$. Also, there is the point $Q \in AB$ such that $d(Q; CD) = \frac{2k}{CD}$.

Now, we prove that the points from inside the quadrilateral $ABCD$ that are on the segment $[PQ]$ belong to the locus.



Let $M \in \text{int}[ABCD] \cap [PQ]$. We denote M_1 and M_2 the projections of M on AB and CD respectively. Also, let P_1 be the projection of P on AB and Q_1 the projection of Q on CD. The triangles PQQ_1 and PMM_2 are alike, which means that

$$\frac{MM_2}{QQ_1} = \frac{MP}{PQ} \quad (2),$$

and the triangles MM_1Q and PP_1Q are alike, which means that

$$\frac{MM_1}{PP_1} = \frac{MQ}{PQ} \quad (3).$$

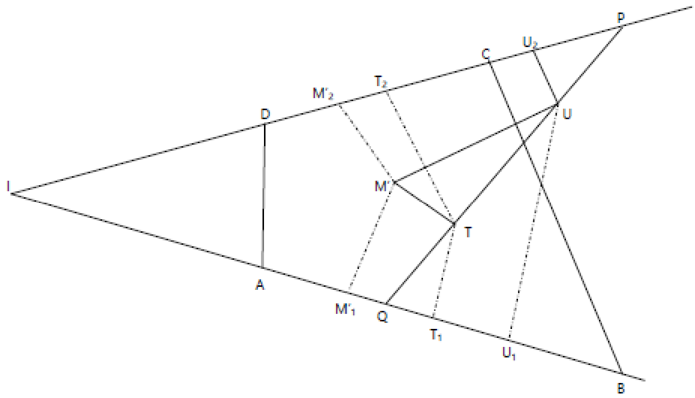
By adding member by member the relations (2) and (3), we obtain

$$\frac{MM_2}{QQ_1} + \frac{MM_1}{PP_1} = \frac{MP + MQ}{PQ} = 1 \quad (4).$$

Substituting in (4), $QQ_1 = \frac{2k}{CD}$ and $P_1 = \frac{2k}{AB}$, we get $AB \cdot MM_1 + CD \cdot MM_2 = 2k$, that is $\sigma[MAB] + \sigma[MCD] = k$.

We prove now by reductio ad absurdum that there is no point inside the quadrilateral $ABCD$ that is not situated on the segment $[PQ]$, built as shown, to verify the relation (1).

Let a point M' inside the quadrilateral $ABCD$ that verifies the relation (1), $M' \notin [PQ]$. We build $M'T \cap AB$, $M'U \parallel CD$, where T and U are situated on $[PQ]$, see *Fig. 3*.



We denote M_1', T_1, U_1 the projections of M_1, T, U on AB and M_2', T_2, U_2 the projections of the same points on CD.

We have the relations:

$$M'M_1' \cdot AB + M'M_2' \cdot CD = 2k \quad (5),$$

$$TT_1 \cdot AB + TT_2 \cdot CD = 2k \quad (6).$$

Because $M'M_1' = TT_1$ and $M'M_2' = UU_2$, substituting in (5), we get:

$$TT_1 \cdot AB + UU_2 \cdot CD = 2k \quad (7).$$

From (6) and (7), we get that $TT_2 = UU_2$, which drives us to $PQ \parallel CD$, false!

Problems in Geometry and Trigonometry

39. Find the locus of the points such that the sum of the distances to two concurrent lines to be constant and equal to l.

Solution to Problem 39

40. Show that in any triangle ABC we have:

    a. $b\,cosC + c\,cosB = a$; b. $b\,cosB + c\,cosC = a\,cos(B - C)$.

Solution to Problem 40

41. Show that among the angles of the triangle ABC we have:

    a. $b\,cosC - c\,cosB = \frac{b^2-a^2}{a}$;

    b. $2(bc\,cosA + ac\,cosB + ab\,cosC = a^2 + b^2 + c^2$.

Solution to Problem 41

42. Using the law of cosines prove that $4m\frac{2}{a} = 2(b^2 + c^2) - a^2$, where m_a is the length of the median corresponding to the side of a length.

Solution to Problem 42

43. Show that the triangle ABC where $\frac{a+c}{b} = \cot\frac{B}{2}$ is right-angled.

Solution to Problem 43

44. Show that, if in the triangle ABC we have $\cot A + \cot B = 2\cot C \Rightarrow a^2 + b^2 = 2c^2$.

Solution to Problem 44

45. Determine the unknown elements of the triangle ABC, given:

    a. A, B and p;

    b. $a + b = m, A$ and B;

    c. $a, A; b - c = a$.

Solution to Problem 45

46. Show that in any triangle ABC we have $\tan\frac{A-B}{2}\tan\frac{C}{2}=\frac{a-b}{a+b}$ (*tangents theorem*).

Solution to Problem 46

47. In triangle ABC it is given $\hat{A}=60°$ and $\frac{b}{c}=2+\sqrt{3}$. Find $\tan\frac{B-C}{2}$ and angles B and C.

Solution to Problem 47

48. In a convex quadrilateral $ABCD$, there are given $\|AD\|=7(\sqrt{6}-\sqrt{2}), \|CD\|=13, \|BC\|=15, C=\arccos\frac{33}{65}$, and $D=\frac{\pi}{4}+\arccos\frac{5}{13}$. The other angles of the quadrilateral and $\|AB\|$ are required.

Solution to Problem 48

49. Find the area of ΔABC when:
   a. $a=17, B=\arcsin\frac{24}{25}, C=\arcsin\frac{12}{13}$;
   b. $b=2, \hat{A}\in 135°, \hat{C}\in 30°$;
   c. $a=7, b=5, c=6$;
   d. $\hat{A}\in 18°, b=4, c=6$.

Solution to Problem 49

50. How many distinct triangles from the point of view of symmetry are there such that $a=15, c=13, s=24$?

Solution to Problem 50

51. Find the area of ΔABC if $a=\sqrt{6}$, $\hat{A}\in 60°$, $b+c=3+\sqrt{3}$.

Solution to Problem 51

52. Find the area of the quadrilateral from problem 48.

Solution to Problem 52

53. If S_n is the area of the regular polygon with n sides, find:

$S_3, S_4, S_6, S_8, S_{12}, S_{20}$ in relation to R, the radius of the circle inscribed in the polygon.

Solution to Problem 53

54. Find the area of the regular polygon $ABCD \dots M$ inscribed in the circle with radius R, knowing that: $\frac{1}{\|AB\|} = \frac{1}{\|AC\|} + \frac{1}{\|AD\|}$.

Solution to Problem 54

55. Prove that in any triangle ABC we have:

a. $r = (p-a)\tan\frac{A}{2}$;
b. $S = (p-a)\tan\frac{A}{2}$;
c. $p = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;
d. $p - a = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;
e. $m_a^2 = R^2(\sin^2 A + 4\cos A\sin B\sin C$;
f. $h_a = 2R\sin B\sin C$.

Solution to Problem 55

56. If l is the center of the circle inscribed in triangle ABC show that $\|AI\| = 4R\sin\frac{B}{2}\sin\frac{C}{2}$.

Solution to Problem 56

57. Prove the law of sine using the analytic method.

Solution to Problem 57

58. Using the law of sine, show that in a triangle the larger side lies opposite to the larger angle.

Solution to Problem 58

59. Show that in any triangle ABC we have:

a. $\dfrac{a\cos C - b\cos B}{a\cos B - b\cos A} + \cos C = 0, a \neq b$;

b. $\dfrac{\sin(A-B)\sin C}{1+\cos(A-B)\cos C} = \dfrac{a^2-b^2}{a^2+b^2}$;

c. $(a+c)\cos\dfrac{B}{4} + a\cos(A+\dfrac{3B}{4}) = 2c\cos\dfrac{B}{2}\cos\dfrac{B}{4}$.

Solution to Problem 59

60. In a triangle ABC, $A \in 45°, \|AB\| = a, \|AC\| = \frac{2\sqrt{2}}{3}a$. Show that $\tan B = 2$.

Solution to Problem 60

61. Let A', B', C' be tangent points of the circle inscribed in a triangle ABC with its sides. Show that $\frac{\sigma[A'B'C']}{\sigma[ABC]} = \frac{r}{2R}$.

Solution to Problem 61

62. Show that in any triangle ABC $\sin\frac{A}{2} \leq \frac{a}{2\sqrt{bc}}$.

Solution to Problem 62

63. Solve the triangle ABC, knowing its elements A, B and area S.

Solution to Problem 63

64. Solve the triangle ABC, knowing $a = 13, \arccos\frac{4}{5}$, and the corresponding median for side $a, m_a = \frac{1}{2}\sqrt{15\sqrt{3}}$.

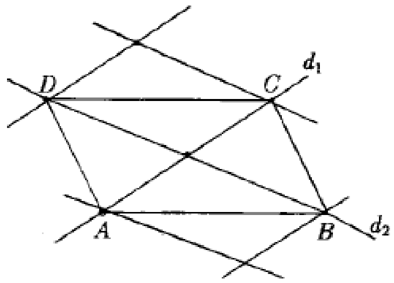
Solution to Problem 64

65. Find the angles of the triangle ABC, knowing that $B - C = \frac{2\pi}{3}$ and $R = 8r$, where R and r are the radii of the circles circumscribed and inscribed in the triangle.

Solution to Problem 65

Solutions

Solution to Problem 39.



Let d_1 and d_2 be the two concurrent lines. We draw 2 parallel lines to d_1 located on its both sides at distance l. These intersect on d_2 at D and B, which will be points of the locus to be found, because the sum of the distances $d(B, d_1) + d(B, d_2) = l + 0$ verifies the condition from the statement.

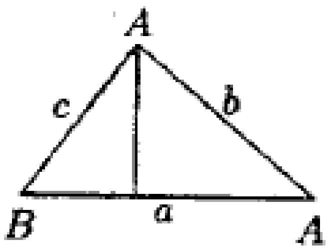
We draw two parallel lines with d_2 located at distance l from it, which cut d_1 in A and C, which are as well points of the locus to be found. The equidistant parallel lines determine on d_2 congruent segments $\Longrightarrow \begin{matrix} |DO| \equiv |OB| \\ |AO| \equiv |OC| \end{matrix}$, in the same way $ABCD$ is a parallelogram.

$$\Delta BOC, \left.\begin{matrix} \|CC'\| = d(C, d_2) \\ \|BB'\| = d(B, d_1) \end{matrix}\right\} \Longrightarrow \|CC'\| = \|BB'\|$$

$\Longrightarrow \Delta BOC$ is isosceles.

$\Longrightarrow ||OC|| = ||OB|| \Longrightarrow ABCD$ is a rectangle. Any point M we take on the sides of this rectangle, we have $||R_1, d_1|| + ||M, d_2|| = l$, using the propriety according to which the sum of the distances from a point on the base of an isosceles triangle at the sides is constant and equal to the height that starts from one vertex of the base, namely l. Thus the desired locus is rectangle $ABCD$.

Solution to Problem 40.



$$\triangle ABC : \cos B = \frac{\|BD\|}{c} \Rightarrow \|BD\| = c\cos B$$

$$\triangle ADC : \cos C = \frac{\|DC\|}{b} \Rightarrow \|DC\| = b\cos C$$

$$a = \|BD\| + \|DC\| = c\cos B + b\cos C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = m \Rightarrow \begin{cases} b = m\sin B \\ c = m\sin C \end{cases}$$

$$b\cos B + c\cos C = m\sin B\cos B + m\sin C\cos C = \frac{m}{2}(2\sin B\cos B + 2\sin C\cos C) =$$

$$= \frac{m}{2}(\sin 2B + \sin 2C) = \frac{m}{2}\cdot 2\sin(B+C)\cos(B-C) = \frac{a}{\cancel{\sin A}}\cancel{\sin}(\pi - A)\cos(B-C) =$$

$$= a\cos(B-C).$$

Solution to Problem 41.

$$\text{a) } \cos C = \frac{a^2+b^2-c^2}{2ab} \text{ și } \cos B = \frac{a^2+b^2-c^2}{2ac}$$

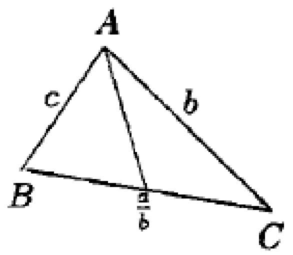
$$b\cos C - c\cos B = b\cdot\frac{a^2+b^2-c^2}{2ab} - c\cdot\frac{a^2+c^2-b^2}{2ac} = \frac{a^2+b^2-c^2-a^2-c^2+b^2}{2a} =$$

$$= \frac{2b^2-2c^2}{2a} = \frac{b^2-c^2}{a}$$

b) $$2bc\cos A + 2ac\cos B + 2ab\cos C = 2bc\frac{b^2+c^2-a^2}{2bc} +$$

$$+2ac\frac{a^2+c^2-b^2}{2ac} + 2ab\frac{a^2+b^2-c^2}{2ab} = b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2 = a^2+b^2+c^2$$

Solution to Problem 42.



$$m_a^2 = c^2 + \frac{a^2}{4} - 2\frac{a}{2}c\cos B\ ;$$

$$4m_a^2 = 4c^2 + a^2 - 4ac\cos B = 4c^2 + a^2 - 4ac\frac{a^2+c^2-b^2}{2ac} = 4c^2 + a^2 - 2a - 2c^2 + 2b^2$$
$$= 2c^2 + 2b^2 - a^2 = 2(b^2+c^2) - a^2\,.$$

Solution to Problem 43.

Using the sine theorem, $a = m \sin A$.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = m \Rightarrow b = m \sin B$$

$$c = m \sin C$$

$$\frac{a+c}{b} = \frac{m \sin A + m \sin C}{m \sin B} = \frac{\sin A + \sin C}{\sin B} = \frac{2 \sin \frac{A+C}{2} \cos \frac{A-C}{2}}{2 \sin \frac{B}{2} \cos \frac{B}{2}} =$$

$$\left.\begin{array}{l} \dfrac{\sin(\frac{\pi}{2} - \frac{B}{2}) \cos \frac{A-C}{2}}{\sin \frac{B}{2} \cos \frac{B}{2}} = \dfrac{\cos \frac{B}{2} \cos \frac{A-C}{2}}{\sin \frac{B}{2} \cos \frac{B}{2}} = \dfrac{\cos \frac{A-C}{2}}{\sin \frac{B}{2}} \\ \operatorname{ctg} \dfrac{B}{2} = \dfrac{\cos \frac{B}{2}}{\sin \frac{B}{2}} \end{array}\right\} \Rightarrow \cos \frac{A-C}{2} = \cos \frac{B}{2} \Rightarrow$$

$$\Rightarrow \frac{A-C}{2} = \frac{B}{2}$$

$$\frac{A-C}{2} = -\frac{B}{2} \Longrightarrow A - B = C \text{ or } A - C = B \Longrightarrow \begin{array}{c} A = B + C \\ \text{or} \Longrightarrow \\ A + B = C \end{array} \begin{array}{c} 2A = 180^0 \\ \text{or} \Longrightarrow \\ 2C = 180^0 \end{array} \begin{array}{c} A = 90^0 \\ \text{or} \\ C = 90^0 \end{array}$$

Solution to Problem 44.

$$\operatorname{ctg} A + \operatorname{ctg} B = 2 \operatorname{ctg} C \Rightarrow \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} = 2 \frac{\cos C}{\sin C}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = m \Rightarrow \sin A = \frac{a}{m}, \sin B = \frac{b}{m}, \sin C = \frac{c}{m};$$

By substitution:

$$2c^2 = 2(a^2 + b^2 - c^2) \Rightarrow 2c^2 = a^2 + b^2$$

Solution to Problem 45.

a. Using the law of sine,

$$\Rightarrow \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{a+b+c}{\sin A+\sin B+\sin C}=$$

$$=\frac{2p}{\sin A+\sin B+\sin C}$$

$$a=\frac{2p\sin A}{\sin A+\sin B+\sin C};b=\frac{2p\sin B}{\sin A+\sin B+\sin C}c=\frac{2p\sin C}{\sin A+\sin B+\sin C}$$

$$\text{iar } C=\pi-(A+B)$$

b.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{a+b}{\sin A+\sin B}=\frac{m}{\sin A+\sin B}\Rightarrow a=\frac{m\sin A}{\sin A+\sin B}$$

$$b=\frac{m\sin B}{\sin A+\sin B}\Rightarrow$$

$$\Rightarrow c=\frac{a\sin C}{\sin A}=\frac{a\sin(A+B)}{\sin A}\text{ dar }\frac{a}{\sin A}=\frac{c}{\sin C}$$

c.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C};\frac{b}{\sin B}=\frac{c}{\sin B}=\frac{b-c}{\sin B-\sin C}=\frac{d}{\sin B-\sin C}=\frac{a}{\sin A}\Rightarrow$$

$$\Rightarrow \sin B-\sin C=\frac{d\sin A}{a}\Rightarrow$$

$$\Rightarrow 2\sin\frac{B-C}{2}\cos\frac{B+C}{2}=\frac{d\sin A}{a}$$

$$B+C=\pi-A\Rightarrow\frac{B+C}{2}=\frac{\pi}{2}-\frac{A}{2}\Rightarrow\cos\frac{B+C}{2}=\sin\frac{A}{2}$$

Therefore,

$$2\sin\frac{B-C}{2}\sin\frac{A}{2}=\frac{d}{a}2\sin\frac{A}{2}\cos\frac{A}{2}\Rightarrow\sin\frac{B-C}{2}=\frac{d}{a}\cos\frac{A}{2}$$

We solve the system, and find B and C. Then we find $b=\frac{a\sin B}{\sin A}$ and $c=b-d$.

Solution to Problem 46.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=m\Longrightarrow \begin{matrix} a=m\sin A \\ b=m\sin B\end{matrix},$$

$$\frac{a-b}{a+b}=\frac{m\sin A-m\sin B}{m\sin A+m\sin B}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\tan\frac{A-B}{2}\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$

$$=\tan\frac{A-B}{2}\tan\frac{C}{2}.$$

Solution to Problem 47.

Using tangents' theorem,

$$\frac{b-c}{b+c} = \operatorname{tg}\frac{B-C}{2}\operatorname{tg}\frac{A}{2}$$

$$\hat{A} \in 60^\circ \Rightarrow m\left(\frac{A}{2}\right) = 30^\circ \Rightarrow \operatorname{tg}\frac{A}{2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\frac{b}{c} = \frac{2+\sqrt{3}}{1} \Rightarrow \frac{b-c}{b+c} = \frac{2+\sqrt{3}-1}{2+\sqrt{3}+1} = \frac{1+\sqrt{3}}{3+\sqrt{3}}$$

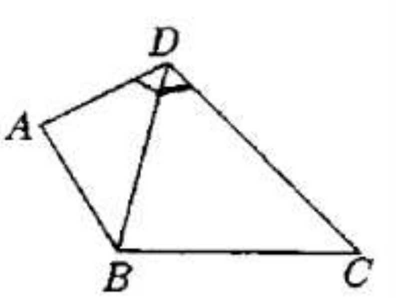
$$\operatorname{tg}\frac{B-C}{2} = \frac{1+\sqrt{3}}{3+\sqrt{3}} \cdot \frac{1}{\frac{1}{\sqrt{3}}} = 1 \Rightarrow \mu\left(\frac{B-C}{2}\right) = 45^\circ \Rightarrow \begin{cases} B-C = 90^\circ \\ B+C = 120^\circ \end{cases}$$

$$2B = 210^\circ \Rightarrow \mu(B) = 105^\circ \Rightarrow \mu(C) = 120^\circ - 105^\circ = 15^\circ$$

So

$$\mu(C) = \frac{\pi}{12} \text{ şi } \mu(B) = \frac{7\pi}{12}.$$

Solution to Problem 48.



$$\|BD\|^2 = 13^2 - 15^2 - 2\cdot 13\cdot 15\cos C = 13^3 + 15^2 - 2\cdot 13\cdot 15\frac{33}{65} =$$

$$= 13^2 + 15^2 - 2\cdot 13\cdot 15\cdot\frac{3\cdot 11}{13\cdot 5} = 13^2 + 15^2 - 18\cdot 11 = 196 \Rightarrow$$

$$\Rightarrow \|BD\| = 14$$

In ΔBDC we have

$$\frac{14}{\sin C} = \frac{15}{\sin\widehat{BDC}} \Rightarrow \sin\widehat{BDC} = \frac{15\cdot\sin C}{14}$$

$$\sin C = \sqrt{1-\frac{33^2}{65^2}} = \sqrt{\frac{(65-33)(65+33)}{65^2}} = \sqrt{\frac{2^6\cdot 7^2}{65^2}} = \frac{56}{65}$$

$$\sin\widehat{BDC} = \frac{15\cdot\frac{56}{65}}{14} = \frac{15\cdot 56}{14\cdot 65} = \frac{3\cdot 5\cdot 14\cdot 4}{14\cdot 5\cdot 13} = \frac{12}{13}$$

$$\cos 2\widehat{BDC} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{169-144}{169}} = \frac{5}{13}$$

$$\sin \widehat{BDC} = \arccos \frac{5}{13}$$

$$\sin \widehat{ADB} = \frac{\pi}{4} + \arccos \frac{5}{13} - \arccos \frac{5}{13} = \frac{\pi}{4}.$$

In ΔADB,

$$\Rightarrow \|AB\|^2 = 49(\sqrt{6}-\sqrt{2})^2 + 14^2 - 2 \cdot 14 \cdot 7(\sqrt{6}-\sqrt{2})\frac{\sqrt{2}}{2} = 49(6+2-2\sqrt{2})+$$

$$+196 - 93(\sqrt{2}-2) = 98(4-\sqrt{12}) - 98(\sqrt{12}-2) + 196 = 98(4-\sqrt{12}-\sqrt{12}+2)+196 =$$

$$= 196(3-\sqrt{12}) + 196 = 196(4-2\sqrt{3}) = 196(\sqrt{3}-1)^2,$$

$$\|AB\| = 14(\sqrt{3}-1)$$

In ΔADB we apply sine's theorem:

$$\frac{\|AD\|}{\sin \widehat{ABD}} = \frac{\|AB\|}{\sin \frac{\pi}{4}} \Rightarrow \frac{7(\sqrt{6}-\sqrt{2})}{\sin \widehat{ABD}} = \frac{14(\sqrt{3}-1)}{\frac{\sqrt{2}}{2}} = \frac{28(\sqrt{3}-1)}{\sqrt{2}}$$

$$\sin ABD = \frac{7(\sqrt{1}-2)}{28(\sqrt{3}-1)} = \frac{14(\sqrt{3}-1)}{28(\sqrt{3}-1)} = \frac{1}{2} = \mu(\widehat{ABD}) = \frac{\pi}{6}$$

$$\mu(A) = A - \frac{\pi}{6} - \frac{\pi}{4} = \frac{12\pi - 2\pi - 3\pi}{12} = \frac{7\pi}{12}$$

$$\mu(D) = 2\pi - \frac{7\pi}{12} - \frac{\pi}{4} - \arccos \frac{5}{13} - \arccos \frac{33}{65} = \frac{14\pi}{12} - (\underbrace{\arccos \frac{5}{13}}_{\alpha} + \underbrace{\arccos \frac{33}{65}}_{\beta})$$

$$\cos \alpha = \frac{5}{15} \Rightarrow \sin \alpha = \frac{12}{13}$$

$$\cos \beta = \frac{33}{65} \Rightarrow \sin \beta = \frac{56}{65}$$

$$\cos(\alpha+\beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{5}{13} \cdot \frac{33}{65} - \frac{12}{13}\frac{56}{65} = -\frac{507}{13 \cdot 65} = -\frac{3 \cdot 13^2}{13 \cdot 13 \cdot 5} = -\frac{3}{5}$$

$$\alpha + \beta = \pi - \arccos \frac{3}{5}$$

$$\mu(D) = \frac{14\pi}{12} - \pi + \arccos \frac{3}{5} = \frac{2\pi}{12} + \arccos 35 = \frac{\pi}{6} + \arccos 35$$

Or we find $\mu(\widehat{DBC})$ and we add it to $\frac{\pi}{6}$.

Solution to Problem 49.

$$\text{a) } B = \arcsin \frac{24}{25} \Rightarrow \sin B = \frac{24}{25} \Rightarrow \cos B = \frac{7}{25}$$

$$C = \arcsin \frac{12}{13} \Rightarrow \sin C = \frac{12}{13} \Rightarrow \cos C = \frac{5}{13}$$

$$\sin A = \sin[\pi - (B+C)] = \sin(B+C) = \sin B \cos C + \sin C \cos B = \frac{24}{25} \cdot \frac{5}{13} + \frac{12}{13} \cdot \frac{7}{25} =$$

$$= \frac{120+84}{325} = \frac{204}{325}$$

$$S = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{289 \cdot \frac{24}{25}\frac{12}{13}}{2 \cdot \frac{204}{325}}$$

b) $b = 2, \hat{A} \in 135°, \hat{C} \in 30° \Rightarrow \hat{B} \in 15°$

$\sin A = \sin 45 = \frac{\sqrt{2}}{2}$

$\sin C = \frac{1}{2}$

$\sin B = \sin \frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{2}} = \frac{1}{2}\left(\sqrt{\frac{2+1}{2}} - \sqrt{\frac{2-1}{2}}\right) =$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \sqrt{\frac{1}{2}}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$$S = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{4\frac{\sqrt{2}}{2}\cdot\frac{1}{2}}{2\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{3}-1} = \frac{\sqrt{2}(\sqrt{3}+1)}{2}$$

d) $\hat{A} \in 18°, b = 4, c = 6$

$\mu(A) = \frac{\pi}{10}0$

$2\hat{A} = 36°, 3\hat{A} \in 54°$

$\sin \alpha = \cos(90° - \alpha)$

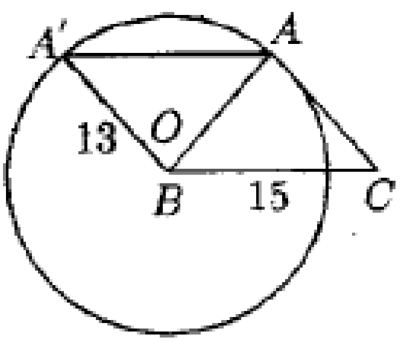
$\sin 30° = \cos 54° \Rightarrow \sin 2A = \cos 3A \Rightarrow 2 \sin A \cos A = \cos(4\cos^2 A - 3) \Rightarrow$

$\Rightarrow 4\sin^2 A + 2 \sin A - 1 = 0$

$$\sin A = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$\sin A = \frac{-1+\sqrt{5}}{4}$, because $m(A) < 180^0$ and $\sin A > 0$.

Solution to Problem 50.



$$\frac{ac\sin B}{2} = 24 \Rightarrow \sin B = \frac{48}{15\cdot 13} = \frac{16}{65}$$

$$\cos B = \sqrt{1-\frac{16^2}{65^2}} = \sqrt{\frac{(65-16)(65-16)}{65^2}} =$$

$$= \sqrt{\frac{49\cdot 81}{65^2}} = \frac{7\cdot 9}{65} = \frac{63}{65}$$

$$b^2 = 13^2+15^2-2\cdot 13\cdot 15\cdot \frac{7\cdot 9}{13\cdot 5} = 13^2+15^2-376 = 394-378 = 16,$$

$$b = 4$$

$$b^2 = 13^2+15^2+2\cdot 13\cdot 15\cdot \frac{7\cdot 9}{13\cdot 5} = 394+378 = 772 = 4\cdot 193$$

$$b = 2\sqrt{193}.$$

Solution to Problem 51.

$$a^2 = b^2+c^2-2bc\cos A \Rightarrow 6 = b^2+c^2-2bc\frac{1}{2}$$

$$\left.\begin{aligned} 6 = (b+c)^2-2bc-bc = (b+c)^2-3bc \\ b+c = 3+\sqrt{3} \end{aligned}\right\} \Rightarrow (3+\sqrt{3})^2-3bc = 6$$

$$9+6\sqrt{3}+3-3bc = 6 \Rightarrow 2(1+\sqrt{3}) = bc$$

$$\left.\begin{aligned} &b+c = 3+\sqrt{3} \\ &bc = 2+2\sqrt{3} \end{aligned}\right\} \Rightarrow x^2-5x+p = 0 \Rightarrow x^2-(3+\sqrt{3})x+2+2\sqrt{3} = 0 \Rightarrow$$

$$\Rightarrow x_{1,2} = \frac{3+\sqrt{3}\pm\sqrt{4-2\sqrt{3}}}{2} = \frac{3+\sqrt{3}\pm\sqrt{\sqrt{(3-1)^2}}}{2} = \frac{3+\sqrt{3}\pm(\sqrt{3}-1)}{2} \Rightarrow$$

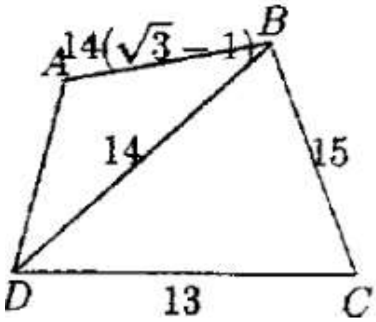
$$\Rightarrow \begin{aligned} x_1 &= 1+\sqrt{3} \\ x_2 &= 2 \end{aligned}$$

$b = 1+\sqrt{3}$ și $c = 2$ sau $b = 2$ și $c = 1+\sqrt{3}$

$$2p = \sqrt{6}+1+\sqrt{3}+2 = 3+\sqrt{3}+2 = 3+\sqrt{3}+\sqrt{6} \Rightarrow p = \frac{3+\sqrt{3}+\sqrt{6}}{2}$$

$$S = \sqrt{p(p-a)(p-b)(p-c)}.$$

Solution to Problem 52.



$$\|AD\| = 7(\sqrt{6}-\sqrt{2})$$

$$\|CD\| = 13, \quad \|BC\| = 15$$

At problem 9 we've found that

$$\|BD\| = 14, \|AB\| = 14(\sqrt{3}-1)$$

With Heron's formula, we find the area of each triangle and we add them up.

$$\sigma[ABCD] = \sigma[ABD] + \sigma[BDC]$$

Solution to Problem 53.

The formula for the area of a regular polygon:

$$S_n = \frac{n}{2}R^2 \sin\frac{2\pi}{n}$$

$$n = 3 \Rightarrow S_3 = \frac{3}{2}R^2 \sin\frac{2\pi}{3} = \frac{3\sqrt{3}R^2}{4}$$

$$n = 4 \Rightarrow S_4 = \frac{4}{2}R^2 \sin\frac{2\pi}{4} = 2R^2$$

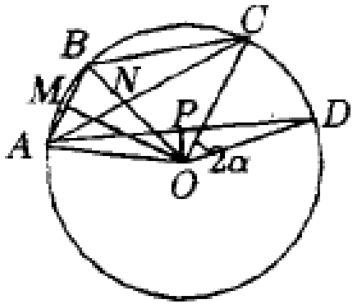
$$n = 6 \Rightarrow S_6 = \frac{6}{2}R^2 \sin\frac{2\pi}{6} = \frac{3\sqrt{3}R^2}{2}$$

$$n = 8 \Rightarrow S_8 = \frac{8}{2}R^2 \sin\frac{2\pi}{8} = 2\sqrt{2}R^2$$

$$n = 12 \Rightarrow S_{12} = \frac{12}{2}R^2 \sin\frac{2\pi}{12} = 3R^2$$

$$n = 20 \Rightarrow S_20 = \frac{20}{2}R^2 \sin\frac{2\pi}{20} = 10R^2\frac{\sqrt{5}-1}{4} = \frac{5}{2}(\sqrt{5}-1)R^2$$

Solution to Problem 54.



$$m(\widehat{AB}) = 2d \Rightarrow m(\widehat{AOB}) = 2d$$

In ΔBOM:

$$\sin\alpha = \frac{\|BM\|}{\|BO\|} \Rightarrow \begin{matrix} \|BM\| = R\sin\alpha \\ \|AB\| = 2R\sin\alpha \end{matrix} \qquad (1)$$

In ΔNOC:

$$\sin 2\alpha = \frac{\|NC\|}{\|OC\|} \Rightarrow \|NC\| = R\sin 2\alpha \Rightarrow \|AC\| = 2R\alpha \qquad (2)$$

In ΔPOD:

$$\sin 3\alpha = \frac{\|DP\|}{\|OD\|} \Rightarrow \|DP\| = R\sin 3\alpha$$

$$\|AD\| = 2R\sin 3\alpha \qquad (3)$$

Substituting (1), (2), (3) in the given relation:

$$\frac{1}{2R\sin\alpha} = \frac{1}{2R\sin 2\alpha} = \frac{1}{2R\sin 3\alpha} \Rightarrow \frac{1}{\sin\alpha} = \frac{1}{\sin 2\alpha} = \frac{1}{\sin 3\alpha}$$

$$\frac{1}{\sin 2\alpha} = \frac{1}{\sin\alpha} = \frac{1}{\sin 3\alpha} \Rightarrow \frac{1}{\sin 2\alpha} = \frac{\sin 3\alpha - \sin\alpha}{\sin\alpha\cdot\sin 3\alpha} = \frac{2\sin\alpha\cdot\cos 2\alpha}{\sin\alpha\cdot\sin 3\alpha} \Rightarrow 2\sin 2\alpha\cos 2\alpha =$$

$$= \sin 3\alpha \Rightarrow \sin 4\alpha = \sin 3\alpha \Rightarrow \sin 4\alpha - \sin 3\alpha = 0 \Rightarrow 2\sin\frac{\alpha}{2}\cos\frac{7\alpha}{2} = 0 \Longleftrightarrow \sin\frac{\alpha}{2} = 0$$

or

$$\cos\frac{7\alpha}{2} = 0.$$

$$\sin\frac{\alpha}{2} \Rightarrow \frac{\alpha}{2} = 0$$

which is impossible.

$$\cos\frac{7\alpha}{2} = 0 \Rightarrow \frac{7\alpha}{0} = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{7} \Rightarrow m(\widehat{AB}) = \frac{2\pi}{7}$$

$$n = \frac{m(\text{complete circle})}{m(\widehat{AB})} = \frac{2\pi}{\frac{2\pi}{7}} = 7.$$

Thus the polygon has 7 sides.

$$S_7 = \frac{7}{2}R^2\sin\frac{2\pi}{7}$$

Solution to Problem 55.

$$\text{a) } \sin\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{bc}},\ \cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}} \Rightarrow \operatorname{tg}\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}$$

$$(p-a)\operatorname{tg}\frac{A}{2}=(p-a)\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\sqrt{\frac{(p-a)^2(p-b)(p-c)}{p(p-a)}}=$$

$$=\sqrt{\frac{p(p-a)(p-b)(p-c)}{p^2}}=\frac{9}{p}=r$$

$$\text{b) } \frac{S}{p}=(p-a)\operatorname{tg}\frac{A}{2} \Rightarrow S=p(p-a)\operatorname{tg}\frac{A}{2}$$

$$\text{c) } \cos\frac{A}{2}=\sqrt{\frac{p(p-a)}{bc}},\ \cos\frac{B}{2}=\sqrt{\frac{p(p-b)}{ac}},\ \cos\frac{C}{2}=\sqrt{\frac{p(p-c)}{ab}},\ R=\frac{abc}{4S}\Rightarrow 4R=\frac{abc}{S}$$

$$4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{abc}{S}\cdot\sqrt{\frac{p^3(p-a)(p-b)(p-c)}{a^2b^2c^2}}=\frac{abc}{S}\frac{\sqrt{p^3()p-a(p-b)(p-c)}}{abc}=\frac{abc}{S}$$

$$\frac{pS}{abc}=p$$

$$\text{d) } 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{abc}{S}\cdot\sqrt{\frac{p(p-a)^3(p-b)(p-c)}{bcacab}}=\frac{abc}{S}\cdot\frac{p-a}{abc}$$

$$S=p-a$$

$$\sin A=\frac{a}{2R},\ \sin B=\frac{b}{2R},\ \sin C=\frac{c}{2R}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

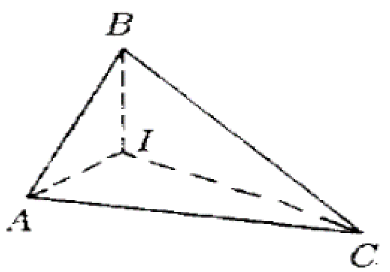
$$R^2(\sin^2 A+4\cos A\sin B\sin C)=R^2\left(\frac{a^2}{4R^2}+4\frac{b^2+c^2-a^2}{2bc}\cdot\frac{b}{2R}\cdot\frac{c}{2R}\right)=$$

$$=R^2\frac{a^2+2b^2+2c^2-2a^2}{4R^2}=\frac{2(b^2+c^2)-a^2}{4}=\mu_a{}^2$$

$$\text{f) } S=\frac{ah_a}{2}\Rightarrow h_a=\frac{2S}{a},\ R=\frac{abc}{4S},\ \frac{b}{2R}=\sin B,\ \frac{c}{2a}=\sin C$$

$$2R\sin B\sin C=2R\cdot\frac{b}{2R}\cdot\frac{c}{2R}=\frac{bc}{2R}=\frac{bc}{2\cdot\frac{abc}{4S}}=\frac{2bcS}{abc}=\frac{2S}{a}=h_a$$

Solution to Problem 56.



We apply the law of sine in ΔABI:

$$\frac{\|AI\|}{\sin\frac{B}{2}}=\frac{\|BI\|}{\sin\frac{A}{2}}=\frac{\|AB\|}{\sin\widehat{BIA}}$$

$$m(\widehat{BIA})=180^\circ-\frac{A+B}{2}=180^\circ-90^\circ+\frac{c}{2}=90^\circ+\frac{c}{2}$$

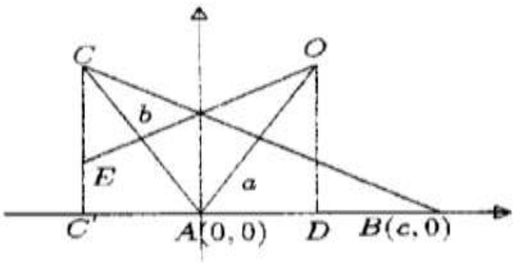
$$\sin\widehat{BIA}=\sin(90^\circ+\frac{c}{2})=\sin(180^\circ-90^\circ-\frac{c}{2})=\sin(90^\circ-\frac{c}{2})=\cos\frac{c}{2}$$

The law of sine applied in ΔABC:

$$\frac{\|AB\|}{\sin C}=2R\Rightarrow\|AB\|=2R\sin C=4R\sin\frac{C}{2}\cos\frac{C}{2}\sin\frac{B}{2}$$

$$\frac{1}{\cos\frac{C}{2}}=4R\sin\frac{B}{2}\sin\frac{C}{2}$$

Solution to Problem 57.



In $\Delta ACC'$: $\sin(180^0-A)=\frac{\|CC'\|}{b}\Longrightarrow\|CC'\|=b\sin A$; $\cos(180^0-A)=b\cos A$.

So the coordinates of C are $(-b\cos A\,, b\sin A)$.

The center of the inscribed circle is at the intersection of the perpendicular lines drawn through the midpoints of sides AB and AC.

$$m_{EO}=-\frac{1}{m_{AC}}=\frac{1}{\text{tg } A}=\text{ctg } A$$

$$E\left(\frac{O-b\cos A}{2},\frac{O+b\sin A}{2}\right)=E\left(-\frac{b\cos A}{2},\frac{b\sin A}{2}\right)$$

The equation of the line EO:

$$y-y_0=m(\lambda-x_0)\Rightarrow y-\frac{b\sin A}{2}=\text{ ctg } A(x+\frac{b\cos A}{2})$$

$$R=\|OA\|=\sqrt{\left(\frac{c}{2}\right)^2+\left(\frac{b}{2\sin A}+\frac{c\cos A}{2\sin A}\right)^2}=$$

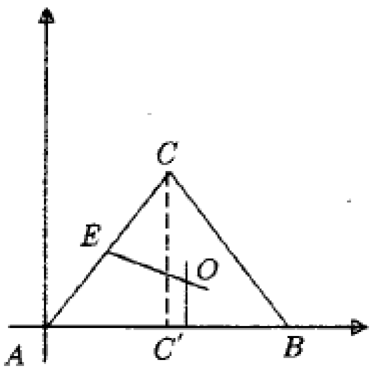
$$=\sqrt{\frac{c^2}{4}+\frac{b^2}{4}\cdot\frac{1}{\sin^2 A}+2\cdot\frac{bc\cos A}{4\sin^2 A}+\frac{c^2}{4}\cdot\frac{\cos^2 A}{\sin^2 A}}=$$

$$= \sqrt{\frac{c^2}{4}\left(1+\frac{\cos^2 A}{\sin^2 A}\right)+\frac{1}{4\sin^2 A}(b^2+2bc\cos A)} =$$

$$= \sqrt{\frac{1}{4\sin^2 A}(c^2+b^2+2bc\cos A)} = \sqrt{\frac{a^2}{4\sin^2 A}} =$$

$$= \frac{a}{2\sin A} \Rightarrow R = \frac{a}{2\sin A}$$

If we redo the calculus for the same draw, we have the following result: $(b\cos A, b\sin A)$.



$$m_{AC} = \text{tg}A \Rightarrow m_{OE} = -\frac{1}{\text{tg}\, A}$$

$$(OE): 2\sin A = -2x\cos A + B$$

$$O\left(\frac{c}{2}; \frac{-c\cos A + b}{2\sin A}\right) \text{ şi } \|OA\| = R =$$

$$= \sqrt{\frac{c^2}{4}+\frac{b^2+c^2\cos A - 2bc\cos A}{4\sin^2 A}} =$$

$$= \sqrt{\frac{1}{4\sin^2 A}\cdot(c^2\sin^2 A + c^2\cos^2 A + b^2 - 2bc\cos A)} = \frac{a}{2\sin A},$$

using the law of cosine.

Solution to Problem 58.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

We suppose that $a > b$. Let's prove that $A > B$.

$$\left.\begin{matrix}\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{b} = \frac{\sin A}{\sin B} \\ a > b \Rightarrow \frac{a}{b} > 1\end{matrix}\right\} \Longrightarrow \frac{\sin A}{\sin B} > 1 \Longrightarrow A, B, C \in (0,\pi) \Longrightarrow \sin B > 0 \Longrightarrow \sin A > \sin B$$

$$\Longrightarrow \sin A - \sin B > 0 \Longrightarrow 2\sin\frac{A-B}{2}\cos\frac{A+B}{2} > 0 \Longrightarrow \frac{A+B}{2} = \frac{180^0 - C}{2}$$

$$= 90^0 - \frac{C}{2}.$$

$\cos\frac{A+B}{2} = \cos\left(90^{\text{circ}} - \frac{C}{2}\right) = \sin\frac{C}{2} > 0$, therefore $\frac{A-B}{2} > 0 \Rightarrow A > B$;

$\left(-\frac{\pi}{2} < \frac{A-B}{2} < \frac{\pi}{2}\right)$.

Solution to Problem 59.

a) $a = 2R\sin A, b = 2R\sin B$

$$\frac{2R\sin A\cos A - 2R\sin B\cos B}{2R\sin A\cos B - 2R\sin B\cos A} + \cos C = \cos C + \frac{\sin A\cos A - \sin B\cos B}{\sin A\cos B - \sin B\cos A} =$$

$$= \frac{\frac{1}{2}\sin 2A - \frac{1}{2}\sin 2B}{\sin(A-B)} + \cos C = \frac{1}{2}\cdot\frac{2\sin(A-B)\cos(A+B)}{\sin(A-B)} + \cos C =$$

$$\cos(A+B) + \cos C = \cos(180^\circ - C) + \cos C = -\cos C + \cos C = 0$$

b. We transform the product into a sum:

$$\sin(A-B)\sin C = \frac{\cos(A-B-C) - \cos(A-B+C)}{2} \doteq \frac{1}{2}[-\cos 2A + \cos 2B] =$$

$$(B+C = 180^\circ - A, \quad A+C = 180^\circ - B)$$

$$= \frac{a^2-b^2}{4R^2}; \qquad (1)$$

$$1 + \cos(A-B)\cos C = 1 + \frac{\cos(A-B+C) + \cos(A-B+C)}{2} \doteq$$

$$= \frac{2 + \cos(180 - 2B) + \cos(2A - 180)}{2}$$

$$(A+B = 180^\circ - B, \quad B+C = 180^\circ - A)$$

$$\doteq \frac{2 - \cos 2B - \cos 2A}{2} = \frac{2 - 1 + 2\sin^2 B - 1 + 2\sin^2 A}{2} = \sin^2 A + \sin^2 B = \left(\frac{a}{2R}\right)^2 +$$

$$+ \left(\frac{b}{2R}\right)^2 = \frac{a^2+b^2}{4R^2} \qquad (2)$$

From (1) and (2) $\Rightarrow$

$$\frac{\frac{a^2-b^2}{4R^2}}{\frac{a^2+b^2}{4R^2}} = \frac{a^2-b^2}{a^2+b^2}$$

$$c)(a+c)\cos\frac{B}{4} + b\cos\left(A + \frac{3B}{4}\right) = a\cos\frac{B}{4} + b\cos\frac{B}{4} + b\cos\left(A + \frac{3B}{4}\right) = c\cos\frac{B}{4} +$$

$$+ a\cos\left(B - \frac{3B}{4}\right) + b\cos\left(A + \frac{3B}{4}\right)$$

We consider the last two terms:

$$a\cos\left(B-\frac{3B}{4}\right)+b\cos\left(A+\frac{3B}{4}\right)=2R\sin A\cos\left(B-\frac{3B}{4}\right)+2R\sin B\cos\left(A-\frac{3B}{4}\right)=$$

$$=2R\left(\frac{\sin\left(A+B-\frac{3B}{4}\right)+\sin\left(A-B+\frac{3B}{4}\right)}{2}+\frac{\sin\left(A+B-\frac{3B}{4}\right)+\sin\left(A-B+\frac{3B}{4}\right)}{2}\right)=$$

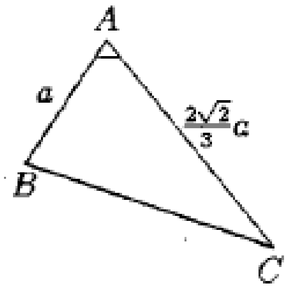
$$=R\left[\sin\left(A+B-\frac{3B}{4}\right)+\sin\left(A+B+\frac{3B}{4}\right)+\sin\left(A-B+\frac{3B}{4}\right)+\sin\left(A-B+\frac{3B}{4}\right)\right]=$$

$$=2R\sin(A+B)\cos\frac{3B}{4}-2R\sin(\pi-C)\cos\frac{3B}{4}+2R\sin c\cos\frac{3B}{4}=c\cos\frac{3B}{4}+c\cos\frac{3B}{4}=$$

$$=c\left(\cos\frac{B}{4}+\cos\frac{3B}{4}\right)=2c\cos\frac{B}{4}+\cos 3B4=2c\cos\frac{B}{2}\cos\frac{B}{2}$$

Solution to Problem 60.

We apply the law of cosines in triangle ABC:



$$\|BC\|^2=a^2+\frac{8}{9}a^2-2a\cdot\frac{2\sqrt{3}}{3}\cdot a\cdot\frac{\sqrt{2}}{2}=a^2+\frac{8a^2}{9}-\frac{4a^2}{3}=\frac{5a^2}{9}$$

$$\|BC\|=\frac{a\sqrt{5}}{3}$$

$$\|AC\|^2=\|AB\|^2+\|BC\|^2-2\|AB\|\|BC\|\cos B\Rightarrow\frac{8a^2}{9}=a^2+\frac{5a^2}{9}-2a\cdot\frac{a\sqrt{5}}{3}\cdot\cos B$$

$$\Rightarrow\frac{2a^2\sqrt{5}\cos B}{2}=a^2+\frac{5a^2}{9}-\frac{8a^2}{9}=\frac{6a^2}{9}\Rightarrow\cos B=\frac{6a^2}{9}\cdot\frac{3}{2a^2}\sqrt{5}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

$$\sin B=\sqrt{1-\cos^2 B}=\sqrt{1-\frac{5}{25}}=\frac{2\sqrt{5}}{5}$$

$$\operatorname{tg} B=\frac{\sin B}{\cos B}=\frac{2\sqrt{5}}{5}\cdot\frac{5}{\sqrt{5}}=2$$

Solution to Problem 61.

$\|IA\|=\|IB\|=\|IC\|=r$

$\left.\begin{array}{l}IC'\perp AB\\ IA'\perp BC\end{array}\right\}\Rightarrow IA'B'C'$ inscribable quadrilateral

$m(A'IC')=180-\hat{B}\Longrightarrow\sin\left(\widehat{A'IC'}\right)=\sin\hat{B}$

Similarly, $\widehat{A'IB'}=\sin C$ and $\widehat{C'IB'}=\sin A$.

$$\sigma[B'C'I] = \frac{r^2 \sin A}{2}$$

$$\frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{a}{2R}$$

$$\sigma[B'C'I] = \frac{r^2 \frac{a}{2R}}{2} = \frac{r^2 a}{4R}$$

In the same way,

$$\sigma[A'B'I] = \frac{r^2}{4R}(a+b+c)$$

$$\sigma[ABC] = s = rp$$

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = \frac{r^2(a+b+c)}{4R} \Rightarrow \frac{1}{rp} = \frac{r}{2R}$$

Solution to Problem 62.

$$0 < A \Rightarrow < \tfrac{A}{2} < \tfrac{\pi}{2} \Rightarrow \sin\tfrac{A}{2} > 0\ ;$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} \Longrightarrow \left.\begin{aligned} \sin^2\frac{A}{2} &= \frac{1-\cos A}{2} \\ \cos A &= \frac{b^2+c^2-a^2}{2bc} \end{aligned}\right\}$$

$$\Longrightarrow \sin^2\frac{A}{2} = \frac{1-\frac{b^2-a^2+c^2}{2bc}}{2} = \frac{a^2-(b-c)^2}{4bc} \le \frac{a^2}{4bc} \Longrightarrow \sin\frac{A}{2} \le \frac{a}{2\sqrt{bc}}.$$

Solution to Problem 63.

$$C = \pi - (A+B)$$

$$\left.\begin{aligned} S &= \tfrac{a^2 \sin B \sin C}{2\sin A} \\ A, B, C &\text{ are known} \end{aligned}\right\} \Longrightarrow \text{We find } a.$$

$\frac{a}{\sin a} = \frac{b}{\sin b} \Longrightarrow b = \frac{a \sin B}{\sin A}$. In the same way, we find c.

Solution to Problem 64.

$$A = \arccos\frac{4}{5} \Rightarrow \cos A = \frac{4}{5} \Rightarrow \sin A = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$$

$$a^2 + b^2 + c^2 - 2bc\cos A \Rightarrow 169 = 841 - 2bc\frac{4}{5} \Rightarrow \frac{8bc}{5} = 841 - 169 \Rightarrow$$

$$\Rightarrow bc = 420$$

$$\left.\begin{array}{r} b^2+c^2=841 \\ bc=420 \end{array}\right\} \Rightarrow \begin{cases} b=21 \\ c=20 \end{cases} \text{or } \begin{cases} b=20 \\ c=21 \end{cases}$$

$$\frac{a}{\sin A}=\frac{b}{\sin B} \Rightarrow \sin B=\frac{b\sin A}{a}$$

We find B.

$$C=180^\circ-(A+B)$$

$$\sin B=\frac{21\cdot\frac{3}{5}}{13}=\frac{63}{65} \Rightarrow B=\arcsin\frac{63}{65}$$

$$C=180^\circ-\left(\arccos\frac{4}{5}+\arcsin\frac{63}{65}\right)=180^\circ-\left(\arcsin\frac{3}{5}+\arcsin\frac{63}{65}\right)$$

We find the sum.

Or

$$\sin B=\frac{20\cdot\frac{3}{5}}{13}=\frac{12}{13} \Rightarrow B=\arcsin\frac{12}{13}$$

$$C=180^\circ-\left(\underbrace{\arcsin\frac{3}{5}}_{\alpha}+\underbrace{\arcsin\frac{12}{13}}_{\beta}\right)$$

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha=\frac{3}{5}\cdot\frac{5}{13}+\frac{12}{13}\cdot\frac{4}{5}=\frac{63}{65} \Rightarrow \alpha+\beta=\arcsin\frac{63}{65}$$

$$C=180^\circ-\arcsin\frac{63}{65}$$

Solution to Problem 65.

$$R=8r \Rightarrow \frac{r}{R}=\frac{1}{8}$$

We already know that

$$\frac{r}{R}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{1}{8}=4\sin\frac{A}{2}\frac{\cos\frac{B-C}{2}-\cos\frac{B+C}{2}}{2} \Rightarrow$$

$$\Rightarrow \frac{1}{8}=2\sin\frac{A}{2}\left(\cos\frac{2\pi}{6}-\cos\frac{180^\circ-A}{2}\right) \Rightarrow \frac{1}{8}=\sin\frac{A}{2}\left(\cos\frac{\pi}{3}-\sin\frac{A}{2}\right) \Rightarrow$$

$$\Rightarrow \frac{1}{8}=2\sin\frac{A}{2}\left(\frac{1}{2}-\sin\frac{A}{2}\right)$$

We write $\sin\frac{A}{2}=t$. We have

$$\frac{1}{8}=2t\left(\frac{1}{2}-t\right)=t-2t^2$$

$$1 = 8t - 16t^2 \Rightarrow 16t^2 - 8t + 1 = 0 \Rightarrow$$

$$4t - 1^2 = 0 \Rightarrow t = \frac{1}{4}$$

$$\sin\frac{A}{2} = \frac{1}{4}; \ \cos\frac{A}{2} = \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{\sqrt{15}}{8} \Rightarrow A = \arcsin\frac{\sqrt{15}}{8}$$

$$\begin{cases} B + C = \pi - \arcsin\frac{\sqrt{15}}{8} \\ B - C = \frac{2\pi}{3} \end{cases}$$

From this system we find B and C.

$$2B = \frac{5\pi}{3} - \arcsin\frac{\sqrt{15}}{8}$$

$$B = \frac{5\pi}{6} - \frac{1}{2}\arcsin\frac{\sqrt{15}}{8}$$

$$C = B - \frac{2\pi}{3} = \frac{5\pi}{6} - \frac{2\pi}{3} - \frac{1}{2}\arcsin\sqrt{15}8 = \frac{\pi}{6} - \frac{1}{2}\arcsin\frac{\sqrt{15}}{8}$$

Other Problems in Geometry and Trigonometry (10th grade)

66. Show that a convex polygon can't have more than three acute angles.

Solution to Problem 66

67. Let ABC be a triangle. Find the locus of points $M \in (ABC)$, for which $\sigma[ABM] = \sigma[ACM]$.

Solution to Problem 67

68. A convex quadrilateral $ABCD$ is given. Find the locus of points $M \in int. ABCD$, for which $\sigma[MBCD] = \sigma[MBAD]$.

Solution to Problem 68

69. Determine a line MN, parallel to the bases of a trapezoid $ABCD$ ($M \in |AD|, N \in |BC|$) such that the difference of the areas of $[ABNM]$ and $[MNCD]$ to be equal to a given number.

Solution to Problem 69

70. On the sides of ΔABC we take the points D, E, F such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2$. Find the ratio of the areas of triangles DEF and ABC.

Solution to Problem 70

71. Consider the equilateral triangle ABC and the disk $\left[C\left(O, \frac{a}{3}\right)\right]$, where O is the orthocenter of the triangle and $a = \|AB\|$. Determine the area $[ABC] - \left[C\left(O, \frac{a}{3}\right)\right]$.

Solution to Problem 71

72. Show that in any triangle ABC we have:

a. $1+\cos A\cos(B-C)=\frac{b^2+c^2}{4R^2}$;

b. $(b^2+c^2=a^2)\tan A=4S$;

c. $\frac{b+c}{2c\cos\frac{A}{2}}=\frac{\sin\left(\frac{A}{2}+C\right)}{\sin(A+B)}$;

d. $p=r\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot C2\right)$;

e. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{p}{r}$.

Solution to Problem 72

73. If H is the orthocenter of triangle ABC, show that:

a. $\|AH\|=2R\cos A$;

b. $a\|AH\|+b\|BH\|+c\|CH\|=4S$.

Solution to Problem 73

74. If O is the orthocenter of the circumscribed circle of triangle ABC and I is the center of the inscribed circle, show that $\|OI\|^2=R(R-2r)$.

Solution to Problem 74

75. Show that in any triangle ABC we have: $\cos^2\frac{B-C}{2}\geq\frac{2r}{R}$.

Solution to Problem 75

76. Find $z^n+\frac{1}{z^n}$ knowing that $z+\frac{1}{z}=2\sin\alpha$.

Solution to Problem 76

77. Solve the equation: $(z+1)^n-(z-1)^n=0$.

Solution to Problem 77

78. Prove that if $z<\frac{1}{2}$ then $|(1+i)z^3+iz|\leq\frac{3}{4}$.

Solution to Problem 78

79. One gives the lines d and d'. Show that through each point in the space passes a perpendicular line to d and d'.

Solution to Problem 79

80. There are given the lines d and d', which are not in the same plane, and the points $A \in d, B \in d'$. Find the locus of points M for which $\text{pr}_d M = A$ and $\text{pr}_{d'} M = B$.

Solution to Problem 80

81. Find the locus of the points inside a trihedral angle $\widehat{abc}$ equally distant from the edges of a, b, c.

Solution to Problem 81

82. Construct a line which intersects two given lines and which is perpendicular to another given line.

Solution to Problem 82

83. One gives the points A and B located on the same side of a plane; find in this plane the point for which the sum of its distances to A and B is minimal.

Solution to Problem 83

84. Through a line draw a plane onto which the projections of two lines to be parallel.

Solution to Problem 84

85. Consider a tetrahedron $[ABCD]$ and centroids L, M, N of triangles BCD, CAD, ABD.

   a. Show that $(ABC) \parallel (LMN)$;
   b. Find the ratio $\frac{\sigma[ABC]}{\sigma[LMN]}$.

Solution to Problem 85

86. Consider a cube $[ABCDA'B'C'D']$. The point A is projected onto $A'B, A'C, A'D$ respectively in A_1, A_2, A_3. Show that:
    a. $A'C \perp (A_1A_2A_3)$;
    b. $AA_1 \perp A_1A_2$, $AA_3 \perp A_3A_2$;
    c. $AA_1A_2A_3$ is an inscribable quadrilateral.

Solution to Problem 86

87. Consider the right triangles BAC and ABD $\left(m(\widehat{BAC})\right) = m((\widehat{ABD}) = 90^0)$ located on perpendicular planes M and N, being midpoints of segments $[AB], [CD]$. Show that $MN \perp CD$.

Solution to Problem 87

88. Prove that the bisector half-plane of a dihedral angle inside a tetrahedron divides the opposite edge in proportional segments with the areas of the adjacent faces.

Solution to Problem 88

89. Let A be a vertex of a regular tetrahedron and P, Q two points on its surface. Show that $m(\widehat{PAQ}) \leq 60^0$.

Solution to Problem 89

90. Show that the sum of the measures of the dihedral angles of a tetrahedron is bigger than 360°.

Solution to Problem 90

91. Consider lines d_1, d_2 contained in a plane α and a line AB which intersects plane α at point C. A variable line, included in α and passing through C all d_1, d_2 respectively at MN. Find the locus of the intersection $AM \cap BN$. In which case is the locus an empty set?

Solution to Problem 91

92. A plane α intersects sides $[AB], [BC], [CD], [DA]$ of a tetrahedron $[ABCD]$ at points L, M, N, P. Prove that $\|AL\| \cdot \|BM\| \cdot \|CN\| \cdot \|PD\| = \|BL\| \cdot \|CM\| \cdot \|DN\| \cdot \|AP\|$.

Solution to Problem 92

93. From a point A located outside a plane α, we draw the perpendicular line AO, $O \in \alpha$, and we take $B, C \in \alpha$. Let H, H_1 be the orthocenters of triangles ABC, OBC; AD and BE heights in triangle ABC; and BE_1 height in triangle OBC. Show that:

a. $HH_1 \perp (ABC)$;

b. $\left\|\frac{OA}{AD}\right\| \cdot \left\|\frac{DH_1}{H_1B}\right\| \cdot \left\|\frac{BE}{EE_1}\right\| = 1$.

Solution to Problem 93

94. Being given a tetrahedron $[ABCD]$ where $AB \perp CD$ and $AC \perp BD$, show that:

a. $\|AB\|^2 + \|CD\|^2 = \|BC\|^2 + \|AD\|^2 = \|CA\|^2 + \|BD\|^2$;

b. The midpoints of the 6 edges are located on a sphere.

Solution to Problem 94

95. It is given a triangular prism $[ABCA'B'C']$ which has square lateral faces. Let M be a mobile point $[AB']$, N the projection of M onto (BCC') and $A"$ the midpoint of $[B'C"]$. Show that $A'N$ and $MA"$ intersect in a point P and find the locus of P.

Solution to Problem 95

96. We have the tetrahedron $[ABCD]$ and let G be the centroid of triangle BCD. Show that if $M \in AG$ then $v[MGBC] = v[MGCD] = v[MGDB]$.

Solution to Problem 96

97. Consider point $M \in$ the interior of a trirectangular tetrahedron with its vertex in O. Draw through M a plane which intersects the edges of the

respective tetrahedron in points A, B, C so that M is the orthocenter of ΔABC.

Solution to Problem 97

98. A pile of sand has as bases two rectangles located in parallel planes and trapezoid side faces. Find the volume of the pile, knowing the dimensions a', b' of the small base, a, b of the larger base, and h the distance between the two bases.

Solution to Problem 98

99. A pyramid frustum is given, with its height h and the areas of the bases B and b. Unite any point σ of the larger base with the vertices A, B, A', B' of a side face. Show that $\upsilon[OA'B'A] = \frac{\sqrt{6}}{\sqrt{B}}\upsilon[OABB']$.

Solution to Problem 99

100. A triangular prism is circumscribed to a circle of radius R. Find the area and the volume of the prism.

Solution to Problem 100

101. A right triangle, with its legs b and c and the hypotenuse a, revolves by turns around the hypotenuse and the two legs, V_1, V_2, V_3; S_1, S_2, S_3 being the volumes, respectively the lateral areas of the three formed shapes, show that:

a. $\frac{1}{V_1^2} = \frac{1}{V_2^2} = \frac{1}{V_3^2};$

b. $\frac{S_2}{S_3} + \frac{S_3}{S_2} = \frac{S_2+S_3}{S_1}$.

Solution to Problem 101

102. A factory chimney has the shape of a cone frustum and $10m$ height, the bases of the cone frustum have external lengths of 3,14m and 1,57m, and the wall is 18cm thick. Calculate the volume of the chimney.

Solution to Problem 102

103. A regular pyramid, with its base a square and the angle from the peak of a side face of measure α is inscribed in a sphere of radius R. Find:

a. the volume of the inscribed pyramid;

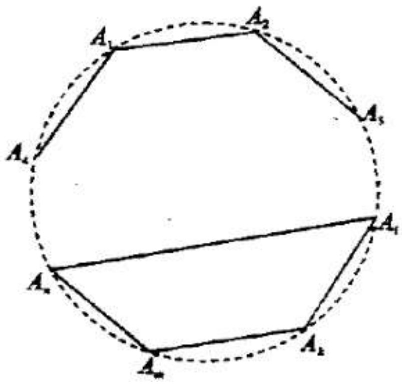
b. the lateral and total area of the pyramid;

c. the value α when the height of the pyramid is equal to the radius of the sphere.

Solution to Problem 103

Solutions

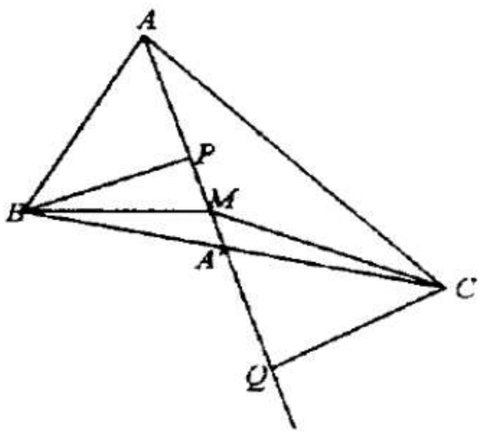
Solution to Problem 66.



Let $A_1, A_2 \dots A_n$ the vertices of the convex polygon. Let's assume that it has four acute angles. The vertices of these angles form a convex quadrilateral $A_l A_k A_m A_n$. Due to the fact that the polygon is convex, the segments $|A_l A_k|$, $|A_k A_m|$, $|A_m A_n|$, $|A_n A_l|$ are inside the initial polygon. We find that the angles of the quadrilateral are acute, which is absurd, because their sum is 360°.

Another solution: We assume that $A_l A_k A_m A_n$ is a convex polygon with all its angles acute ⟹ the sum of the external angles is bigger than 360°, which is absurd (the sum of the measures of the external angles of a convex polygon is 360°).

Solution to Problem 67.



Let $|AA'|$ be the median from A and $CQ \perp AA'$, $BP \perp AA'$.

$\Delta BA'P \equiv CA'Q$ because:

$$\begin{cases} \widehat{PBC} \equiv \widehat{BCQ} & \text{alternate interior} \\ \widehat{PA'B} \equiv \widehat{CA'Q} & \text{vertical angles} \\ BA' \equiv A'C & \end{cases}$$

$\Longrightarrow ||BP|| = ||QC||$ and by its construction $BP \perp AA', CQ \perp AA'$.

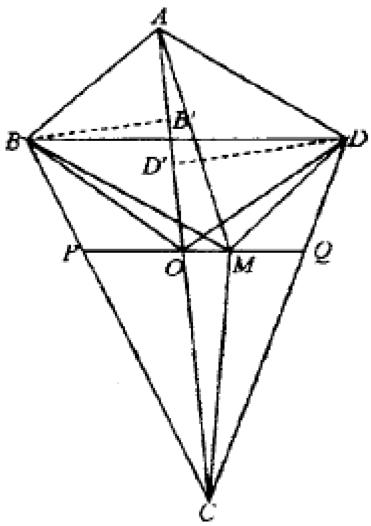
The desired locus is median $|AA'|$. Indeed, for any $M \in |AA'|$ we have $\sigma[ABM] = \sigma[ACM]$, because triangles ABM and ACM have a common side $|AM|$ and its corresponding height equal $||BP|| = ||QC||$.

Vice-versa. If $\sigma[ABM] = \sigma[AC'M]$, let's prove that $M \in |AA'|$.

Indeed: $\sigma[ABM]\,\sigma[ACM] \Rightarrow d\,(B, AM) = d(C, AM)$, because $|AM|$ is a common side, $d\,(B, AM) = ||BP||$ and $d(C, AM) = ||CQ||$ and both are perpendicular to $AM \Longrightarrow PBQC$ is a parallelogram, the points P, M, Q are collinear (P, Q the feet of the perpendicular lines from B and C to AM).

In parallelogram $PBQC$ we have $|PQ|$ and $|BC|$ diagonals $\Longrightarrow AM$ passes through the middle of $|BC|$, so $M \in |AA'|$, the median from A.

Solution to Problem 68.



Let O be the midpoint of diagonal $|AC| \Longrightarrow ||AO|| = ||OC||$.

$\sigma[AOD] = \sigma[COD]$ (1)

Because $\begin{cases} ||AO|| = ||OC|| \\ ||OD'|| \text{ common height} \end{cases}$

$\sigma[AOB] = \sigma[COB]$ (2)

the same reasons; we add up (1) and (2) $\Longrightarrow$

$\sigma[ADOB] = \sigma[DCBO]$ (3),

so O is a point of the desired locus.

We construct through O a parallel to BD until it cuts sides $|BC|$ and $|DC|$ at P respectively Q. The desired locus is $|PQ|$.

Indeed $(\forall)M \in |PQ|$ we have:

$$\sigma[MBAD] = \sigma[ABD] + \sigma[BDM] = \sigma[ABD] + \sigma[BOD] = \sigma[ABOD].$$

$\sigma[BDO] = \sigma[BDM]$ because M and Q belongs to a parallel to BD.

$$\sigma[BCDM] = \sigma[PQC] + \sigma[PMB] + \sigma[MQD] = \sigma[PQC] + \sigma[PBO] + \sigma[OMB] + \sigma[MQD] \stackrel{*}{=}$$

$$\sigma[OMB] = \sigma[OMD]$$

$$\stackrel{*}{=} \sigma[PQC] + \sigma[PBO] + \sigma[OMD] + \sigma[MQD] = \sigma[OBCD]$$

$B, D \in$ a parallel to OM.

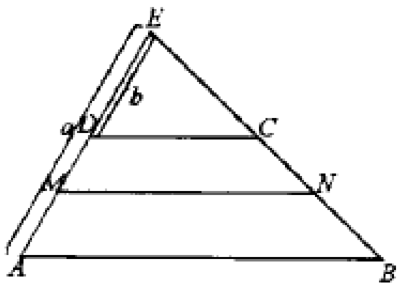
So $\left.\begin{array}{c}\sigma[MBAD] = \sigma[ABOD] \\ \text{and } \sigma[BCDM] = \sigma[OBCD] \\ \text{and from (3)}\end{array}\right\} \Longrightarrow \sigma[MBAD] = \sigma[BCDM].$

Vice-versa: If $\sigma[MBCD] = \sigma[MBAD]$, let's prove that $M \in$ parallel line through O to BD. Indeed:

$$\left.\begin{array}{c}\sigma[BCDM] = \sigma[MBAD] \\ \text{and because } \sigma[BCDM] + \sigma[MBAD] = \sigma[ABCD]\end{array}\right\} \Longrightarrow \sigma[MBCD] = \sigma[MBAD] = \frac{\sigma[ABCD]}{2} \text{ (2).}$$

So, from (1) and (2) $\Longrightarrow \sigma[MBAD] = \sigma[ABOD] \Longrightarrow \sigma[ABD] + \sigma[BDO] = \sigma[ABD] + \sigma[BDM] \Longrightarrow \sigma[BDM] \Longrightarrow M$ and O are on a parallel to BD.

Solution to Problem 69.



We write $||EA|| = a$ and $||ED|| = b, ||EM|| = x$.

$$\frac{\sigma[EMN]}{\sigma[EDC]} = \frac{x^2}{b^2} \Rightarrow \frac{[MNCD]}{\sigma[EDC]} = \frac{x^2 - b^2}{b^2} \qquad (2)$$

$$\frac{\sigma[EAB]}{\sigma[EDC]} = \frac{a^2}{b^2} \Rightarrow \frac{[ABCD]}{\sigma[EDC]} = \frac{a^2 - b^2}{b^2} \qquad (3)$$

We subtract (2) from (3)

$$\Rightarrow \frac{\sigma[ABCD]-\sigma[MNCD]}{\sigma[ECD]}=$$

$$=\frac{a^2-x^2}{b^2}\Rightarrow\frac{\sigma[ABNM]}{\sigma[ECD]}=\frac{a^2-x^2}{b^2} \qquad (4)$$

We subtract (2) from (4)

$$\left.\begin{array}{l}\Rightarrow \dfrac{\sigma[ABNM]-\sigma[MNCD]}{\sigma[EDC]}=\dfrac{a^2-x^2-x^2-b^2}{b^2}\\ \text{from the hypothesis}\quad \sigma[ABNM]-\sigma[MNCD]=k\end{array}\right\}\Rightarrow\frac{k}{\sigma[EDC]}=$$

$$=\frac{a^2+b^2-2x^2}{b^2}\Longrightarrow\frac{kb^2}{\sigma[ECD]}-a^2-b^2=-2x^2\Rightarrow x^2=\frac{(a^2+b^2)\sigma[ECD]-kb^2}{\sigma[ECD]}$$

From the relation (3), by writing $[ABCD]-S\Longrightarrow\sigma[ECD]=\frac{Sb^2}{a^2-b^2}$.

We substitute this in the relation of x^2 and we obtain:

$$x^2=a^2+b^2-kb^2\frac{a^2-b^2}{Sb^2}\Rightarrow x^2=\frac{(a^2+b^2)s-k(a^2-b^2)}{S}\Rightarrow$$

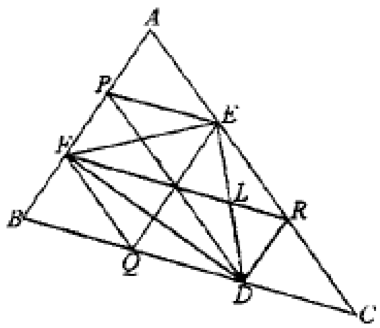
$$\left.\begin{array}{l}\Rightarrow\quad x^2=\dfrac{(s-k)a^2+(s+k)b^2}{S}\\ \qquad x=\|EM\end{array}\right\}\Rightarrow\|EM\|=\sqrt{\frac{(s-k)a^2+(s+k)b^2}{S}}$$

and taking into consideration that $||EM||=||DM||+b$, we have

$$\|DM\|=\sqrt{\frac{(s-k)a^2+(s+k)b^2}{S}}$$

so we have the position of point M on the segment $|DA|$ (but it was sufficient to find the distance $||EM||$).

Solution to Problem 70.



We remark from its construction that $EQ||AB||RD$, more than that, they are equidistant parallel lines. Similarly, EQ, PD, AC and AB, EQ, RD are also equidistant parallel lines.

$(\|AP\| = \|PF\| = \|FB\|; \|AE\| = \|ER\| = \|RC\|,$

$\|BQ\| = \|QD\| = \|DC\|).$

We write $\sigma[BFQ] = S$.

Based on the following properties:

- two triangles have equal areas if they have equal bases and the same height;
- two triangles have equal areas if they have the same base and the third peak on a parallel line to the base,

we have:

$$\sigma[ABC] = \sigma[AFE] + \sigma[FER] + \sigma[FBQ] + \sigma[FRD] + \sigma[DRC] = 9S$$

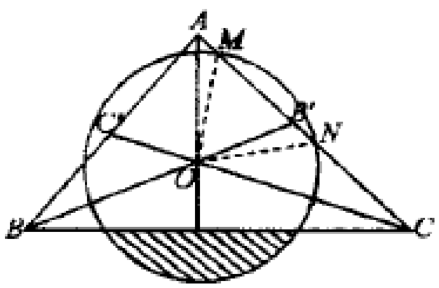
$$\left.\begin{array}{l} \sigma[FEL] = \sigma[FER] - \sigma[ELR] = 2S - \sigma[ELR] \\ \sigma[FDL] = \sigma[FRD] - \sigma[RLD] = 2S - \sigma[RLD] \end{array}\right\} \xRightarrow{\text{by addition}}$$

$$\Rightarrow \sigma[DEF] = 4S - (\sigma[ELR] + \sigma[RLD]) = 4S - S = 3S.$$

So $\frac{\sigma[DEF]}{\sigma[ABC]} = \frac{3S}{9S} = \frac{1}{3}$.

(If necessary the areas S can be arranged).

Solution to Problem 71.



$\|OB\| = \frac{a\sqrt{3}}{6}$ (BB' median)

In $\Delta MOB'$:

$$\cos \widehat{MOB'} = \frac{\frac{a\sqrt{3}}{6}}{\frac{a}{3}} = \frac{\sqrt{3}}{2} \Rightarrow \mu(\widehat{MOB'}) = \frac{\pi}{6}.$$

So $(\widehat{MON}) = \frac{\pi}{3}$.

We mark with Σ the disk surface bordered by a side of the triangle outside the triangle.

$$\sigma[\Sigma] = \sigma[\text{circle sector } MON] - \sigma[MON]$$
$$= \frac{\pi a^2}{9\cdot 6} - \frac{a^2}{9\cdot 2}\sin 60^0 = \frac{\pi a^2}{9\cdot 6} - \frac{a^2\sqrt{3}}{4\cdot 9} = \frac{a^2}{18}\cdot\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right).$$

If through the disk area we subtract three times $\sigma[\Sigma]$, we will find the area of the disk fraction from the interior of ABC. So the area of the disk surface inside ABC is:

$$\frac{\pi a^2}{9} - 3\frac{a^2}{18}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi a^2}{9} - \frac{\pi a^2}{18} + \frac{a^2\sqrt{3}}{12} = \frac{\pi a^2}{18} + \frac{a^2\sqrt{3}}{12}.$$

The desired area is obtained by subtracting the calculated area form $\sigma[ABC]$.

So:

$$\frac{a^2\sqrt{3}}{4} - \frac{\pi a^2}{18} - \frac{2a^2\sqrt{3}}{12} = \frac{2a^2\sqrt{3}}{12} - \frac{\pi a^2}{18} = \frac{a^2\sqrt{3}}{6} - \frac{\pi a^2}{18} = \frac{a^2}{18}(3\sqrt{3} - \pi).$$

Solution to Problem 72.

a. $1 + \cos A \cdot \cos(B - C) = \frac{b^2+c^2}{4R^2}$

$$1 + \cos A\cos(B - C) = 1 + \cos[\pi - (C + B)]\cos(B - C) = 1 - \cos(B + C)\cdot\cos(B - C) =$$
$$1 - \frac{1}{2}[\cos 2B + \cos 2C] = 1 - \frac{1}{2}[2\cos^2 B - 1 + 2\cos^2 C - 1] = 2 - \cos^2 B - \cos^2 C = \sin^2 B +$$
$$+ \sin^2 C \overset{\text{din } T.\ \sin}{=} \frac{b^2}{4R^2} + \frac{C^2}{4R^2} = \frac{b^2 + c^2}{4R^2}.$$

b. We prove that $\tan A = \frac{4S}{b^2+c^2-a^2}$.

$$\text{tg } A = \frac{\sin A}{\cos A} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2} - 1} = \frac{2\sqrt{\frac{p(p-a)(p-b)(p-c)}{(bc)^2}}}{2\frac{p(p-a)}{bc} - 1} = \frac{2S}{bc\left(\frac{2p(p-a) - bc}{bc}\right)} =$$
$$= \frac{2S}{bc\left(\frac{2p(p-a) - bc}{bc}\right)} = \frac{2S}{2\frac{a+b+c}{2}\cdot\frac{b+c-a}{2} - bc} =$$
$$\frac{4S}{ab + ac - a^2 + b^2 + bc - ba + bc + c^2 - ac - 2bc} = \frac{4S}{b^2 + c^2 - a^2}.$$

c) $$\frac{b+c}{2c\cos\frac{A}{2}} = \frac{\sin(\frac{A}{2} + C)}{\sin(A + B)} \overset{A+B=\pi-C}{\Longrightarrow} \frac{b+c}{2c\cos\frac{A}{2}} = \frac{\sin(\frac{A}{2} + C)}{\sin C} \Longrightarrow$$

$$\Longrightarrow \frac{b+c}{2c\cos\frac{A}{2}} = \frac{\sin A2\cos C + \sin C\cos\frac{A}{2}}{\sin C} \Longrightarrow (b+c)\sin C = 2C\sin\frac{A}{2}\cos\frac{A}{2}\cos C +$$
$$2c\sin C\cos^2\frac{A}{2} \Longrightarrow (b+c)\sin C = c\sin A\cos C + 2c\sin C\cos^2\frac{A}{2}$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R\Longrightarrow(b+c)\frac{ac}{2k}\cos C=\frac{2c^2\cos^2\dfrac{A}{2}}{2R}\Longrightarrow$$

$$\Longrightarrow b+c=a\cos C+2\cos^2\frac{A}{2}\Longrightarrow b+c=a\frac{a^2+b^2-c^2}{2ab}+2c\frac{p(p-a)}{bc}\Longrightarrow b+c=$$

$$=\frac{a^2+b^2-c^2}{2b}+\frac{2p(p-a)}{b}\Longrightarrow 2b^2+2bc=a^2+b^2-c^2+(a+b+c)(c+b-a)\Longrightarrow$$

$\Rightarrow$ All terms reduce.

d. Let's prove that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot C2=\frac{p}{r}$.

Indeed

$$\operatorname{ctg}\frac{A}{2}+\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}=\operatorname{ctg}\frac{A}{2}\cdot\operatorname{ctg}\frac{B}{2}\cdot\operatorname{ctg}\frac{C}{2}=\sqrt{\frac{p(p-a)}{(p-b)(p-c)}}\cdot$$

$$\cdot\sqrt{\frac{p(p-b)}{(p-a)(p-c)}}\cdot\sqrt{\frac{p(p-c)}{(p-a)(p-b)}}=p\sqrt{\frac{p(p-a)(p-b)(p-c)}{(p-a)^2(p-b)^2(p-c)^2}}=\frac{p^2}{S}\overset{S=pr}{=}\frac{p^2}{pr}=\frac{p}{r}.$$

We now have to prove that:

$$\operatorname{ctg}\frac{A}{2}+\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}=\operatorname{ctg}\frac{A}{2}\cdot\operatorname{ctg}\frac{B}{2}\cdot\operatorname{ctg}\frac{C}{2}\Longleftrightarrow$$

$$\frac{1}{\operatorname{tg}\frac{A}{2}}+\frac{1}{\operatorname{tg}\frac{B}{2}}+\operatorname{tg}\frac{A+B}{2}=\frac{1}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}\cdot\operatorname{tg}\frac{A+B}{2}\Leftrightarrow\frac{\operatorname{tg}\frac{A}{2}+\operatorname{tg}\frac{B}{2}}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}+$$

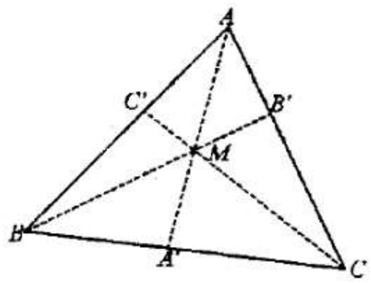
$$+\frac{\operatorname{tg}\frac{A}{2}+\operatorname{tg}\frac{B}{2}}{1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}=\frac{1}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}\cdot\frac{\operatorname{tg}\frac{A}{2}+\operatorname{tg}\frac{B}{2}}{1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}\Longleftrightarrow\frac{1}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}+\frac{1}{1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}=$$

$$\frac{1}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}\cdot(1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2})}\Longleftrightarrow\frac{1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}+\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}\cdot(1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2})}=$$

$$=\frac{1}{\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2}\cdot(1-\operatorname{tg}\frac{A}{2}\cdot\operatorname{tg}\frac{B}{2})}\qquad(a)$$

q.e.d.

Solution to Problem 73.



a. In triangle ABB': $\|AB'\| = c\cos A$

In triangle AHB':

$$\cos \widehat{HAB'} = \|AB'\| : \|AH\| \Rightarrow \|AH\| =$$

$$= \frac{\|AB'\|}{\cos \widehat{HAB'}} = \frac{\|AB'\|}{\cos\left(\frac{\pi}{2} - c\right)} = \frac{c\cos A}{\sin C} \overset{T.\sin}{=} \frac{c\cos A}{\frac{c}{2R}} =$$

$$= 2R\cos A \Rightarrow \|AH\| = 2R\cos A.$$

b) $a\|AH\| + b\|BH\| + c\|CH\| = 2R(a\cos A + b\cos B + c\cos C) \overset{T.\sin}{=} 4R^2(\sin A\cos A +$

$+ \sin B\cos B + \sin C\cos C) = 2R^2(\sin 2A + \sin 2B + \sin 2C).$

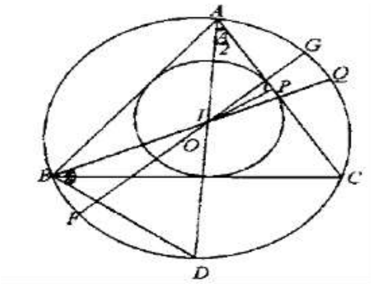
We used:

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

$$\sin 2A + \sin 2B + \sin 2C = \underbrace{2\sin(A+B)}_{\sin C}\cos(A-B) + 2\sin C\cos C =$$

$$= 2\sin C[\cos(A-B) - \cos(A+B)] = 2\sin C \cdot 2\sin A\sin B = 4\sin A\sin B\sin C.$$

Solution to Problem 74.



Using the power of point I in relation to circle $C(O,R)$

$\Rightarrow \|IG\| \cdot \|IF\| = \|AI\| \cdot \|ID\| \quad (1)$

$$\|IG\| \cdot \|IF\| = (R - \|OI\|)(\|R + \|OI\|) \Rightarrow$$

$\|IG\| \cdot \|IF\| = R^2 - \|OI\|^2.$

Taking into consideration (1), we have $\|IA\| \cdot \|ID\| = R^2 - \|OI\|^2$.

We now find the distances ||IA|| and ||ID||

In triangle ΔIAP,

$$\|IA\| = \frac{r}{\sin\frac{A}{2}} \qquad (2)$$

We also find $\|ID\|$: $\mu(\widehat{BID}) = \mu(\widehat{DBI})$ have the same measure, more exactly:

$$\mu(\widehat{BID}) = \frac{m(\widehat{BD}) + m(\widehat{AQ})}{2} = \frac{m(\hat{A}) + m(\hat{B})}{2}$$

$$m\left(\frac{\widehat{DCQ}}{2}\right) = m(\widehat{DBI}) \text{ Deci } \|ID\| = \|BD\|. \qquad (3)$$

In ΔABD according to the law of sine, we have:

$$\frac{\|BD\|}{\sin\frac{A}{2}} = 2R \Rightarrow \|BD\| = 2R\sin\frac{A}{2}.$$

So taking into consideration (3),

$$\|ID\| = 2R\sin\frac{A}{2}. \qquad (4)$$

Returning to the relation $\|IA\| \cdot \|ID\| = R^2 - \|OI\|^2$. with (2) and (4) we have:

$$\frac{r}{\sin\frac{A}{2}} \cdot 2R\sin\frac{A}{2} =$$

$$R^2 - \|IO\|^2 \Rightarrow \|IO\|^2 = R^2 - 2Rr \Rightarrow \|IO\|^2 = R(R-2r).$$

Solution to Problem 75.

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \Rightarrow r = 2R\sin\frac{A}{2}(\cos\frac{B-C}{2} - \underbrace{\cos\frac{B+C}{2}}_{\sin\frac{A}{2}}) \Rightarrow$$

$$r = 2R\sin\frac{A}{2}\cos\frac{B-C}{2} - 2R\sin^2\frac{A}{2} \Rightarrow 2R\sin^2\frac{A}{2} - 2R\sin\frac{B}{2}\cdot\cos\frac{B-C}{2} + r \geq 0 \Rightarrow$$

$$\Rightarrow \Delta \geq 0 \Rightarrow 4R^2\cos^2\frac{B-C}{2} - 8Rr \geq 0 \Rightarrow R^2\cos^2\frac{B-C}{2} - 2Rr \geq 0 \Rightarrow \cos^2\frac{B-C}{2} \geq \frac{2r}{R}$$

Note. We will have to show that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}.$$

Indeed:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}\cdot\frac{(p-a)(p-c)}{ac}\cdot\frac{(p-a)(p-b)}{ab}} =$$

$$= \frac{p(p-a)(p-b)(p-c)}{pabc} = \frac{S^2}{p4RS} = \frac{pr}{p4R} = \frac{r}{4R}$$

(by Heron's formula).

Solution to Problem 76.

$$z+\frac{1}{z}=2\sin a \Rightarrow z^2-2(\sin a)z+1=0 \Rightarrow z_{1,2}=\frac{\sin \pm\sqrt{\sin^2 a-1}}{1}$$

$$\Rightarrow z_{1,2}=\sin a \pm \sqrt{-\cos^2 a} \Rightarrow z_{1,2}=\sin a \pm i\cos a$$

So:

$z_1=\sin a+i\cos a$

$z_2=\sin a-i\cos a=\bar{z}_1$

We calculate for z_1 and z_2:

$$z_1^n+\frac{1}{z_1^n}=z_1^n+\left(\frac{1}{z_1}\right)^n=z_1^n+z_2^n, \qquad z^n+\frac{1}{z^n}$$

so $z^n+\frac{1}{z^n}$ takes the same value for z_1 and for z_2 and it is enough if we calculate it for z_1.

$$z_1^n+\frac{1}{z_1^n}=(\sin a+i\cos a)^n+\frac{1}{(\sin a+i\cos a)}=[\cos\left(\frac{\pi}{2}-a\right)+i\sin\left(\frac{\pi}{2}-a\right)]+$$

$$+\frac{1}{[\cos\left(\frac{\pi}{2}-a\right)+i\sin\left(\frac{\pi}{2}-a\right)]^n}=\cos[n\left(\frac{\pi}{2}-a\right)]+i\sin[n\left(\frac{\pi}{2}-a\right)]+$$

$$+\cos[n\left(\frac{\pi}{2}-a\right)]-i\sin[n\left(\frac{\pi}{2}-a\right)]=2\cos[n\left(\frac{\pi}{2}-a\right)]\overset{\cos.\ par}{=}2\cos\left(na-\frac{a\pi}{2}\right).$$

Analogously:

$$z_2^n+\bar{z}_2^n=2\cos[n\left(a-\frac{\pi}{2}\right)].$$

Solution to Problem 77.

$$(z+1)^n-(z-1)^n=a \Rightarrow (z+1)^n=(z-1)^n \Rightarrow \left(\frac{z+1}{z-1}\right)^n=1 \Rightarrow$$

$$\Rightarrow \frac{z+1}{z-1}=\sqrt[n]{1} \Rightarrow \frac{z+1}{z-1}=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n} \Rightarrow$$

$$\Rightarrow z+1=\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)z-\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right) \Rightarrow$$

$$\Rightarrow \left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}-1\right)z=\left(1+\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right) \Rightarrow$$

$$\Rightarrow \left(-2\sin^2\frac{k\pi}{n}+2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}\right)z=2\cos^2\frac{k\pi}{n}+2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n} \Rightarrow$$

$$\Rightarrow z=\frac{2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)}{-2\sin^2\frac{k\pi}{n}+2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}}=$$

(we substitute -1 with i^2 at denominator)

$$=\frac{2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)}{2i\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)}=\frac{\cos k\pi}{i\sin\frac{k\pi}{n}}=\frac{i}{i^2}\operatorname{ctg}\frac{k\pi}{n}=-i\operatorname{ctg}\frac{k\pi}{n}.$$

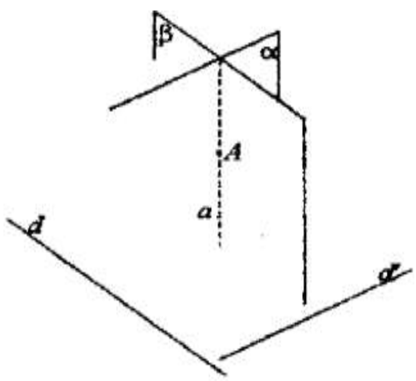
Solution to Problem 78.

$$|(1+i)z^3+iz| \leq |(1+i)z^3|+|iz| = |1+i|\cdot|z^3|+|i|\cdot|z| =$$

$$\overset{\substack{|1+i|=2\\|i|=1}}{=} 2|z^3|+|z| \overset{\text{ip.}}{\leq} 2\cdot\frac{1}{8}+\frac{1}{2}=\frac{1}{4}+\frac{1}{2}=\frac{3}{4} \Rightarrow |(1+i)z^3+iz| \leq \frac{3}{4}.$$

Solution to Problem 79.

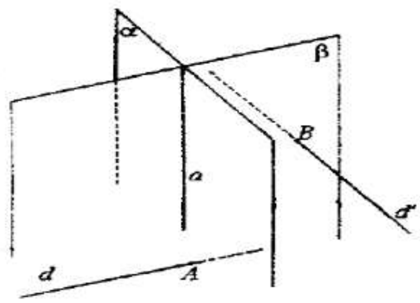
We construct $\alpha \perp d$ and $A \in \alpha$. The so constructed plane is unique. Similarly we construct $\beta \perp d'$ and $A \perp \beta$, $\alpha \cap \beta = a \ni A$.



From $\left.\begin{matrix}\alpha \perp d \Rightarrow d \perp a\\ \beta \perp d' \Rightarrow d' \perp a\end{matrix}\right\} \Longrightarrow a$ is a line which passes through A and is perpendicular to d and d'. The line a is unique, because α and β constructed as above are unique.

Solution to Problem 80.

We construct plane α such that $A \in \alpha$ and $d \perp \alpha$. We construct plane β such that $B \in \beta$ and $d' \perp \beta$.



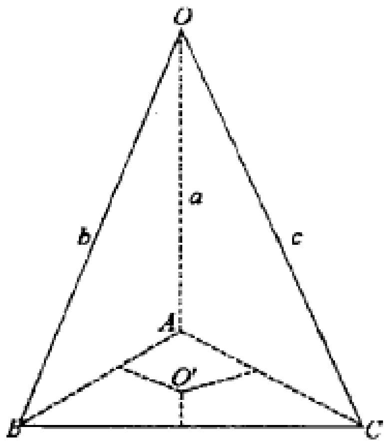
The so constructed planes α and β are unique.

Let $a = \alpha \cap \beta \Longrightarrow a \subset \alpha$ so $(\forall)$ $M \in a$ has the property $\mathrm{pr}_d M = A$.

$\alpha \subset \beta \Longrightarrow (\forall)$ $M \in a$ has the property $pr_{d'} M = B$.

Vice-versa. If there is a point M in space such that $\mathrm{pr}_d M = A$ and $pr_{d'} M = B \Longrightarrow M \in \alpha$ and $M \in \beta \Longrightarrow M \in \alpha \cap \beta \Longrightarrow M \in a$ (α and β previously constructed).

Solution to Problem 81.

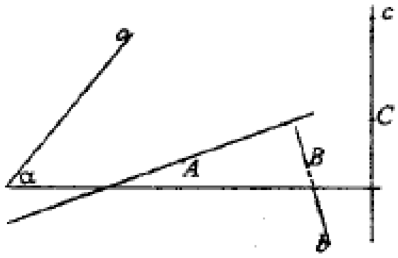


Let $A \in a$, $B \in b$, $C \in c$ such that $\|OA\| = \|OB\| = \|OC\|$. Triangles OAB, OBC, OAC are isosceles. The mediator planes of segments $\|AB\|, \|AC\|, \|BC\|$ pass through O and O' (the center of the circumscribed circle of triangle ABC). Ray $|OO'|$ is the desired locus.

Indeed $(\forall)$ $M \in |OO'| \Longrightarrow M \in$ mediator plane of segments $|AB|, |AC|$ and $|BC| \Longrightarrow M$ is equally distant from a, b and c.

Vice-versa: $(\forall)$ M with the property: $d(M,a) = d(M,b) = d(M,c) \Longrightarrow M \in$ mediator plan, mediator planes of segments $|AB|, |AC|$ and $|BC| \Longrightarrow M \in$ the intersection of these planes $\Longrightarrow M \in |OO'|$.

Solution to Problem 82.



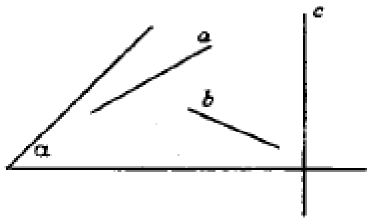
Let a, b, c be the 3 lines in space.

I. We assume $a \not\perp c$ and $b \not\perp c$. Let α be a plane such that:

$$\alpha \cap c = \{C\}$$
$$\alpha \cap a = \{A\} \text{ şi } \alpha \perp c$$
$$\alpha \cap b = \{B\}$$

The construction is possible because $\not\perp c$ and $b \not\perp c$. Line AB meets a on p and it is perpendicular to c, because $AB \subset \alpha$ and $c \perp \alpha$.

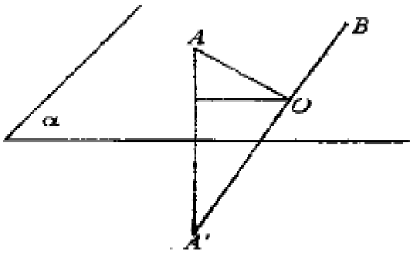
II. If $a \perp c$ or $b \perp c$, the construction is not always possible, only if plane $p(a, b)$ is perpendicular to c.



III. If $a \perp c$ and $b \not\perp c$, we construct plane $a \perp c$ so that $a \subset \alpha$ and $b \subset \alpha \neq \emptyset$. Any point on line a connected with point $b \cap \alpha$ is a desired line.

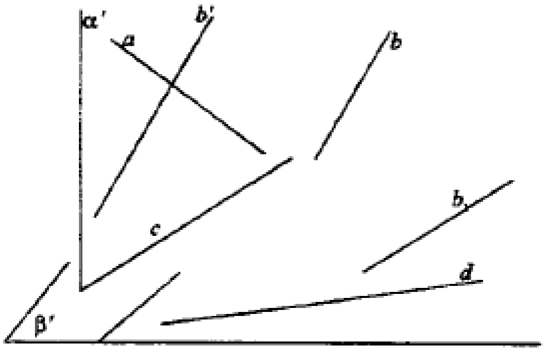
Solution to Problem 83.

We construct A' the symmetrical point of A in relation to α. A' and B are on different half-spaces, $\alpha \cap |A'B| = O$.



O is the desired point, because $||OA|| + ||OB|| = ||OA'|| + ||OB||$ is minimal when $O \in |A'B|$, thus the desired point is $O = |A'B| \cap \alpha$.

Solution to Problem 84.



Let a, b, d be the 3 given lines and through d we construct a plane in which a and b to be projected after parallel lines.

Let A be an arbitrary point on a. Through A we construct line $b'||b$. It results from the figure $b||\alpha, \alpha = p(a, b')$.

Let β such that $d \subset \beta$ and $\beta \perp \alpha$.

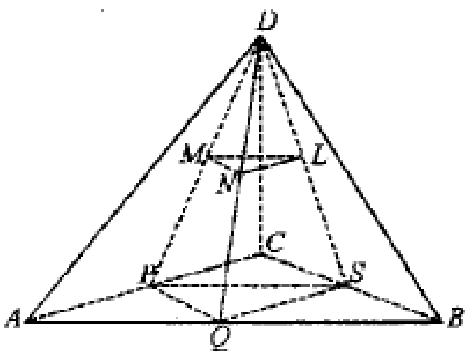
Lines a and b' are projected onto β after the same line c. Line b is projected onto β after b_1 and $b_1 \parallel c$.

If $b_1 \nparallel c$,

$b_1 \nparallel c \Rightarrow s \cap b_1 = \{N\} \Rightarrow \alpha \cap p(b, b_1) \neq \emptyset \Rightarrow b \cap \alpha \neq \emptyset$,

absurd because $b||\alpha(b||b')$.

Solution to Problem 85.



M is the centroid in $\Delta ACD \Longrightarrow$

$$\Longrightarrow \frac{|MD|}{|MP|} = 2 \quad (1)$$

N is the centroid in $\Delta ABD \Longrightarrow$

$$\Longrightarrow \frac{|ND|}{|NQ|} = 2 \quad (2)$$

L is the centroid in $\Delta BCD \Longrightarrow$

$$\Longrightarrow \frac{|LD|}{|LS|} = 2 \quad (3)$$

From 1 and 2,

and from 2 and 3

$$\left.\begin{array}{l} \Rightarrow MN \parallel PQ \\ \Rightarrow ML \parallel QS \end{array}\right\} \Rightarrow (LMN) \parallel (PQS) = (ABC) \Rightarrow (LMN) \parallel (ABC).$$

$$\frac{\sigma[SPQ]}{\sigma[ABC]} = \frac{s}{4s} = \frac{1}{4}$$

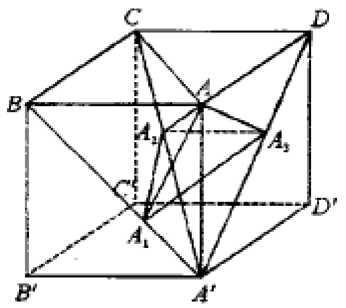
because:

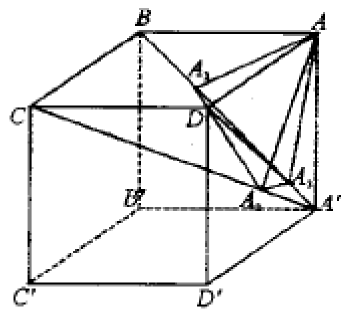
$$\sigma[AQP] = \sigma[PQS] = \sigma[QBS] = \sigma[PSC] = s$$

So

$$\frac{\sigma[ABC]}{\sigma[LMN]} = \frac{4}{1} \cdot \frac{9}{4} = 9.$$

Solution to Problem 86.





$BD \perp (AA'C)$ from the hypothesis $ABCDA'B'C'D'$ cube (1).

$$\left.\begin{array}{r} A_1 \text{ midpoint of segment } |BA'| \\ (ABA') \text{ isosceles and } AA_1 \perp BA' \\ A_3 \text{ midpoint of } |A'D| \end{array}\right\} |A_1A_3| \text{ mid-side in } \Delta A'BD \Longrightarrow A_1A_3 \parallel BD \ (2)$$

From (1) and (2) $\Longrightarrow A_1A_3 \parallel (AA'C) \Longrightarrow A'C \perp A_1A_3$ (3)

From $\Delta ACA'$:

$$\|AA_2\| = \frac{\|AC\| \cdot \|AA'\|}{\|A'C\|} = \frac{a\sqrt{2 \cdot a}}{a\sqrt{3}} = \frac{a\sqrt{6}}{3}.$$

From $\Delta ABA'$:

$$\|AA_1\| = \frac{a^2}{a\sqrt{2}} = \frac{a^2\sqrt{2}}{2};$$

Similarly

$$\|AA_3\| = \frac{a\sqrt{2}}{2}\|.$$

In $\Delta ACA'$:

$$\|AA'\|^2 = \|A'A_2\| \cdot \|A'C\| \Rightarrow a^2 = \|A'A_2\| \cdot a\sqrt{3} \Rightarrow \|A'A_2\| = \frac{a\sqrt{3}}{3}$$

and

$$\|A'A_1\| = \|\frac{a\sqrt{2}}{2}\|.$$

$$(\cos\alpha = \frac{a\sqrt{2}}{a\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}(\triangle A'BC))$$

$$\|A_1A_2\|^2 = \frac{a^2}{2} + \frac{a^2}{3} - 2\frac{a^2}{\sqrt{6}} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{5a^2}{6} - \frac{2a^2}{3} = \frac{a^2}{6}.$$

$$\|AA_2\|^2 = \|a_1A_2\|^2 + \|AA-1\|^2 \Rightarrow \frac{a^2 6}{9} = \frac{a^2}{6} + \frac{a^2}{2} \Rightarrow \frac{2a^2}{3} = \frac{2a^2}{3} \overset{pct.(B)}{\Rightarrow} AA_1 \perp A_1A_2 \; pct.(B)$$

$$AA_3 \perp A_2A_3.$$

A_1A_2A' right with $m(A'A_2A_1) = 90$ because

$$\|A'A_1\|^2 = \|A'A_2\|^2 + \|A_1A_2\|^2 \Leftrightarrow \frac{a^2}{2} =$$

$$= \frac{a^2}{3} + \frac{a^2}{6} \Leftrightarrow \frac{a^2}{2} = \frac{3a^2}{6}(a) \Rightarrow A'C \perp A_1A_2 \quad (4)$$

From (4) and (3) $\Longrightarrow A'C \perp (A_1A_2A_3)$.

As $\left.\begin{array}{c} A'C \perp (A_1A_2A_3) \\ A'C \perp A_2A \text{ (by construction)} \end{array}\right\} \Longrightarrow A_1A_2A_3A$ coplanar $\Longrightarrow A_1A_2A_3A$ quadrilateral with opposite angles A_1 and A_3 right $\Longrightarrow A_1A_2A_3A$ inscribable quadrilateral.

Solution to Problem 87.

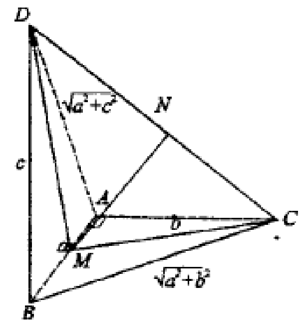
The conclusion is true only if $||BD|| = ||AC||$ that is $b = c$.

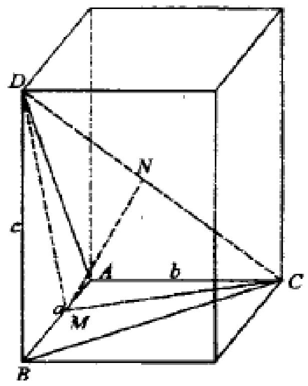
$$\|NC\| = \frac{\sqrt{a^2 + b^2 + c^2}}{2}; \; \|MC\| = \sqrt{b^2 + \frac{a^2}{4}}; \; \|MN\| = \frac{b^2 + c^2}{3}$$

$MN \perp DC$ if

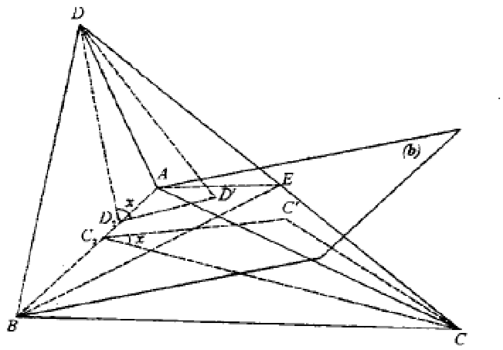
$$\|MC\|^2 = \|MN\|^2 + \|NC\|^2 \Leftrightarrow \frac{a^2 + b^2 + c^2}{4} + \frac{b^2 + c^2}{4} = b^2 + \frac{a^2}{4} \Rightarrow a^2 + 2b^2 +$$

$$+2c^2 = 4b^2 + a^2 \Rightarrow c^2 = b^2 \Rightarrow b = c.$$





Solution to Problem 88.



bisector plane

$$\left.\begin{array}{l}\left.\begin{array}{l} DD' \perp (ABE) = b \\ DD_1 \perp AB \end{array}\right\} \Rightarrow D'D_1 \perp AB \\ \left.\begin{array}{l} CC' \perp (ABE) \\ CC_1 \perp AB \end{array}\right\} \Rightarrow C'C_1 \perp AB \qquad C', C_1, D', D_1 \end{array}\right\} \Rightarrow D_1D' \| C_1C' \|.$$

$$m(\widehat{DD_1D'}) = m(\widehat{C'C_1C}) = x$$

(b bisector half-plane)

In triangle DD_1D':

$$\left.\begin{array}{l} \sin x = \dfrac{\|DD'\|}{\|DD_1\|} \\ \sin x = \dfrac{\|CC'\|}{\|CC_1\|} \end{array}\right\} \Rightarrow \frac{\|DD'\|}{\|DD_1\|} = \frac{\|CC'\|}{\|CC_1\|} \Rightarrow \frac{\|DD'\|}{\|CC'\|} = \frac{\|DD_1\|}{\|CC_1\|} \Rightarrow$$

$$\Rightarrow \frac{\|DD'\|}{\|CC'\|} = \frac{\sigma[ABD]}{\sigma[ABC]} \qquad (1)$$

$$\left.\begin{array}{l} v[ABED] = \dfrac{\sigma[ABE] \cdot \|DD'\|}{3} \\ v[ABEC] = \dfrac{\sigma[ABE] \cdot \|CC'\|}{3} \end{array}\right\} \Rightarrow \frac{\|DD'\|}{\|CC'\|} = \frac{v[ABED}{v[ABEC]} \qquad (2)$$

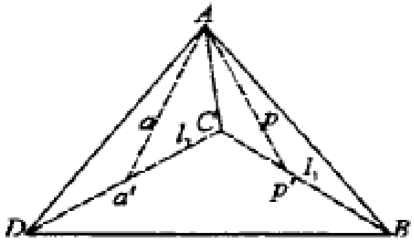
But

$$\left.\begin{array}{l} v[ABED] = \dfrac{\sigma[DEC]\cdot d(A,(DEC))}{3} \\ v[ABEC] = \dfrac{\sigma[BEC]\cdot d(A,(DBC))}{3} \end{array}\right\} \Rightarrow \frac{v[ABED]}{v[ABEC]} = \frac{\sigma[BDE]}{\sigma[BEC]} = \frac{\|DE\|\cdot d(B,DC)}{\|EC\|\cdot d(B,DC)} =$$

$$= \frac{\|DE\|}{\|EC\|} \Rightarrow \left|\frac{v[ABED]}{v[ABEC]}\right| = \frac{\|DE\|}{\|EC\|} \qquad (3)$$

From 1, 2, 3 $\Longrightarrow \frac{\sigma[ABD]}{\sigma[ABC]} = \frac{\|DE\|}{\|EC\|}$ q.e.d.

Solution to Problem 89.



Because the tetrahedron is regular $AB = \ldots =$

$\|BD\| = l$

$\|CP'\| = l_1$

$\|CQ'\| = l_2$

$$\cos\widehat{QAP} = \cos(\widehat{Q'AP'}) = \frac{\|AQ'\|^2 + \|AP'\|^2 - \|Q'P'\|^2}{2\|AQ'\|\cdot\|AP'\|} \geq$$

we increase the denominator

$$\geq \frac{l^2 + l_2^2 - ll_2 + l^2 - ll_1 - l_1^2 - l_2^2 + l_1 l_2}{2l^2} = \frac{l^2 + l^2 - ll_1 - ll_2 + l_1 l_2}{2l^2} = \frac{1}{2} + \frac{(l-l_1)(l-l_2)}{2l^2} \geq \frac{1}{2} \Rightarrow$$

$$\Rightarrow \cos\widehat{QAP} \geq \frac{1}{2} \Rightarrow m(\widehat{QAP}) \leq 60^\circ.$$

If one of the points P or Q is on face CBD the problem is explicit.

Solution to Problem 90.

We consider tetrahedron $Oxyz$, and prove that the sum of the measures of the dihedral angles of this trihedron is bigger than 360°. Indeed: let $100'$ be the internal bisector of trihedron $Oxyz$ ($1000'$ the intersection of the bisector planes of the 3 dihedral angles) of the trihedron in A, B, C.

The size of each dihedron with edges ox, oy, oz is bigger than the size of the corresponding angles of ABC, the sum of the measures of the dihedral angles of trihedron $Oxyz$ is bigger than 180°.

Let (a, b) be a plane $\perp$ to oz at C; $a \perp oz, b \perp oz$, but $|CA$ and $|CB$ are on the same half-space in relation to $(ab) \Rightarrow m(\hat{C}) < m(\widehat{ab})$.

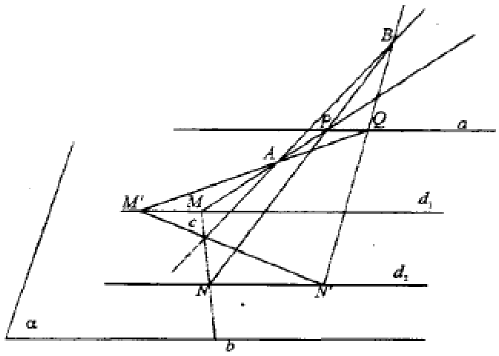
In tetrahedron $ABCD$, let a_1, a_2, a_3, a_4, a_5 and a_6 be the 6 dihedral angles formed by the faces of the tetrahedron.

$$\left.\begin{array}{l} m(\alpha_1+\alpha_2+\alpha_3)>180 \\ m(\alpha_1)+m(\alpha_3)+m(\alpha_6)>180 \\ m(\alpha_2)+m(\alpha_4)+m(\alpha_5)>180 \\ m(\alpha_4)+m(\alpha_5)+m(\alpha_6)>180 \end{array}\right\}$$

according to the inequality previously established.

$$2(m(\alpha_1)+m(\alpha_2)+\ldots+m(\alpha_6))>4\cdot 180 \Rightarrow m(\alpha_1)+\ldots+m(\alpha_6)>360°.$$

Solution to Problem 91.



We mark with a the intersection of planes (A, d_1) and (B, d_2). So

$$(A, d_1) \cap (B, d_2) = a.$$

Let b be a variable line that passes through C and contained in α, which cuts d_1 and d_2 at M respectively N. We have: $MA \subset (A, d_1), MA \cap NB = P(MA$ and NB intersect because they are contained in the plane determined by $(AM, b))$.

Thus $P \in (A, d_1)$ and $P \in (B, d_2), \Longrightarrow P \in a$, so P describes line a the intersection of planes (A, d_1) and (B, d_2).

Vice-versa: let $Q \in a$.

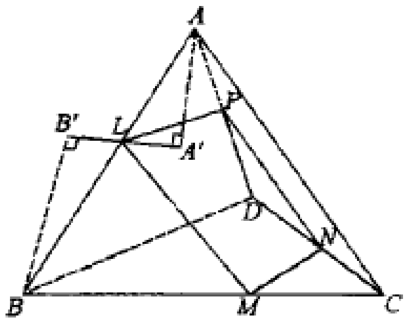
In the plane (A, d_1): $QA \cap d_1 = M'$

In the plane (B, d_2): $QB \cap d_2 = N'$

Lines $N'M'$ and AB are coplanar (both are on plane (Q, A, B)). But because $N'M' \subset \alpha$ and AB has only point C in common with $\alpha \Longrightarrow M'N' \cap AB = C$.

So $M'N'$ passes through C. If planes (A, d_1) and (B, d_2) are parallel, the locus is the empty set.

Solution to Problem 92.



Remember the theorem: If a plane γ intersects two planes α and β such that $\sigma||\alpha \Longrightarrow (\gamma \cap \alpha)||(\gamma \cap \beta)$. If plane $(LMNP)||BD$ we have:

$$\left.\begin{array}{l} LP\|MN\|BD \Rightarrow \dfrac{\|LA\|}{\|AP\|} = \dfrac{\|LB\|}{\|PD\|} \Rightarrow \|LA\|\cdot\|PD\| = \|AP\|\cdot\|LB\| \\ MN\|BD \Rightarrow \dfrac{\|NC\|}{\|MC\|} = \dfrac{\|ND\|}{\|MB\|} \Rightarrow \|BM\|\cdot\|NC\| = \|ND\|\cdot\|MC\| \end{array}\right\} \Rightarrow$$

$$\Rightarrow \|AL\|\cdot\|BM\|\cdot\|CN\|\cdot\|DP\| = \|BL\|\cdot\|CM\|\cdot\|DN\|\cdot\|AP\|.$$

If $(LMNP)||AC$ we have:

$$LM\|PN\|AC \Rightarrow \frac{\|PD\|}{\|DN\|} \text{ şi } \frac{\|AL\|}{\|MC\|} = \frac{\|LB\|}{\|BM\|} \Rightarrow$$

$$\Rightarrow \begin{cases} \|CN\|\cdot\|DP\| = \|DN\|\cdot\|AP\| \\ \|AL\|\cdot\|BM\| = \|BL\|\cdot\|CM\| \end{cases}$$

$\Longrightarrow$ relation a.

Solution:

Let A', B', C', D' the projections of points A, B, C, D onto plane $(MNPL)$.

For ex. points B', L, A' are collinear on plane $(LPMN)$ because they are on the projection of line AB onto this plane.

$$\triangle ALA' \sim \triangle BLB'(U,U) \Rightarrow \frac{\|AL\|}{\|LB\|} = \frac{\|AA'\|}{BB'}.$$

Similarly we obtain:

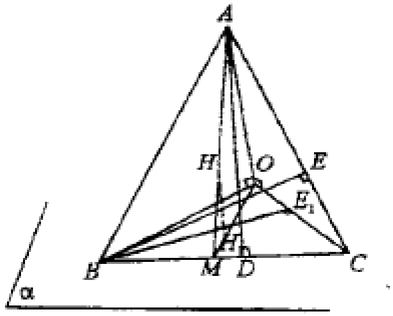
$$\frac{\|PD\|}{\|AP\|} = \frac{\|DD'\|}{\|AA'\|};\ \frac{\|CN\|}{\|ND\|} = \frac{\|CC'\|}{\|DD'\|};\ \frac{\|BM\|}{\|MC\|} = \frac{\|BB'\|}{\|CC'\|}.$$

By multiplying the 4 relations,

$$\frac{\|AL\|\cdot\|PD\|\cdot\|CN\|\cdot\|BM\|}{\|LB\|\cdot\|AP\|\cdot\|DN\|\cdot\|MC\|}=1\Rightarrow$$

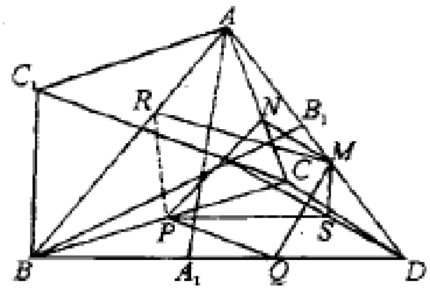
$\Longrightarrow$ relation (a) from d.

Solution to Problem 93.



M – Midpoint of $|BC|$.

Solution to Problem 94.



(1) $AB \perp CD$ (hypothesis)

$DH\perp(ABC)\Rightarrow DH\perp Ab\Rightarrow AB\perp DH$ (2)

From 1 and 2

$\Rightarrow AB\perp(CDH)\Rightarrow AB\perp CH\Rightarrow CH$

height in ΔABC a

$AC\perp BD$ (3)

$DH\perp(ABC)\Rightarrow DH\perp AC\Rightarrow AC\perp DH$ (4)

From 3 and 4 $AC\perp(BDH)\Longrightarrow AC\perp BH\Longrightarrow BH$ height in ΔABC b

From a and b $\Longrightarrow H$ orthocenter ΔABC. Let C_1 be the diametrical opposite point to C in circle $C(ABC)CC_1$ diameter $m(\widehat{C_1BC})=90^0$ but $AH\perp BC\Longrightarrow AH\parallel BC_1$. Similarly $BH||C_1A$, so $AHBC_1$ parallelogram, we have:

$$\|AH\|^2 + \|BC\|^2 = \|BC_1\|^2 + \|BC\|^2 = \|CC_1\|^2 = (2R)^2$$

similarly

$$\|BH\|^2 + \|AC\|^2 = (2R)^2\ (\|BH\| = \|AB_1\|)B_1$$

diametrical opposite to B

$$\|CH\|^2 + \|AB\|^2 = (2R)^2\ (\|CH\| = \|BA_1\|)$$

but

$$DH \perp (ABC) \Rightarrow \left.\begin{array}{l} \|AH\|^2 = \|AD\|^2 - \|DH\|^2 \\ \|BH\|^2 = \|BD\|^2 - \|DH\|^2 \\ \|CH\|^2 = \|DC\|^2 - \|DH\|^2 \end{array}\right\}$$

by substituting above, we have:

$$\|AD\|^2 + \|BC\|^2 = (2R)^2 + \|DH\|^2$$

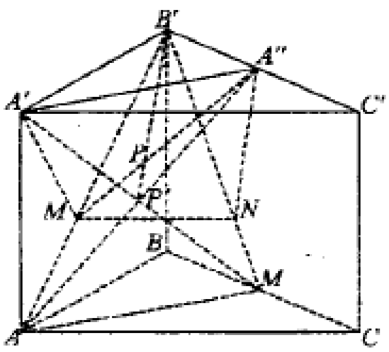
$$\|BD\|^2 + \|AC\|^2 = (2R)^2 + \|DH\|^2$$

$$\|DC\|^2 + \|AB\|^2 = (2R)^2 + \|DH\|^2$$

Let N, M, Q, P, S, R midpoints of the edges of the quadrilateral $NMPQ$ because: $NM||CD||PQ$ (median lines), $QM||AB||PN$ (median lines), but $CD \perp AB \Rightarrow MNQP$ rectangle $|NQ| \cap |PM| = \{0\}$.

Similarly $MSPR$ rectangle with $|MP|$ common diagonal with a, the first rectangle, so the 6 points are equally distant from "O" the midpoint of diagonals in the two rectangles $\Longrightarrow$ the 6 points are on a sphere.

Solution to Problem 95.



M arbitrary point on $|AB'|$

$N = pr_{(BCC')}M$

$A"$ midpoint of segment $[B'C']$

When $M = B'$, point P is in the position B'.

When $M = A$, point P is in the position $\{P_1\} = [A'A_1] \cap [AA'']$

($A'A''A_1A$ rectangle, so P_1 is the intersection of the diagonals of the rectangle)

[The locus is $[B'P_1]$].

Let M be arbitrary point $M \in |AB'|$.

$N = pr_{(BCC')}M \in |A_1B'|$

because:

$(B'AA_1) \perp (B'CC')$.

By the way it was constructed

$AA_1 \perp BC, AA_1 \perp CC' \Rightarrow AA_1 \perp (B'C'C) \Rightarrow$

$\Rightarrow$ ($\forall$) plane that contains AA_1 is perpendicular to $(B'CC'')$, particularly to $(B'AA_1) \perp (B'C'C)$.

(1) $[B'P_1] \subset (B'A''A)$

because $B', P \in (B'AA'')$

(2) $[B'P_1] \subset (A'B'A_1)$

from this reason $B', P_1 \in (A'B'A_1)$.

From 1 and 2

$\Rightarrow B'P_1 = (A'B'A_1) \cap (B'A''A)$

Let

$$\left.\begin{array}{r} \{P\} = |MA''| \cap |A'N| \\ |MA''| \subset (B'A''A) \text{ şi} \\ |A'N| \subset (A'B'A_1) \end{array}\right\} \Rightarrow P \in (B'A''A) \cap (A'B'A_1) \Rightarrow P \in |B'P_1| \quad N = pr_{(B'C'C)}M$$

So ($\forall$) $M \in |B'A|$

and we have

$|MA''| \cap |A'N| \in |B'P_1|$.

Vice-versa. Let P arbitrary point, $P \in |B'P_1|$ and

In plane

$(B'A''A) : \{M\} = |B'A| \cap |A''P|$

In plane

$(B'A'A_1) : \{N\} = |B'A_1| \cap |A'P|$

Indeed: $A'A''$ || $(B'AA_1)$ thus any plane which passes through $A'A''$ will intersect $(B'AA_1)$ after a parallel line to $A'A''$. Deci $MN || A'A''$ or $MN || AA_1$ as $M \in (B'AA_1) \Rightarrow MN \perp (B'CC'')$.

We've proved

$(\forall) M \in |B'A|$

and

$N = pr_{(B'CC')}$

we have

$\{P\} = |MA''| \cap |A'N|$

describes $|B'P_1|$ and vice-versa,

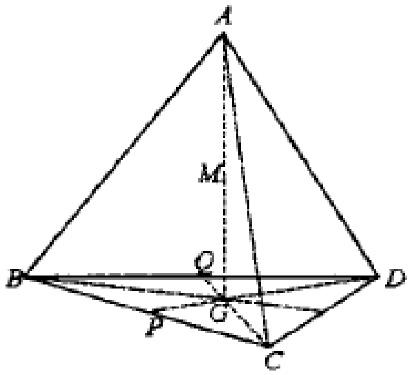
$(\forall)P \in [B'P_1]$

there is $M|B'A|$ and $N|B'A_1|$ such that

$N = pr_{(B'C'C)}M$

and P is the intersection of the diagonals of the quadrilateral $A'NMA''$.

Solution to Problem 96.



$$\sigma[CDG] = \sigma[BDG] = \sigma[BCG] = \frac{\sigma[BCD]}{3}$$

known result

$$\begin{cases} v[MGCD] = \dfrac{\sigma[GCD]d(M,(BCD))}{3} \\ v[MGDB] = \dfrac{\sigma[BDG]d(M,(BCD))}{3} \\ v[MGBC] = \dfrac{\sigma[BCG]d(M,(BCD))}{3} \end{cases}$$

From 1 and 3

$\Rightarrow v[MGCD] = v[MGDB] = v[MBC]$.

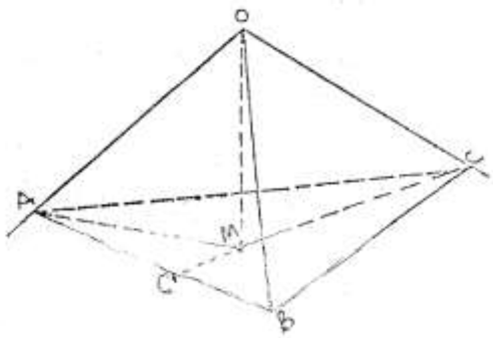
Solution to Problem 97.

From the hypothesis:

$OA \perp OB \perp OC \perp OA$

We assume the problem is solved.

Let M be the orthocenter of triangle ABC.



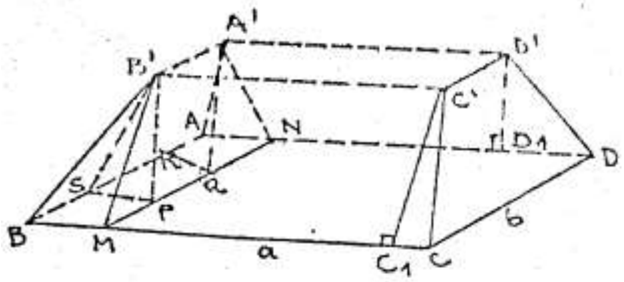
But $CC' \perp AB \Rightarrow \left.\begin{matrix} AB \perp (OCC') \\ MO \perp (COC') \end{matrix}\right\} \Rightarrow AB \perp OM \Rightarrow MO \perp AB$ (1)

$AO \perp (COB) \Rightarrow AO \perp BC$, but $AA' \perp BC \Rightarrow \left.\begin{matrix} BC \perp (AOA') \\ MO \subset (AOA') \end{matrix}\right\} \Rightarrow BC \perp MO \Rightarrow MO \perp BC$ (2)

From (1) and (2) $\Rightarrow MO \perp (ABC)$

So the plane (ABC) that needs to be drawn must be perpendicular to OM at M.

Solution to Problem 98.



$A'N \perp AD$, $B'M \perp BC$

$\|BM\| = \frac{a-a'}{2}$, $\|PM\| = \frac{b-b'}{2}$

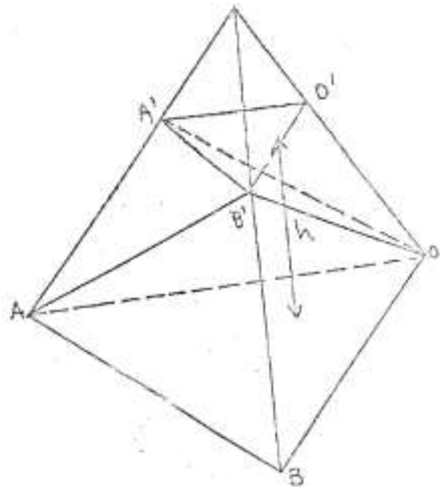
$$v[BMPSB'] = \frac{a-a'}{2} \cdot \frac{b-b'}{2} \cdot \frac{h}{3}$$

$$v[SPWRA'B'] = \frac{\sigma[SPB'] \cdot \|B'A'\|}{3} = \frac{a-a'}{2} \cdot \frac{h}{2} \cdot b'$$

$$v[B'A'NMC'D'D_1C_1] = \frac{(b+b')h}{2} \cdot a'$$

$v[ABA'B'CDC'D'] = 2\left[2 \cdot \frac{a-a'}{2} \cdot \frac{b-b'}{2} \cdot \frac{h}{3} + \frac{a-a'}{2} \cdot \frac{h}{2} \cdot b'\right] + \left(\frac{(b+b')h}{2}\right) \cdot a' =$
$\frac{h}{6}(2ab - 2ab' - 2a'b' + 3ab' - 3a'b' + 3a'b + 3a'b') = \frac{h}{6}[ab + a'b' +$
$(a+a')(b+b')]$.

Solution to Problem 99.



$$v[OABB'] = \frac{B \cdot h}{2}$$

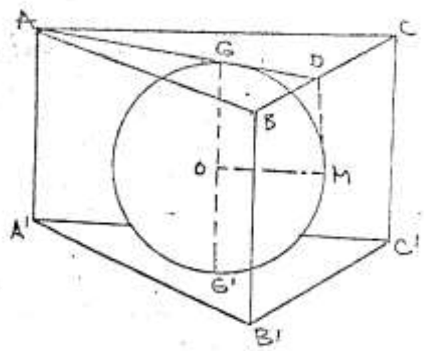
$v[OA'B'A] = v[ABOO'A'B'] - v[ABB'O] - v[A'B'O'O] = \frac{h}{3}\left(B + b + \sqrt{Bb}\right) - \frac{Bh}{3} - \frac{bh}{3} = \frac{h}{3}\sqrt{bB}$.

So:

$$\frac{v[OA'B'A]}{v[DABB']} = \frac{\frac{h}{3}\sqrt{Bb} \cdot \sqrt{Bb}}{\frac{Bh}{3} \cdot B} = \frac{\sqrt{b}}{\sqrt{B}} \Longrightarrow v[OA'B'A] = \frac{\sqrt{b}}{\sqrt{B}} \cdot v[OABB']$$

For the relation above, determine the formula of the volume of the pyramid frustum.

Solution to Problem 100.



$d(GG') = h = 2R$

Let $l = \|AC\| \Longrightarrow \|AD\| = \frac{l\sqrt{3}}{2} \Longrightarrow \|GD\| = \frac{l\sqrt{3}}{6}$

Figure $GDMO$ rectangle $\Longrightarrow \|GD\| = \|OM\| \Longrightarrow \frac{l\sqrt{3}}{6} = R = 2\sqrt{3}R$

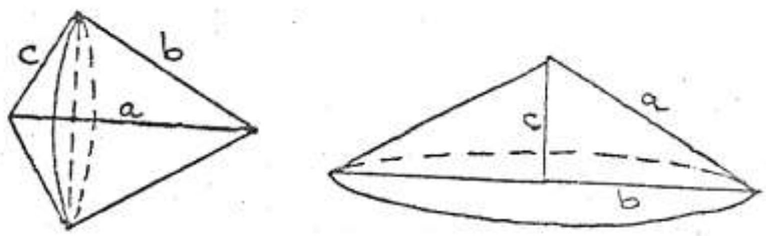
So, the lateral area is $S_l = 3 \cdot 2\sqrt{3}R \cdot R = 12\sqrt{3}R^2$.

$v[ABCA'B'C'] = \sigma[ABC] \cdot 2R = 2\sqrt{3}R \cdot \frac{2\sqrt{3}R\sqrt{3}}{4} \cdot 2R = 6\sqrt{3}R^2$.

The total area:

$S_t = S_l + 2\sigma[ABC] = 12\sqrt{3}R^2 + 2 \cdot 3R\sqrt{3}R^2 = 18\sqrt{3}R^2$

Solution to Problem 101.



Let V_1 and S_1 be the volume, respectively the area obtained revolving around a.

V_2 and S_2 be the volume, respectively the area obtained after revolving around b.

V_3 and S_3 be the volume, respectively the area obtained after revolving around c.

So:

$$V_1 = \frac{\pi \cdot i^2(\|CD\| + \|DB\|)}{3} = \frac{\pi \cdot i^2 \cdot a}{3}$$

$$S_1 = \pi \cdot i \cdot c + \pi \cdot i \cdot b = \pi \cdot i \cdot (b + c)$$

$$V_2 = \frac{\pi c^2 b}{3} = \frac{\pi c^2 b^2}{3b} = \frac{\pi b^2 c^2 a}{3a^2}$$

$$S_2 = \pi \cdot c \cdot a$$

$$V_3 = \frac{\pi b^2 c}{3} = \frac{\pi b^2 c^2}{3c} = \frac{\pi b^2 c^2}{3a}$$

$$S_3 = \pi \cdot b \cdot a$$

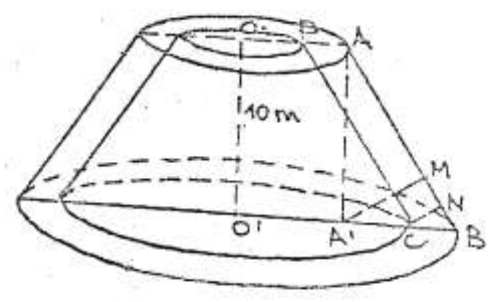
Therefore:

$$\frac{1}{V_1^2} = \frac{1}{V_2^2} + \frac{1}{V_3^2} \Leftrightarrow \frac{9a^2}{(\pi b^2 c^2)^2} = \frac{9c^2}{(\pi b^2 c^2)^2} + \frac{9c^2}{(\pi b^2 c^2)^2}$$

$$\frac{S_2}{S_3} + \frac{S_3}{S_2} = \frac{S_2 + S_3}{S_1} \Leftrightarrow \frac{c}{b} + \frac{b}{c} = \frac{\pi a(b + c)}{\pi i(b + c)} \Leftrightarrow \frac{c^2 + b^2}{b \cdot c} = \frac{a}{i}$$

But $i \cdot a = b \cdot c \Longrightarrow \frac{c^2+b^2}{bc} = \frac{a^2}{bc}$, $\|AD\| = i$.

Solution to Problem 102.



$$r = \|OA\| = 25\ cm$$

$$R = \|O'B\| = 50\ cm$$

$$2\pi r = 1{,}57 \Longrightarrow r = 0{,}25\ m$$

$$2\pi R = 3{,}14 \Longrightarrow R = 0{,}50\ m$$

$$\|CN\| = 18\ cm = 0{,}18\ m$$

$$\|A'B\| = 25\ cm$$

$$\|AB\| = \sqrt{100 + 0{,}0625} = 10{,}003125$$

$$\|A'M\| = \frac{\|AA'\| \cdot \|A'B\|}{\|AB\|} = \frac{10 \cdot 0{,}25}{10{,}003125} \approx 0{,}25$$

$$\frac{\|CN\|}{\|A'M\|} = \frac{\|CB\|}{\|BA'\|} \Longrightarrow \frac{0{,}18}{0{,}25} = \frac{\|CB\|}{0{,}25} \Longrightarrow \|CB\| = 0{,}18$$

$$\|O'C\| = R' = 0{,}50 - 0{,}18 = 0{,}32$$

$$\|OP\| = r' = 0{,}25 - 0{,}18 = 0{,}07$$

$$V = \frac{\pi 1}{3}(R^2 + r^2 + Rr)$$

$$V = \frac{\pi 10}{3}(0{,}50^2 + 0{,}25^2 + 0{,}50 \cdot 0{,}25 - 0{,}32^2 - 0{,}07^2 - 0{,}32 \cdot 0{,}07)$$
$$= \frac{\pi 10}{3}(0{,}4375 - 0{,}1297) = 1{,}026\pi m^3$$

Solution to Problem 103.

$$\|VP\| = \frac{a}{2\sin\frac{\alpha}{2}}\cos\frac{\alpha}{2}$$

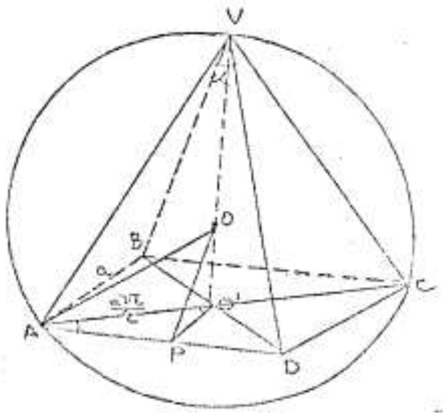
In ΔVAP: $\|VA\| = \frac{A}{2\sin\frac{\alpha}{2}}$.

In $\Delta VAO'$: $\|VO'\|^2 = \frac{a^2}{4\sin^2\frac{\alpha}{2}} - \frac{a^2}{2}$.

$$\|VO'\| = \frac{a\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}}$$

In $\Delta VOO'$: $\|OO'\| = \frac{a\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}} - R$.

$$R^2 = \frac{a^2}{2} + \left(\frac{a\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}} - R\right)^2 \Rightarrow \frac{a^2}{2} + \frac{a^2\cos\alpha}{4\sin^2\frac{\alpha}{2}} - \frac{2aR\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}} \Rightarrow a = \frac{4R\sqrt{\cos\alpha}\sin^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\left(2\cos^2\frac{\alpha}{2} + \cos\alpha\right)}$$
$$\Rightarrow a = 4R\sqrt{\cos\alpha}\cdot\sin\frac{\alpha}{2}$$



$$A_l = 4\frac{a^2\cos\frac{\alpha}{2}}{2\cdot 2\sin\frac{\alpha}{2}} = \frac{a^2\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = 16R^2\cos\alpha\sin^2\frac{\alpha}{2}\cdot\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = 8R^2\cos\alpha\sin\alpha = 4R^2\sin 2\alpha$$

$$A_t = A_l + a^2 = 4R^2\sin 2\alpha + 16R^2\cos\alpha\sin^2\frac{\alpha}{2}$$

$$\|VO'\| = R \Rightarrow R = \frac{a\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}} \Rightarrow R = 4R\sqrt{\cos\alpha}\sin\frac{\alpha}{2}\cdot\frac{\sqrt{\cos\alpha}}{2\sin\frac{\alpha}{2}} \Rightarrow 2\cos\alpha = 1 \Rightarrow \alpha = 60^0.$$

Various Problems

104. Determine the set of points in the plane, with affine coordinates z that satisfy:

   a. $|z| = 1$;
   b. $\pi < \arg z \leq \frac{3\pi}{2}; z \neq 0$;
   c. $\arg z > \frac{4\pi}{3}, z \neq 0$;
   d. $|z + i| \leq 2$.

Solution to Problem 104

105. Prove that the n roots of the unit are equal to the power of the particular root ε_1.

Solution to Problem 105

106. Knowing that complex number z verifies the equation $z^n = n$, show that numbers 2, $-iz$ and iz verify this equation.

   Application: Find $(1 - 2i)^4$ and deduct the roots of order 4 of the number $-7 + 24i$.

Solution to Problem 106

107. Show that if natural numbers m and n are coprime, then the equations $z^m - 1 = 0$ and $z^n - 1 = 0$ have a single common root.

Solution to Problem 107

108. Solve the following binomial equation: $(2 - 3i)z^6 + 1 + 5i = 0$.

Solution to Problem 108

109. Solve the equations:

a) $z^6 - 9z^3 + 8 = 0$

$z^3 = y$

b) $z^8 - 2z^4 + 2 = 0$

$z^4 = y$

c) $z^4 + 6(1+i)z^2 + 5 + 6i = 0$

$z^2 = y$

Solution to Problem 109

110. Solve the equation $\bar{z} = z^{n-1}, n \in N$, where $\bar{z}$ the conjugate of z.

Solution to Problem 110

111. The midpoints of the sides of a quadrilateral are the vertices of a parallelogram.

Solution to Problem 111

112. Let $M_1M_2M_3M_4$ and $N_1N_2N_3N_4$ two parallelograms and P_i the midpoints of segments $[M_iN_i]$, $i \in \{1,2,3,4\}$. Show that $P_1P_2P_3P_4$ is a parallelogram or a degenerate parallelogram.

Solution to Problem 112

113. Let the function $f: C \to C, f(z) = az + b$; $(a, b, c \in C, a \neq 0)$. If M_1 and M_2 are of affixes z_1 and z_2, and M_1' and M_2' are of affixes $f(z_1), f(z_2)$, show that $\|M_1'M_2'\| = |a| \cdot \|M_1M_2\|$. We have $\|M_1'M_2'\| = \|M_1M_2\| \Leftrightarrow |a| = 1$.

Solution to Problem 113

114. Prove that the function $\mathrm{z} \to \bar{\mathrm{z}}, \mathrm{z} \in \mathrm{C}$ defines an isometry.

Solution to Problem 114

115. Let $\mathrm{M_1M_2}$ be of affixes $z_1, z_2 \neq 0$ and $\mathrm{z_2} = \alpha \mathrm{z_1}$. Show that rays $|\mathrm{OM_1}$, $|\mathrm{OM_2}$ coincide (respectively are opposed) $\Leftrightarrow$ $\alpha > 0$ (respectively $\alpha < 0$).

Solution to Problem 115

116. Consider the points $\mathrm{M_1M_2M_3}$ of affixes $z_1 z_2 z_3$ and $M_1 \neq M_2$. Show that:

a. $\mathrm{M_3} \in |\mathrm{M_1M_2} \Leftrightarrow \frac{\mathrm{z_3-z_1}}{\mathrm{z_2-z_1}} > 0$;

b. $\mathrm{M_3} \in \mathrm{M_1M_2} \Leftrightarrow \frac{\mathrm{z_3-z_1}}{\mathrm{z_2-z_1}} \in \mathrm{R}$.

Solution to Problem 116

117. Prove Pompeiu's theorem. If the point M from the plane of the equilateral triangle $M_1M_2M_3$ $\notin$ the circumscribed circle $\Delta\, M_1M_2M_3 \Rightarrow$ there exists a triangle having sides of length $\|MM_1\|, \|MM_2\|, \|MM_3\|$.

Solution to Problem 117

Solutions

Solution to Problem 104.

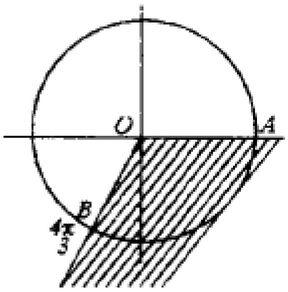
a. $\left.\begin{array}{c}|z| = 1 \\ |z| = \sqrt{x^2+y^2}\end{array}\right\} \Rightarrow x^2+y^2=1$, so the desired set is the circle $C_{(0,1)}$.

b. $\pi < \arg z \leq \frac{3\pi}{2}$.

The desired set is given by all the points of quadrant III, to which ray $|Oy$ is added, so all the points with $x<0, y<0$.

c. $\left.\begin{array}{c}\arg z > \frac{4\pi}{3}, z \neq 0 \\ \arg z \in [0, 2\pi]\end{array}\right\} \Rightarrow \frac{4\pi}{2} < \arg z < 2\pi$

$$\left.\begin{array}{l}\arg z > \frac{4\pi}{3}, z \neq 0 \\ \arg z \in [0, 2\pi]\end{array}\right\} \Rightarrow \frac{4\pi}{3} < \arg z < 2\pi$$



$$x_B = \cos\frac{4\pi}{3} = \cos\left(\pi+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$y_B = \sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$m_{OB} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3} \Rightarrow OB : y = \sqrt{3}x$$

The desired set is that of the internal points of the angle with its sides positive semi-axis and ray $|OB$.

d. $|z+i| \leq 2$; $z = x+yi$, its geometric image M.

$$z+i = x+yi+i = x+(y+1)i$$

$$|z+i| = \sqrt{x^2+(y+1)^2} \leq 2 \Rightarrow \|O'M\| \leq 2 \Rightarrow x^2+(y+1)^1 \leq 4 \Rightarrow \|O'M\|^2 \leq 4$$

where $O'(0,-1)$.

Thus, the desired set is the disk centered at $O'_{(0,-1)}$ and radius 2.

Solution to Problem 105.

$$\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1$$

$$\varepsilon_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$\varepsilon_k = \cos\, k\frac{2\pi}{n} + i\sin k\frac{2\pi}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k = \varepsilon_1{}^k, \quad k = 2, 3, \ldots, n-1$$

Solution to Problem 106.

Let the equation $z^4 = n$. If $z^4 = 4$ (z is the solution) then: $(-z)^4 = (-1)^4 z^4 = 1 \cdot n = n$, so – z is also a solution.

$$(iz)^4 = i^4 z^4 = 1 \cdot n = n$$

$\Longrightarrow$ iz is the solution;

$$(-iz)^4 = (-i)^4 z^4 = 1 \cdot n = n$$

$\Longrightarrow$ $-iz$ is the solution;

$$(1-2i)^4 = [(1-2i)^2]^2 = (1-4i+4i^2)^2 = (1-4-4i)^2 = (-3+4i)^2 = 9+24i-16 =$$

$$= -7+24i \Rightarrow z = 1-2i$$

$\Longrightarrow$ is the solution of the equation $z^4 = -7+24i$.

The solutions of this equation are:

$$z_k = \sqrt[4]{-7+24i}, \quad k = 0, 1, 2, 3$$

but based on the first part, if $z - 1 - 2i$ is a root, then

$$-z = -1+2i, iz = 2+i, -iz = -2-i$$

are solutions of the given equation.

Solution to Problem 107.

$$z^m - 1 = 0 \Rightarrow z = \sqrt[m]{1} \Rightarrow z_k = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}, \quad k = 0, \ldots, m-1$$

$$z^n - 1 = 0 \Rightarrow z = \sqrt[n]{1} \Rightarrow z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, \ldots, n-1$$

If there exist k and k' with $z_k = z_{k\prime}$, then

$$\frac{2k\pi}{m} - \frac{2k'\pi}{n} = 2p\pi = mn|k'm - kn \Rightarrow n|k',\ m|k,$$

because $(m,n) = 1$. Because $k' < n, k < m$, we have $k' = 0, k = 0$.

Thus the common root is z_0.

Solution to Problem 108.

$$(2-3i)z^6 + 1 + 5i = 0 \Rightarrow z^6 = \frac{-1-5i}{2-3i} = 1 - i$$
$$r = \sqrt{2}$$
$$\operatorname{tg} t = -1, t \in \left(\frac{3\pi}{2}, 2\pi\right) \Rightarrow t = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$
$$z_k = \sqrt[6]{2}\left(\cos\frac{\frac{7\pi}{4} + 2k\pi}{6} + i\sin\frac{\frac{7\pi}{4} + 2k\pi}{6}\right);\quad k \in 0, \ldots, 5$$

Solution to Problem 109.

$$a)\ \left.\begin{array}{c} z^6 - 9z^3 + 8 = 0 \\ z^3 = y \end{array}\right\} \Rightarrow y^2 - 9y + 8 = 0 \Rightarrow \left\{\begin{array}{l} z^3 = 8 \\ z^3 = 1 \end{array}\right. \quad \text{etc.}$$

$$b)\ \left.\begin{array}{c} z^8 - 2z^4 + 2 = 0 \\ z^4 = y \end{array}\right\} \Rightarrow y^2 - 2y + 2 = 0 \Rightarrow \left\{\begin{array}{l} z^4 = 1 + i \\ z^4 = 1 - i \end{array}\right. \quad \text{etc.}$$

$$c)\ \left.\begin{array}{c} z^4 + 6(1+i)z^2 + 5 + 6i = 0 \\ z^2 = y \end{array}\right\} \Rightarrow y^2 + 6(1+i)y + 5 + 6i = 0$$
$$y_{1,2} = \frac{-6(1+i) \pm \sqrt{-20+48i}}{2} = \frac{-6(1+i) \pm \sqrt{(4+6i)2}}{2} \quad \text{etc.}$$

Solution to Problem 110.

$$\left.\begin{array}{l} z = x + iy \Rightarrow |z| = \sqrt{x^2+y^2} \\ \bar{z} = x - iy \Rightarrow |\bar{z}| = \sqrt{x^2+y^2} \end{array}\right\} \Rightarrow |z| = |\bar{z}|$$

As:

$$\left.\begin{array}{r} \bar{z} = z^{n-1} \Rightarrow |\bar{z}| = |z|^{n-1} \\ |z| = |\bar{z}| \end{array}\right\} \Rightarrow |z| = |z|^{n-1} \Rightarrow |z|\cdot(1-|z|^{n-2}) = 0 \Rightarrow \left[\begin{array}{l} |z| = 0 \\ |z|^{n-2} - 1 = 0 \end{array}\right.$$

From:

$$|z| = 0 \Rightarrow \sqrt{x^2+y^2} = 0 \Rightarrow x^2+y^2 = 0 \Rightarrow x = 0 \text{ şi } y = 0 \Rightarrow z = 0 + 0i$$

$$|z|^{n-2} - 1 = 0 \Rightarrow (|z|-1)(\underbrace{|z|^{n-3} + |z|^{n-4} + \ldots + 1}_{\text{positive}}) = 0 \Rightarrow |z| = 1 \Rightarrow x^2+y^2 = 1$$

$$\left.\begin{array}{l} z = x+iy \\ \bar{z} = x-iy \end{array}\right\} \Rightarrow z\bar{z} = x^2+y^2 = 1$$

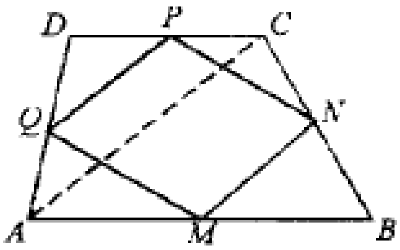
The given equation becomes

$$\bar{z}\cdot z = z^n \Rightarrow z^n = 1 \Rightarrow z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 0, 1, \ldots, n-1$$

Solution to Problem 111.

$$A(z_1), B(z_2), C(z_3), D(z_4)$$

$$\Rightarrow M\left(\frac{z_1+z_2}{2}\right), N\left(\frac{z_2+z_3}{2}\right), P\left(\frac{z_3+z_4}{2}\right), Q\left(\frac{z_4+z_1}{2}\right)$$



We find the sum of the abscissa of the opposite points:

$$\frac{z_1+z_2}{2} + \frac{z_1+z_4}{2} = \frac{z_1+z_2+z_3+z_4}{2}$$

$$\frac{z_2+z_3}{2} + \frac{z_4+z_1}{2} = \frac{z_1+z_2+z_3+z_4}{2}$$

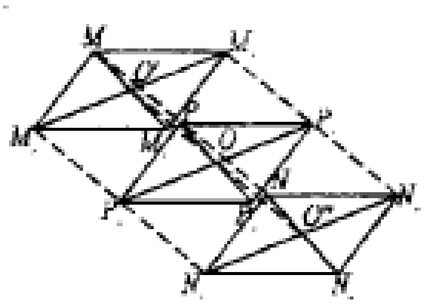
$$\frac{z_1+z_2}{2} + \frac{z_3+z_4}{2} = \frac{z_2+z_3}{2} + \frac{z_4+z_1}{2}$$

$\Longrightarrow MNPQ$ a parallelogram.

Solution to Problem 112.

In the quadrilateral $M_1M_3N_3N_1$ by connecting the midpoints we obtain the parallelogram $O'P_1O''P_3$, with its diagonals intersecting at O, the midpoint of $|O'O''|$ and thus $|P_1O| \equiv |OP_3|$. (1)

In the quadrilateral $M_4M_2N_2N_4$ by connecting the midpoints of the sides we obtain the parallelogram $O'P_2O''P_4$ with its diagonals intersecting in O, the midpoint of $|O'O''|$ and thus $|P_2O| \equiv |OP_4|$. (2)



From (1) and (2) $P_1P_2P_3P_4$ a parallelogram.

Solution to Problem 113.

$$\|M'M\| = |z - z'|, \text{ deci } \|M_1M_2\| = |z_2 - z_1|$$

$$\|M_1M_2\| = |f(z_2) - f(z_1)| = |az_2 + b - az_2 - b| = |az_2 - az_1| = |a(z_2 - z_1)| = |a| \cdot |z_2 - z_1| =$$
$$= |a| \cdot \|M_1M_2\|$$

If:

$$|a| = 1 \Rightarrow \|M_1'M_2'\| = \|M_1M_2\|$$

If:

$$\left.\begin{array}{r} \|M_1'M_2'\| = \|M_1M_2\| \\ \|M_1'M_2'\| = |a| \cdot \|M_1M_2\| \end{array}\right\} \Rightarrow \|M_1M_2\| = |a| \cdot \|M_1M_2\| \Rightarrow |a| = 1$$

Solution to Problem 114.

$$z = x + iy,\ \bar{z} = x = iy$$

Let M_1 and M_2 be of affixes z_1 and z_2. Their images through the given function M_1' and M_2' with affixes $\bar{z}_1$ and $\bar{z}_2$, so

$$f(z_1) = \bar{z}_1,\ f(z_2) = \bar{z}_2$$

$$\|M_1M_2\| = |z_2 - z_1| = |x_2 + iy_2 - x_1 - iy_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (1)$$

$$\|M_1'M_2'\| = |\bar{z}_2 - \bar{z}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2)$$

From (1) and (2) $\Longrightarrow \|M_1M_2\| = \|M_1'M_2'\|$ or $\|M_1'M_2'\| = |\bar{z}_2\bar{z}_1| = |\sqrt{z_2 - z_1}| = |z_2 - z_1| = \|M_1M_2\|$.

So $f: C \to C, f(z) = \bar{z}$ defines an isometry because it preserves the distance between the points.

Solution to Problem 115.

We know that the argument $(az_1) = \arg z_1 + \arg z_\alpha - 2k\pi$, where $k = 0$ or $k = 1$. Because $\arg z_2 = \arg(az_1), \arg z_2 = \arg z_1 + \arg z_\alpha - 2k\pi$.

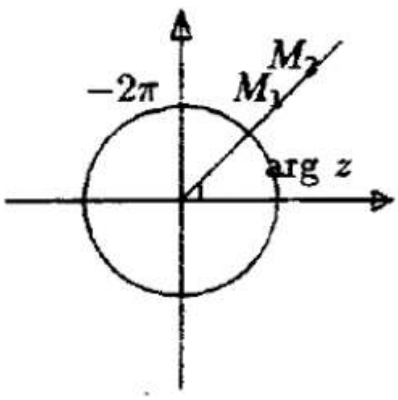
a. We assume that

$$|OM_1 = |OM_2 \Rightarrow \arg z_1 = \arg z_2 \Rightarrow \arg z_1 = \arg z_1 + \arg \alpha - 2k\pi \Rightarrow$$

$$\Rightarrow \arg \alpha = 2k\pi,\ \arg \in [0, 2\pi] \Rightarrow \arg \alpha = 0 \Rightarrow \alpha \in |Ox \text{ (poz.)} \Rightarrow \alpha > 0.$$

Vice versa,

$$\alpha > 0 \Rightarrow \arg \alpha = 0 \Rightarrow \arg z_2 = \arg z_1 - 2k\pi \Rightarrow \arg z_1 = \arg z_2 \text{ sau } \arg z_2 =$$
$$= \arg z_1 - 2\pi \Rightarrow |OM_1 = |OM_2.$$



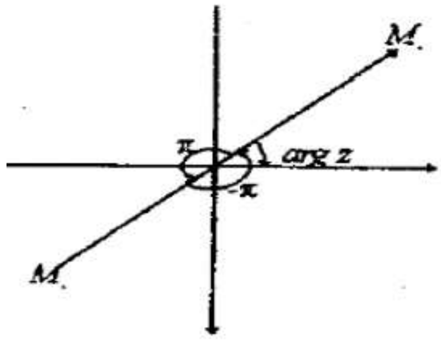
b. Let $|OM_1$ and $|OM_2$ be opposed $\Longrightarrow \arg z_2 = \arg z_1 + \pi$

$$\Rightarrow \arg z_1 + \pi = \arg z_1 + \arg \alpha - 2k\pi \Rightarrow \arg \alpha = \pi$$

$\in$ to the negative ray $|Ox' \Longrightarrow \alpha < 0$. Vice versa,

$$\alpha < 0 \Rightarrow \arg \alpha = \pi \Rightarrow \arg z_2 = \arg z_1 + \pi - 2k\pi$$

$$k = 0 \text{ or } k = 1 \Longrightarrow \left.\begin{matrix} \arg z_2 = \arg z_1 + \pi \\ \text{or} \\ \arg z_2 = \arg z_1 - \pi \end{matrix}\right\} \Longrightarrow |OM_1 \text{ and } |OM_2 \text{ are opposed.}$$

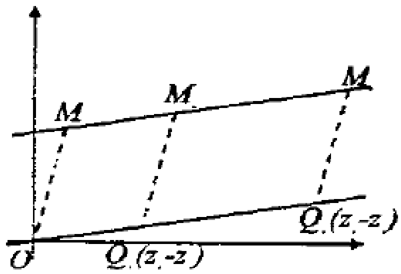


Solution to Problem 116.

If n and n' are the geometric images of complex numbers z and z', then the image of the difference $z-z'$ is constructed on $|\mathrm{OM}_1|$ and $|M'M|$ as sides.

We assume that $M_3 \in |M_1M_2$

We construct the geometric image of z_2-z_1. It is the fourth vertex of the parallelogram $OM_1M_2Q_1$. The geometric image of z_3-z_1 is Q_2, the fourth vertex of the parallelogram $OM_1M_3Q_2$.



$$\left.\begin{array}{r} OQ_1 \parallel M_1M_2 \\ OQ_2 \parallel M_1M_3 \\ M_1M_2M_3 \text{collinear} \end{array}\right\} \Rightarrow Q_1, Q_2, Q_3 \text{ collinear } \Rightarrow$$

$$|OQ_1 = |OQ_2 \Rightarrow z_3 - z_1 = \alpha(z_2 - z_1) \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \alpha \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} > 0$$

Vice versa, we assume that

$$\frac{z_3 - z_1}{z_2 - z_1} > 0 \Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = k > 0 \Rightarrow z_3 - z_1 = k(z_2 - z_1),\ k > 0$$

$$\left.\begin{array}{l} OQ_1 = |OQ_2 \\ M_1M_2 \parallel OQ_1 \\ M_1M_3 \parallel OQ_2 \end{array}\right\} \Rightarrow |M_1M_2 = |M_1M_3 \Rightarrow M_3 \in |M_1M_2$$

If M_3 and M_2 $\in$ the opposite ray to O, then $z_3-z_1 = \alpha(z_2-z_1)$ with $\alpha < 0$.

We repeat the reasoning from the previous point for the same case.

Thus, when $M_3 \in M_1M_2M_3 + M_2$ we obtain for the respective ratio positive, negative or having $M_3 = M_1$, so $\frac{z_3 - z_1}{z_2 - z_1} \in \mathrm{R}$.

Solution to Problem 117.

The images of the roots of order 3 of the unit are the peaks of the equilateral triangle.

$$\varepsilon_0 = 1, \varepsilon = \frac{-1 + i\sqrt{3}}{2}, \varepsilon_2 = \frac{-1 - i\sqrt{3}}{2}$$

But $\varepsilon_1 = \varepsilon_2^2$, so if we write $\varepsilon_2 = \varepsilon$, then $\varepsilon_1 = \varepsilon_2$.

Thus $M_1(1), M_2(\varepsilon), M_3(\varepsilon^2)$.

We use the equality:

$$(z-1)(\varepsilon^2 - \varepsilon) + (z - \varepsilon)(1 - \varepsilon^2) = (2 - \varepsilon^2)(1 - \varepsilon)$$

adequate $(\forall) z \in \mathbb{C}$.

$$|(z-1)(\varepsilon^2 - \varepsilon) + (z - \varepsilon)(1 - \varepsilon^2)| = |(z - \varepsilon^2)(1 - \varepsilon)|$$

But

$$|(z-1)(\varepsilon^2 - \varepsilon) + (z - \varepsilon)(1 - \varepsilon^2)| \le |(z-1)(\varepsilon^2 - \varepsilon)| + |(z - \varepsilon)(1 - \varepsilon^2)|$$

$$|(z-1)(\varepsilon^2 - \varepsilon) + (z - \varepsilon)(1 - \varepsilon^2)| \ge |(z-1)(\varepsilon^2 - \varepsilon)| - |-(z - \varepsilon)(1 - \varepsilon^2)|$$

Therefore,

$$|(z-1)(\varepsilon^2 - \varepsilon)| + |(z - \varepsilon)(1 - \varepsilon^2)| \ge |(z - \varepsilon^2)(1 - \varepsilon)|$$

$$|(z-1)(\varepsilon^2 - \varepsilon)| + |(z - \varepsilon)(1 - \varepsilon^2| \ge |z - \varepsilon^2| \cdot |1 - \varepsilon|$$

$$\varepsilon = \frac{-1 - i\sqrt{3}}{2}, \varepsilon^2 = \frac{-1 + \sqrt{3}}{2} \Rightarrow \varepsilon^2 - \varepsilon = i\sqrt{3} = 0 + i\sqrt{3} \Rightarrow |\varepsilon^2 - \varepsilon| = \sqrt{3}$$

$$1 - \varepsilon^2 = -1 - \frac{-1 + i\sqrt{3}}{2} = \frac{3}{2} - i\frac{\sqrt{3}}{2} \quad |1 - \varepsilon^2| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$$

$$1 - \varepsilon = 1 - \frac{-1 - i\sqrt{3}}{2} = \frac{3}{2} = \frac{i\sqrt{3}}{2}$$

$$|1 - \varepsilon| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}.$$

By substitution:

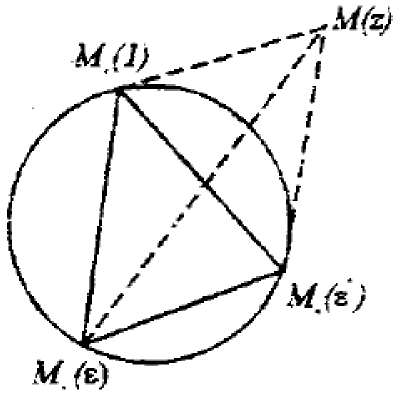
$$|z - 1| \cdot \sqrt{3} + |z - \varepsilon| \cdot \sqrt{3} \ge |z - \varepsilon^2| \cdot \sqrt{3}$$

but

$$\|MM_1\| = |z - 1|; \ \|MM_2\| = |z - \varepsilon|; \ \|MM_3\| = |z - \varepsilon^2|,$$

thus

$$\|MM_1\| + \|MM_2\| \geq \|MM_3\|$$



Therefore $\|MM_1\|, \|MM_2\|, \|MM_3\|$ sides of a Δ.

Then we use $||x| - |y|| \leq |x - y|$ and obtain the other inequality.

Problems in Spatial Geometry

118. Show that if a line d is not contained in plane α, then $d \cap \alpha$ is $\emptyset$ or it is formed of a single point.

Solution to Problem 118

119. Show that $(\forall)$ α, $(\exists)$ at least one point which is not situated in α.

Solution to Problem 119

120. The same; there are two lines with no point in common.

Solution to Problem 120

121. Show that if there is a line d $(\exists)$ at least two planes that contain line d.

Solution to Problem 121

122. Consider lines d, d', d'', such that, taken two by two, to intersect. Show that, in this case, the 3 lines have a common point and are located on the same plane.

Solution to Problem 122

123. Let A, B, C be three non-collinear points and D a point located on the plane (ABC). Show that:
    a. The points D, A, B are not collinear, and neither are D, B, C; D, C, A.
    b. The intersection of planes $(DAB), (DBC), (DCA)$ is formed of a single point.

Solution to Problem 123

124. Using the notes from the previous exercise, take the points E, F, G distinct from A, B, C, D, such that $E \in AD, F \in BD, G \in CD$. Let $BC \cap FG = \{P\}, GE \cap CA = \{Q\}, EF \cap AB = \{R\}$. Show that P, Q, R are collinear (*T. Desarques*).

Solution to Problem 124

125. Consider the lines d and d' which are not located on the same plane and the distinct points $A, B, C \in d$ and $D, E \in d'$. How many planes can we draw such that each of them contains 3 non-collinear points of the given points? Generalization.

Solution to Problem 125

126. Show that there exist infinite planes that contain a given line d.

Solution to Problem 126

127. Consider points A, B, C, D which are not located on the same plane.
    a. How many of the lines AB, AC, AD, BC, BD, CD can be intersected by a line that doesn't pass through A, B, C, D?
    b. Or by a plane that doesn't pass through A, B, C, D?

Solution to Problem 127

128. The points α and β are given, $A, B \in \alpha$. Construct a point $M \in \alpha$ at an equal distance from A and B, that $\in$ also to plan β.

Solution to Problem 128

129. Determine the intersection of three distinct planes α, β, γ.

Solution to Problem 129

130. Given: plane α, lines d_1, d_2 and points $A, B \notin \alpha \cup d_1 \cup d_2$. Find a point $M \in \alpha$ such that the lines MA, MB intersect d_1 and d_2.

Solution to Problem 130

131. There are given the plane α, the line $d \notin \alpha$, the points $A, B \notin \alpha \cup d$, and $C \in \alpha$. Let $M \in d$ and A', B' the points of intersection of the lines MA, MB with plane α (if they exist). Determine the point M such that the points C, A', B' to be collinear.

Solution to Problem 131

132. If points A and B of an open half-space σ, then $[AB] \subset \sigma$. The property is as well adherent for a closed half-space.

Solution to Problem 132

133. If point A is not situated on plane α and $B \in \alpha$ then $|BA \subset |\alpha A$.

Solution to Problem 133

134. Show that the intersection of a line d with a half-space is either line d or a ray or an empty set.

Solution to Problem 134

135. Show that if a plane α and the margin of a half-space σ are secant planes, then the intersection $\sigma \cap \alpha$ is a half-plane.

Solution to Problem 135

136. The intersection of a plane α with a half-space is either the plane α or a half-plane, or an empty set.

Solution to Problem 136

137. Let A, B, C, D four non coplanar points and α a plane that doesn't pass through one of the given points, but it passes trough a point of the line $|AB|$. How many of the segments $|AB|, |AC|, |AD|, |BC|, |BD|, |CD|$ can be intersected by plane α?

Solution to Problem 137

138. Let d be a line and α, β two planes such that $d \cap \beta = \emptyset$ and $\alpha \cap \beta = \emptyset$. Show that if $A \in d$ and $B \in \alpha$, then $d \subset |\beta A$ and $\alpha \subset |\beta B$.

Solution to Problem 138

139. Let $|\alpha A$ and $|\beta B$ be two half-spaces such that $\alpha \neq \beta$ and $|\alpha A \subset |\beta B$ or $|\alpha A \ \cap |\beta B \ = \emptyset$. Show that $\alpha \cap \beta = \emptyset$.

Solution to Problem 139

140. Show that the intersection of a dihedral angle with a plane α can be: a right angle, the union of two lines, a line, an empty set or a closed half-plane and cannot be any other type of set.

Solution to Problem 140

141. Let d be the edge of a proper dihedron $\angle\alpha'\beta'$, $A \in \alpha' - d$, $b \in \beta' - d$ and $P \in \text{int.} \angle\alpha'\beta'$. Show that:

a. $(Pd) \cap \text{int.} \angle\alpha'\beta' = |dP$;

b. If $M \in d$, $\text{int.} \angle AMB = \text{int.} \alpha'\beta' \cap (AMB)$.

Solution to Problem 141

142. Consider the notes from the previous problem. Show that:

a. The points A and B are on different sides of the plane (Pd);

b. The segment $|AB|$ and the half-plane $|dP$ have a common point.

Solution to Problem 142

143. If $\angle abc$ is a trihedral angle, $P \in \text{int.} \angle abc$ and A, B, C are points on edges a, b, c, different from O, then the ray $|OP$ and $\text{int.} ABC$ have a common point.

Solution to Problem 143

144. Show that any intersection of convex sets is a convex set.

Solution to Problem 144

145. Show that the following sets are convex planes, half-planes, any open or closed half-space and the interior of a dihedral angle.

Solution to Problem 145

146. Can a dihedral angle be a convex set?

Solution to Problem 146

147. Which of the following sets are convex:

a. a trihedral angle;
b. its interior;
c. the union of its faces;
d. the union of its interior with all its faces?

Solution to Problem 147

148. Let σ be an open half-space bordered by plane α and M a closed convex set in plane α. Show that the set $M \cap \sigma$ is convex.

Solution to Problem 148

149. Show that the intersection of sphere $S(O, r)$ with a plane which passes through O, is a circle.

Solution to Problem 149

150. Prove that the int. $S(O, r)$ is a convex set.

Solution to Problem 150

151. Show that, by unifying the midpoints of the opposite edges of a tetrahedron, we obtain concurrent lines.

Solution to Problem 151

152. Show that the lines connecting the vertices of a tetrahedron with the centroids of the opposite sides are concurrent in the same point as the three lines from the previous example.

Solution to Problem 152

153. Let $ABCD$ be a tetrahedron. We consider the trihedral angles which have as edges $[AB, [AE, [AD, [BA, [BC, [BD, [CA, [CB, [CD, [DA, [DB, [DC$. Show that the intersection of the interiors of these 4 trihedral angles coincides with the interior of tetrahedron $[ABCD]$.

Solution to Problem 153

154. Show that $(\forall)$ $M \in \mathrm{int.}[ABCD]$ $(\exists)$ $P \in |AB|$ and $Q \in |CD|$ such that $M \in \| PQ$.

Solution to Problem 154

155. The interior of tetrahedron $[ABCD]$ coincides with the union of segments $|PQ|$ with $P \in |AB|$ and $Q \in |CD|$, and tetrahedron $[ABCD]$ is equal to the union of the closed segments $[PQ]$, when $P \in [AB]$ and $Q \in [CD]$.

Solution to Problem 155

156. The tetrahedron is a convex set.

Solution to Problem 156

157. Let M_1 and M_2 convex sets. Show that by connecting segments $[PQ]$, for which $P \in M_1$ and $Q \in M_2$ we obtain a convex set.

Solution to Problem 157

158. Show that the interior of a tetrahedron coincides with the intersection of the open half-spaces determined by the planes of the faces and the opposite peak. Define the tetrahedron as an intersection of half-spaces.

Solution to Problem 158

Solutions

Solution to Problem 118.

We assume that $d \cap \alpha = \{A, B\} \Rightarrow d \subset \alpha$.

It contradicts the hypothesis $\Longrightarrow d \cap \alpha = \{A\}$ or $d \cap \alpha = \emptyset$.

Solution to Problem 119.

We assume that all the points belong to the plane $\alpha \Longrightarrow (\nexists)$ for the points that are not situated in the same plane. False!

Solution to Problem 120.

$\exists\ A, B, C, D$, which are not in the same plane. We assume that $AB \cap CD = \{0\} \Longrightarrow$ AB and CD are contained in the same plane and thus A, B, C, D are in the same plane. False, it contradicts the hypothesis $\Rightarrow AB \cap CD = \emptyset \Longrightarrow (\exists) \Longrightarrow$ lines with no point in common.

Solution to Problem 121.

$(\exists)\ A \notin\ d$ (if all the points would $\in d$, the existence of the plane and space would be negated). Let $\alpha = (dA)$, $(\exists) B \notin \alpha$ (otherwise the space wouldn't exist). Let $\beta = (Bd), \alpha \neq \beta$ and both contain line d.

Solution to Problem 122.

We show that $d \neq d' \neq d'' \neq d$.

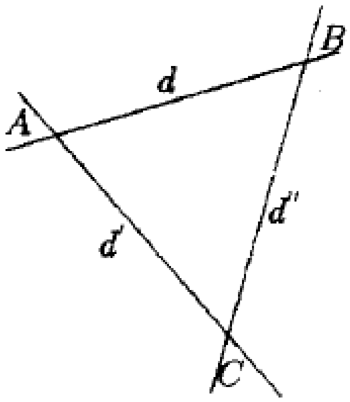
Let $$d \cap d' = \{A\} = (d, d') \Longrightarrow \begin{cases} d \subset \alpha \\ d' \subset \alpha \end{cases}$$

$$\begin{matrix} d \cap d' = \{B\} \\ B \neq A \end{matrix} \Longrightarrow \begin{cases} B \in d \\ d \subset \propto \end{cases} \Longrightarrow B \in a, B \in d'$$

$$\begin{array}{r} d''\cap d'=\{C\}\\ C\neq B\\ C\neq A\end{array} \Longrightarrow \begin{array}{c} C\in d'\\ d'\subset\propto\end{array} \Longrightarrow C\in\alpha, C\in d''$$

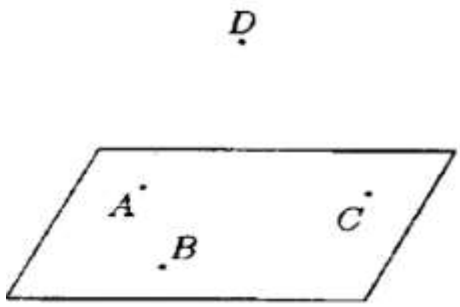
$$\Longrightarrow \begin{array}{c} d=d'\\ \text{or}\\ d=d''\end{array}$$

$\Longrightarrow d''\subset\alpha$, so the lines are located on the same plane α.



If $\left.\begin{array}{r} d\cap d'=\{A\}\Longrightarrow A\in d'\\ d''\cap d=\{A\}\Longrightarrow A\in d''\end{array}\right\}\Longrightarrow d'\cap d''=\{A\}$, and the three lines have a point in common.

Solution to Problem 123.



a. $D\notin(ABC)$.

We assume that D, A, B collinear $\Longrightarrow(\exists)d$ such that $\left.\begin{array}{c} D\in d, A\in d, B\in d\\ A\in(ABC), B\in(ABC)\end{array}\right\}\overset{T_2}{\Rightarrow} d\subset(ABC)\Longrightarrow D\in(ABC)$ – false. Therefore, the points D, A, B are not collinear.

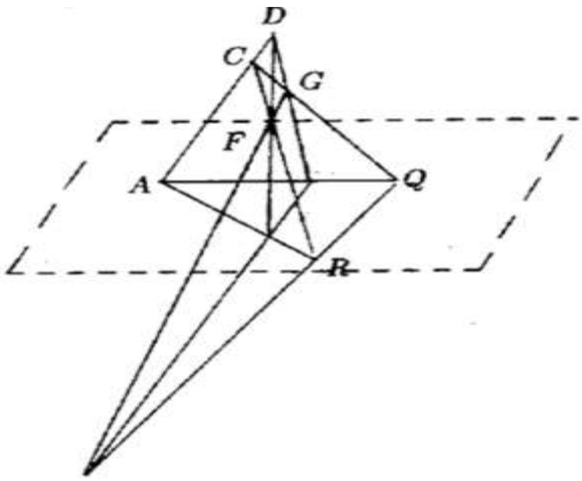
b. Let $(DAB)\cap(BCD)\cap(DCA)=E$.

As the planes are distinct, their intersections are:

$$\left.\begin{array}{l}(DAB)\cap(DBC)=DB\\(DAB)\cap(DCA)=DA\\(DBC)\cap(DCA)=DC\end{array}\right\}\begin{array}{l}\Longrightarrow \text{ If } (DAB)=(DBC)\\ \Longrightarrow A,B,C,D \text{ coplanar, contrary to the hypothesis.}\end{array}$$

We suppose that $(\exists)M\in E, M\neq D\Longrightarrow \left.\begin{array}{l}M\in DB\\M\in DA\end{array}\right\}\Longrightarrow\left.\begin{array}{l}B\in MD\\A\in MD\end{array}\right\}\Longrightarrow A,B,D$ are collinear (false, contrary to point a.). Therefore, set E has a single point $E=\{D\}$.

Solution to Problem 124.



We showed at the previous exercise that if $D\notin(ABC),(DAB)\neq(DBC)$. We show that E,F,G are not collinear. We assume the opposite. Then,

$$\left.\begin{array}{l}G\in EF\\EF\subset(DAB)\end{array}\right\}\Rightarrow\left\{\begin{array}{l}G\in(DAB)\\G\in(DBC)\end{array}\right.\Rightarrow$$
$$\Rightarrow(DAB)=(DBC)$$

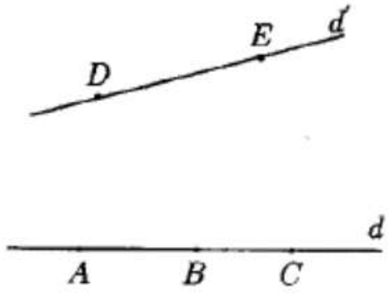
Having three common points D,B and $G\Longrightarrow$ false. So E,F,G are not collinear and determine a plane (EFG).

$$\left.\begin{array}{l}P\in BC\Rightarrow P\in(ABC)\\P\in FG\Rightarrow P\in(EFG)\\R\in AB\Rightarrow R\in(ABC)\\R\in EF\Rightarrow R\in(EFG)\\Q\in CA\Rightarrow P\in(ABC)\\Q\in GE\Rightarrow P\in(EFG)\end{array}\right\}P,Q,R\in(ABC)\cap(EFG)\Rightarrow$$

$\Longrightarrow P,Q,R$ are collinear because $\in$ to the line of intersection of the two planes.

Solution to Problem 125.

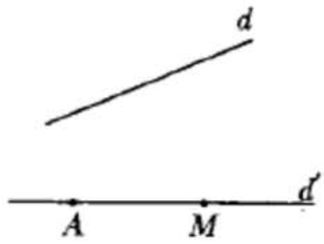
The planes are (A, d'); (B, d'); (C, d').



Generalization: The number of planes corresponds to the number of points on line d because d' contains only 2 points.

Solution to Problem 126.

Let line d be given, and A any point such that $A \notin d$.

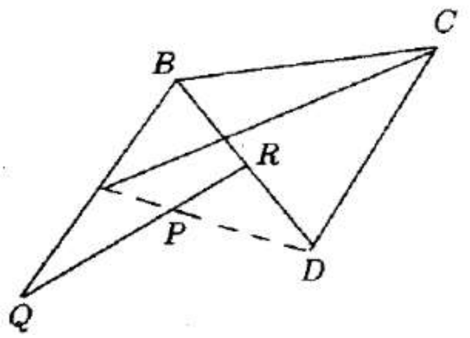


We obtain the plane $\alpha = (A, d)$, and let $M \notin \alpha$. The line $d' = AM, d' \not\subset \alpha$ is not thus contained in the same plane with d. The desired planes are those of type $(Md), M \in d'$, that is an infinity of planes.

Solution to Problem 127.

a. $(\forall)$ 3 points determine a plane. Let plane (ABD). We choose in this plane $P \in |AD|$ and $Q \in |AB|$ such that $P \in |BQ|$, then the line PQ separates the points A and D, but does not separate A and B, so it separates P and $D \Rightarrow PQ \cap |BD| = R$, where $R \in |BD|$.

Thus, the line PQ meets 3 of the given lines. Let's see if it can meet more.



We assume that

$$PQ \cap BC = \{E\} \Rightarrow E \in PQ \subset (ABD) \Rightarrow \left.\begin{array}{l} E \in (ABD) \\ E \in BC \end{array}\right\} \Rightarrow$$

it has two points in common with the plane.

$$\Rightarrow \left\{\begin{array}{l} B \in (ABD) \\ B \in BQ \end{array}\right. \Rightarrow BC \subset (ABD)$$

$\Longrightarrow A, B, C, D$ coplanar – false.

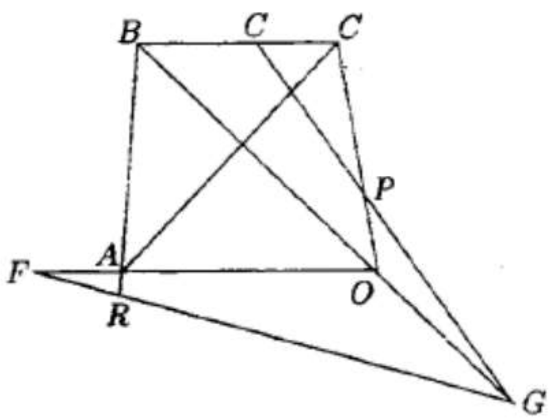
Thus,

$$\left.\begin{array}{l} BC \cap (ABD) = \{E\} \\ BC \cap (ABD) = \{B\} \end{array}\right\} \Rightarrow E = B \Rightarrow B \in PQ$$

false.

We show in the same way that PQ does not cut AC or DC, so a line meets at most three of the given lines.

b. We consider points E, F, G such that $E \in |BC|, A \in |DF|, D \in |BG|$. These points determine plane (EFG) which obviously cuts the lines BC, BD and BD. FG does not separate A and D or $BD \Longrightarrow$ it does not separate A or $B \Longrightarrow A \in |BR|$.



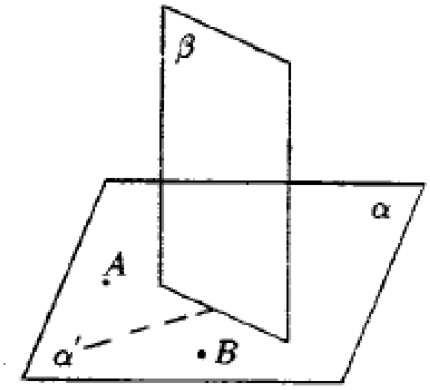
Let's show that (EFG) meets as well the lines AB, CD, AC. In the plane (ABD) we consider the triangle FDG and the line AB.

As this line cuts side $|FD|$, but it does not cut $|DG|$, it must cut side $|FG|$, so $AB \cap |FG| = \{R\} \Rightarrow R \in |FG| \subset (EFG)$, so $AB \cap (EFG) = \{R\}$. In the plane (BCD), the line EG cuts $|BC|$ and does not cut $|BD|$, so EG cuts the side |CD|, $EG \cap |CD| = \{P\} \Rightarrow P \in EG \subset (EFG) \Rightarrow CD \cap (EFG) = \{P\}$.

$$\left.\begin{array}{c} R \in (EFG), R \text{ does not separate } A \text{ and } B \\ E \text{ separates } B \text{ and } C \end{array}\right\} \Rightarrow R \in\cap |AC| = Q \Rightarrow Q \in RE \Rightarrow Q \in (EFG) \cap AC = \{Q\}.$$

Solution to Problem 128.

We assume problem is solved, if $\left.\begin{array}{c} M \in \alpha \\ M \in \beta \end{array}\right\} \Rightarrow \alpha \cap \beta \neq \emptyset, \Rightarrow \alpha \cap \beta = d$.



As $||MA|| = ||MB|| \Rightarrow M \in$ the bisecting line of the segment $[AB]$.

So, to find M, we proceed as follows:

1. We look for the line of intersection of planes α and β, d. If $\alpha \parallel \beta$, the problem hasn't got any solution.
2. We construct the bisecting line d' of the segment $[AB]$ in the plane α.
3. We look for the point of intersection of lines d and d'. If $d \parallel d'$, the problem hasn't got any solution.

Solution to Problem 129.

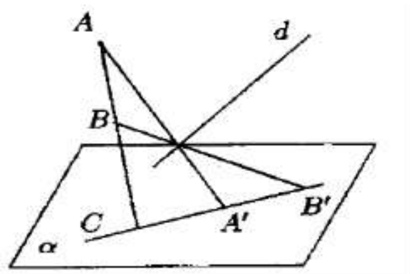
If $\alpha \cap \beta = \emptyset \Longrightarrow \alpha \cap \beta \cap \gamma = \emptyset$. If $\alpha \cap \beta = d$, the desired intersection is $d \cap \gamma$, which can be a point (the 3 planes are concurrent), the empty set (the line of intersection of two planes is || with the third) or line d (the 3 planes which pass through d are secant).

Solution to Problem 130.

To determine M, we proceed as follows:

1. We construct plane (Ad_1) and we look for the line of intersection with α_1, d_1. If d_1 (/∃), ∄ neither does M.
2. We construct plane (Bd_2) and we look for the line of intersection with α, $d_2{}'$. If $d_2{}'$ does not exist, neither does M.
3. We look for the point of intersection of lines $d_1{}'$ and $d_2{}'$. The problem has only one solution if the lines are concurrent, an infinity if they are coinciding lines and no solution if they are parallel.

Solution to Problem 131.



We assume the problem is solved.

a. First we assume that $A.B,C$ are collinear. As AA' and BB' are concurrent lines, they determine a plane β, that intersects α after line $A'B'$.

As

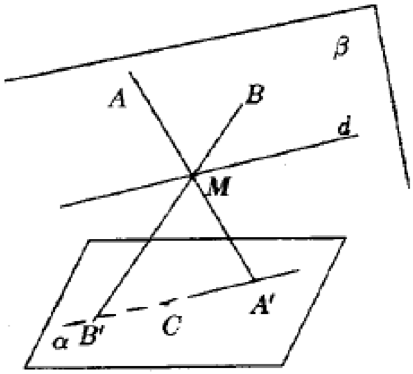
$$\left.\begin{array}{r} C \in AB \Rightarrow C \in \beta \\ C \in \alpha \end{array}\right\} \Rightarrow C \in \alpha \cup \beta \Rightarrow C \in A'B'$$

and points C, A, B' are collinear $(\forall) M \in d$.

b. We assume that A, B, C are not collinear.

We notice that: $(AA', BB') = \beta$ (plane determined by 2 concurrent lines).

$\beta \cup \alpha = d'$ and $C \in d'$.



To determine M we proceed as follows:

1) We determine plane (ABC);

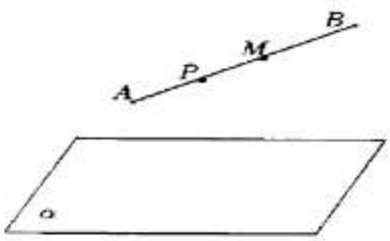
2) We look for the point of intersection of this plane with line d, so $d \cap (ABC) = \{M\}$ is the desired point.

Then $(ABC) \cap \alpha = d'$.

$$\left.\begin{array}{l}\left.\begin{array}{r} AM \cap \alpha = \{A'\} \\ BM \cap \beta = \{B'\} \end{array}\right\} \Rightarrow A'B \in d' \\ \left.\begin{array}{r} C \in \{A'\} \\ C \in \alpha \end{array}\right\} \Rightarrow \qquad C \in d' \end{array}\right\} \Rightarrow$$

$\Longrightarrow A', B', C'$ are collinear.

Solution to Problem 132.



$A \in \sigma$ and $B \in \sigma \Longrightarrow [AB] \cap \alpha \neq \emptyset$.

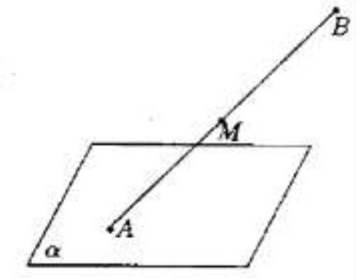
Let $\sigma = |\alpha A = |\alpha B$.

Let $M \in |AB|$ and we must show that $M \in \sigma (\forall) M$ inside the segment.

We assume the contrary that $M \notin \sigma \Longrightarrow (\exists) P$ such that $[AM] \cap d = \{P\} \Longrightarrow P \in [AM] \Longrightarrow P \in [AB] \Longrightarrow [AB] \cap \alpha \neq \emptyset$ false.

$P \in \alpha$, so $M \in \sigma$. The property is also maintained for the closed half-space.

Compared to the previous case there can appear the situation when one of the points A and $B \in \alpha$ or when both belong to α.



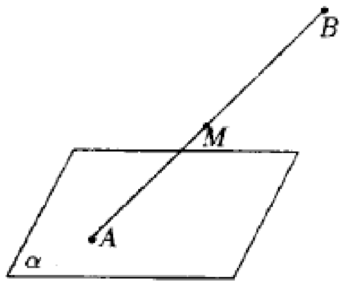
If $A \in \alpha, B \in \sigma, |AB| \cap \alpha \neq \emptyset$ and we show as we did above that:

$$\left.\begin{array}{l} |AB| \subset \sigma \\ A \in \alpha \\ B \in \sigma \end{array}\right\} \Rightarrow [AB] \subset \sigma \bigcup \alpha$$

If:

$A, B \in \alpha \Rightarrow AB \subset \alpha \Rightarrow [AB] \subset \alpha \Rightarrow [AB] \subset \alpha \bigcup \sigma.$

Solution to Problem 133.



Let

$M \in |BA \Rightarrow B \notin [MA] \Rightarrow [MA] \bigcap \alpha = \oslash$

$\Rightarrow M \in |\alpha A$

So

$|BA \subset |\alpha A$

Solution to Problem 134.

Let α be a plane and σ_1, σ_2 the two half-spaces that it determines. We consider half-space σ_1.

$d \bigcap \alpha = \oslash$

1) $d \cap \sigma_1 = \varnothing$

2) $d \cap \sigma_1 \neq \varnothing$, fie $A \in d \cap \sigma_1 \Rightarrow \begin{cases} A \in \sigma_1 \\ A \in d \end{cases}$

$$\left.\begin{array}{r} M \in d \Rightarrow [AM] \subset d \\ d \cap \alpha = \varnothing \end{array}\right\} \Rightarrow \left.\begin{array}{l} [AM] \cap \alpha = \varnothing \\ A \in \sigma_1 \end{array}\right\} \Rightarrow M \in \sigma_1,\ (\forall) M \in d \Rightarrow d \subset \sigma_1 \Rightarrow$$

$\Rightarrow d \cap \sigma_1 = d$

3) $d \cap \alpha \neq \varnothing \Rightarrow d \cap \alpha = \{P\} \Rightarrow$

P determines on d two rays, $|PA$ and $|PB$ where

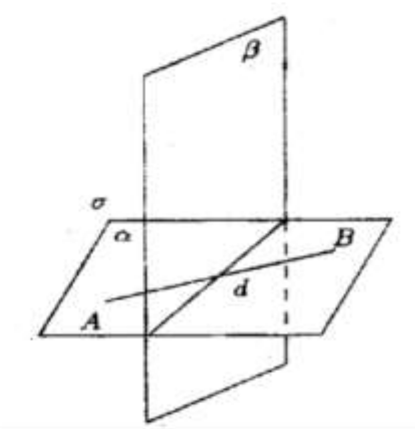
$$\left.\begin{array}{l} P \in |AB| \\ P \in \alpha \end{array}\right\} \Rightarrow |AB| \cap \alpha \neq \varnothing \Rightarrow$$

A and B are in different half-spaces.

We assume

$$\left.\begin{array}{l} A \in \sigma_1 \\ P \in \alpha \end{array}\right\} \overset{pr.2}{\Longrightarrow} |PA \subset |\alpha A \Rightarrow |PA \subset \sigma_1 \Rightarrow \sigma_1 \cap d = |PA.$$

Solution to Problem 135.



Let σ be an open half-space and p its margin and let $d = \alpha \cap \beta$.

We choose points A and $B \in \alpha - d$, on both sides of line $d \Longrightarrow$

$$\left.\begin{array}{l} \Rightarrow [AB] \cap \beta \neq \varnothing \\ d \subset \beta \end{array}\right\} \Rightarrow [AB] \cap \beta \neq \varnothing$$

$\Longrightarrow A, B$ are on one side and on the other side of β and it means that only one of them is on σ.

We assume that $A \in \sigma \Longrightarrow B \in \sigma$. We now prove $\alpha \cap \sigma = |dA$.

$$\alpha \cap \sigma \subset |dA$$

Let

$$\left.\begin{array}{r} M \in \alpha \cap \sigma \Rightarrow M \in \alpha, M \in \sigma \\ A \in \sigma, B \neq \sigma \end{array}\right\} \Rightarrow [MB] \cap B \neq \varnothing \quad \left.\begin{array}{r} M \in \alpha \\ B \in \alpha \end{array}\right\} \Rightarrow$$

$[MB] \in d \neq \emptyset \Longrightarrow M$ and B are on one side and on the other side of line $d \Longrightarrow M$ is on the same side of line d with $A \Longrightarrow M \in |dA$

$$\left.\begin{array}{r} \Rightarrow M \in |dA \Rightarrow [MA] \cap d = \varnothing \\ \alpha \cap \beta = d \\ [MA] \subset \alpha \end{array}\right\} \Rightarrow [MA] \cap B = \varnothing \Rightarrow M \in |\beta A \Rightarrow \left.\begin{array}{r} M \in \sigma \\ M \in \alpha \end{array}\right\} \Rightarrow$$

$$\Longrightarrow M \in a \cap \sigma, \text{ so } |dA \subset a \cap \sigma.$$

Solution to Problem 136.

Let σ be the considered half-space and β its margin. There are more possible cases:

1) $\alpha \cap \beta = \varnothing$

In this case it is possible that:

a) $a \cap \sigma = \varnothing$

b) $\alpha \subset \sigma \neq \varnothing \Rightarrow (\exists) A \in \alpha \cap \sigma \Rightarrow \begin{array}{l} A \in \alpha \\ A \in \sigma \end{array}$

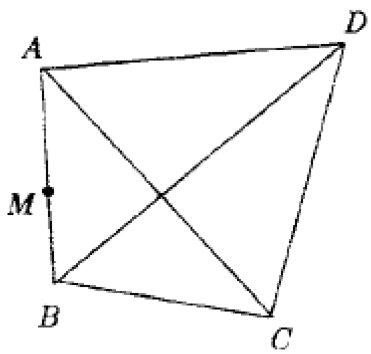
Let

$$\left.\begin{array}{r} M \in \alpha \Rightarrow [MA] \subset \alpha \\ a \cap \beta = \varnothing \end{array}\right\} \Rightarrow \begin{array}{c} [MA] \cap \beta = \varnothing \\ A \in \sigma \end{array} \Rightarrow M \in \sigma, (\forall) M \in \alpha \Rightarrow \alpha \subset \sigma \Rightarrow \alpha \cap \sigma = \alpha$$

2) $a \cap \beta \neq \varnothing \Rightarrow \alpha \cap \beta = d \Rightarrow \alpha \cap \sigma$

is a half-plane according to a previous problem.

Solution to Problem 137.



The intersection of two planes is a line and it cuts only two sides of a triangle.

There are more possible cases:

1. d cuts $\underline{|AB|}$ and $\underline{|BC|}$

d' cuts $|AB|$ and $\underline{|AD|}$, α cuts $|AD|$ so it has a point in common with (ADC) and let $(ADC) \cap \alpha = d''$.

d'' cuts $|AD|$ and does not cut $|AC| \Longrightarrow d''$ cuts $|DC|$

α cuts $|DC|$ and $|BC| \Longrightarrow$ it does not cut $|BD|$. In this case α cuts 4 of the 6 segments (the underlined ones).

2. d cuts $\underline{|AB|}$ and $\underline{|AC|}$, it does not cut $|BD|$

d' cuts $|AB|$ an $|AD|$, it does not cut $|BD|$

d'' cuts $|AD|$ an $|AC|$, it does not cut $|DC|$

$\Longrightarrow \alpha$ does not intersect plane (BCD). In this case α intersects only 3 of the 6 segments.

3. d cuts $\underline{|AB|}$ and $|BC|$, it does not cut $|AC|$

d' cuts $|AB|$ an $|BD|$, it does not cut $|DC|$

α intersects $|BD|$ and $|BC|$, so it does not cut $|DC|$

In $\Delta BDC \Longrightarrow \alpha$ does not intersect plane (ADC)

In this case α intersects only three segments.

4. d cuts $\underline{|AB|}$ and $\underline{|AC|}$, it does not cut $|BC|$

d' cuts $|AB|$ an $\underline{|BD|}$, it does not cut $|AD|$

d'' cuts $|AC|$ an $\underline{|DC|}$

α does not cut $|BC|$ in triangle BDC. So α intersects 4 or 3 segments.

Solution to Problem 138.

$$d \cap \beta = \varnothing$$

Let

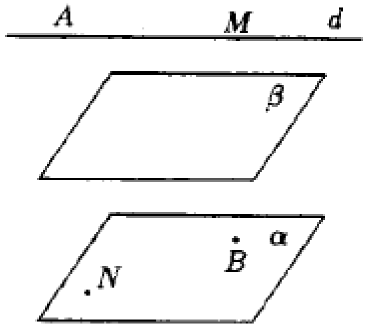
$$\left.\begin{aligned} M \in d \\ A \in d \end{aligned}\right\} \Rightarrow \left.\begin{aligned} [AM] \subset d \\ d \cap \beta = \varnothing \end{aligned}\right\} \Rightarrow = [AM] \cap \beta = \varnothing$$

$$\Rightarrow M \in |\beta A, (\forall) M \in d \Rightarrow d \subset |\beta A$$

Let

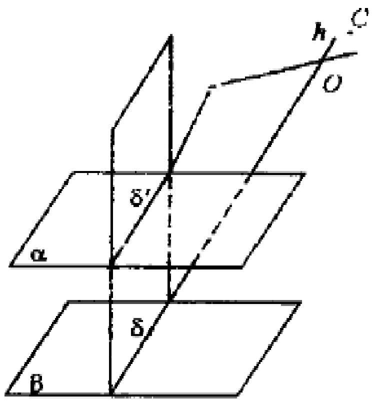
$$\left.\begin{aligned} N \in \alpha \\ B \in \alpha \end{aligned}\right\} \Rightarrow \left.\begin{aligned} [NB] \subset \alpha \\ \alpha \cap \beta = \varnothing \end{aligned}\right\} \Rightarrow = [NB] \cap \beta = \varnothing$$

$$\alpha \subset |\beta B.$$



Solution to Problem 139.

We first assume that $\alpha \neq \beta$ and $|\alpha A \subset |\beta B$.



As

$$\left.\begin{array}{r} A \in |\alpha A \\ |\alpha A \subset |\beta B \end{array}\right\} \Rightarrow A \in |\beta B \Rightarrow |\beta B = |\beta A$$

The hypothesis can then be written as $\alpha \neq \beta$ and $|\alpha A \subset |\beta B$. Let's show that $\alpha \cap \beta = \emptyset$. By reductio ad absurdum, we assume that $\alpha \cap \beta \neq \emptyset \Longrightarrow (\exists) d = \alpha \cap \beta$ and let $O \in d$, so $O \in \alpha$ and $O \in \beta$. We draw through A and O a plane r, such that $d \in r$, so the three planes α, β and r do not pass through this line. As r has the common point O with α and β, it is going to intersect these planes.

$$\left.\begin{array}{r} r \cap \alpha = \delta' \\ r \cap \beta = \delta \end{array}\right\} \Rightarrow \delta \cap \delta' = O$$

which is a common point of the 3 planes. Lines δ and δ' determine 4 angles in plane r, having O as a common peak, $A \in$ the interior of one of them, let $A \in$ int. $\widehat{hk}$. We consider $C \in$ int. $\widehat{hk}$.

Then C is on the same side with A in relation to δ', so C is on the same side with A in relation to $\alpha \Longrightarrow C \in |\alpha A$.

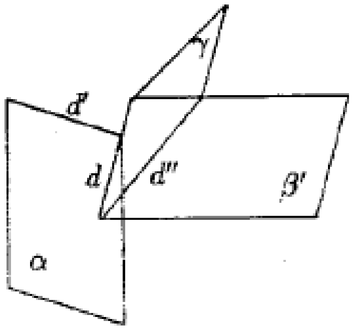
But C is on the opposite side of A in relation to δ, so C is on the opposite side of A in relation to $\beta \Longrightarrow C \notin |\beta A$. So $|\alpha A \not\subset |\beta A$ – false – it contradicts the hypothesis $\Longrightarrow$ So $\alpha \cap \beta = \emptyset$.

Solution to Problem 140.

Let d be the edge of the given dihedral angle. Depending on the position of a line in relation to a plane, there can be identified the following situations:

1) $d \cap \alpha = \{O\}$

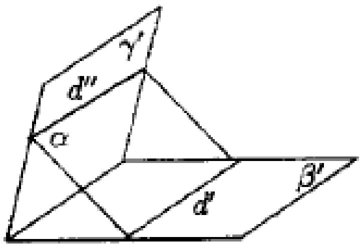
$$\Rightarrow \begin{cases} O \in d \\ O \in \alpha \end{cases} \Rightarrow \begin{cases} O \in \gamma \\ O \in \beta' \end{cases} \Rightarrow \begin{cases} \gamma' \cap \alpha = d' \\ \beta' \cap \alpha = d'' \end{cases}$$



The ray with its origin in O, so $\alpha \subset \widehat{\beta'\gamma'} = \widehat{d'd''}$ thus an angle.

2) $d \cap \alpha = \emptyset$

$$\text{a)} \left.\begin{array}{l} \alpha \cap \beta' \neq \emptyset \Rightarrow \alpha \cap \beta = d' \\ \alpha \cap \gamma' \neq \emptyset \Rightarrow \alpha \cap \gamma' = d'' \end{array}\right\} \Rightarrow d' \parallel d''$$



Indeed, if we assumed that $d' \cap d'' \neq \emptyset \Longrightarrow (\exists) O \in d' \cap d''$.

$$\Rightarrow \begin{cases} O \in d' \\ O \in d'' \end{cases} \Rightarrow \begin{cases} O \in \beta' \\ O \in \gamma' \\ O \in a \end{cases} \Rightarrow \left.\begin{array}{l} O \in d \\ O \in \alpha \end{array}\right\} \Rightarrow O \in d \cap \alpha,$$

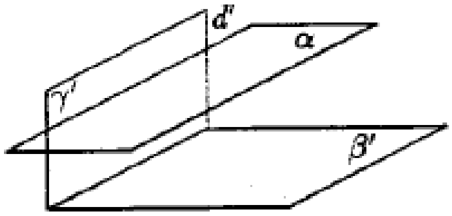
false – it contradicts the hypothesis.

$$\text{b)} \begin{cases} \alpha \cap \beta' = d' \\ \alpha \cap \gamma' = \emptyset \end{cases}$$

Or

$$\begin{cases} \alpha \cap \beta' = \emptyset \\ \alpha \cap \gamma' = d'' \end{cases} \Rightarrow$$

in this case $\alpha \cap \widehat{\beta'\gamma'} = d''$ - a line.

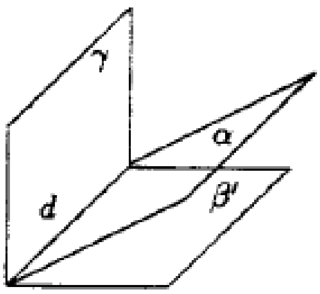


c) $\left.\begin{matrix} \alpha \cap \beta' = \emptyset \\ \alpha \cap \gamma' = \emptyset \end{matrix}\right\}$

Then $\alpha \cap \widehat{\beta'\gamma'} = \emptyset$.

3) $d \cap \alpha = d$

a)



$d \cap \alpha = d$, but $\alpha \neq \beta$, $\alpha \neq \gamma$

$\alpha \cap \widehat{\beta'\gamma'} = d$ thus the intersection is a line.

b) $\alpha = \beta$ $\quad \alpha = \gamma$.

In this case the intersection is a closed half-plane.

Solution to Problem 141.

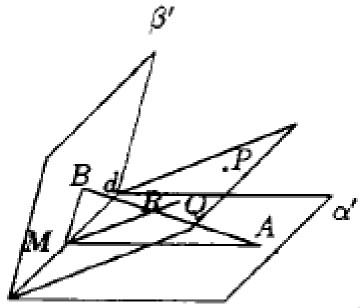
1) int. $\widehat{\alpha'\beta'} = |\alpha B \cap |\beta A$

$P \in \text{int.}\alpha'\beta' \Rightarrow P \in \alpha B$

$P \in |\beta A$

$a \cap (Pd) = d \overset{\text{pr.4}}{\Longrightarrow} (Pd) \cap (\alpha B)$ is a half-plane $\quad P \in |\alpha B$

$\Rightarrow (Pd) \cap |\alpha B = dP \qquad (*)$



$$\left.\begin{array}{r} (Pd)\cap\beta = d \Rightarrow (Pd)\cap|\beta A \text{ is a half-plane} \\ P\in|\beta A \end{array}\right\} \Rightarrow (Pd)\cap|\beta A = |dP \quad (**)$$

From (*) and (**),

$$\Rightarrow (Pd)\cap|\alpha B\cap|\beta A = |dP \Rightarrow (Pd)\cap \text{ int. } \widehat{\alpha'\beta'} = |dP$$

2) $(ABM)\cap\alpha = AM$ so they are secant planes $\overset{\text{pr.4}}{\Longrightarrow} (AMB)\cap\beta A = |MB_1A$

$(AMB)\cap \text{ int. } \widehat{\alpha'\beta'} = (AMB)\cap|\alpha B\cap|\beta A = [(AMB)\cap|\alpha B]\cap[(AMB)\cap|\beta A] =$

$= |MB_1A\cap|MA_1B = \text{int.}\widehat{AMB}.$

Solution to Problem 142.

$$\left.\begin{array}{r} M\in d \Rightarrow M\in dP \\ M\in(ABM) \end{array}\right\} \Rightarrow (AMB)\cap(dP) = d' \text{ şi } M\in d'$$

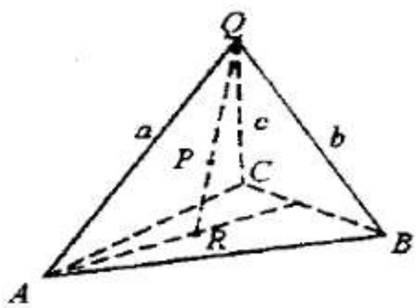
$$\left.\begin{array}{l} d'\subset(dP) \\ d'\cap d = M \end{array}\right\} \Rightarrow \left.\begin{array}{r} |dP\cap d' = |MQ \\ Q\in|dP \end{array}\right\} \Rightarrow \left.\begin{array}{l} |MQ\subset|dP\subset \text{ int.}\widehat{\alpha'\beta'} \\ |MQ\subset d'\subset(AMB) \end{array}\right\} \Rightarrow$$

$$\Rightarrow |MQ\subset \text{int.}(\widehat{\alpha'\beta'})\cap(ABM) \Rightarrow |MQ\subset \text{ int. } \widehat{AMB} \Rightarrow \begin{array}{l} |MQ\subset|AB| = \{R\} \\ |MQ\subset|dP \end{array}$$

$$\Rightarrow |dP\cap|AB| = \{R\} \qquad |dP\subset(dP) \qquad (dP)\cap|AB| = \{R\}$$

$\Longrightarrow$ points A and B are on different sides of (dP).

Solution to Problem 143.



Let rays $\alpha' = |OA_1B,\ \beta = |OA_1C,\ \gamma' = |OA_1P$.

As P is interior to the dihedron formed by any half-plane passing through O of the trihedral, so

$$\left.\begin{array}{r} P \in \text{ int } \widehat{\alpha'\beta'} \\ B \in \alpha' \\ C \in \gamma \end{array}\right\} \Rightarrow |BC| \cap \gamma' = \{Q\}$$

So

$$\left.\begin{array}{r} P \in |OA_1P \\ Q \in |OA_1P = \gamma' \end{array}\right\} \Rightarrow$$

P and Q in the same half-plane det. $OA \Rightarrow P$ and Q on the same side of OA (1)

$P \in \text{int } \widehat{abc} \Rightarrow P \in |OAC,\ A \Rightarrow$

$\Longrightarrow P$ and A are on the same side of $(OBC) \cap \gamma' \Longrightarrow$ P and A are on the same side of OQ (2).

From (1) and (2) $\Longrightarrow$

$$P \in \text{int.}\widehat{AOQ} \overset{\text{T. transv.}}{\Longrightarrow} |AQ| \cap |OP = \{R\} \Rightarrow R \in |AQ|,\ |AQ| \subset \text{int.}ABC$$

$$\Rightarrow R \in \text{int.}ABC \Rightarrow |OP \cap \text{ int.}ABC = \{R\}.$$

Solution to Problem 144.

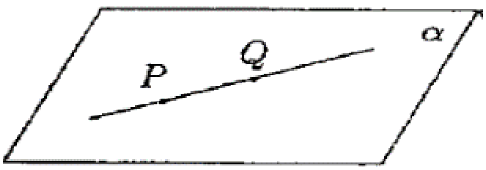
Let M and M' be two convex sets and $M \cap M'$ their intersection. Let

$$\left.\begin{array}{r} \Rightarrow PQ \in M \cap M', P \neq Q \Rightarrow P \in M \cap Q \in M \\ P \in M' \cap Q \in M' \end{array}\right\} \overset{M, M' \text{convex}}{\Longrightarrow} \left\{\begin{array}{l} |PQ| \subset M \\ |PQ| \subset M' \end{array}\right. \Rightarrow$$

$$\Rightarrow |PQ| \subset M \cap M'$$

so the intersection is convex.

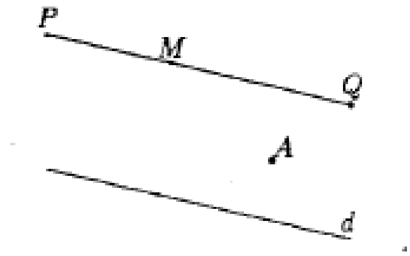
Solution to Problem 145.



a. Let $P, Q \in \alpha$; $P \neq Q \Longrightarrow |PQ = PQ$ (the line is a convex set)

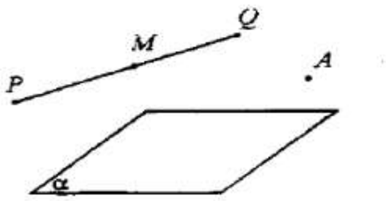
$PQ \subset \alpha$, so $|PQ| \subset \alpha$, so the plane is a convex set.

b. Half-planes: Let $S = |dA$ and $P, Q \in S \Rightarrow |PQ| \cap d = \emptyset$. Let $M \in |PQ| \Rightarrow |PM| \subset |PQ| \Rightarrow |PM| \cap d = \emptyset \Rightarrow$ P and M are in the same half-plane $\Rightarrow M \in S$. So $|PQ| \subset S$ and S is a convex set.



Let $S' = [dA$. There are three situations:

1) $P, Q \in |dA$ – previously discussed;
2) $P, Q \in d \Rightarrow |PQ| \subset d \subset S'$;
3) $P \in d, Q \notin d \Rightarrow |PQ| \subset |dQ \Rightarrow |PQ| \subset |dA \subset [dA$ so $[dA$ is a convex set.



c. Half-spaces: Let $\sigma = |\alpha A$ and let $P, Q \in \sigma \Rightarrow |PQ| \cap \alpha = \emptyset$.

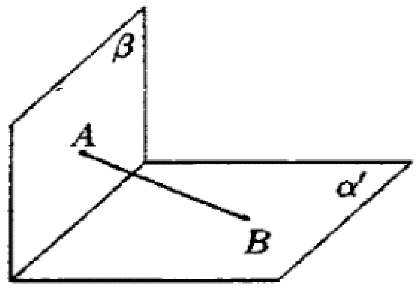
Let $M \in |PQ| \Rightarrow |PM| \subset |PQ| \Rightarrow |PM| \cap \alpha = \emptyset$.

Let $\sigma' = [\alpha A$. There are three situations:

1) $P, Q \in |\alpha A$ previously discussed;
2) $P, Q \in \alpha \Rightarrow |PQ| \subset \alpha \subset \sigma'$;
3) $P \in \alpha,\ Q \notin \alpha$.

$P, Q \in \sigma' \Rightarrow [PQ] \subset \sigma' \Rightarrow |PQ \subset \sigma'|$

and so σ' is a convex set.



d. the interior of a dihedral angle:

$int.\alpha'\beta' = |\alpha A \cap |\beta B$ and as each half-space is a convex set and their intersection is the convex set.

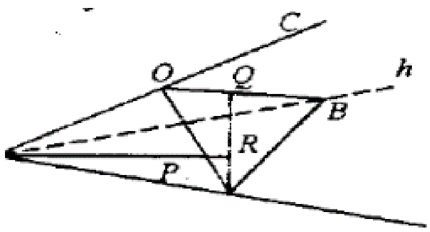
Solution to Problem 146.

No. The dihedral angle is not a convex set, because if we consider it as in the previous figure $A \in \beta'$ and $B \in \alpha'$.

$$(\forall)P \in \text{int.}\widehat{\alpha'\beta'}, |dP \cap |AB| \neq \emptyset$$
$$\left.\begin{array}{r}\Rightarrow (\exists)M \in |AB| \text{ a.i.} M \in \text{int.}\widehat{\alpha'\beta'} \\ M \notin \alpha', M \notin \beta'\end{array}\right\} \Rightarrow |AB| \not\subset \widehat{\alpha'\beta'}.$$

Only in the case of the null or straight angle, when the dihedral angle becomes a plane or closed half-plane, is a convex set.

Solution to Problem 147.



a. No. The trihedral angle is not the convex set, because, if we take $A \in a$ and $Q \in$ the int. $\widehat{bc}$ determined by $P \in$ the int. $\widehat{abc}$, $(\exists)R$ such that $|OP \cap$ the int. $ABC = \{R\}$, $R \in |AQ|, R \notin \widehat{abc}$.

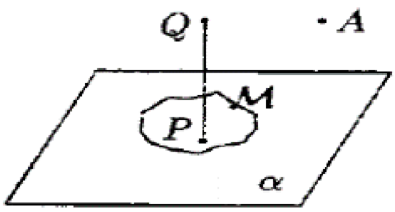
So $A, Q \in \widehat{abc}$, but $|AQ| \notin abc$.

b. $B = (OCA), \gamma = (OAB)$ is a convex set as an intersection of convex sets.

C) It is the same set from a. and it is not convex.

D) The respective set is $[\alpha A \cap [\beta B \cap [\gamma C$, intersection of convex sets and, thus, it is convex.

Solution to Problem 148.



Let $\sigma = |\alpha A$ and $M \subset \alpha$. Let $P, Q \in M \cap \sigma$.

We have the following situations:

a) $P, Q \in \sigma \Rightarrow |PQ| \subset \sigma \subset \sigma \subset \mathcal{M}$;

b) $P.Q \in \mathcal{M} \Rightarrow |PQ| \subset \mathcal{M} \subset \sigma \cap \mathcal{M}$;

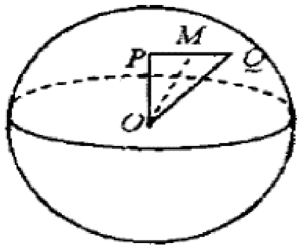
c) $P \in \mathcal{M}, Q \in \sigma \Longrightarrow |PQ| \subset \sigma \subset \sigma \cap \mathcal{M}$.

Solution to Problem 149.

$$S(O,r) = \{M \in S \big/ \|OM\| = r\}.$$

$$S(O,r) \cap \alpha = \{M \in S \big/ M \in S(O,r) \cap M\alpha \cap \|OM\| = r\} = \{M \in \alpha \big/ \|OM\| = r\} =$$
$$= C(O,r)$$

Solution to Problem 150.



Let

$P, Q \in \text{int}.S(O,r) \Rightarrow \|OP\| \subset r, \|OQ\| \subset r.$

In plane (OPQ), let $M \in (PQ)$.

$$\left.\begin{array}{c} \Rightarrow \|OM\| < \|OP\| < r \\ \text{or} \\ \|OM\| < \|OQ\| < r \end{array}\right\} \Rightarrow \|OM\| < r$$

$\Rightarrow M \in S(O,r)(\forall)M \subset |PQ| \Rightarrow |PQ| \subset S(O,r)$

Solution to Problem 151.

Let: P midpoint of $|AB|$

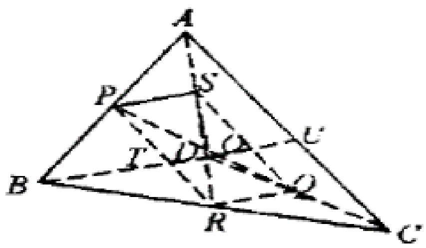
R midpoint of $|BC|$

Q midpoint of $|DC|$

S midpoint of $|AD|$

T midpoint of $|BD|$

U midpoint of $|AC|$



In triangle ABC:

$|RP|$ l.m. $\|PR\| = \dfrac{\|AC\|}{2}$; $PR \parallel AC$.

In triangle DAC:

$|SQ|$ l.m. $\|SQ\| = \dfrac{\|AC\|}{2}$; $SQ \parallel AC$.

$\Rightarrow \|PR\| = \|SQ\|, PR \parallel SQ \Rightarrow PRSQ$

$\Rightarrow$ parallelogram $\Rightarrow |PQ|$ and $|SR|$ intersect at their midpoint O.

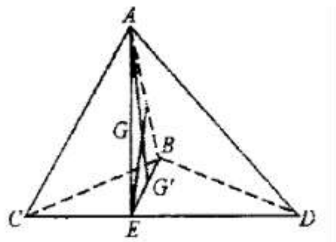
$$\left.\begin{matrix} \|ST\| = \dfrac{\|AB\|}{2}, & \|OR\| = \dfrac{\|AB\|}{2} \\ ST \parallel AB & UR \parallel AB \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \|ST\| = \|UR\| \\ ST \parallel UR \end{matrix}\right\} \Rightarrow$$

$\Rightarrow STRU$ parallelogram.

$\Rightarrow |TU|$ passes through midpoint O of $|SR|$.

Thus the three lines PR, SR, TU are concurrent in O.

Solution to Problem 152.



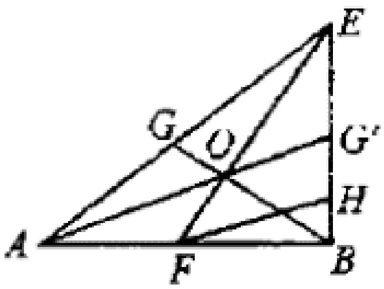
Let tetrahedron $ABCD$ and E be the midpoint of $|CD|$. The centroid G of the face ACD is on $|AE|$ at a third from the base. The centroid G' of the face BCD is on $|BE|$ at a third from the base $|CD|$.

We separately consider ΔAEB. Let F be the midpoint of AB, so EF is median in this triangle and, in the previous problem, it was one of the 3 concurrent segments in a point located in the middle of each.

Let O be the midpoint of $|EF|$. We write $AO \cap EB = \{G\}$ and $BG \cap EA = \{G\}$.

$$|AF| \equiv |FB| \Rightarrow |FH| \text{ m. in } ABG' \Rightarrow |BH| \equiv |HG'| \qquad (1)$$

$$\left.\begin{matrix} OG' \parallel FH \\ |EO| \equiv |OF| \end{matrix}\right\} \Rightarrow |OG'| \text{ l.m. in } \Delta EFH \Rightarrow |EG'| \equiv G'H \qquad (2)$$



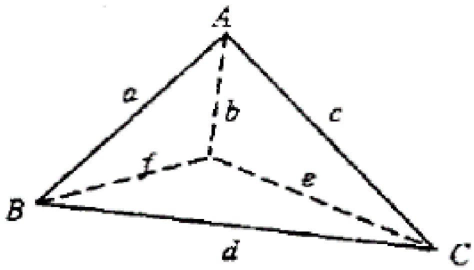
From (1) and (2)

$\Rightarrow |EG'| \equiv |G'H| \equiv |HB| \Rightarrow \frac{\|EG\|}{\|EB\|} = \frac{1}{3} \Rightarrow$

$\Longrightarrow G'$ is exactly the centroid of face BCD, because it is situated on median $|EB|$ at a third from E. We show in the same way that G is exactly the centroid of face ACD. We've thus shown that BG and AG' pass through point O from the previous problem.

We choose faces ACD and ACB and mark by G'' the centroid of face ACB, we show in the same way that BG and DG'' pass through the middle of the segment $|MN|$ ($|AM| \equiv |MC|, |BN| \equiv |ND|$) thus also through point O, etc.

Solution to Problem 153.



We mark planes $(ABC) = \alpha, (ADC) = \beta, (BDC) = \gamma, (ABO) = \delta$.

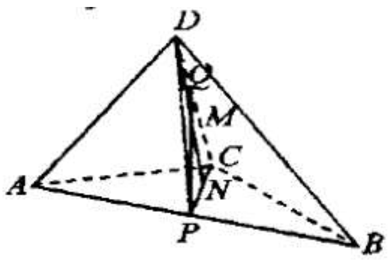
Let M be the intersection of the interiors of the 4 trihedral angles.

We show that:

$M = \text{int.}[ABCD]$, by double inclusion.

1. $P \in M \Longrightarrow P \in \text{int.}\widehat{abc} \cap \text{int.}\widehat{afd} \cap \text{int.}\widehat{dec} \cap \text{int.}\widehat{bfc} \Longrightarrow P \in |\alpha D \cap |\gamma C \cap \beta B$ and $P \in |\delta A \cap |\gamma C \cap |\beta B \Longrightarrow P \in |\alpha D$ and $P \in |\beta B$ and $P \in |\gamma C$ and $P \in |\delta C \Longrightarrow P \in |\alpha D \cap |\gamma C \cap \beta B \cap \delta A \Longrightarrow P \in \text{int.}[ABCD]$. So $M \in [ABCD]$.
2. Following the inverse reasoning we show that $[ABCD] \subset M$ from where the equality.

Solution to Problem 154.



$M \in \text{int.}[ABCD] \Rightarrow (\exists) N \in \text{int. } ABC$

such that

$M \in |DN|,\ N \in \text{int.} ABC \Rightarrow (\exists) P \in |AB|$

such that $N \in |CP|, (ADB) \cap (DPC) = DP$.

From $N \in |CP|$ and $\in |DN| \xrightarrow{lemma} \text{int.} DPC \Rightarrow M \in \text{int.} DPC \Rightarrow (\exists) Q \in |DC|$.

So we showed that $(\exists) P \in |AB|$ and $Q \in |DC|$ such that $M \in |PQ|$.

Solution to Problem 155.

Let $\mathcal{M}$ be the union of the open segments $|PQ|$. We must prove that: int. $[ABCD] = \mathcal{M}$ through double inclusion.

1. Let $M \in \text{int.}[ABCD] \Rightarrow (\forall) P \in |AB$ and $Q \in |CD$ such that $M \in |PQ| \Rightarrow M \in \mathcal{M}$ so int. $[ABCD] \subset \mathcal{M}$.
2. Let $M \in \mathcal{M} \Rightarrow (\exists) P \in |AB|$ and $Q \in |CD|$ such that $M \in |PQ|$. Points D, C and P determine plane (PDC) and $(PDC) \cap (ACB) = PC,\ (PUC) \cap (ADB) = PD$.

As $(\forall) Q \in |CD|$ such that $M \in |PQ| \Rightarrow M \in [PCD] \Rightarrow |(\forall) R \in |PC|$ such that $M \in |DR|$.

If $P \in |AB|$ and $R \in |PC| \Rightarrow R \in$ int. ACB such that $M \in |DR| \Rightarrow M\ \text{int.}[ABCD] \Rightarrow \mathcal{M} \subset \text{int.}[ABCD]$.

Working with closed segments we obtain that $(\forall) R \in [ACB]$ such that $M \in [DR]$, thus obtaining tetrahedron $[ABCD]$.

Solution to Problem 156.

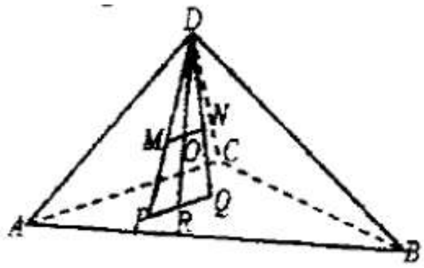
Let $M \in [ABCD]\ \Rightarrow (\exists)\ P \in [ABC]$ such that $M \in [DP]$.

Let $N \in [ABCD] \Rightarrow\ (\exists)\ Q \in [ABC]$ such that $N \in [DQ]$.

The concurrent lines DM and DN determine angle DMN.

The surface of triangle DPQ is a convex set.

$$\Rightarrow \left.\begin{array}{l} M \in [DPQ] \\ N \in [DPQ] \end{array}\right\} \Rightarrow |MN| \subset [DPQ].$$



Let

$O \in |MN| \Rightarrow O \in [DPQ] \Rightarrow (\exists)R \in [PQ]$

such that $O \in [DR]$. But $[PQ] \subset [ABC]$ because $P \in [ABC] \cap Q \in [ABC]$ and the surface of the triangle is convex. So $(\exists)R \in [ABC]$ such that

$O \in [DR] \Rightarrow O \in [ABCD], (\forall)O \in |MN| \Rightarrow |MN| \subset [ABCD]$

and the tetrahedron is a convex set.

Note: The tetrahedron can be regarded as the intersection of four closed half-spaces which are convex sets.

Solution to Problem 157.

Let $\mathcal{M}$ be the union of the segments $[PQ]$ with $P \in \mathcal{M}_1$ and $Q \in \mathcal{M}_2$.

Let $x, x' \in \mathcal{M} \Longrightarrow (\forall)P \in \mathcal{M}_1$ and $Q \in \mathcal{M}_2$ such that $x \in [PQ]$;

$(\exists)P' \in \mathcal{M}_1$ and $Q' \in \mathcal{M}_2$ such that $x' \in [P'Q']$.

From $P, P' \in \mathcal{M}_1 \Longrightarrow [PP']' \in \mathcal{M}_1$ which is a convex set.

From $Q, Q' \in \mathcal{M}_2 \Longrightarrow [QQ'] \in \mathcal{M}_2$ which is a convex set.

The union of all the segments $[MN]$ with $M \in [PP']$ and $N \in [QQ']$ is tetrahedron $[PP'QQ'] \subset \mathcal{M}$.

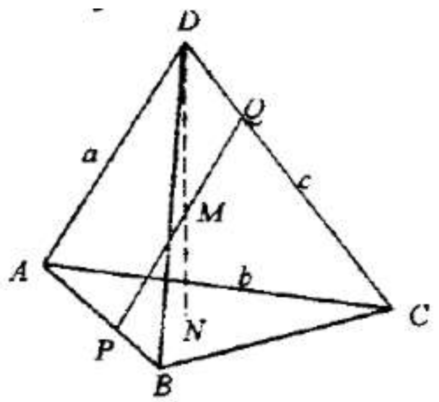
So from $x, x' \in \mathcal{M} \Longrightarrow |xx'| \subset \mathcal{M}$, so set $\mathcal{M}$ is convex.

Solution to Problem 158.

The interior of the tetrahedron coincides with the union of segments $|PQ|, P \in |AB|$ and $Q \in |CD|$, that is $\text{int.}[ABCD] = \{|PQ| \setminus P \in |AB|, Q \in |CD|\}$.

Let's show that:

$\text{int.}[ABCD] = |(ABC), D \cap |(ABD), C \cap |(ADC), B \cap$
$\cap |(DBC), A.$



1. Let

$$M \in \text{int.}[ABCD] \Rightarrow P \in |AB| \;\; Q \in |DC| \qquad M \in |PQ|.$$

$$P \in |AB| \Rightarrow B \notin |AP| \Rightarrow |AP| \cap (BDC) = \emptyset \Rightarrow \left.\begin{array}{l} P \in |(BDC), A \\ P \in (BDC) \end{array}\right\} \Rightarrow$$

$$\Rightarrow |PQ| \subset |(BDC), A; M \in |PQ| \Rightarrow M \in |(BDC), A \qquad (1)$$

$$P \in |AB| \Rightarrow A \notin |PB| \Rightarrow |PB| \cap (ADC) = \emptyset \Rightarrow P \in |(ADC),\; B;\; Q \in (ADC) \Rightarrow$$

$$\Rightarrow |PQ| \subset |(ADC), B \Rightarrow M \in |(ADC), B \qquad (2)$$

$$Q \in |DC| \Rightarrow D \notin |QC| \Rightarrow |QC| \cap (ABD) = \emptyset \Rightarrow \left.\begin{array}{l} Q \in |(ABD), C \\ P \in (ABD) \end{array}\right\} \Rightarrow$$

$$\Rightarrow |PQ| \subset |(ABD), C \Rightarrow M \in |(ABD), C \qquad (3)$$

$$Q \in |DC| \Rightarrow C \notin |DQ| \Rightarrow |DQ| \cap (ABC) = \emptyset \Rightarrow \left.\begin{array}{l} Q \in |(ABC), D \\ P \in (ABC) \end{array}\right\} \Rightarrow$$

$$\Rightarrow |PQ| \subset |(ABC), D \Rightarrow M \in |(ABC), D \qquad (4)$$

$$(1),(2),(3),(4) \Rightarrow M \in |(BDC), A \cap |(ADC), B \cap |(ABD), C \cap |(ABC), D$$

$$\text{int.}[ABCD] \subset |(BDC), A \cap |(ADC), B \cap |(ABD), C \cap |(ABC), D.$$

2. Let

$$M \in |(BDC), A \cap |(ADC), B \cap |(ABD), C \cap |(ABC), D \Rightarrow M \in |(BDC), A \cap |(ADC),$$

$$B \cap |(ABD), C \Rightarrow M \in \text{int. } \widehat{abc} \Rightarrow |DM| \cap \text{int.}ABC = \{N\}.$$

If we assume $N \in |DM| \Longrightarrow |DM| \cap (ABC) \neq \emptyset \Longrightarrow M$ and D are in different half-spaces in relation to $(ABC) \Longrightarrow M \notin (ABC), D$, false (it contradicts the hypothesis).

So

$$M \in |DN|, (\exists) N \in \text{int.}ABC \text{ a.i. } M \in |DN| \Rightarrow M \in \text{int. } [ABCD]$$

and the second inclusion is proved.

As regarding the tetrahedron: $[ABCD] = \{[PQ] \setminus P \in [AB] \text{ and } Q \in [CD]\}$.

$P = A, Q \in [CD], [PQ]$ describes face $[ADC]$

If $P = B, Q \in [CD], [PQ]$ describes face $[BDC]$

$Q = C, P \in [AB], [PQ]$ describes face $[ABC]$.

Because the triangular surfaces are convex sets and along with their two points P, Q, segment $[PQ]$ is included in the respective surface.

So, if we add these two situations to the equality from the previous case, we obtain:

$$[ABCD] = [(BCD), A \cap [(ACD), B \cap [(ABD), C \cap [(ABC), D.$$

Lines and Planes

159. Let d, d' be two parallel lines. If the line d is parallel to a plane α, show that $d'||\alpha$ or $d' \subset \alpha$.

Solution to Problem 159

160. Consider a line d, parallel to the planes α and β, which intersects after the line a. Show that $d\|a$.

Solution to Problem 160

161. Through a given line d, draw a parallel plane with another given line d'. Discuss the number of solutions.

Solution to Problem 161

162. Determine the union of the lines intersecting a given line d and parallel to another given line d' $(d \nparallel d')$.

Solution to Problem 162

163. Construct a line that meets two given lines and that is parallel to a third given line. Discuss.

Solution to Problem 163

164. If a plane α intersects the secant planes after parallel lines, then α is parallel to line $\beta \cap \gamma$.

Solution to Problem 164

165. A variable plane cuts two parallel lines in points M and N. Find the geometrical locus of the middle of segment $[MN]$.

Solution to Problem 165

166. Two lines are given. Through a given point, draw a parallel plane with both lines. Discuss.

Solution to Problem 166

167. Construct a line passing through a given point, which is parallel to a given plane and intersects a given line. Discuss.

Solution to Problem 167

168. Show that if triangles ABC and $A'B'C'$, located in different planes, have $AB \parallel A'B', AC \parallel A'C'$ and $BC \parallel B'C'$, then lines AA', BB', CC' are concurrent or parallel.

Solution to Problem 168

169. Show that, if two planes are parallel, then a plane intersecting one of them after a line cuts the other one too.

Solution to Problem 169

170. Through the parallel lines d and d' we draw the planes α and α' distinct from (d, d'). Show that $\alpha \parallel \alpha'$ or $(\alpha \cap \alpha') \parallel d$.

Solution to Problem 170

171. Given a plane α, a point $A \in \alpha$ and a line $d \subset \alpha$.

a. Construct a line d' such that $d' \subset \alpha$, $A \in d'$ and $d' \parallel d$.
b. Construct a line through A included in α, which forms with d an angle of a given measure a. How many solutions are there?

Solution to Problem 171

172. Show that relation $\alpha \parallel \beta$ defined on the set of planes is an equivalence relation. Define the equivalence classes.

Solution to Problem 172

173. Consider on the set of all lines and planes the relation "$x \parallel y$" or $x = y$, where x and y are lines or planes. Have we defined an equivalence relation?

Solution to Problem 173

174. Show that two parallel segments between parallel planes are concurrent.

Solution to Problem 174

175. Show that through two lines that are not contained in the same plane, we can draw parallel planes in a unique way. Study also the situation when the two lines are coplanar.

Solution to Problem 175

176. Let α and β be two parallel planes, $A, B \in \alpha$, and CD is a parallel line with α and β. Lines CA, CB, DB, DA cut plane β respectively in M, N, P, Q. Show that these points are the vertices of a parallelogram.

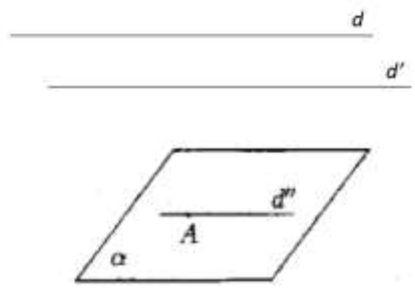
Solution to Problem 176

177. Find the locus of the midpoints of the segments that have their extremities in two parallel planes.

Solution to Problem 177

Solutions

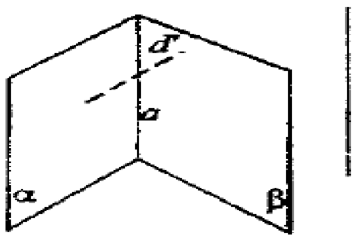
Solution to Problem 159.



Let $\left.\begin{array}{r}\left.\begin{array}{r}A\in\alpha\\ d\parallel\alpha\\ d''\parallel d\end{array}\right\}\Longrightarrow d''\parallel\alpha\\ \left.\begin{array}{r}d'\parallel d\\ d''\parallel d\end{array}\right\}\Longrightarrow d'\parallel d''\end{array}\right\}\Longrightarrow d''\parallel\alpha \text{ or } d'\subset\alpha.$

Solution to Problem 160.

Let $A\in a\Rightarrow A\in\alpha\cap A\in\beta$. We draw through A, $d'\parallel d$.

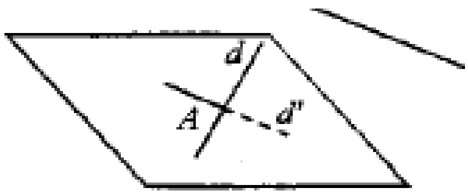


$$\left.\begin{array}{r}A\in\alpha,\left.\begin{array}{r}d\parallel\alpha\\ d'\parallel d\end{array}\right\}\Longrightarrow d'\subset\alpha\\ A\in\beta,\left.\begin{array}{r}d\parallel\beta\\ d'\parallel d\end{array}\right\}\Longrightarrow d'\subset\beta\end{array}\right\}\Longrightarrow\left.\begin{array}{r}d'\subset\alpha\subset\beta\\ \alpha\cap\beta=a\end{array}\right\}\Longrightarrow\left.\begin{array}{r}d'=a\\ d'\parallel d\end{array}\right\}\Longrightarrow a\parallel d$$

Solution to Problem 161.

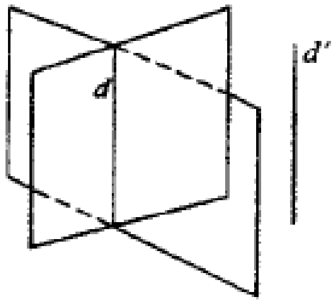
a. If $d\nparallel d'$ there is only one solution and it can be obtained as it follows:

Let $A\in d$. In the plane (A,d'') we draw $d''\parallel d'$. The concurrent lines d and d'' determine plane α. As $d''\parallel d\Longrightarrow d\parallel\alpha$, in the case of the non-coplanar lines.

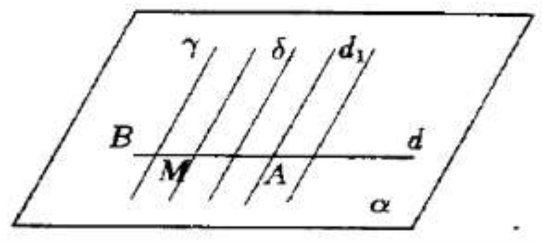


b. If $d \parallel d' \parallel$ d'', ($\exists$) infinite solutions. Any plane passing through d is parallel to d'', with the exception of plane (d, d').

c. $d \nparallel d'$, but they are coplanar ($\nexists$) solutions.



Solution to Problem 162.



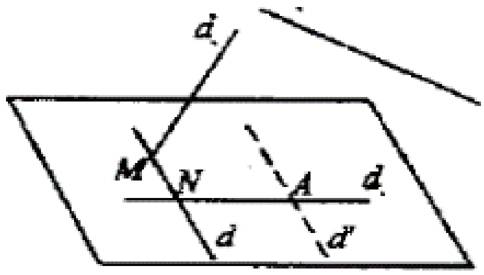
Let $A \in d$, we draw through $A, d_1 \parallel d'$. We write $\alpha = (d, d_1)$. As $d_1 \parallel d' \Rightarrow d' \parallel \alpha$.

Let $M \in d$, arbitrary $\Rightarrow M \in \alpha$.

We draw $\left.\begin{matrix} \delta \parallel \delta', M \in \delta \\ d' \parallel a \end{matrix}\right\} \Rightarrow \delta \subset \alpha$, so all the parallel lines to d' intersecting d are contained in plane α.

Let $\gamma \subset \alpha, \gamma \parallel d' \Rightarrow \gamma \cap d = B$, so ($\forall$) parallel to d' from α intersects d. Thus, the plane α represents the required union.

Solution to Problem 163.



We draw d through M such that $\left.\begin{matrix} d \parallel d' \\ d' \parallel d_3 \end{matrix}\right\} \Rightarrow d \parallel d_3$. According to previous problem: $d \cap d_1 = \{N\}$. Therefore,

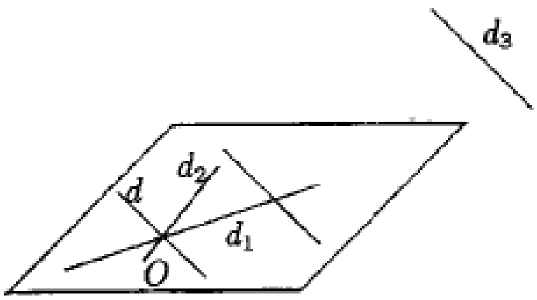
$$d \cap d_2 = \{M\}$$
$$d \cap d_1 = \{N\}$$
$$d \parallel d_3$$

a. If $d_3 \parallel d_1$, the plane α is unique, and if $d_2 \cap \alpha \neq \emptyset$, the solution is unique.

b. If $d_1 \parallel d_3$, $(\nexists)$ $\begin{matrix} d \parallel d_1 \\ d \cap d_1 \neq \emptyset \end{matrix}$, because it would mean that we can draw through a point two parallel lines d, d_1 to the same line d_3. So there is no solution.

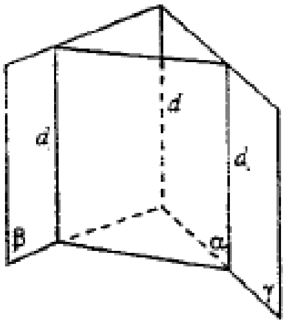
c. If $d_1 \nparallel d_3$ and $d_2 \cap \alpha \neq \emptyset$, all the parallel lines to d_2 cutting d_1 are on the plane α and none of them can intersect d_2, so the problem has no solution.

d. If $d_2 \subset \alpha$, $d_1 \cap d_2 \neq \emptyset$, let $d_1 \cap d_2 = \{O\}$, and the required line is parallel to d_3 drawn through $O \Rightarrow$ one solution.



e. If $d_2 \subset \alpha$, $d_1 \parallel d_2$. The problem has infinite solutions, $(\forall)$ || to d_3 which cuts d_1, also cuts d_2.

Solution to Problem 164.

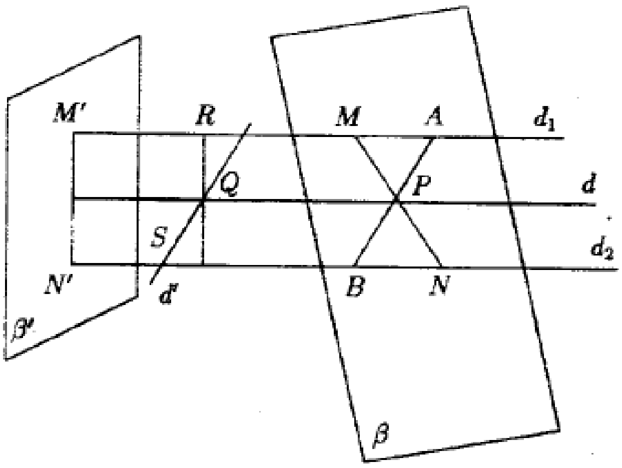


$$\beta \cap \gamma = d$$
$$\alpha \cap \beta = d_1$$
$$\alpha \cap \gamma = d_2$$
$$d_1 \parallel d_2$$

$$\left.\begin{array}{r} d_1 \parallel d_2 \Rightarrow d_1 \parallel \gamma \\ d_1 \subset \beta \end{array}\right\} \Rightarrow \beta \cap \gamma = d$$

$$\left.\begin{array}{r} d = d_1 \\ d_1 \subset \alpha \end{array}\right\} \Rightarrow d \parallel \alpha$$

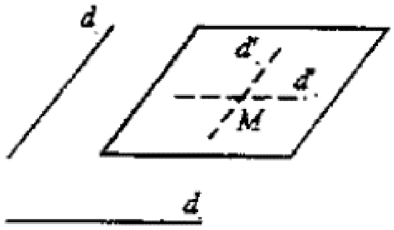
Solution to Problem 165.



$$d_1 \parallel d_2 \Rightarrow (\exists)\alpha = (d_1, d_2)$$

The problem is reduced to the geometrical locus of the midpoints of the segments that have extremities on two parallel lines. P is such a point $|MP| = |PN|$. We draw $AB \perp d1 \Rightarrow AB?\widehat{MPA} = \widehat{BPN} \Longrightarrow \Delta MAP = \Delta NBP \Longrightarrow |PA| \equiv |PB| \Longrightarrow ||AP||? \Longrightarrow$ the geometrical locus is the parallel to d_1 and d_2 drawn on the mid-distance between them. It can also be proved vice-versa.

Solution to Problem 166.



$M \notin d_1, M \notin d_2.$

Let $d_1 \nparallel d_2$. In plane (d, M) we draw $d_1' \parallel d_1$, $M \in d_1'$. In plane $(d_2 M)$ we draw $d_2' \parallel d_2$, $M \in d_2'$. We note $\alpha = d_1' d_2'$ the plane determined by two concurrent lines.

$d_1 \parallel d_1'' \Rightarrow d_1 \parallel \alpha$

$d_1 \parallel d_2'' \Rightarrow d_2 \parallel \alpha$

$M \in \alpha$ the only solution.

Let $d_1 \parallel d_2$, $N \notin d_1, M \notin d_2$.

$d_1 = d_1' = d_2'$

In this case $d_1' = d_2' = d$ and infinite planes pass through d;

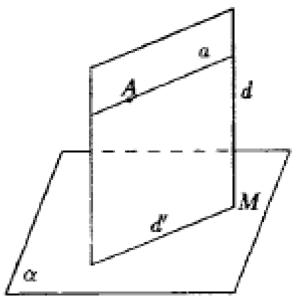
$\left.\begin{matrix} d_2 \parallel d \\ d_1 \parallel d \end{matrix}\right\} \Rightarrow d_1, d_2$ are parallel lines with (∀) of the planes passing through d.

The problem has infinite solutions. But $M \in d_1$ or $M \in d_2$, the problem has no solution because the plane can't pass through a point of a line and be parallel to that line.

Solution to Problem 167.

Let A be the given point, α the given plane and d the given line.

a. We assume that $d \nparallel \alpha, d \cap \alpha = \{M\}$. Let plane (dA) which has a common point M with $\alpha \Rightarrow (dA) \cap \alpha = d'$.



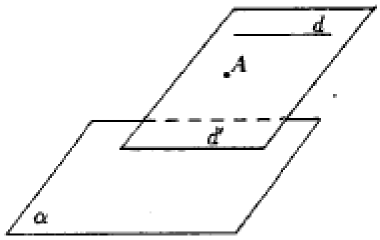
We draw in plane (dA) through point A a parallel line to d'.

$$\left.\begin{array}{l}\left.\begin{array}{l} a \parallel d' \\ d' \cap d = \{M\} \end{array}\right\} \Rightarrow a \cap d \neq \emptyset \\ \left.\begin{array}{l} a \parallel d \\ d' \in \alpha \end{array}\right\} \qquad \Rightarrow a \parallel \alpha, A \in a \end{array}\right\}$$

$\Longrightarrow a$ is the required line.

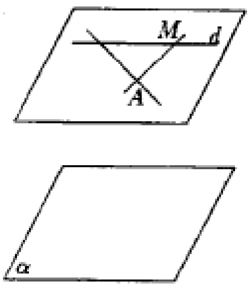
b. $d \parallel \alpha, (dA) \cap \alpha \neq \emptyset$.

Let $\left.\begin{array}{r} (dA) \cap \alpha = \alpha' \\ d \parallel \alpha \end{array}\right\} \Rightarrow d' \parallel d$



All the lines passing through A and intersecting d are contained in plane (dA). But all these lines also cut $d' \parallel d$, so they can't be parallel to α. There is no solution.

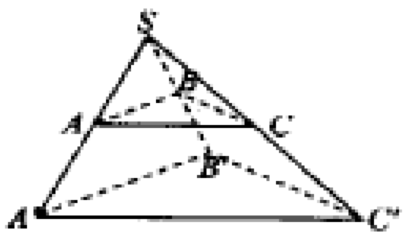
c. $d \parallel \alpha, (dA) \cap \alpha = \emptyset$.



Let $M \in d$ and line $AM \subset (dA)$; $(dA) \cap \alpha = \emptyset \Longrightarrow AM \cap \alpha = \emptyset \Longrightarrow AM \parallel \alpha, (\forall) M \in d$.

The problem has infinite solutions.

Solution to Problem 168.



(ABC) and $(A'B'C')$ are distinct planes, thus the six points A, B, C, A', B', C' can't be coplanar.

$AB \parallel A'B' \Longrightarrow A, B, A', B'$ are coplanar.

The points are coplanar four by four, that is $(ABB'A)$, $(ACC'A')$, $(BCC'B')$, and determine four distinct planes. If we assumed that the planes coincide two by two, it would result other 6 coplanar points and this is false.

In plane $ABB'A'$, lines AA', BB' can be parallel or concurrent.

First we assume that:

$$AA' \cap BB' = \{S\} \Rightarrow S \in AA' \cap S \in BB'$$

$$\left.\begin{array}{l} S \in AA' \Rightarrow \begin{cases} S \in (ABB'A') \\ S \in (ACC'A') \end{cases} \\ S \in BB' \Rightarrow \begin{cases} S \in (BCC'B') \\ S \in (ABB'A') \end{cases} \end{array}\right\}$$

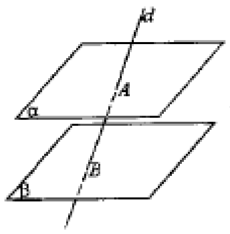
is a common point to the 3 distinct planes, but the intersection of 3 distinct planes can be only a point, a line or $\emptyset$. It can't be a line because lines

$$\left.\begin{array}{l} (ABB'A') \cap (ACC'A') = AA' \\ (ABB'A') \cap (BCC'B') = BB' \\ (ACC'A') \cap (BCC'B') = CC' \end{array}\right\}$$

$\Longrightarrow$ are distinct if we assumed that two of them coincide, the 6 points would be coplanar, thus there is no common line to all the three planes. There is one possibility left, that is they have a common point S and from

$$\left.\begin{array}{l} S \in (ACC'A) \\ S \in (CC'BB') \end{array}\right\} \Rightarrow S \in CC' \Rightarrow$$

$$\alpha \parallel \beta \Rightarrow \alpha \cap \beta = \emptyset$$



We assume $d \cap \beta = \emptyset \Longrightarrow d \parallel \beta \Longrightarrow d \in$ plane $\parallel \beta$ drawn through $A \Longrightarrow d \subset \alpha$, false. So $d \cap \beta = \{B\}$.

Solution to Problem 169.

Hypothesis: $\alpha \parallel \beta, \gamma \cap \alpha = d_1$.

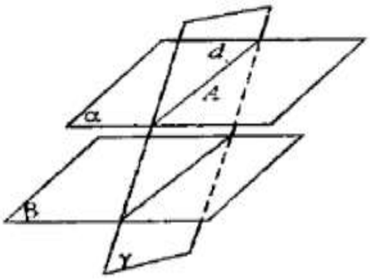
Conclusion: $\gamma \cap \beta = d_2$.

We assume that $\gamma \cap \beta = \emptyset \Longrightarrow \gamma \parallel \beta$.

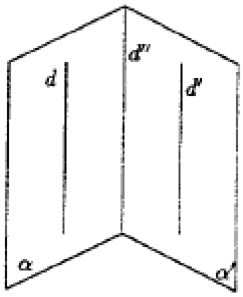
Let

$$A \in d_1 \Rightarrow \begin{cases} A \in \alpha, \ \alpha \parallel \beta \\ A \in \gamma, \ \gamma \parallel \beta \end{cases} \Rightarrow \alpha = \gamma$$

because from a point we can draw only one parallel plane with the given plane. But this result is false, it contradicts the hypothesis $\gamma \cap \alpha = d_1$ so $\gamma \cap \beta = d_2$.



Solution to Problem 170.



Hypothesis: $d \parallel d'$; $d \subset \alpha; d' \ \subset \alpha'$; $\alpha, a' \neq (dd')$.

Conclusion: $\alpha \parallel \alpha'$ or $d'' \parallel d$.

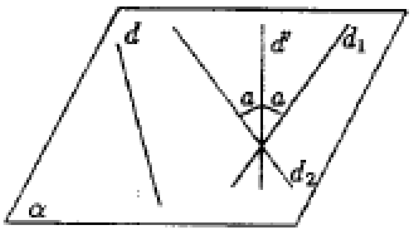
As $\alpha, a' \ \neq (dd') \Longrightarrow \alpha \ \neq \ a'$.

If $a \cap a' = \emptyset \Longrightarrow a \parallel a'$.

If $a \cap a' = \emptyset \Longrightarrow a \cap a' = d''$.

If $\begin{matrix} d \parallel d'' \\ d' \subset a' \end{matrix} \Longrightarrow \left. \begin{matrix} d \parallel a' \\ d \subset a \end{matrix} \right\} \Longrightarrow d'' \| d$.

Solution to Problem 171.



a. If $A \in d$, then $d' = d$. If $A \notin d$, we draw through $A, d' \parallel d$.

b. We draw $d_1 \subset \alpha, A \in d$, such that $m(\widehat{d_1 d'}) = a$ and $d \subset \alpha, A \in d_2$, such that $m(\widehat{d_2 d'}) = a$, a line in each half-plane determined by d'. So (∃) 2 solutions excepting the situation $a = 0$ or $a = 90$ when (∃) only one solution.

Solution to Problem 172.

$\alpha \parallel \beta$ or $\alpha = \beta \Leftrightarrow \alpha \sim \beta$

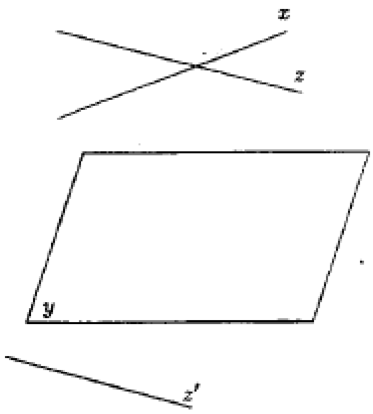
1. $\alpha = \alpha \Longrightarrow \alpha \sim \alpha$, the relation is reflexive;
2. $\alpha \sim \beta \Longrightarrow \beta \sim \propto$, the relation is symmetric.
   $\alpha \parallel \beta$ or $\alpha = \beta \Longrightarrow \beta \parallel \alpha$ or $\beta = \alpha \Longrightarrow \beta \sim \propto$;
3. $\alpha \sim \beta \cap \beta \sim \gamma \Longrightarrow \alpha \sim \gamma$.

If $\alpha = \beta \cap \beta \sim \gamma \Longrightarrow \alpha \sim \gamma$.

If $\left.\begin{array}{c} \alpha \neq \beta \text{ and } \alpha \sim \beta \Longrightarrow \alpha \parallel \beta \\ \beta \sim \gamma \Longrightarrow \beta = \gamma \text{ or } \beta \parallel \gamma \end{array}\right\} \Longrightarrow \alpha \parallel \gamma \Longrightarrow \alpha \sim \gamma$.

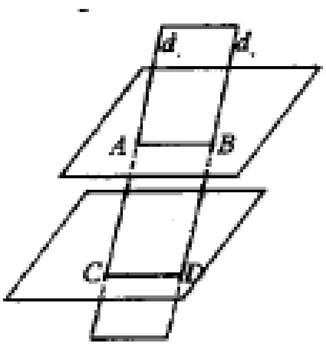
The equivalence class determined by plane α is constructed of planes α' with $\alpha' \sim \alpha$, that is of α and all the parallel planes with α.

Solution to Problem 173.



No, it is an equivalence relation, because the transitive property is not true. For example, x is a line, y a plane, z a line. From $x \,||\, y$ and $||\, z \nRightarrow x \,||\, z$, lines x and z could be coplanar and concurrent or non-coplanar.

Solution to Problem 174.



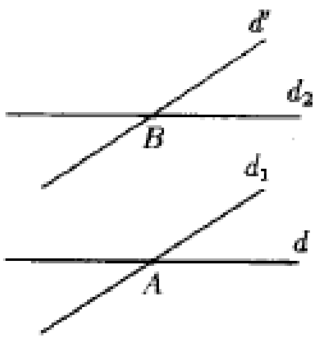
$d_1 \parallel d_2 \Rightarrow (\exists)\gamma = (d_1 d_2)$

$$\left.\begin{array}{l} \alpha \cap \gamma = AB \\ \beta \cap \gamma = CD \\ \alpha \parallel \beta \end{array}\right\} \Rightarrow \begin{array}{l} AB \parallel CD \\ AC \parallel BD \end{array} \Rightarrow$$

$\Longrightarrow ABCD$ parallelogram.

So $||AC|| = ||BD||$.

Solution to Problem 175.



We consider $A \in d$ and draw through it $d_1 \parallel d'$. We consider $B \in d'$ and draw $d_2 \parallel d$. Plane $(d_1 d_2) \parallel (d d_1)$, because two concurrent lines from the first plane are parallel with two concurrent lines from the second plane.

When d and d' are coplanar, the four lines d, d_1, d_2 and d' are coplanar and the two planes coincide with the plane of the lines d and d'.

Solution to Problem 176.

Let planes:

$$\left.\begin{array}{r}(CDA)\\ CD \parallel \beta\end{array}\right\} \Rightarrow CD \parallel QM \qquad (1)$$

$$\left.\begin{array}{r}(CDB)\\ CD \parallel \beta\end{array}\right\} \Rightarrow CD \parallel PN \qquad (2)$$

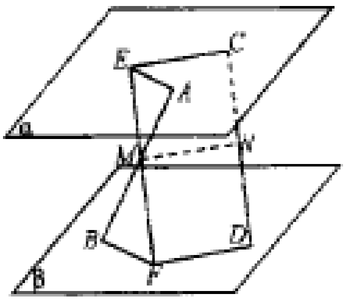
$$\left.\begin{array}{r}(CAB)\\ \alpha \parallel \beta\end{array}\right\} \Rightarrow AB \parallel MN \qquad (3)$$

$$\left.\begin{array}{r}(DAB)\\ \alpha \parallel \beta\end{array}\right\} \Rightarrow AB \parallel QP \qquad (4)$$

From (1), (2), (3), (4) $\Longrightarrow MNPQ$ parallelogram.

Solution to Problem 177.

Let $[AB]$ and $[CD]$ be two segments, with $A, C \in \alpha$ and $B, D \in \beta$ such that $|AM| = |MB|$ and $|CN| = |ND|$.



In plane (MCD) we draw through $M, EF||CD \Longrightarrow EC||DF \Longrightarrow EFDC$ parallelogram $\Longrightarrow |EF| \equiv |CD|$.

Concurrent lines AB and EF determine a plane which cuts planes α after 2 parallel lines $\Longrightarrow EA||BF$.

In this plane, $|AM| \equiv |BM|$.

$$\left.\begin{array}{l}\widehat{EMA} \equiv \widehat{BMF} \text{ (angles opposed at peak)}\\ \widehat{EAM} \equiv \widehat{FBM} \text{ (alternate interior angles)}\end{array}\right\}$$

$$\Rightarrow \triangle AME \equiv \triangle BMF \Rightarrow |EM| \equiv |MF|$$

In parallelogram $ECDF$,

$|CN| \equiv |ND|$, $|EM| \equiv |MF| \Rightarrow$

$$\left\{\begin{array}{l}MN \parallel EC\\ MN \parallel FD\end{array}\right. \Rightarrow \begin{array}{l}MN \parallel \alpha\\ MN \parallel \beta\end{array} \qquad (1)$$

So the segment connecting the midpoints of two of the segments with the extremity in α and β is parallel to these planes. We also consider $[GH]$ with $G \in \alpha, H \in \beta$ and $|GQ| \equiv |QH|$ and we show in the same way that $OM||\alpha$ and $OM||\beta$. (2)

From (1) and (2) $\Longrightarrow M, N, Q$ are elements of a parallel plane to α and β, marked by γ.

Vice-versa, let's show that any point from this plane is the midpoint of a segment, with its extremities in α and β.

Let segment $[AB]$ with $A \in \alpha$ and $B \in \beta$ and $|AM| = |BM|$. Through M, we draw the parallel plane with α and β and in this plane we consider an arbitrary point $O \in \gamma$.

Through O we draw a line such that $d \cap \alpha = \{I\}$ and $d \cap \beta \ = \{I\}$.

In plane (OAB) we draw $A'B' \,||\, AB$. Plane $(AA'B'B)$ cuts the three parallel planes after parallel lines $\Longrightarrow$

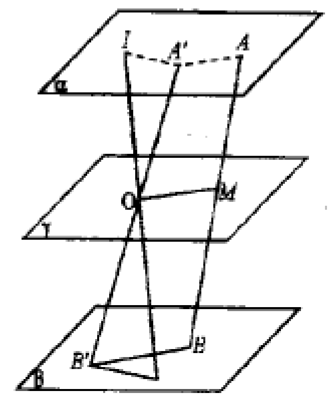
$$\left.\begin{array}{l} A'A \parallel OM \parallel B'B \\ B \in \beta \end{array}\right\} \Rightarrow$$

In plane $(A'B'I) \Longrightarrow |A'O| \equiv |OB'| \Longrightarrow IA' \,||\, B'I$ and thus $\dfrac{|A'O| \equiv |OB'|}{\widehat{IOA'} \equiv \widehat{IOB'}}$ and $\widehat{IA'O} \equiv \widehat{IB'O} \Longrightarrow$

$$\Rightarrow \triangle IOA' \equiv \triangle IOB' \Rightarrow |OI| \equiv |IO| \Rightarrow$$

O is the midpoint of a segment with extremities in planes α and β.

Thus the geometrical locus is plane γ, parallel to α and β and passing through the mid-distance between α and β.



Projections

178. Show that if lines d and d' are parallel, then $\mathrm{pr}_\alpha d \parallel \mathrm{pr}_\alpha d'$ or $\mathrm{pr}_\alpha d = \mathrm{pr}_\alpha d'$. What can we say about the projective planes of d and d'?

Solution to Problem 178

179. Show that the projection of a parallelogram on a plane is a parallelogram or a segment.

Solution to Problem 179

180. Knowing that side $[OA$ of the right angle AOB is parallel to a plane α, show that the projection of $\widehat{AOB}$ onto the plane α is a right angle.

Solution to Problem 180

181. Let $A'B'C'$ be the projection of ΔABC onto a plane α. Show that the centroid of ΔABC is projected onto the centroid of $\Delta A'B'C'$. Is an analogous result true for the orthocenter?

Solution to Problem 181

182. Given the non-coplanar points A, B, C, D, determine a plane on which the points A, B, C, D are projected onto the peaks of parallelogram.

Solution to Problem 182

183. Consider all triangles in space that are projected onto a plane α after the same triangle. Find the locus of the centroid.

Solution to Problem 183

184. Let A be a point that is not on line d. Determine a plane α such that $\mathrm{pr}_\alpha d$ passes through $\mathrm{pr}_\alpha A$.

Solution to Problem 184

185. Determine a plane onto which three given lines to be projected after concurrent lines.

Solution to Problem 185

186. Let α, β be planes that cut each other after a line a and let d be a perpendicular line to a. Show that the projections of line d onto α, β are concurrent.

Solution to Problem 186

187. Consider lines $OA, OB, OC \perp$ two by two. We know that $||OA|| = a$, $||OB|| = b$, $||OC|| = c$. Find the measure of the angle of planes (ABC) and (OAB).

Solution to Problem 187

188. A line cuts two perpendicular planes α and β at A and B. Let A' and B' be the projections of points A and B onto line $\alpha \cap \beta$.

a. Show that $||AB||^2 = ||AA'||^2 + ||A'B'||^2 + ||B'B||^2$;

b. If a, b, c are the measures of the angles of line AB with planes α, β and with $\alpha \cap \beta$, then $\cos c \frac{\|A'B'\|}{\|AB\|}$ and $\sin^2 a + \sin^2 b = \sin^2 c$.

Solution to Problem 188

189. Let ABC be a triangle located in a plane α, $A'B'C'$ the projection of $\Delta A'B'C'$ onto plane α. We mark with S, S', S'' the areas of ΔABC, $\Delta A'B'C', \Delta A"B"C"$, show that S' is proportional mean between S and S''.

Solution to Problem 189

190. A trihedral $[ABCD]$ has $|AC| \equiv |AD| \equiv |BC| \equiv |BD|$. M, N are the midpoints of edges $[AB], [CD]$, show that:

a. $MN \perp AB, MN \perp CD, AB \perp CD$

b. If A', B', C', D' are the feet of the perpendicular lines drawn to the peaks A, B, C, D on the opposite faces of the tetrahedron, points B, A', N are collinear and so are A, B', N; D, C', M; C, D', M.

c. AA', BB', MN and CC', DD', MN are groups of three concurrent lines.

Solution to Problem 190

191. If rays $[OA$ and $[OB$ with their origin in plane α, $OA \perp \alpha$, then the two rays form an acute or an obtuse angle, depending if they are or are not on the same side of plane α.

Solution to Problem 191

192. Show that the 6 mediator planes of the edges of a tetrahedron have a common point. Through this point pass the perpendicular lines to the faces of the tetrahedron, drawn through the centers of the circles of these faces.

Solution to Problem 192

193. Let d and d' be two non-coplanar lines. Show that (∃) unique points $A \in d$, $A' \in d'$ such that $AA' \perp d$ and $AA' \perp d'$. The line AA' is called the common perpendicular of lines d and d'.

Solution to Problem 193

194. Consider the notations from the previous problem. Let $M \in d$, $M' \in d'$. Show that $\|AA'\| \leq \|MM'\|$. The equality is possible only if $M = A$, $M' = A'$.

Solution to Problem 194

195. Let AA' be the common ⊥ of non-coplanar lines d, d'' and $M \in d$, $M' \in d'$ such that $|AM| \equiv |A'M'|$. Find the locus of the midpoint of segment $[MM']$.

Solution to Problem 195

196. Consider a tetrahedron $VABC$ with the following properties. ABC is an equilateral triangle of side a, $(ABC) \perp (VBC)$, the planes (VAC) and (VAB)

form with plane (ABC) angles of $60°$. Find the distance from point V to plane (ABC).

Solution to Problem 196

197. All the edges of a trihedral are of length a. Show that a peak is projected onto the opposite face in its centroid. Find the measure of the dihedral angles determined by two faces.

Solution to Problem 197

198. Let DE be a perpendicular line to the plane of the square $ABCD$. Knowing that $\|BE\| = l$ and that the measure of the angle formed by $[BE$ and (ABC) is β, determine the length of segment AE and the angle of $[AE$ with plane (ABC).

Solution to Problem 198

199. Line $CD \perp$ plane of the equilateral ΔABC of side a, and $[AD$ and $[BD$ form with plane (ABC) angles of measure β. Find the angle of planes (ABC) and ABD.

Solution to Problem 199

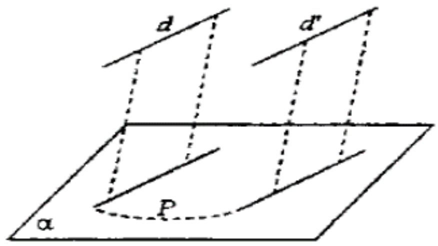
200. Given plane α and ΔABC, $\Delta A'B'C'$ that are not on this plane. Determine a ΔDEF, located on α such that on one side lines AD, BE, CF and on the other side lines $A'D, B'E, C'F$ are concurrent.

Solution to Problem 200

Solutions

Solution to Problem 178.

Let $d \parallel d'$, β the projective plane of d.



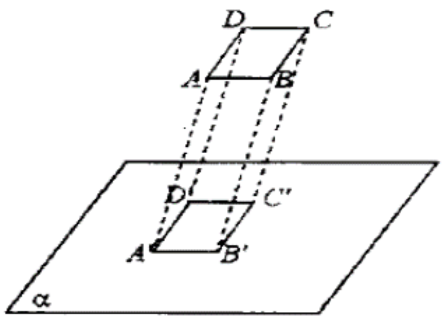
We assume that $d' \not\subset \beta$, which means that is plane $d, d' \not\perp \alpha$, $\Longrightarrow$ the projective plane of d' is β'. We want to show that $\mathrm{pr}_a d \parallel \mathrm{pr}_a d'$. We assume that $\mathrm{pr}_a d \cap \mathrm{pr}_a d' = \{P\} \Rightarrow (\exists)\ M \in d$ such that $\mathrm{pr}_a M = P$ and $(\exists) M' \in d'$ such that $\mathrm{pr}_a M' = P$.

$\Longrightarrow \left.\begin{matrix} PM \perp \alpha \\ PM' \perp \alpha \end{matrix}\right\} \Longrightarrow$ in the point P on plane α we can draw two distinct perpendicular lines. *False*.

If β is the projective plane of d and β of d', then $\beta \parallel \beta'$, because if they had a common point their projections should be elements of $\mathrm{pr}_a d$ and $\mathrm{pr}_a d'$, and thus they wouldn't be anymore parallel lines.

If $d' \subset \beta$ or $d \subset \beta'$, that is $(d, d') \perp \alpha \Longrightarrow d$ and d' have the same projective plane $\Longrightarrow \mathrm{pr}_a d = \mathrm{pr}_a d'$.

Solution to Problem 179.

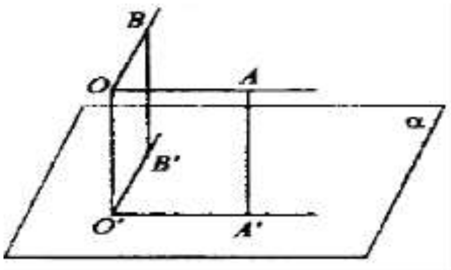


We assume that $ABCD \not\perp \alpha$. Let A', B', C', D' be the projections of points A, B, C, D.

$$\left.\begin{array}{l} AB \parallel DC \overset{pr1}{\Longrightarrow} A'B' \parallel D'C' \\ AD \parallel BC \Longrightarrow A'D' \parallel B'D' \end{array}\right\} \Longrightarrow A', B', C', D' \text{ parallelogram.}$$

If $(ABCD) \perp \alpha \Rightarrow$ the projection $A', B', C', D' \in$ the line $(ABCD) \cap \alpha \Longrightarrow$ the projection of the parallelogram is a segment.

Solution to Problem 180.



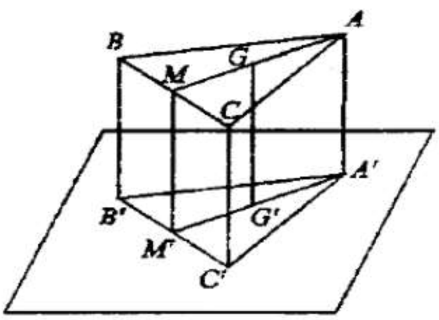
If $OA || \alpha \Longrightarrow \mathrm{proj}_{\alpha} OA || OA \Longrightarrow O'A' || OA$ because (∀) a plane which passes through OA cuts the plane α after a parallel to OA.

$$\left.\begin{array}{l} OO' \perp O'A' \\ O'A' \parallel OA \end{array}\right\} \Rightarrow \left.\begin{array}{l} OA \perp OO' \\ OA \perp OB \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} OA \perp (OO'B) \\ O'A' \parallel OA \end{array}\right\} \Rightarrow O'A' \perp (OO'B) \Rightarrow$$

$\Longrightarrow O'A' \perp O'B' \Longrightarrow \widehat{A'O'B'}$ is a right angle.

Solution to Problem 181.



In the trapezoid $BCC'B'$ $(BB' || CC')$,

$$\left.\begin{array}{l} |BM| \equiv |MC| \\ MM' \parallel BB' \end{array}\right\} \Rightarrow |B'M'| \equiv |MC'|$$

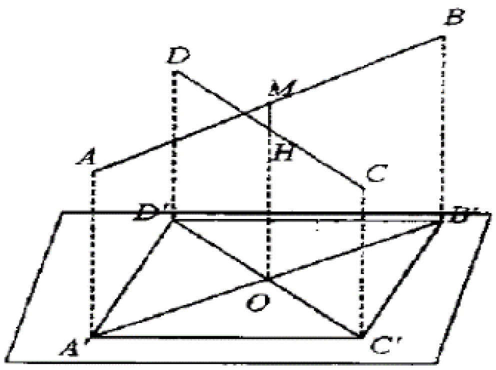
$\Longrightarrow A'M'$ is a median.

$$\left.\begin{array}{r} MM' \parallel AA' \Longrightarrow MM'A'A \text{ trapezoid} \\ \dfrac{\|AG\|}{\|GM\|} = 2, GG' \parallel AA' \end{array}\right\} \dfrac{\|A'G\|}{\|G'M'\|} = \dfrac{\|AG\|}{\|GM\|} = 2$$

$\Longrightarrow G'$ is on median A'M' at 2/3 from the peak and 1/3 from the base.

Generally *no*, because the right angle AMC should be projected after a right angle. The same thing is true for another height. This is achieved if the sides of the Δ are parallel to the plane.

Solution to Problem 182.



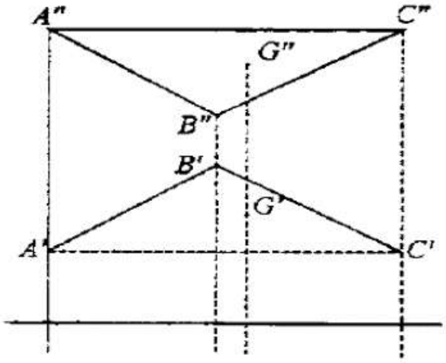
Let A, B, C, D be the 4 non-coplanar points and M, N midpoints of segments $|AB|$ and $|CD|$.

M and N determine a line and let a plane $\alpha \perp MN$, M and N are projected in the same point O onto α.

$$\left.\begin{array}{l} \left.\begin{array}{c} AA' \parallel MO \parallel BB' \\ |AM| \equiv |MB| \end{array}\right\} \Rightarrow |A'O| \equiv |OB'| \\ \left.\begin{array}{c} CC' \parallel DD' \parallel NO \\ |DN| \equiv |NC| \end{array}\right\} \Rightarrow |D'O| \equiv |OC'| \end{array}\right\} \Rightarrow$$

$\Longrightarrow A'B'C'D'$ a parallelogram.

Solution to Problem 183.



Let $A'B'C'$, $A''B''C''$ two triangles of this type, with the following property:

$\mathrm{pr}_\alpha A'B'C' = ABC$, $\mathrm{pr}_\alpha A''B''C'' = ABC$.

$$\Rightarrow \left.\begin{array}{l} \mathrm{pr}_\alpha G' = G \Rightarrow G'G \perp \alpha \\ \mathrm{pr}_\alpha G'' = G \Rightarrow G''G \perp \alpha \end{array}\right\} \Rightarrow$$

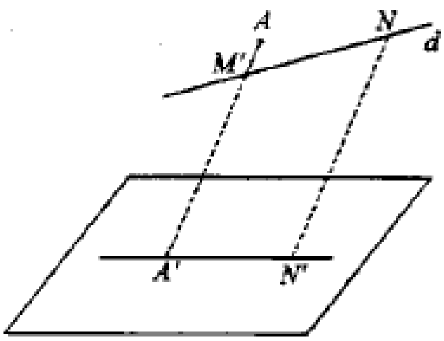
$\Longrightarrow G'G = GG'' \Longrightarrow G'', G', G$ are collinear.

Due to the fact that by projection the ratio is maintained, we show that G'' is the centroid of A, B, C.

$$\frac{\|BG\|}{\|GM\|} = \frac{\|B_1G\|}{\|G_1M_1\|} = 2.$$

Solution to Problem 184.

Let $M \in d$ and $A \notin d$. The two points determine a line and let α be a perpendicular plane to this line, $AM \perp \alpha \Longrightarrow A$ and M are projected onto α in the same point A' through which also passes $\mathrm{proj}_\alpha d = \mathrm{proj}_\alpha A \in \mathrm{proj}_\alpha d$.



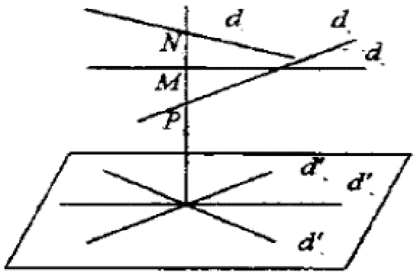
Solution to Problem 185.

We determine a line which meets the three lines in the following way.

Let

$M \in d_2 \Rightarrow \beta = (M, d_1),\ \gamma = (M, d_3)$ şi $\beta \cap \gamma =$

$= d \Rightarrow d \subset \beta_1,\ d \cap d_1 = \{N\},\ d \subset \gamma_1,\ d \cap d_3 = \{P\}.$



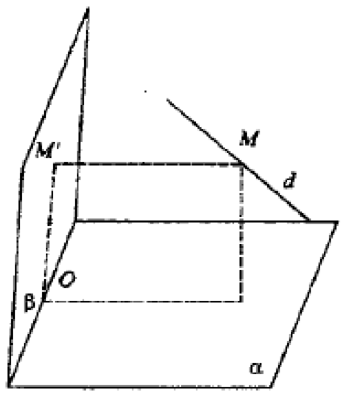
Let now a plane

$\alpha \perp d \Rightarrow \mathrm{pr}_\alpha M = \mathrm{pr}_\alpha N = \mathrm{pr}_\alpha P =$

$= O \Rightarrow \mathrm{pr}_\alpha d_1 \cap \mathrm{pr}_\alpha d_2 \cap \mathrm{pr}_\alpha d_3 \neq \varnothing \Rightarrow d_1' \cap d_2' \cap d_3' =$

$= \{O\}.$

Solution to Problem 186.



Let $\alpha \cap \beta = a$ and $M \in d$. We project this point onto α and β:

$$MM' \perp \alpha \Rightarrow \left.\begin{array}{r} MM' \perp a \\ d \perp a \end{array}\right\} \Rightarrow a \perp (MM', d)$$

$$MM'' \perp \beta \Rightarrow \left.\begin{array}{r} MM'' \perp a \\ d \perp a \end{array}\right\} \Rightarrow a \perp (MM'', d)$$

$\Longrightarrow a \perp$ onto the projective plane of d onto β.

$$\left.\begin{array}{l}(MM',d)\perp d\\(NM'',d)\perp a\end{array}\right\}\Rightarrow\quad (MM',d)\parallel(MM'',d)\quad\Big\}\Rightarrow$$

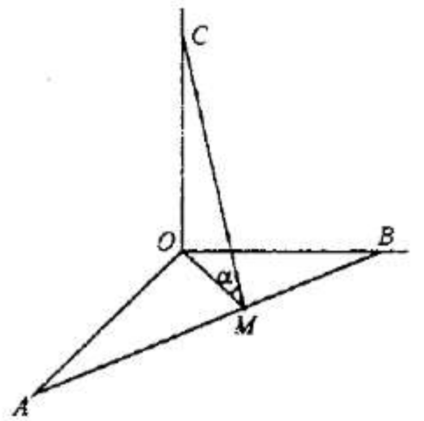
$$\Rightarrow (MM',d)=(MM'',d)=(MM'M'')$$

Let

$$a\cap(MM'M'')=\{O\}\Rightarrow\left.\begin{array}{l}\mathrm{pr}_\alpha d=OM'\\\mathrm{pr}_\alpha d=OM''\end{array}\right\}\Rightarrow$$

$\Longrightarrow OM'\cap OM''=\{O\}$, so the two projections are concurrent.

Solution to Problem 187.



$$\left.\begin{array}{l}OC\perp OA\\OC\perp OB\end{array}\right\}\Rightarrow\left.\begin{array}{l}OC\perp(OAB)\\OM\perp AB\end{array}\right\}\Rightarrow CM\perp AB\Rightarrow$$

the angle of planes (ABC) and (OAB) is $\widehat{OMC}=\alpha$.

$$\|AB\|=\sqrt{a^2+b^2};\quad \|OM\|=\frac{ab}{\sqrt{a^2+b^2}}$$

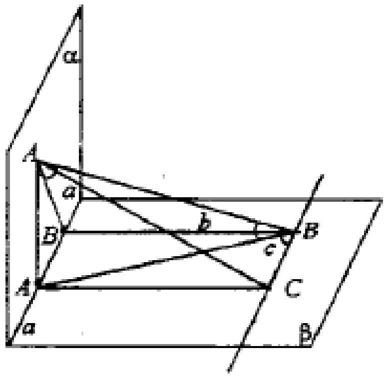
$$\mathrm{tg}\ \alpha=\frac{\|OC\|}{\|OM\|}=\frac{c}{\frac{ab}{\sqrt{a^2+b^2}}}=\frac{c\sqrt{a^2+b^2}}{ab}.$$

Solution to Problem 188.

Let $\alpha\cap\beta$ = a and $AA'\perp a$, $BB'\perp a$.

$$\left.\begin{array}{l}\alpha\perp\beta\\A\in\alpha\\AA'\perp a\end{array}\right\}\Rightarrow AA'\perp\beta\Rightarrow AA'\perp A'B\Rightarrow$$

$$\Rightarrow\|AB\|^2=\|AA'\|^2+\|A'B\|^2.$$



As

$$\left.\begin{array}{r} BB'\perp a \\ \beta\perp\alpha \end{array}\right\} \Rightarrow BB'\perp\alpha \Rightarrow \mathrm{pr}_\alpha AB = AB' \Rightarrow$$

$\Longrightarrow \measuredangle$ of line AB with α, $\widehat{BAB'} = a$

$$AA'\perp\beta \Rightarrow \mathrm{pr}_\beta AB = A'B \Rightarrow$$

$\Longrightarrow \measuredangle$ of line AB with β, $ABA' = b$

In the plane β we draw through B a parallel line to a and through A' a parallel line to BB'. Their intersection is C, and $||A'B'|| = ||BC||, ||BB'|| = ||A'C||$. The angle of line AB with α is $\widehat{ABC} = c$.

As $AA' \perp \beta \Longrightarrow AA' \perp A'C \Longrightarrow \|AC\|^2 = \|AA'\|^2 + \|A'C\|^2 = \|AA'\|^2 + \|B'B\|^2$(1)

$B'BCA$ rectangle

$$\left.\begin{array}{r} \Rightarrow A'C\perp\|CB \\ AA'\perp\beta \end{array}\right\} \Rightarrow AC\perp CB.$$

$\Longrightarrow \Delta ACB$ is right in C.

We divide the relation (1) with $||AB||^2$:

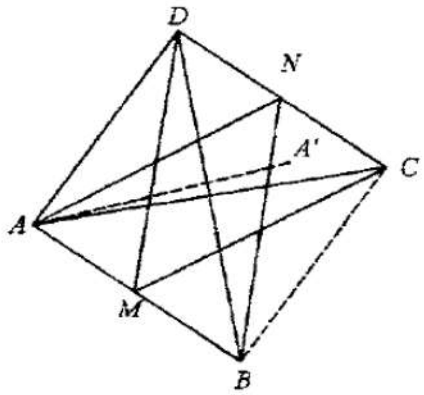
$$\frac{\|AC\|^2}{\|AB\|^2} = \frac{\|AA'\|^2}{\|AB\|^2} + \frac{\|B'B\|^2}{\|AB\|^2} \Rightarrow \sin^2 c = \sin^2 b + \sin^2 a.$$

Solution to Problem 189.

$$\left.\begin{array}{l} S' = S\cos a \\ S'' = S'\cos a \end{array}\right., \ a \in \left(\widehat{\alpha,\beta}\right)$$

$$\left.\begin{array}{l} \cos a = \dfrac{S'}{S} \\ \cos a = \dfrac{S''}{S'} \end{array}\right\} \Rightarrow \frac{S'}{S} = \frac{S''}{S'} \Rightarrow S'^2 = SS'' \Rightarrow S' = \sqrt{SS''}.$$

Solution to Problem 190.



a. $\left.\begin{array}{r} |AC| \equiv |BC| \Longrightarrow \Delta ACB \text{ isosceles} \\ CM \text{ median} \end{array}\right\} \Longrightarrow CM \perp AB$ (1)

$\left.\begin{array}{r} |AD| \equiv |BC| \Longrightarrow \Delta ABD \text{ isosceles} \\ DM \text{ median} \end{array}\right\} \Longrightarrow DM \perp AB$ (2)

From (1) and (2) $\Longrightarrow AB \perp (DMC) = \left.\begin{array}{c} AB \perp MN \\ AB \perp DC \end{array}\right\}$

$$\left.\begin{array}{l} \left.\begin{array}{l} |BC| \equiv |BD| \\ BN \text{ median} \end{array}\right\} \Longrightarrow BN \perp DC \\ \left.\begin{array}{l} |AD| \equiv |AC| \\ AN \text{ median} \end{array}\right\} \Longrightarrow AN \perp DC \end{array}\right\} \Longrightarrow DC \perp (ABN) \Longrightarrow DC \perp MN$$

b.

$$\left.\begin{array}{l} DC \perp MN \\ DC \perp AB \end{array}\right\} \Rightarrow \left.\begin{array}{l} DC \perp (ABN) \\ DC \subset (BDC) \end{array}\right\} \Rightarrow \left.\begin{array}{l} (BDC) \perp (ABN) \\ AA' \perp (BDC) \end{array}\right\} \Rightarrow \left.\begin{array}{l} AA' \subset (ABN) \\ A' \in (BDC) \end{array}\right\} \Rightarrow$$

$\Longrightarrow A' \in BN \Longrightarrow B, A', N$ are collinear.

In the same way:

From $(ADC) \perp (ABN) \Longrightarrow A, B', N$ collinear

$(ABC) \perp (DMC) \Longrightarrow M, D', C$ collinear

$(ABD) \perp (DMC) \Longrightarrow D, C', M$ collinear

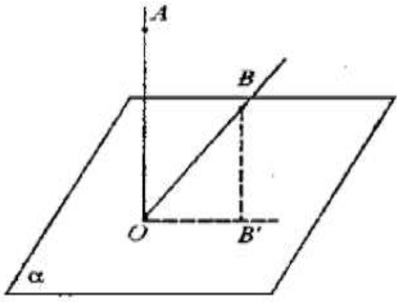
C. At point a. we've shown that $MN \perp AB$

$$\left.\begin{array}{l} \left.\begin{array}{r} AA' \perp (BDC) \\ BN \subset (DBC) \end{array}\right\} \Rightarrow \begin{array}{c} AA' \perp BN \\ A' \in BN \end{array} \\ \left.\begin{array}{r} BB' \perp (ADC) \\ AN \subset (ADC) \end{array}\right\} \Rightarrow \begin{array}{c} BB' \perp AN \\ B' \in AN \end{array} \end{array}\right\} \Rightarrow$$

AA', BB' and MN are heights in ΔABN, so they are concurrent lines.

In the same way, DD', CC', MN will be heights in ΔDMC.

Solution to Problem 191.



We assume that $[OA, [OB$ are on the same side of plane α.

We draw

$$\left.\begin{matrix} BB'\perp\alpha \\ AO\perp\alpha \end{matrix}\right\} \Rightarrow BB' \parallel AO \Rightarrow$$

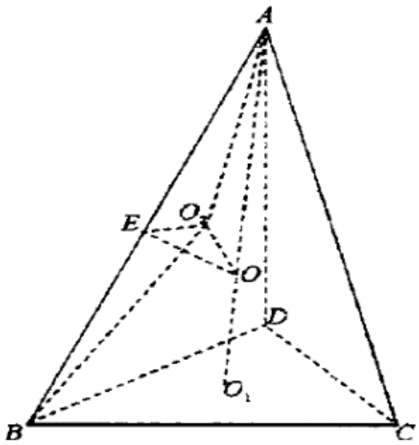
($\exists$) plane $(AD, BB') = \beta \Longrightarrow |OA, |OB$ are in the same half-plane.

$AO\perp\alpha \;\Rightarrow\; AO\perp OB',\; m(\widehat{BOB'}) \;<\; 90^{\circ} \;\Rightarrow\; |OB \;\subset$ int $\widehat{AOB'}$.

In plane β we have $m(\widehat{AOB}) = 90^0 = m(\widehat{BOB'}) < 90^0 \Longrightarrow \widehat{AOB}$ acute.

We assume that $[OA$ and $[OB$ are in different half-planes in relation to $\alpha \Longrightarrow A$ and B are in different half-planes in relation to OB' in plane $\beta \Longrightarrow |OB' \subset \text{int}.\widehat{BOA} \Longrightarrow m\big(\widehat{AOB}\big) = 90^0 + m\big(\widehat{BOB'}\big) > 90^0 \Longrightarrow AOB$ obtuse.

Solution to Problem 192.



We know the locus of the points in space equally distant from the peaks of ΔBCD is the perpendicular line d to the pl. Δ in the center of the circumscribed circle of this Δ, marked with O. We draw the mediator plane of side $|AC|$, which intersects this $\perp d$ at point O. Then, point O is equally distant from all the peaks of the tetrahedron $||OA|| = ||OB|| = ||OC|| = ||OD||$. We connect O with midpoint E of side $|AB|$. From $|OA| \equiv |OB| \Longrightarrow \Delta OAB$ isosceles $\Longrightarrow OC \perp AB$ (1).

We project O onto plane (ABD) in point O_2.

As $\left.\begin{matrix} |OA| \equiv |OB| \equiv |OD| \\ |O)O_2 \text{ common side} \end{matrix}\right\} \Longrightarrow \Delta OAO_2 = \Delta OBO_2 = \Delta ODO_2$

$\Longrightarrow |O_2A| \equiv |BO_2| \equiv |DO_2| \Longrightarrow$

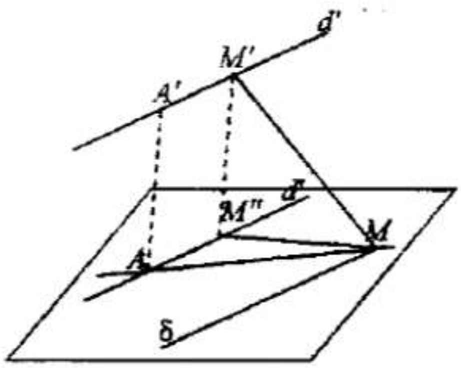
$\Longrightarrow O_2$ is the center of the circumscribed circle of ΔABD. We show in the same way that O is also projected on the other faces onto the centers of the circumscribed circles, thus through O pass all the perpendicular lines to the faces of the tetrahedron. These lines are drawn through the centers of the circumscribed circles. *So* b. *is proved.*

From $|O_2A| \equiv |OB_2| \Longrightarrow \Delta O_2AB$ isosceles $O_2E \perp AB$ (2)

From (1) and (2) $\Longrightarrow$ $\left.\begin{matrix} AB \perp (EO_2O) \\ |AE| \equiv |EB| \end{matrix}\right\} \Longrightarrow$

$\Longrightarrow (EO_2O)$ is a mediator plane of side $|AB|$ and passes through O and the intersection of the 3 mediator planes of sides $|BC|, |CD|, |BD|$ belongs to line d, thus O is the common point for the 6 mediator planes of the edges of a tetrahedron.

Solution to Problem 193.



Let $M \in d$ and $\delta || d'$, $M \in \delta$. Let $= (d, \delta) \Longrightarrow d' || \alpha$.

Let

$\left.\begin{matrix} d'' = \text{pr}_\alpha d' \\ d' \|, \alpha \end{matrix}\right\} \Rightarrow d'' \| d' \Rightarrow d'' \| d' \Rightarrow d' \cap d =$

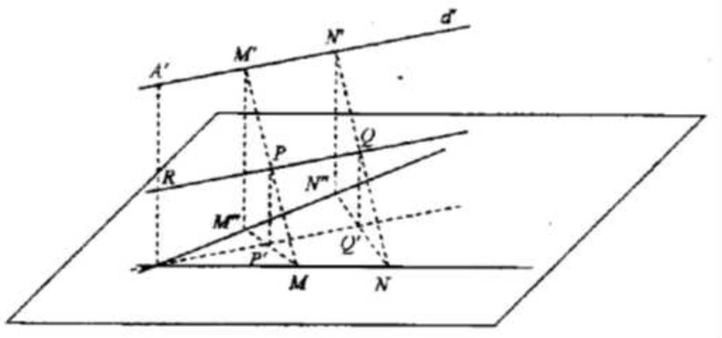
$= \{A\}$ otherwise d and d' would be parallel, thus coplanar. Let β be the projective plane of line

$d' \Rightarrow \begin{matrix} \beta \perp \alpha \\ \beta \cap \alpha = d'' \end{matrix}$

In plane β we construct a perpendicular to d'' in point A and

$\left.\begin{matrix} AA' \perp d'' \\ d' \| d'' \end{matrix}\right\} \Rightarrow AA' \perp d'.$

Solution to Problem 194.



We draw

$M'M''\perp d'' \Rightarrow M'M''\perp d \Rightarrow M'M''\perp M''M \Rightarrow \|M'M\| \geq \|M'M''\| = \|A'A\|$.

We can obtain the equality only when $M = A$ and $M' = A'$.

Solution to Problem 195.

Let $M \in d$, $M' \in d'$ such that $|AM| \equiv |A'M'|$. Let $d'' = \text{pr}_\alpha d'$ and $M'M'' \perp d'' \Longrightarrow M'M'' \perp \alpha \Longrightarrow M'M'' \perp M''M$.

$$\left.\begin{array}{l} M'' \parallel A'A \\ A'M' \parallel AM'' \end{array}\right\} \Rightarrow \left.\begin{array}{l} |A'M'| \equiv |AM''| \\ |A'M'| \equiv |AM| \end{array}\right\} \Rightarrow |AM| \equiv |AM''| \Rightarrow$$

$\Longrightarrow \Delta AMM'$ isosceles.

Let P be the midpoint of $|MM'|$ and $P' = \text{pr}_\alpha P \Longrightarrow PP' \parallel M'M'' \Longrightarrow P'$ is the midpoint of MM'', $\Delta AMM''$ isosceles $\Longrightarrow [AP'$ the bisector of $\widehat{M'AM}$. (PP') is midline in $\Delta M'M''M \Longrightarrow \|PP'\| = \frac{1}{2}\left\|M'M''\right. = \frac{1}{2}\|A'A\| =$ constant.

Thus, the point is at a constant distance from line AP', thus on a parallel line to this line, located in the $\perp$ plane α, which passes through AP'.

When $M = A$ and $M' = A' \Longrightarrow ||AM|| = ||N'A' = 0 \Longrightarrow P = R$, where R is the midpoint of segment $|AA'|$. So the locus passes through R and because

$$\left.\begin{array}{l} AA'\perp AP' \\ RP \parallel AP' \end{array}\right\} \Rightarrow RP\perp AA' \Rightarrow$$

$\Longrightarrow RP$ is contained in the mediator plane of segment $|AA'|$.

So RP is the intersection of the mediator plane of segment $|AA'|$ with the $\perp$ plane to α, passing through one of the bisectors of the angles determined by d and d', we obtain one more line contained by the mediator plane of $[AA']$, the parallel line with the other bisector of the angles determined by d and d''.

So the locus will be formed by two perpendicular lines.

Vice-versa, let $Q \in RP$ a $(\forall)$ point on this line and $Q' = \mathrm{pr}_{\alpha} Q \Longrightarrow Q' \in |AP'$ bisector. We draw $NN'' \perp AQ'$ and because AQ' is both bisector and height $\Longrightarrow \Delta ANN''$ isosceles $\Longrightarrow |AQ'|$ median $\Longrightarrow |NQ'| \equiv |Q'N''|$.

We draw

$$N'N'' \perp d'' \Rightarrow \left.\begin{array}{l} |A'N'| \equiv |AN''| \\ |AN''| \equiv |AN| \end{array}\right\} \Rightarrow |AN| \equiv |A'N'|$$

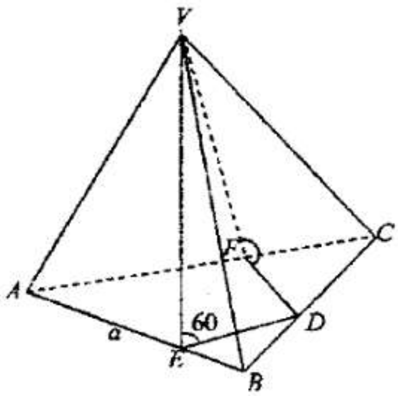
$$QQ'' \parallel N'N'' \Rightarrow Q,\ Q',\ N',\ N'' \text{ coplanar } \Rightarrow N''Q' \in (QQ'N'N'') \Rightarrow N \in (QQ'N'N'')$$

As

$$\left.\begin{array}{l} \|Q'Q\| = \frac{1}{2}\|AA'\| = \frac{1}{2}\|N'N''\| \\ QQ' \parallel N'N'' \end{array}\right\} \Rightarrow$$

$\Longrightarrow |Q'Q|$ midline in $\Delta NN'N'' \Longrightarrow Q, N', N$ collinear and $|QN'| \equiv |QN|$.

Solution to Problem 196.



$D = \mathrm{pr}_{BC} V$ şi $E = \mathrm{pr}_{AB} V$ $FD = \mathrm{pr}_{AC} V$

$(VBC) \perp (ABC) \Rightarrow VD \perp (ABC) \Rightarrow d(V, \alpha) = \|VD\|$,

where α = (ABC).

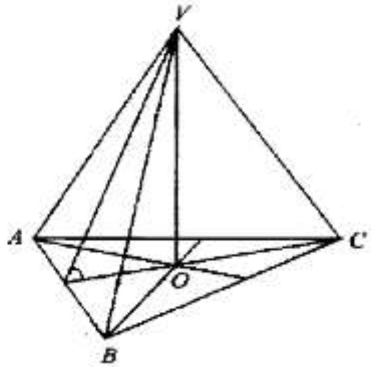
$$\left.\begin{array}{l} \left.\begin{array}{l} VD \perp (ABC) \\ VE \perp AB \end{array}\right\} \Rightarrow DE \perp AB \Rightarrow m(\widehat{VED}) = 60^\circ \\ \left.\begin{array}{l} VD \perp (ABC) \\ VF \perp AC \end{array}\right\} \Rightarrow \begin{array}{c} DF \perp AC \Rightarrow m(\widehat{VFD}) = 60^\circ \\ VD \text{ common side} \end{array} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{r} \Delta VDE = \Delta VDF \Rightarrow |ED| \equiv |FD| \\ m(\hat{B}) = m(\hat{C}) = 60^\circ \end{array}\right\} \Rightarrow \Delta EDB \equiv \Delta FDC \Rightarrow |BD| \equiv |DC| \Rightarrow \|BD\| = \frac{a}{2}$$

$$\|ED\| = \frac{a}{2} \cdot \sin 60^\circ = \frac{a}{2}\frac{\sqrt{3}}{2} = \frac{a\sqrt{3}}{4}$$

$$\|VD\| = \|ED\| \cdot \mathrm{tg}\, 60^\circ = \frac{a\sqrt{3}}{4} \cdot \sqrt{3} = \frac{3a}{4}$$

Solution to Problem 197.



Let

$$\left.\begin{array}{r} O = \text{pr}_{(BAC)}V;\ \|VA\| = \|VB\| = \|VC\| = a \\ \|VO\| \text{ common} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \Delta VAO \equiv \Delta VBO \equiv \Delta VBO \equiv \Delta VCO \Rightarrow |OA| \equiv$$

$$\equiv |BO| \equiv |CO| \Rightarrow$$

$\Longrightarrow O$ is the center of the circumscribed circle and as ΔABC is equilateral $\Longrightarrow O$ is the centroid $\Longrightarrow$

$$\Rightarrow \|OM\| = \frac{1}{3}\|MC\| = \frac{1}{3}\cdot\frac{a\sqrt{3}}{2} = \frac{a\sqrt{3}}{6}$$

$$\|VM\| = \|MC\| = \sqrt{a^2 - \frac{a^2}{4}} = \frac{a\sqrt{3}}{2}$$

$$\Delta VOM \Rightarrow \cos(\widehat{VMO}) = \frac{\|OM\|}{\|NM\|} = \frac{\frac{a\sqrt{3}}{6}}{\frac{a\sqrt{3}}{2}} = \frac{1}{3} \Rightarrow \widehat{VMO} = \arccos\frac{1}{3}.$$

Solution to Problem 198.

$$\|DE\| = l$$

$$m(DBE) = \beta$$

$$\Delta EDB:\ \|DE\| = l\sin\beta$$

$$\|DB\| = l\cos\beta.$$

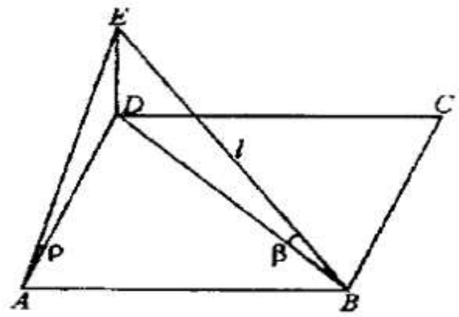
$$\|AB\| = a \Rightarrow \|DB\| = a\sqrt{2},$$

$$\|AB\| = \frac{\|DB\|}{\sqrt{2}} = \frac{l\cos\beta}{\sqrt{2}}.$$

In ΔAEB, right in A:

$$\|AE\| = \sqrt{l^2 - \frac{l^2\cos^2\beta}{2}} = l\sqrt{\frac{2-\cos^2\beta}{2}} = l\sqrt{\frac{1+\sin^2\beta}{2}}$$

$$\Delta ADE\ (m(\widehat{D}) = 90°) \qquad \text{tg}\,\rho = \frac{\|DE\|}{\|AD\|} = \frac{\rho\sin\beta}{\frac{\rho\cos\beta}{\sqrt{2}}} = \sqrt{2}\,\text{tg}\,\beta.$$



Solution to Problem 199.

$\left.\begin{matrix} CE \perp BA \\ DE \perp AB \end{matrix}\right\} \Rightarrow \sphericalangle \text{pl.}\,(ABC)$ and ABD are $m(\widehat{DEC})$.

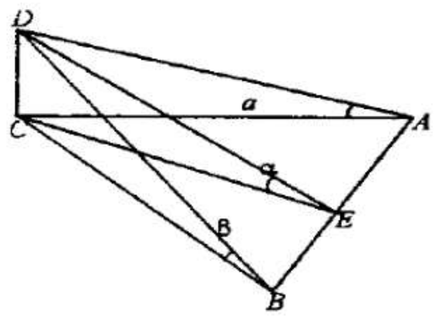
ABC equilateral $\Rightarrow \|CE\| = \frac{a\sqrt{3}}{2}$.

$\Delta CBD:\ \|DB\| = \frac{a}{\cos\beta} = \|AD\|$

$$DE^2 = \sqrt{\frac{a^2}{\cos^2\beta} - \frac{a^2}{2}} = a\sqrt{\frac{\sin\beta}{\cos\beta}}$$

$$\cos\alpha = \frac{\|CO\|}{\|CE\|} = \frac{\frac{a\sqrt{3}}{2}}{a\sqrt{\frac{\sin\beta}{\cos\beta}}} = \frac{\sqrt{3}\cos\beta}{2\cdot\sqrt{\sin\beta}}$$

$$\text{tg}\,\alpha \cdot \frac{\|CD\|}{\|CE\|} = \frac{\frac{a\cdot\sin\beta}{\cos\beta}}{\frac{a\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}\cdot \text{tg}\,\beta.$$



Solution to Problem 200.

We consider the problem solved and we take on plane α, ΔDEF, then points O and O' which are not located on α.

We also construct lines $|DO, |FO, |EO$ respectively $|DO', |FO', |EO'$. On these rays we take ΔABC and $\Delta A'B'C'$. Obviously, the way we have constructed the lines AD, BE, CF shows that they intersect at O. We extend lines BA, BC, CA until they intersect plane

α at points B, C respectively A. Then, we extend lines $C'A', C'B', A'B'$ until they intersect plane α at points A_2, C_2 respectively B_2.

Obviously, points A_1, B_1, C_1 are collinear (because $\in \alpha \cap (ABC)$) and A_2, B_2, C_2 are as well collinear (because $\in \alpha \cap (A'B'C')$).

On the other side, points D, F, A_1, A_2 are collinear because:

$$\left.\begin{aligned} A_1 \in AC(ACOD) \\ A, D, F \subset \alpha \end{aligned}\right\} \Rightarrow O, F, A_1 \in \alpha \cap (ACO),$$

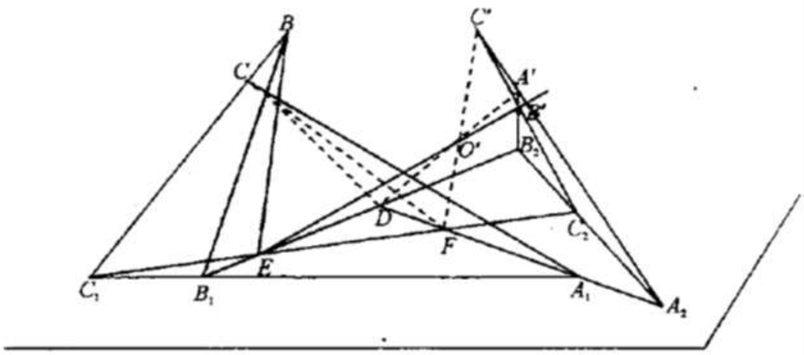
thus collinear (1)

$$A_2 \in O'A' \subset (A'C'ODF)$$
$$D, F, A_2 \in \alpha$$
$$\Rightarrow D, F, A_2 \in \alpha \cap (C'A'O) \Rightarrow$$

$\Longrightarrow D, F, A_2$ collinear (2)

From (1) and (2) $\Longrightarrow D, F$, A_1, A_2 collinear. Similarly C, E, F, C_2 collinear and B_1, E, D, D_2 collinear.

Consequently, DEF is at the intersection of lines A_1A_2, C_1C_2 , B_1B_2 on plane α, thus uniquely determined.



Review Problems

201. Find the position of the third peak of the equilateral triangle, the affixes of two peaks being $z_1 = 1, z_2 = 2 + i$.

Solution to Problem 201

202. Let z_1, z_2, z_3 be three complex numbers, not equal to 0, + two by 2, and of equal moduli. Prove that if $z_1 + z_2 z_3, z_2 + z_3 z_1, z_2 + z_{\ 1} z_3 \in R \Rightarrow z_1 z_2 z_3 = 1$.

Solution to Problem 202

203. We mark by G the set of n roots of the unit, $G = \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}\}$. Prove that:

   a. $\varepsilon_i \cdot \varepsilon_j \in G, (\forall)\ i, j \in \{0, 1, \dots, n-1\}$;
   b. $\varepsilon_i^{-1} \in G, (\forall\ )\ i \in \{0, 1, \dots, n-1\}$.

Solution to Problem 203

204. Let the equation $az^2 + bz^2 + c = 0$, $a, b, c \in C$ and $\arg a + \arg c = 2\arg b$, and $|a| + |c| = |b|$. Show that the given equation has at list one root of unity.

Solution to Problem 204

205. Let z_1, z_2, z_3 be three complex numbers, not equal to 0, such that $|z_1| = |z_2| = |z_3|$.

   a. Prove that $(\exists)$ complex numbers α and β such that $z_2 = \alpha z_1$, $z_3 = \beta z_2$ and $|\alpha| = |\beta| = 1$;
   b. Solve the equation $\alpha^2 + \beta^2 - \alpha \cdot \beta - \alpha - \beta + 1 = 0$ in relation to one of the unknowns.
   c. Possibly using the results from $a.$ and $b.$, prove that if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$, then we have $z_1 = z_2 = z_3$ or the numbers z_1, z_2, z_3 are affixes of the peaks of an equilateral Δ.

Solution to Problem 205

206. Draw a plane through two given lines, such that their line of intersection to be contained in a given plane.

Solution to Problem 206

207. Let a, b, c be three lines with a common point and P a point not located on any of them. Show that planes $(Pa), (Pb), (Pc)$ contain a common line.

Solution to Problem 207

208. Let A, B, C, D be points and α a plane separating points A and B, A and C, C and D. Show that $\alpha \cap |BD| \neq \emptyset$ and $\alpha \cap |AD| = \emptyset$.

Solution to Problem 208

209. On edges a, b, c of a trihedral angle with its peak O, take points A, B, C; let then $D \in |BC|$ and $E \in |AD|$. Show that $|OE \subset int. \angle abc$.

Solution to Problem 209

210. Show that the following sets are convex: the interior of a trihedral angle, a tetrahedron without an edge (without a face).

Solution to Problem 210

211. Let A, B, C, D be four non-coplanar points and E, F, G, H the midpoints of segments $[AB], [BC], [CD], [DA]$. Show that $EF \,||\, (ACD)$ and points E, F, G, H are coplanar.

Solution to Problem 211

212. On lines d, d' consider distinct points A, B, C; A', B', C' respectively. Show that we can draw through lines AA', BB', CC' three parallel planes if and only if $\frac{\|AB\|}{\|A'B'\|} = \frac{\|BC\|}{\|B'C'\|}$.

Solution to Problem 212

213. Let M, M' be each mobile points on the non-coplanar lines d, d'. Find the locus of points P that divide segment $|MM'|$ in a given ratio.

Solution to Problem 213

214. Construct a line that meets three given lines, respectively in M, N, P and for which $\frac{\|MN\|}{\|NP\|}$ to be given ratio.

Solution to Problem 214

215. Find the locus of the peak P of the triangle M, N, P if its sides remain parallel to three fixed lines, the peak M describes a given line d, and the peak $N \in$ a given plane α.

Solution to Problem 215

216. On the edges $[OA, [OB, [OC$ of a trihedral angle we consider points M, N, P such that $\|OM\| = \lambda\|OA\|, \|ON\| = \lambda\|OB\|, \|OP\| = \lambda\|OC\|$, where λ is a positive variable number. Show the locus of the centroid of triangle MNP.

Solution to Problem 216

217. $ABCD$ and $A_1B_1C_1D_1$ are two parallelograms in space. We take the points A_2, B_2, C_2, D_2 which divide segments $[AA_1], [BB_1], [CC_1], [DD_1]$ in the same ratio. Show that $A_2B_2C_2D_2$ is a parallelogram.

Solution to Problem 217

218. The lines d, d' are given, which cut a given plane α in A and A'. Construct the points M, M' on d, d' such that $MM' \parallel \alpha$ and segment $[MM']$ to have a given length l. Discuss.

Solution to Problem 218

219. Construct a line which passes through a given point A and that is perpendicular to two given lines d and d'.

Solution to Problem 219

220. Show that there exist three lines with a common point, perpendicular two by two.

Solution to Problem 220

221. Let $a\ b, c, d$ four lines with a common point, d is perpendicular to $a\ b, c$. Show that lines a, b, c are coplanar.

Solution to Problem 221

222. Show that there do not exist four lines with a common point that are perpendicular two by two.

Solution to Problem 222

223. Let $d \perp \alpha$ and $d' \parallel d$. Show that $d' \perp \alpha$.

Solution to Problem 223

224. Show that two distinct perpendicular lines on a plane are parallel.

Solution to Problem 224

225. Let $d \perp \alpha$ and $d'[\parallel \alpha$. Show that $d' \perp d$.

Solution to Problem 225

226. Show that two perpendicular planes on the same line are parallel with each other.

Solution to Problem 226

227. Show that the locus of the points equally distant from two distinct points A and B is a perpendicular plane to AB, passing through midpoint O of the segment $[AB]$ (called mediator plane of $[AB]$).

Solution to Problem 227

228. Find the locus of the points in space equally distant from the peaks of a triangle ABC.

Solution to Problem 228

229. The plane α and the points $A \in \alpha, B \notin \alpha$ are given. A variable line d passes through A and it is contained in plane α. Find the locus of the $\perp$ feet from B to d.

Solution to Problem 229

230. A line α, and a point $A \notin \alpha$ are given. Find the locus of the feet of the perpendicular lines from A to planes passing through α.

Solution to Problem 230

231. Consider a plane α that passes through the midpoint of segment $[AB]$. Show that points A and B are equally distant from plane α.

Solution to Problem 231

232. Through a given point, draw a line that intersects a given line and is $\perp$ to another given line.

Solution to Problem 232

233. Let α and β be two distinct planes and the line d their intersection. Let M be a point that is not located on $\alpha \cup \beta$. We draw the lines MM_1 and MM_2 $\perp$ on α and β. Show that the line d is $\perp$ to (MM_1MM_2).

Solution to Problem 233

234. A plane α and a point $A, A \notin \alpha$ are given. Find the locus of points $M \in \alpha$ such that segment $|AM|$ has a given length.

Solution to Problem 234

235. Let O, A, B, C be four points such that $OA \perp OB \perp OC \perp DA$ and we write $a = \|OA\|$, $b = \|OB\|$, $c = \|OC\|$.

a. Find the length of the sides of ΔABC in relation to a, b, c;
b. Find $\sigma[ABC]$ and demonstrate the relation $\sigma[ABC]^2 = \sigma[DAB]^2 + \sigma[OBC]^2 + \sigma[OCA]^2$;
c. Show that the orthogonal projection of point O on plane (ABC) is the orthocenter H of ΔABC;
d. Find the distance $\|OH\|$.

Solution to Problem 235

236. Consider non-coplanar points A, B, C, D and lines AA', BB', CC', DD' perpendicular to $(BCD), (ACD), (ABD)$. Show that if lines AA' and BB' are concurrent, then lines CC', DD' are coplanar.

Solution to Problem 236

237. Let A, B, C, D four non-coplanar points. Show that $AB \perp CD$ and $AC \perp BD$ $\Longrightarrow AD \perp BC$.

Solution to Problem 237

238. On the edges of a triangle with its peak O, take the points A, B, C such that $|OA| \equiv |OB| \equiv |OC|$. Show that the $\perp$ foot in O to the plane (ABC) coincides with the point of intersection of the bisectors ΔABC.

Solution to Problem 238

239. Let a peak A of the isosceles triangle ABC ($|AB| \equiv |AC|$) be the orthogonal projection onto A' on a plane α which passes through BC. Show that $\widehat{BA'C} > \widehat{BAC}$.

Solution to Problem 239

240. With the notes of *Theorem 1*, let $[AB'$ be the opposite ray to $[AB''$. Show that for any point $M \in \alpha - [AB''$ we have $\widehat{B''AB} > \widehat{MAB}$.

Solution to Problem 240

241. Let α be a plane, $A \in \alpha$ and B and C two points on the same side of α such that $AC \perp \alpha$. Show that $\widehat{CAB}$ is the complement of the angle formed by $[AB$ with α.

Solution to Problem 241

242. Let $\alpha'\beta'$ be a trihedral angle with edge m and $A \in m$. Show that of all the rays with origin at A and contained in half-plane β', the one that forms with plane α the biggest possible angle is that $\perp p \in m$ (its support is called the line with the largest slope of β in relation to α).

Solution to Problem 242

243. Let α be a plane, σ a closed half-plane, bordered by α, α' a half-plane contained in α and a a real number between 0^0 and 180^0. Show that there is only one half-space β' that has common border with α' such that $\beta' \subset \sigma$ and $m(\alpha'\beta') = a$.

Solution to Problem 243

244. Let $(\widehat{\alpha'\beta'})$ be a proper dihedral angle. Construct a half-plane γ' such that $m(\widehat{\alpha'\beta'}) = m(\widehat{\gamma'\beta'})$. Show that the problem has two solutions, one of which is located in the $\mathrm{int}.\widehat{\alpha'\beta'}$ (called bisector half-plane of $\widehat{\alpha'\beta'}$).

Solution to Problem 244

245. Show that the locus of the points equally distant from two secant planes α, β is formed by two $\perp$ planes, namely by the union of the bisector planes of the dihedral angles α and β.

Solution to Problem 245

246. If α and β are two planes, $Q \in \beta$ and $d \perp$ through Q on α. Show that $d \subset \beta$.

Solution to Problem 246

247. Consider a line $d \subset \alpha$. Show that the union of the $\perp$ lines to α, which intersect line d, is a plane $\perp \alpha$.

Solution to Problem 247

248. Find the locus of the points equally distant from two concurrent lines.

Solution to Problem 248

249. Show that a plane $\alpha \perp$ to two secant planes is $\perp$ to their intersection.

Solution to Problem 249

250. Let A be a point that is not on plane α. Find the intersection of all the planes that contain point A and are $\perp$ to plane α.

Solution to Problem 250

251. From a given point draw a $\perp$ plane to two given planes.

Solution to Problem 251

252. Intersect a dihedral angle with a plane as the angle of sections is right.

Solution to Problem 252

253. Show that a line d and a plane α, which are perpendicular to another plane, are parallel or line d is contained in α.

Solution to Problem 253

254. If three planes are $\perp$ to a plane, they intersect two by two after lines a, b, c. Show that $a \parallel b \parallel c$.

Solution to Problem 254

255. From a point A we draw perpendicular lines AB and AC to the planes of the faces of a dihedral angle $\widehat{\alpha'\beta'}$. Show that $m(\widehat{BAC}) = m(\widehat{\alpha'\beta'})$ or $m(\widehat{BAC}) = 180^0 - m(\widehat{\alpha'\beta'})$.

Solution to Problem 255

Solutions

Solution to Problem 201.

$M_1 - z_1 = 1$
$M_2 - z_1 = 2 + i$
$M_1 - z_1 = x + yi$
$\Delta M_1M_2M_3$ equilateral $\Longrightarrow \|M_1M_2\| = \|M_1M_3\| = \|M_2M_3\| \Rightarrow |z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$

$$\Rightarrow \sqrt{2} = \sqrt{(x-2)^2 + (y-1)^2} \Rightarrow \begin{cases} (x-2)^2 + (y-1)^2 = 2 \\ (1-x)^2 + y^2 = 2 \end{cases} \Rightarrow \begin{cases} x + y = 2 \\ x^2 + y^2 - 2x = 1 \end{cases}$$

$$\Rightarrow y = 2 - x$$

$$x^2 + 4 + x^2 - 4x - 2x = 1 \Rightarrow x_{1,2} = \frac{3 \pm \sqrt{3}}{2} \Rightarrow \left[\begin{array}{l} y_1 = \dfrac{1-\sqrt{3}}{2} \\ y_2 \dfrac{1+\sqrt{3}}{2} \end{array}\right.$$

Thus: $M_3\left(\frac{3+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ or $M_3\left(\frac{3-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$.

There are two solutions!

Solution to Problem 202.

$z_1 = r(\cos t_1 + i \sin t_1)$
$z_2 = r(\cos t_2 + i \sin t_2)$
$z_3 = r(\cos t_3 + i \sin t_3)$
$z_1 \neq z_2 \neq z_3 \Rightarrow t_1 \neq t_2 \neq t_3$

$$\begin{cases} z_1 + z_2z_3 \in \mathbb{R} \Rightarrow \sin t_1 + r \sin(t_2 + t_3) = 0 \\ z_2 + z_3z_1 \in \mathbb{R} \Rightarrow \sin t_2 + r \sin(t_1 + t_3) = 0 \Longrightarrow \\ z_3 + z_1z_2 \in \mathbb{R} \Rightarrow \sin t_3 + r \sin(t_1 + t_2) = 0 \end{cases}$$

$$\begin{cases} \sin t_1(1 - r\cos t) + r \sin t \cdot \cos t_1 = 0 \\ \sin t_2(1 - r\cos t) + r \sin t \cdot \cos t_2 = 0 \\ \sin t_3(1 - r\cos t) + r \sin t \cdot \cos t_3 = 0 \end{cases}$$

$t_1 \neq t_2 \neq t_3$

These equalities are simultaneously true only if $1 - r \cdot \cos t = 0$ and $r \cdot \sin t = 0$, as $r \neq 0 \Rightarrow \sin t = 0 \Rightarrow t = 0 \Rightarrow \cos t = 1 \Rightarrow 1 - r = 0 \Rightarrow r = 1$, so $z_1z_2z_3 = 1 \cdot (\cos 0 + \sin 0) = 1$.

Solution to Problem 203.

a. $\varepsilon_k = \frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ $, k \in \{0, 1, \dots, n-1\}$.

So $\left.\begin{matrix} \varepsilon_i = \cos\frac{2i\pi}{n} + i\sin\frac{2i\pi}{n} \\ \varepsilon_j = \cos\frac{2j\pi}{n} + i\sin\frac{2j\pi}{n} \end{matrix}\right\} \Longrightarrow \varepsilon_i\varepsilon_j = \cos\frac{2\pi(i+j)}{n} + i\sin\frac{2\pi(i+j)}{n}, i, j \in \{0, 1, \dots, n-1\}$.

1) $i + j < n - 1 \Longrightarrow i + j = k \in \{0, 1, \dots, n-1\} \Longrightarrow \varepsilon_i\varepsilon_j = \varepsilon_k \in G$;

2) $i + j = n \Longrightarrow \varepsilon_i\varepsilon_j = \cos 2\pi + i\sin 2\pi = 1 = \varepsilon_o \in G$;

3) $i + j > n \Longrightarrow i + j = n \cdot m + r, 0 \le r < n, \varepsilon_i\varepsilon_j = \cos\frac{2\pi(n \cdot m + r)}{n} + i\sin\frac{2\pi(n \cdot m + r)}{n} =$ $\cos\left(2\pi m + \frac{2\pi r}{n}\right) + i\sin\left(2\pi m + \frac{2\pi r}{n}\right) = \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n} = \varepsilon_r \in G$.

b. $\varepsilon_i = \cos\frac{2\pi i}{n} + i\sin\frac{2\pi i}{n}$

$\varepsilon_i^{-1} = \frac{1}{\varepsilon_i} = \frac{\cos 0 + i\sin 0}{\cos\frac{2\pi i}{n} + i\sin\frac{2\pi i}{n}} = \cos\left(-\frac{2\pi i}{n}\right) + i\sin\left(-\frac{2\pi i}{n}\right) = \cos\left(2\pi - \frac{2\pi i}{n}\right) +$ $i\sin\left(2\pi - \frac{2\pi i}{n}\right) = \cos\frac{2\pi n - 2\pi i}{n} + i\sin\frac{2\pi n - 2\pi i}{n} = \cos\frac{2\pi(n-1)}{n} + i\sin\frac{2\pi(n-1)}{n}$,

$i \in \{0, 1, \dots, n-1\}$.

If $i = 0 \Longrightarrow n - i = n \Longrightarrow \varepsilon_0^{-1} = \varepsilon_0 \in g$.

If $i \neq 0 \Longrightarrow n - i \le n - 1 \Longrightarrow h = n - i \in \{0, 1, \dots, n-1\} \Longrightarrow \epsilon^{-1} = \cos\frac{2\pi h}{n} +$ $i\sin\frac{2\pi h}{n} \in G$.

Solution to Problem 204.

$$\begin{cases} a = r_1(\cos t_1 + i\sin t_1) \\ b = r_2(\cos t_2 + i\sin t_2) \\ c = r_3(\cos t_3 + i\sin t_3) \end{cases}$$

$\arg a + \arg c = 2\arg b \Longrightarrow t_1 + t_3 = 2t_2$

and $|a| + |c| = |b| \Longrightarrow r_1 + r_3 = r_2$

$$az^2 + bz + c = 0 \Longrightarrow z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-r_2(\cos t_2 + i\sin t_2) \pm \sqrt{r_2^2(\cos 2t_2 + i\sin 2t_2) - 4r_1r_3(\cos(t_1 + t_3) + i\sin(t_1 + t_3))}}{2r_1(\cos t_1 + i\sin t_1)}$$
$$= \frac{-r_2(\cos t_2 + i\sin t_2) \pm \sqrt{(\cos 2t_2 + i\sin 2t_2)(r_2^2 - 4r_1r_3)}}{2r_1(\cos t_1 + i\sin t_1)}$$

But $r_1 + r_3 = r_2 \Longrightarrow r_2^2 = r_1^2 + r_1^2 + r_3^2 + 2r_1r_3 \Longrightarrow r_2^2 - 4r_1r_3 = r_1^2 + r_1^2 + r_3^2 + 2r_1r_3 -$ $4r_1r_3 = (r_1 - r_3)^2$.

Therefore:

$$z_{1,2} = \frac{-r_2(\cos t_2 + i\sin t_2) \pm (\cos t_2 + i\sin t_2)(r_1 - r_3)}{2r_1(\cos t_1 + i\sin t_1)}$$

We observe that:

$z_2 = \frac{(\cos t_2 + i\sin t_2)(-2r_1)}{2r_1(\cos t_1 + i\sin t_1)} = -[\cos(t_2 - t_1) + i\sin(t_2 - t_1)] = \cos[\pi + (t_2 - t_1)] + i\sin[\pi + t_2 - t_1]$ and $t_2 = 1$.

Solution to Problem 205.

Let

$$\begin{cases} z_1 = r(\cos t_1 + i\sin t_1) \\ z_2 = r(\cos t_2 + i\sin t_2), \qquad r \neq 0 \\ z_3 = r(\cos t_3 + i\sin t_3) \end{cases}$$

Let

$\alpha = r_4(\cos t_4 + i\sin t_4)$

$\beta = r_5(\cos t_5 + i\sin t_5)$

$z_2 = \alpha z_1 \Rightarrow r(\cos t_2 + i\sin t_2) = r \cdot r_4[\cos(t_1 + t_4) + i\sin(t_1 + t_4)] \Rightarrow$

$$\Rightarrow \begin{cases} r = r \cdot r_4 \\ t_4 + t_1 = t_2 + 2k\pi \end{cases} \Rightarrow \begin{cases} r_4 = 1 \\ t_4 = t_2 - t_1 + 2k\pi \end{cases} \Rightarrow \begin{cases} |\alpha| = 1 \\ t_4 = t_2 - t_1 + 2k\pi \end{cases}$$

So α is determined.

$z_3 = \beta z_1 \Rightarrow r(\cos t_3 + i\sin t_3) = r \cdot r_3[\cos(t_1 + t_5) + i\sin(t_1 + t_5)] \Rightarrow$

$$\Rightarrow \begin{cases} r = r \cdot r_5 \\ t_1 + t_5 = t_3 + 2k\pi \end{cases} \Rightarrow \begin{cases} r_5 = 1 \\ t_5 = t_3 - t_1 + 2k\pi \end{cases} \Rightarrow \begin{cases} |\beta| = 1 \\ t_5 = t_3 - t_1 + 2k\pi \end{cases}$$

So β is determined.

If we work with reduced arguments, then $t_4 = t_2 - t_1$ or $t_4 = t_2 - t_1 + 2\pi$, in the same way t_5.

b) $\alpha^2 + \alpha(-\beta - 1) + \beta^2 - \beta + 1 = 0$

$$\alpha_{1,2} = \frac{\beta + 1 \pm \sqrt{\beta^2 + 2\beta + 1 - 4\beta^2 + 4\beta - 4}}{2} = \frac{\beta + 1 \pm \sqrt{-3\beta^2 + 6\beta - 3}}{2} = \frac{\beta + 1 + i(\beta - 1)\sqrt{3}}{2}$$

$$\alpha_1 = \frac{\beta + i(\beta - 1)\sqrt{3}}{2}, \qquad \alpha_2 = \frac{\beta - i(\beta - 1)\sqrt{3}}{2}$$

c) $z_1{}^2 + z_2{}^2 + z_3{}^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$

According to *a*. (∃) the complex numbers of modulus 1, α and β such that $z_2 = \alpha z_1$ and $z_3 = \beta z_1$.

In the given relation, by substitution we obtain:

$$\left.\begin{matrix} z_1{}^2 + \alpha^2 z_1{}^2 + \beta^2 z_1{}^2 = \alpha z_1{}^2 + \alpha\beta z_1{}^2 + \beta z_1{}^2 = \\ z_1 \neq 0 \end{matrix}\right\} \Rightarrow 1 + \alpha^2 + \beta^2 - \alpha - \beta - \alpha \cdot \beta = 0$$

$\alpha = 1$ and $\beta = 1$ verify this equality, so in this case $z_2 = z_3 = z_1$.

According to point *b*.,

$$\alpha_1 = \frac{\beta + i(\beta - 1)\sqrt{3}}{2},$$

where $\beta = x + iy$, when

$$|\beta| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow$$

$$x^2 + y^2 = 1 \qquad (1)$$

$$\left.\begin{matrix} \alpha = \dfrac{x + iy + 1 + i(x + iy - 1)\sqrt{3}}{2} = \dfrac{(x + 1 - y\sqrt{3}) + i(y + x\sqrt{3} - \sqrt{3})}{2} \\ |\alpha| = 1 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow |\alpha| = \sqrt{\left(\frac{x + 1 - y\sqrt{3}}{2}\right)^2 + \left(\frac{x + x\sqrt{3} - y\sqrt{3}}{2}\right)^2} = \sqrt{x^2 + y^2 - x + 1y\sqrt{3} = 1}$$

We construct the system:

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 - x - y\sqrt{3} = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 = 1 \\ 1 - x - y\sqrt{3} = 0 \end{cases} \Rightarrow \begin{cases} x = 1 - y\sqrt{3} \\ y(2y - \sqrt{3}) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \beta = 1 \end{cases} \Rightarrow$$

$$\Rightarrow 1 + \alpha^2 + 1 - \alpha - 1 - \alpha = 0 \Rightarrow \alpha = 1$$

The initial solution leads us to $z_1 = z_2 = z_3$.

$$y = \frac{\sqrt{3}}{2} \Rightarrow x = -\frac{1}{2}$$

and gives

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

By substituting,

$$1 + \frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}i + \frac{1}{2} - \frac{i\sqrt{3}}{2} + \alpha^2 - \alpha - \alpha\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0 \Rightarrow$$

$$\Rightarrow 2\alpha^2 - \alpha(1 + \sqrt{3}i) + 2(1 - \sqrt{3}i) = 0$$

$$\alpha_{1,2} = \frac{(1 + \sqrt{3}) \pm 3\sqrt{-2 + 2\sqrt{3}i}}{4} = \frac{(1 + \sqrt{3}i) \pm 3(1 + \sqrt{3}i)}{4}$$

$$\alpha_1 = \frac{4(1 + \sqrt{3}i)}{4},$$

$|\alpha| = 2$ does not comply with the condition $|\alpha| = 1$.

But

$$\alpha_2 = \frac{-2(1+\sqrt{3}i)}{4}, = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad |\alpha| = 1$$

so

$$\begin{cases} \alpha = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \\ \beta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \end{cases}$$

If

$$z_1 = r(\cos t_1 + i\sin t_1),$$

then

$$t_2 = \alpha t_1 = r \cdot \left[\cos\left(t_1 + \frac{4\pi}{3}\right) + i\sin\left(t_1 + \frac{4\pi}{3}\right)\right]$$

and then

$$t_3 = \beta z_2 = r \cdot \left[\cos\left(t_1 + \frac{2\pi}{3}\right) + i\sin\left(t_1 + \frac{2\pi}{3}\right)\right].$$

If

$$M_1(t_1), M_2(t_2), M_3(t_3)$$

are on the circle with radius r and the arguments are

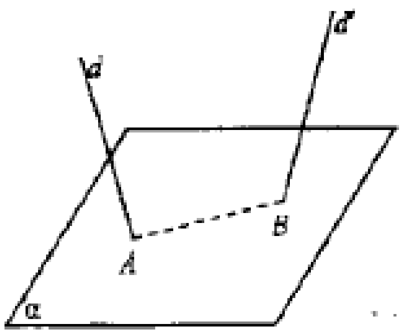
$$t_1, t_1 + \frac{2\pi}{3},$$

$$t_1 + \frac{4\pi}{3} \Rightarrow$$

they are the peaks of an equilateral triangle.

Solution to Problem 206.

a. We assume that $d \cap \alpha \neq \emptyset$ and $d' \cap \alpha = \{B\}$.



Let $d \cap \alpha = \{A\}$ and $d' \cap \alpha = \{B\}$ and the planes determined by pairs of concurrent lines (d, AB); (d', AB).

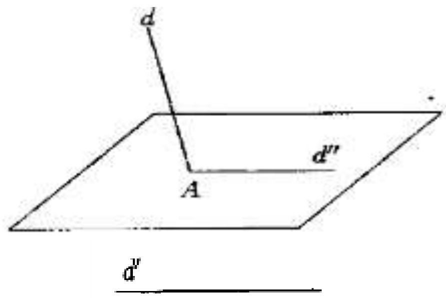
We remark that these are the required planes, because

$d \subset (d, AB)$, $d' \subset (d', AB)$ și $(d, AB) \cap (d', AB) = AB \subset \alpha$.

b. We assume $d \cap \alpha = \{A\}$ and $d' || \alpha$.

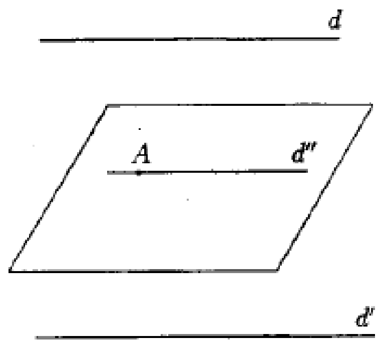
We draw through A, in plane α, line $d' || d$ and we consider planes (d, d'') and (d', d'') and we remark that

$(d, d'') \cap (d', d'') = d'' \subset \alpha$

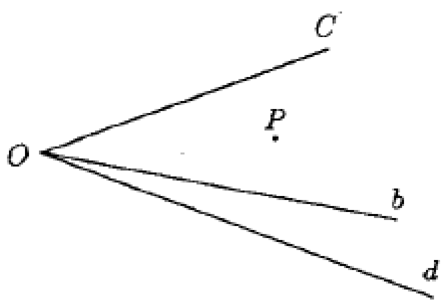


c. We assume $d \cap \alpha = \emptyset$ and $d' \cap \alpha = \emptyset$ and $d' \in$ direction d.

Let $A \in \alpha$ and $d'' || d \Rightarrow d'' || d'$ and the planes are (d, d'') and (d', d''). The reasoning is the same as above.



Solution to Problem 207.



$$a \cap b \cap c = \{O\}$$

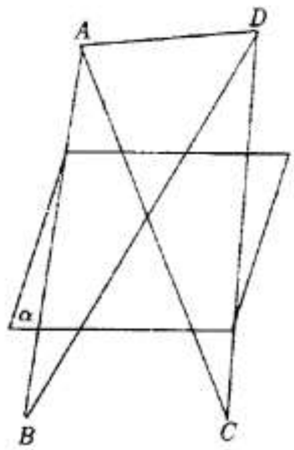
$$\left.\begin{array}{l} P \in (Pa) \\ O \in (Pa) \end{array}\right\} \Rightarrow OP \subset (Pa)$$

$$\left.\begin{array}{l} P \in (Pb) \\ O \in (Pb) \end{array}\right\} \Rightarrow OP \subset (Pb)$$

$$\left.\begin{array}{l} P \in (Pc) \\ O \in (Pc) \end{array}\right\} \Rightarrow OP \subset (Pc)$$

$$\Rightarrow (Pa) \cap (Pb) \cap (Pc) = OP$$

Solution to Problem 208.



If α separates points A and B, it means they are in different half-spaces and let $\sigma = |\alpha A$ and $\sigma' = |\alpha B$.

Because α separates A and $C \Longrightarrow C \in \sigma'$.

Because α separates C and $D \Longrightarrow D \in \sigma$.

From $B \in \sigma'$ and $D \in \sigma \Longrightarrow \alpha$ separates points B and D

$|BD| \cap \alpha \neq \emptyset.$

From $A \in \sigma$ and $D \in \sigma \Longrightarrow |BD| \cap \alpha = \emptyset$.

Solution to Problem 209.

$$\left.\begin{array}{l} B \in (ab) \\ C|(a,b),c \end{array}\right\} \Longrightarrow [BC] \subset |(a,b),c \Rightarrow$$

$\Rightarrow |OE \subset |(bc), A$ (2)

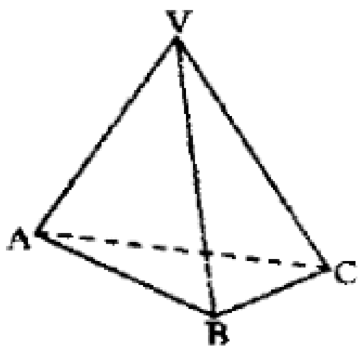
$\left.\begin{array}{l} C \in (ac) \\ B \in |(ac), c \end{array}\right\} \Rightarrow [CB] \subset |(ac), B \Rightarrow \left.\begin{array}{l} D \in |(ac), B \\ A \in (a, c) \end{array}\right\} \Rightarrow [AD] \subset |(ac), B$

$\Rightarrow \left.\begin{array}{l} E \in |(ac), B \\ O \in (a, c) \end{array}\right\} \Rightarrow [AD] \subset |(ac), B \Rightarrow \left.\begin{array}{l} E \in |(ac), B \\ O \in (ac) \end{array}\right\} \Rightarrow |OE \subset |(ac), B$ (3)

From (1), (2), (3)

$\Rightarrow |OE \subset |(ab), c \cap |(bc), A \cap |(ac), B = \text{int. } \widehat{abc}$.

Solution to Problem 210.



a. $\text{int.}(|VA, |\widehat{VB}, |VC) = |(VAB), C \cap |(VBC), A \cap |(VAC), B$ is thus an intersection of convex set and thus the interior of a trihedron is a convex set.
b. Tetrahedron $[VABC]$ without edge $[AC]$. We mark with $\mathcal{M}_1 = [ABC] - [AC] = [AB, C \cap [BC, A \cap |AC, B$ is thus a convex set, being intersection of convex sets.

   $|(ABC), V \cap \mathcal{M}_1$

   is a convex set.

   In the same way

   $|(VAC), B \cup \mathcal{M}_2$

   is a convex set, where

   $\mathcal{M}_2 = [VAC] - [AC]$.

   But $[VABC] - [AC] =$

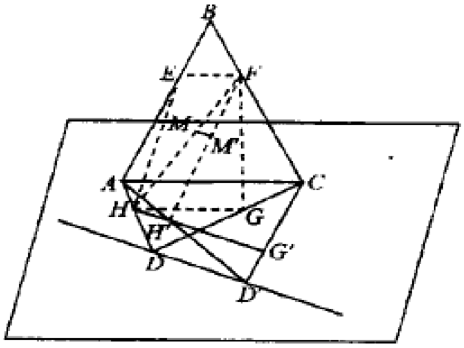
   $[(VAB), C \cap [(VBC), A \cap (|(ABC), V \cup \mathcal{M}_1) \cap (|(VAC), B \cup \mathcal{M}_2)$

   and thus it is a convex set as intersection of convex sets.
c. Tetrahedron $[VABC]$ without face $[ABC]$

   $$[VABC] - [ABC] = [(VAB), C \cap [(VBC), A \cap [(VAC), B \cap |(ABC),$$

   V is thus intersection of convex sets $\Rightarrow$ is a convex set.

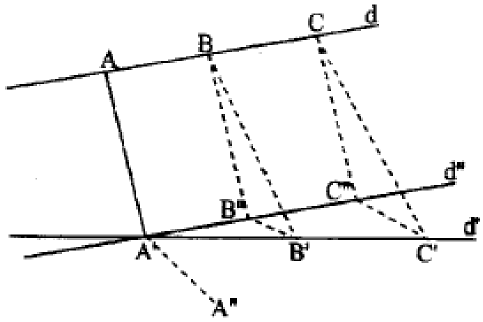
Solution to Problem 211.



In plane (BAC) we have $EF||AC$. In plane (DAC) we have $AC \subset (DAC) \Longrightarrow EF||(DAC)$. In this plane we also have $HG||AC$. So $EF||HG \Longrightarrow E, F, G, H$ are coplanar and because $\|EF\| = \frac{\|AC\|}{2} = \|HG\| \Longrightarrow EFGH$ is a parallelogram.

Solution to Problem 212.

We assume we have $\alpha||\beta||\gamma$ such that $AA' \subset \alpha, BB' \subset , CC' \subset \gamma$.



We draw through A' a parallel line with d: $d''||d$. As d intersects all the 3 planes $A' \subset d''$ at $A, B, C \Longrightarrow$ and its $\|$ d'' cuts them at A', B'', C''.

Because

$$\left.\begin{array}{r}\alpha \parallel \beta \parallel \gamma \\ d \parallel d''\end{array}\right\} \Rightarrow \begin{array}{l}\|AB\| = \|AB''\| \\ \|BC\| = \|B''C''\|\end{array} \qquad (1)$$

Let plane (d', d''). Because this plane has in common with planes α, β, γ the points A', B'', C'' and because $\alpha \parallel \beta \parallel \gamma \Longrightarrow$ it intersects them after the parallel lines

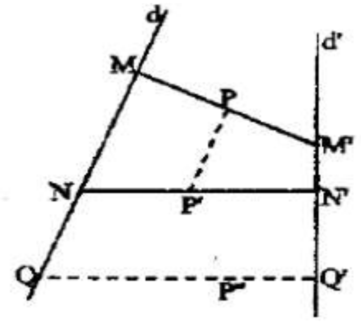
$$\| \; C'C'' \overset{Tales}{\Longrightarrow} \frac{\|A'B''\|}{\|A'B'\|} = \frac{\|B''C''\|}{\|B'C'\|}. \qquad (2)$$

Taking into consideration (1) and (2)

$$\Rightarrow \frac{\|AB\|}{\|A'B'\|} = \frac{\|BC\|}{\|B'C'\|}.$$

The vice-versa can be similarly proved.

Solution to Problem 213.



Let

$P \in |MM'|$

such that

$\frac{\|MP\|}{\|PM'\|} = k$

and

$P' \in |NN'|$

such that

$\frac{\|NP'\|}{\|P'N'\|} = k$

So

$\frac{\|MP\|}{\|PM'\|} = \frac{\|NP'\|}{\|P'N'\|} \Rightarrow \frac{\|MP\|}{\|NP'\|} = \frac{\|PM'\|}{\|P'N'\|}$

according to problem 7, three planes can be drawn $\| \beta \| \alpha \| \gamma$ such that

$MN \subset \beta,\ PP' \subset \alpha,\ M'N' \subset \gamma.$

$\left.\begin{matrix} \beta \parallel \alpha \\ MN \subset \beta \end{matrix}\right\} \Rightarrow MN \parallel \alpha \Rightarrow d \parallel \alpha$

$\left.\begin{matrix} \alpha \parallel \gamma \\ M'N' \subset \gamma \end{matrix}\right\} \Rightarrow M'N' \parallel \alpha \Rightarrow d' \parallel \alpha$

and $PP' \subset \alpha$.

So by marking P and letting P' variable, $P' \in$ a parallel plane with the two lines, which passes through P. It is known that this plane is unique, because by drawing through P parallel lines to d and d' in order to obtain this plane, it is well determined by 2 concurrent lines.

Vice-versa: Let $P \in \alpha$, that is the plane passing through P and it is parallel to d and d'.

(P'', d) determines a plane, and (P'', d') determines a plane $\Longrightarrow$ the two planes, which have a common point, intersect after a line$(P'', d) \cap (P'', d') = QQ'$ where $Q \in d$ and $Q' \in d'$.

Because

$d \parallel \alpha \Rightarrow MQ \parallel \alpha \Rightarrow (\exists)\beta$

such that $MQ \subset \beta, \beta || \alpha$.

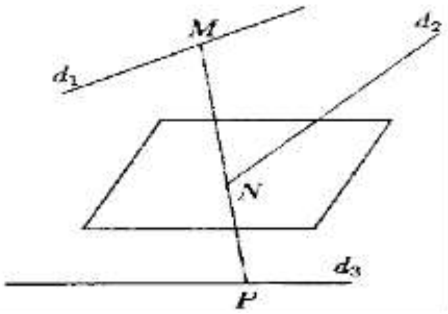
Because

$d' \parallel \alpha \Rightarrow M'Q' \parallel \alpha \Rightarrow (\exists)\gamma$

such that $M'Q' \subset \gamma, \gamma || \alpha$.

So the required locus is a parallel plane with d and d'.

Solution to Problem 214.



We consider the plane, which according to a previous problem, represents the locus of the points dividing the segments with extremities on lines d_1 and d_3 in a given ratio k. To obtain this plane, we take a point $A \in d_1$, $B \in d_3$ and point $C \in AB$ such that $\frac{\|AC\|}{\|CB\|} = k$. Through this point C we draw two parallel lines d_1 and d_3 which determine the above mentioned plane α.

Let $d_2 \cap \alpha = \{N\}$. We must determine a segment that passes through N and with its extremities on d_1 and d_3, respectively at M and P. As the required line passes through N and M

$$\left.\begin{array}{l} N \in (N, d_1) \\ M \in (N, d_1) \end{array}\right\} \Rightarrow MN \subset (Nd_1) \qquad (1)$$

The same line must pass through N and P and because

$$\left.\begin{array}{l} N \in (N, d_1) \\ P \in (N, d_3) \end{array}\right\} \Rightarrow NP \subset (Nd_3) \quad (2)$$

M, N, P collinear.

From (1) and (2)

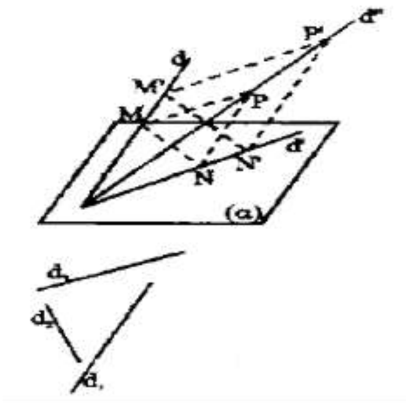
$$\Rightarrow MP \subset (N, d) \cap (N, d_3) \Rightarrow MP = (Nd_1) \cap (d_2 N)$$

Then, according to previous problem 8:

$$\frac{\|MN\|}{\|NP\|} = k$$

and the required line is MP.

Solution to Problem 215.



Let ΔMNP such that

$$MP \parallel d_1,\ MN \parallel d_2,\ PN \parallel d_3.\ M \in d, N \in \alpha$$

Let $\Delta M'N'P'$ such that

$$M' \in d,\ \ N' \in \alpha,\ \ M'P' \parallel d_1;\ M'N' \parallel d_2;$$

$P'N' \parallel d_3.$

Line MP generates a plane β, being parallel to a fixed direction d_1 and it is based on a given line d. In the same way, the line MN generates a plane γ, parallel to a fixed direction d_2, and based on a given line d. As d is contained by $\gamma \Longrightarrow O$ is a common point for α and $\gamma \Longrightarrow \alpha \cap \gamma \neq \emptyset \Longrightarrow \alpha \cap \gamma = d', O \in d'$.

$$\left.\begin{array}{l} N \in \alpha \\ N \in \gamma \end{array}\right\} \Rightarrow N \in \alpha \cap \gamma$$

($\forall$) the considered Δ, so N also describes a line $d' \subset \alpha$.

Because plane γ is well determined by line d and direction d_2, is fixed, so $d' = \alpha \cap \gamma$ is fixed.

In the same way, PN will generate a plane δ, moving parallel to the fixed direction d_3 and being based on the given line d'.

As

$$\left.\begin{array}{l} O \in d' \\ O \in d \end{array}\right\} \Rightarrow \left\{\begin{array}{l} O \in \delta \\ O \in \beta \end{array}\right. \Rightarrow \beta \cap \delta \neq \emptyset \Rightarrow O \in \beta \cap \delta = d''$$

$$\left.\begin{array}{l} P \in MP \Rightarrow P \in \beta \\ P \in PN \Rightarrow P \in \delta \end{array}\right\} \Rightarrow P \in \beta \cap \delta$$

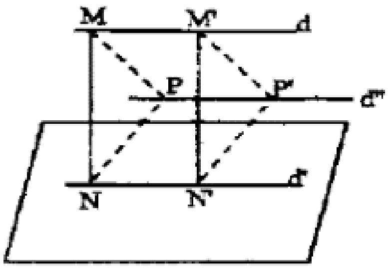
$(\forall)P$ variable peak, $P \in d''$.

Thus, in the given conditions, for any ΔMNP, peak $P \in d''$.

Vice-versa, let $P' \in d''$. On plane (d', d'') we draw $P', M' || PM \Longrightarrow (M'P'N') || (PMN) \Longrightarrow$ (dd') the intersection of two parallel planes after parallel lines $M'N' || MN$ and the so constructed $\Delta M'P'N'$ has its sides parallel to the three fixed lines, has $M' \in d$ and $N' \in \alpha$, so it is one of the triangles given in the text.

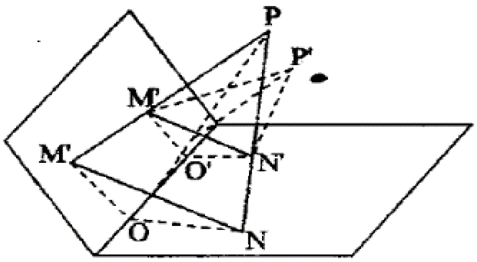
So the locus is line d''. We've seen how it can be constructed and it passes through O.

In the situation when $D || \alpha$ we obtain



$$\left.\begin{array}{l} d \subset \beta \\ \delta' \subset \delta \\ d \parallel d' \end{array}\right\} \Rightarrow \beta \cap \delta = d'' \text{ şi } d'' \parallel d$$

In this case the locus is a parallel line with d.



Let MNP and $M'N'P'$ such that

$$\left.\begin{array}{lll} MP \parallel d_1 & M'P' \parallel d_1 & MP \parallel M'P' \\ MN \parallel d_2 & M'N' \parallel d_2 \quad \Rightarrow & MN \parallel M'N' \\ NP \parallel d_3 & N'P' \parallel d_3 & NP \parallel N'P' \end{array}\right\} \Rightarrow \triangle MNP \sim \triangle M'N'P' \Rightarrow$$

$M \in \beta,\ N \in \alpha \quad M' \in \beta,\ N' \in \alpha$

$\Rightarrow (MNP) \parallel (M'N'P')$.

We assume $\alpha \cap \beta = d$ and let $d \cap (MNP) = \{O\}$ and $d \cap (M'N'P') = \{O'\} \Longrightarrow$

$$\left.\begin{array}{l}(MNP) \cap \beta = MO \\ (M'N'P') \cap \beta = M'O'\end{array}\right\} \Rightarrow MO \parallel M'O'$$

a plane cuts the parallel planes after parallel lines.

In the same way, $ON||O'N'$ and because

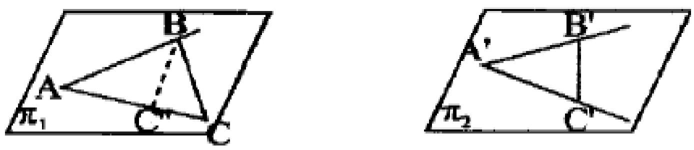
$$MN \parallel M'N' \Rightarrow \triangle OMN \sim \triangle O'M'N'$$

$$\Rightarrow \left.\begin{array}{c}\dfrac{\|OM\|}{\|O'M'\|} = \dfrac{\|ON\|}{\|O'N'\|} = \dfrac{\|MN\|}{\|M'N'\|} \\ O\widehat{M}N = O'\widehat{M'}N'\end{array}\right\} \Rightarrow \triangle MNP \sim M'N'P' \Rightarrow \left.\begin{array}{c}\dfrac{\|MP\|}{\|M'P'\|} = \dfrac{\|MN\|}{\|M'N'\|} \\ M\widehat{N}P = M'\widehat{N'}P'\end{array}\right\} \Rightarrow$$

$$\Rightarrow \triangle OMP \sim O'M'P' \qquad (1) \quad \Rightarrow \begin{array}{l}M\widehat{O}P \equiv M'\widehat{O'}P' \\ M\widehat{P}O \equiv M'\widehat{P'}O'\end{array}$$

We use the property: Let π_1 and π_2 2 parallel planes and $A, B, C \subset \pi_1$ and $A'B'C' \subset \pi_2$, $AB \parallel A'B'$,

$AC \parallel A'C'$, $A'\widehat{B'}C' \equiv A'\widehat{B'}C'$, $\|AB\| = \|A'B'\|$, $\|AC\| = \|A'C'\|$.

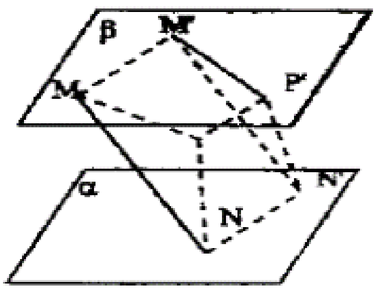


Let's show that $BC||B'C'$. Indeed $(BB'C')$ is a plane which intersects the 2 planes after parallel lines.

$$\Rightarrow \left.\begin{array}{l}B'C' \parallel BC'' \\ AB \parallel A'B'\end{array}\right\} \Rightarrow \left.\begin{array}{l}A\widehat{B}C'' \equiv A'\widehat{B'}C' \\ A\widehat{B}C \equiv A'\widehat{B'}C'\end{array}\right\} \Rightarrow A\widehat{B}C \equiv A\widehat{B}C'' \Rightarrow |BC = |BC'' \Rightarrow$$

$B'C' \parallel BC$.

Applying in (1) this property $\Longrightarrow OP||O'P'$. Maintaining OP fixed and letting P' variable, always $OP||O'P'$.=, so $O'P'$ generates a plane which passes through d. We assume $\beta||\alpha$.



$$\left.\begin{array}{l}\alpha \parallel \beta \\ MN \parallel M'N'\end{array}\right\} \Rightarrow |MN| \equiv |M'N'|$$

$MNN'M'$ parallelogram

$\Rightarrow MM' \parallel NN'$

$\Rightarrow MM' \parallel (NN'P'P)$

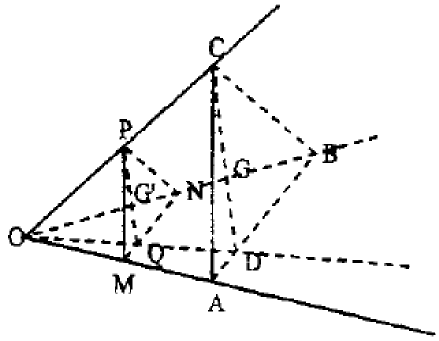
$\Rightarrow (MPP'M') \cap (MN'P'P) = PP'$

$PP' \parallel MM' \Rightarrow PP' \parallel NN'$

Considering P' fix and P variable $\Longrightarrow PP'||\alpha$ and the set of parallel lines drawn $PP'||\beta$ to a plane through an exterior point is a parallel plane with the given plane.

So the locus is a parallel plane with α and β.

Solution to Problem 216.



$\|OM\| = \lambda\|OA\| \Rightarrow \|\frac{\|OM\|}{\|OA\|} = \lambda;\ \|ON\| = \lambda\|OB\| \Rightarrow \|\frac{\|ON\|}{\|OB\|} = \lambda;$

$\|OP\| = \lambda\|OC\|, \Rightarrow \frac{\|OP\|}{\|OC\|} = \lambda$

In plane DAC we have:

$\frac{\|OM\|}{\|OA\|} = \frac{\|OP\|}{\|OC\|} \overset{?}{\Longrightarrow} PM \parallel AC.$

In plane DAB we have:

$\frac{\|OM\|}{\|OA\|} = \frac{\|ON\|}{\|OB\|} \Rightarrow MN \parallel AB.$

In plane OBC we have:

$\frac{\|ON\|}{\|OB\|} = \frac{\|OP\|}{\|OC\|} \Rightarrow PN \parallel BC.$

From $PM||AC$ and $PN||BC \Longrightarrow (MNR)||(ABC)$.

Let Q and D be midpoints of sides $|MN|$ and $|AB|$.

$$\left.\begin{array}{r}\triangle OMN \sim OAB \Rightarrow \frac{\|OM\|}{\|OA\|} = \frac{\|MN\|}{\|AB\|} = \frac{\frac{1}{2}\|MN\|}{\frac{1}{2}\|AB\|} = \frac{\|MN\|}{\|AB\|} \\ \widehat{OMQ} = \widehat{OAB}\end{array}\right\} \Rightarrow \triangle OMQ \sim OAD \Rightarrow$$

$\Rightarrow \widehat{MOQ} = \widehat{AOD} \Rightarrow O, Q, D$

are collinear.

Concurrent lines OD and OC determine a plane which cuts the parallel planes

$$\Rightarrow \left.\begin{array}{c} PQ \parallel CD \\ (OCD) \end{array}\right\} \Rightarrow \triangle OPQ \sim \triangle OCD \Rightarrow \frac{\|OP\|}{\|OC\|} = \frac{\|PQ\|}{\|CD\|} = \frac{\frac{2}{3}\|PQ\|}{\frac{2}{3}\|CD\|} = \frac{\|PG'\|}{\|CB\|} \Rightarrow$$

$$\Rightarrow \widehat{OPG'} \equiv \widehat{OCQ} \Rightarrow \triangle OPG' \sim \triangle OCQ \Rightarrow \widehat{POG'} \equiv \widehat{COG} \Rightarrow O, G', G$$

are collinear.

So $G' \in |OG \Longrightarrow$ the required locus is ray $|OG$.

Vice-versa: we take a point on $|OG, G''$, and draw through it a parallel plane to (ABC), plane (M'', N'', P''), similar triangles are formed and the ratios from the hypothesis appear.

Solution to Problem 217.

Let A_2, B_2, C_2, D_2 such that

$$\frac{\|AA_2\|}{\|A_2A_1\|} = \frac{\|BB_2\|}{\|B_2B_1\|} = \frac{\|CC_2\|}{\|C_2C_1\|} = \frac{\|DD_2\|}{\|D_2D_1\|} = k.$$

Mark on lines AD_1 and BC_1 points M and N such that

$$M \in |AD_1|, \frac{\|AM\|}{\|MD_1\|} = k$$

$$N \in |BC_1|, \frac{\|BN\|}{\|NC_1\|} = k$$

From

$$\frac{\|AA_2\|}{\|A_2A_1\|} = \frac{\|AM\|}{\|MD_1\|} = k \overset{T.Tales}{\Longrightarrow} A_2M \parallel A_1D_1 \qquad (1)$$

$$A_2M \parallel A_1D_1 \Rightarrow \triangle AA_2M \sim \triangle AA_1D_1 \Rightarrow \frac{\|AA_2\|}{\|AA_1\|} = \frac{\|A_2M\|}{\|A_1D_1\|}$$

Next is

$$\frac{\|A_1M_1\|}{\|A_2D_1\|} = \frac{k}{k+1} \Rightarrow \|A_2M\| = \frac{k}{k+1}\|A_1D_1\| \quad (2)$$

The same,

$$\frac{\|BB_2\|}{\|B_2B_1\|} = \frac{\|BN\|}{\|NC_1\|} = k \Rightarrow B_2N \parallel B_1C_1 \qquad (3)$$

$$B_2N \parallel B_1C_1 \Rightarrow \triangle BB_1N \sim \triangle BB_1C,$$

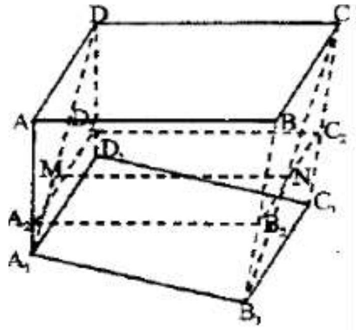
$$\Rightarrow \frac{\|BB_2\|}{\|BB_1} = \frac{\|B_2N\|}{\|B_1C_1\|}.$$

As

$$\frac{\|BB_2\|}{\|BB_1\|} = \frac{\|AA_2\|}{\|AA_1\|} = \frac{k}{k+1}.$$

we obtain

$$\frac{\|B_2N\|}{B_1C_1} = \frac{k}{k+1} \Rightarrow \|B_2N\| = \frac{k}{k+1}\|B_1C_1\| \qquad (4)$$



From

$A_1D_1 \parallel B_1C_1 \Rightarrow \|A_1D_1\| \parallel \|B_1c_1\|$

(1), (2), (3), (4) $\Rightarrow A_2M \parallel B_2N$ şi $\|A_2M\| = \|B_2N\| \Rightarrow A_2D_2NM$

is a parallelogram.

$\Rightarrow A_2B_2 \parallel MN, \|A_2B_2\| \parallel \|MN\| \Rightarrow D_2C_2NM$

is parallelogram.

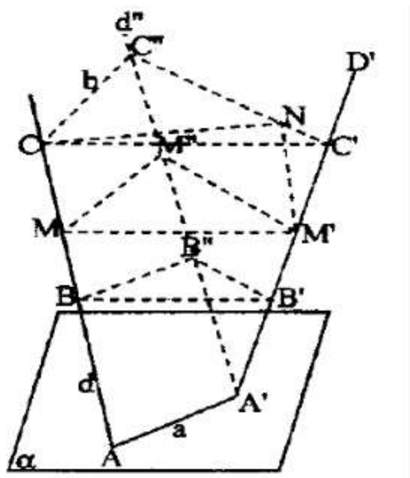
$\Rightarrow D_2C_2 \parallel MN, \|D_2C_2\| \parallel \|MN\|$

So

$A_2B_2 \parallel D_2C_2$ şi $\|A_2D_2\| \parallel \|D_2C_2\| \Rightarrow A_2B_2C_2D_2$

is a parallelogram.

Solution to Problem 218.



We draw through A' a line $d''||d$. We draw two parallel planes with α, which will intersect the three lines in B', B^*, B and C', C^*, C. Plane (d, d^*) intersects planes α, $(B'B^*B)$, $(C'C^*C)$ after parallel lines

$$\left.\begin{array}{c} AA' \parallel BB^* \parallel CC^* \\ d \parallel d^* \end{array}\right\} \Rightarrow \|AA'\| = a =$$

$= \|BB^*\| = \|C'C^*\|$.

Plane (d', d^*) intersects parallel planes $(B'B^*B)$, $(C'C^*C)$ after parallel lines

$$\Rightarrow B'B^* \parallel C'C^*,\ BB^* \parallel CC^* \Rightarrow \widehat{BB'B^*} \equiv \widehat{CC'C^*}.$$

So $(\forall)$ parallel plane with α we construct, the newly obtained triangle has a side of α length and the corresponding angle to $\widehat{B'B^*B}$ is constant. We mark with a line that position of the plane, for which the opposite length of the required angle is l. With the compass spike at C and with a radius equal with l, we trace a circle arc that cuts segment $|C'C^*|$ at N or line $C'C^*$. Through N we draw at (d', d^*) a parallel line to d^* which precisely meets d' in a point M'. Through M', we draw the $\parallel$ plane to α, which will intersect the three lines in M, M', M^*.

$$\Rightarrow \left.\begin{array}{c} NM' \parallel C^*M^* \\ C^*N \parallel M^*M' \end{array}\right\} \Rightarrow M^*M'NC^*$$

is a parallelogram.

$$\Rightarrow \left.\begin{array}{l} \left.\begin{array}{l} \|NM'\| = \|C^*M^*\| \\ NM' \parallel C^*M \end{array}\right\} (1) \\ \left.\begin{array}{l} CM \parallel C^*M^* \\ \|CM\| = \|C^*M^*\| \end{array}\right\} (2) \end{array}\right\} NM' \parallel CM, \|NM'\| = \|CM\|$$

$\Rightarrow CNM'M$ is a parallelogram.

$$\Rightarrow \|MN'\| = \|CN\| = l$$

and line MM', located in a parallel plane to α, is parallel to α.

Discussion:

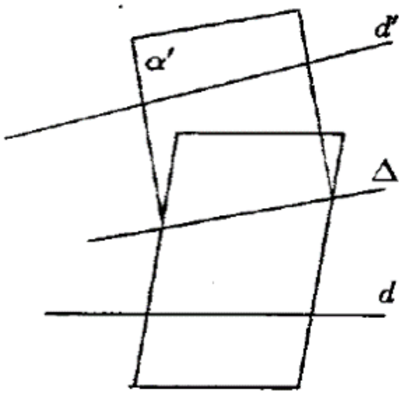
Assuming the plane $(C'C^*C)$ is variable, as $|CC^*|$ and $\widehat{CC'C^*}$ are constant, then $d(C'C^*C) = b =$ also constant

If $l < d$ we don't have any solution.

If $l = d$ $(\exists)$ a solution, the circle of radius l, is tangent to $C'C^*$.

If $l > d$ $(\exists)$ two solutions: circle of radius l, cuts $C'C^*$ at two points N and P.

Solution to Problem 219.



We draw through A planes $\alpha \perp d$ and $\alpha' \perp d'$.

As A is a common point

$\Rightarrow \alpha \cap \alpha' \neq \emptyset \Rightarrow$

$\Longrightarrow \alpha \cap \alpha' = \Delta \Longrightarrow A \in \Delta.$

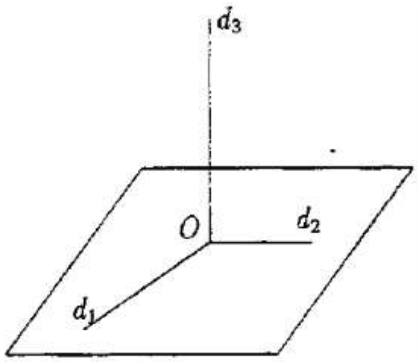
$$\left.\begin{array}{l} d \perp \alpha \Rightarrow d \perp D \\ d' \perp \alpha' \Rightarrow d' \perp D \end{array}\right\} \Rightarrow$$

$\Longrightarrow \Delta$ the required line

If $\alpha \neq \alpha'$ - we have only one solution.

If $\alpha = \alpha'$ $(\forall)$ line from α which passes through A corresponds to the problem, so $(\exists)$ infinite solutions.

Solution to Problem 220.



Let $d_1 \perp d_2$ two concurrent perpendicular lines, $d_1 \cap d_2 = \{O\}$. They determine a plane $\alpha = (d_1, d_2)$ and $O \in \alpha$. We construct on α in O.

$d_3 \perp \alpha,\ O \in d_3 \to d_3 \perp d_1,\ d_3 \perp d_2.$

Solution to Problem 221.

We use the reductio ad absurdum method.

Let $d \perp a, d \perp a, d \perp c$. We assume that these lines are not coplanar. Let $\alpha = (b, c), \alpha' = (a, b), \alpha \neq \alpha'$. Then $d \perp \alpha$, $d \perp \alpha'$.

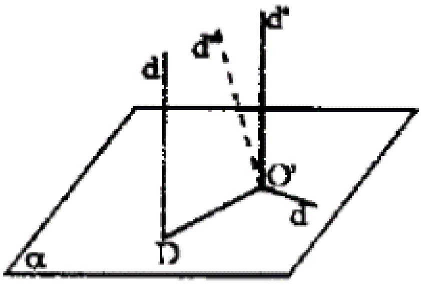
Thus through point O, 2 perpendicular planes to d can be drawn. False $\Longrightarrow a, b, c$ are coplanar.

Solution to Problem 222.

By reductio ad absurdum:

Let $a \cap b \cap c \cap d = \{O\}$ and they are perpendicular two by two. From $d \perp a, d \perp a, d \perp c \Longrightarrow a, b, c$ are coplanar and $b \perp a$, $c \perp a$, so we can draw to point O two distinct perpendicular lines. False. So the 4 lines cannot be perpendicular two by two.

Solution to Problem 223.



We assume that $d \not\perp \alpha$.

In $d' \cap \alpha = \{O\}$ we draw line $d'' \perp \alpha$. Lines d' and d'' are concurrent and determine a plane $\beta = (d', d'')$ and as $O' \in \beta$, $O' \in \alpha \Longrightarrow$

$$\alpha \cap \beta = a \Rightarrow \begin{cases} a \subset \alpha \\ a \subset \beta \end{cases} \Rightarrow \begin{array}{ll} d'' \perp \alpha \Rightarrow d'' \perp a & (1) \\ d \perp \alpha \Rightarrow d \perp a & \\ d' \parallel d \Rightarrow d' \perp a & (2) \end{array}$$

From (1) and (2) $\Longrightarrow$ in plane β, on line a, at point O' two distinct perpendicular lines had been drawn. False. So $d' || \alpha$.

Solution to Problem 224.

$$\left.\begin{array}{r} d\perp\alpha \\ d'\perp\alpha \\ d\neq d' \end{array}\right\} \Rightarrow d \parallel d'$$

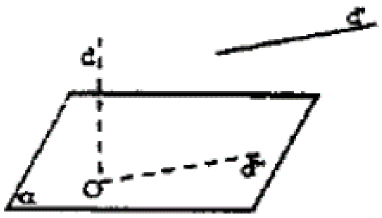
Reductio ad absurdum. Let $d \nparallel d'$. We draw $d''||d$ through O'.

$$\left.\begin{array}{r} d'' \parallel d \\ d\perp\alpha \end{array}\right\} \Rightarrow \left.\begin{array}{r} d''\perp\alpha \\ d'\perp\alpha \end{array}\right\} \Rightarrow$$

$\Longrightarrow$ at point O' we can draw two perpendicular lines to plane α. False.

So $d||d'$.

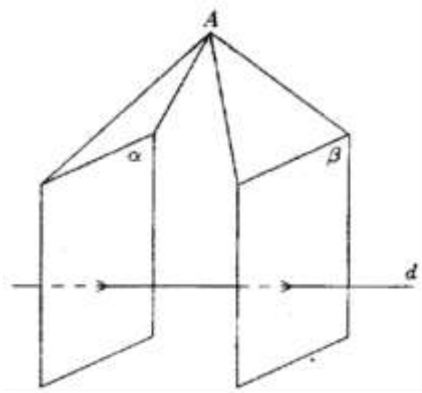
Solution to Problem 225.



Let $d \perp \alpha$ and $d \cap \alpha = \{O\}$. We draw through O a parallel to d', which will be contained in α, then $d||\alpha$.

$$d'' \parallel d',\ O \in d'' \Rightarrow d'' \subset \alpha,\ d\perp\alpha \Rightarrow d\perp d'' \Rightarrow d\perp d'.$$

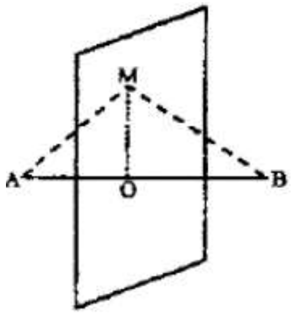
Solution to Problem 226.



We assume $\beta \Longrightarrow \alpha \cap \beta \neq \emptyset$ and let $A \in \alpha \cap \beta \Longrightarrow$ through a point A there can be drawn two distinct perpendicular planes on this line. False.

$$\Longrightarrow \alpha \mid\mid \beta.$$

Solution to Problem 227.



Let M be a point in space with the property $||MA|| = ||MB||$.

We connect M with the midpoint of segment $[AB]$, point O.

$\Rightarrow \triangle AMO = \triangle BMO \Rightarrow MO \perp AB$.

So M is on a line drawn through O, perpendicular to AB.

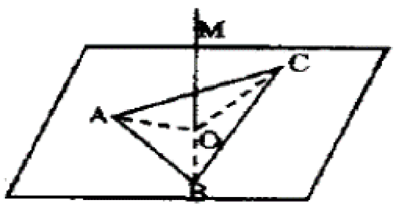
But the union of all perpendicular lines drawn through O to AB is the perpendicular plane to AB at point O, marked with α, so $M \in \alpha$.

Vice-versa: let $M \in \alpha$,

$d = AB \perp \alpha \Rightarrow d \perp MO$

$$\left.\begin{array}{ll} |AO| \equiv |OM| & \\ |MO| & \text{common side} \\ MO \perp AB & \end{array}\right\} \Rightarrow \triangle AMO \equiv \triangle BMO \Rightarrow ||MA|| = ||MB||.$$

Solution to Problem 228.



Let M be a point in space with this property:

$||MA|| = ||MB|| = ||MC||$

Let O be the center of the circumscribed circle $\Delta ABC \Longrightarrow ||OA|| = OB|| = ||OC||$, so O is also a point of the desired locus.

According to the previous problem the locus of the points in space equally distant from A and B is in the mediator plane of segment $[AB]$, which also contains M. We mark with α this plane. The locus of the points in space equally distant from B and C is in the mediator plane of segment $[BC]$, marked β, which contains both O and M. So $\alpha \cap \beta = OM$.

$$\left.\begin{array}{l} AB \perp \alpha \Rightarrow AB \perp OM \\ BC \perp \alpha \Rightarrow BC \perp OM \end{array}\right\} \Rightarrow OM \perp (ABC)$$

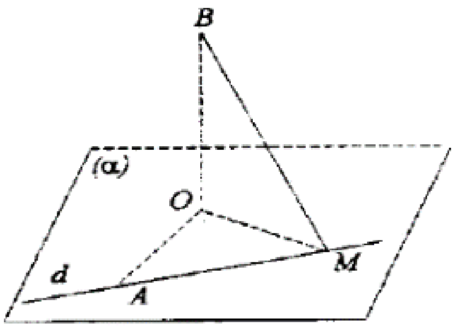
so $M \in$ the perpendicular line to plane (ABC) in the center of the circumscribed circle ΔABC.

Vice-versa, let $M \in$ this perpendicular line

$$\left.\begin{array}{c} OM \perp OA \\ OM \perp OB \\ OM \perp OC \\ \|OA\| = \|OB\| = \|OC\| \end{array}\right\} \Rightarrow \Delta OMA \equiv \Delta OMB \equiv \Delta OMC \Rightarrow \|AM\| = \|BM\|$$

$= ||CM||$, so M has the property from the statement.

Solution to Problem 229.



We draw $\perp$ from B to the plane. Let O be the foot of this perpendicular line.

Let

$$\left.\begin{array}{l} BM \perp d \\ BO \perp \alpha \end{array}\right\} \Rightarrow MO \perp d \Rightarrow$$

$\Rightarrow m(\widehat{OMA}) = 90° \Rightarrow M \in$

the circle of radius OA. *Vice-versa,* let $M \in$ this circle

$$\Rightarrow OM \perp AM,\ BO \perp \alpha \Rightarrow BM \perp AM \Rightarrow BM \perp d,$$

so M represents the foot from B to AM.

Solution to Problem 230.

Let α be a plane that passes through a and let M be the $\perp$ foot from A to $\alpha \Longrightarrow$ $AM \perp \alpha$.

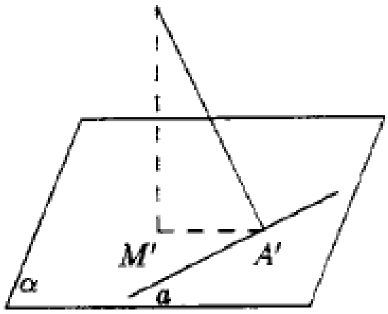
From

$$\left.\begin{array}{l} AM \perp \alpha \\ AA' \perp a \end{array}\right\} \Rightarrow MA' \perp a,$$

so $M \in$ a perpendicular line to a in A', thus it is an element of the perpendicular plane to a in A', which we mark as π and which also contains A.

$AM \perp \alpha \Rightarrow AM \perp MA' \Rightarrow$

$M \in$ the circle of radius AA' from plane π.

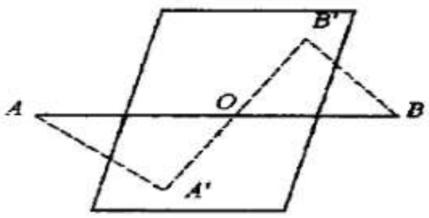


Vice-versa, let M be a point on this circle of radius AA' from plane π.

$$\Rightarrow \left.\begin{array}{l} MA' \perp a \\ AA' \perp a \\ AM \perp MA' \end{array}\right\} \Rightarrow AM \perp (M, a)$$

$\Longrightarrow M$ is the foot of a $\perp$ drawn from A to a plane that passes through a.

Solution to Problem 231.



Let A' and B' be the feet of the perpendicular lines from A and B to α

$$\left.\begin{array}{l} AA' \perp \alpha \\ BB' \perp \alpha \end{array}\right\} \Rightarrow AA' \parallel BB' \Rightarrow$$

$(\exists)$ a plane $\beta = (AA', BB')$ and $AB \subset \beta$

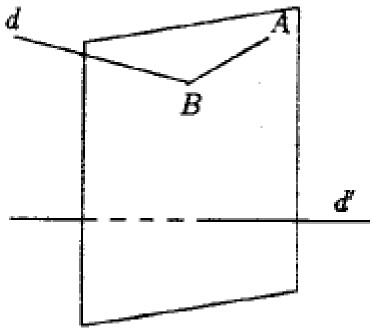
$$\Rightarrow \left.\begin{array}{l} O \in \beta \\ O \in \alpha \end{array}\right\} \Rightarrow O \in \alpha \cap \beta = A'B' \Rightarrow O,\ A',\ B'$$

are collinear.

In plane β we have

$$\left.\begin{array}{l} \|OA\| = \|OB\| \\ \widehat{AOA'} \equiv \widehat{BOB'} \\ \triangle \quad \text{right} \end{array}\right\} \Rightarrow \Delta AOA' \equiv BOB' \Rightarrow \|AA'\| = \|BB'\|$$

Solution to Problem 232.

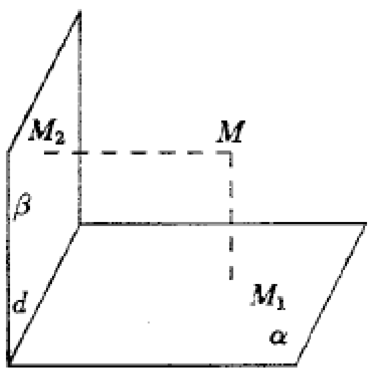


Let d, d' be given lines, A given point. We draw through A plane $\alpha \perp$ to d'.

If $a \cap \alpha = \{B\}$, then line AB is the desired one, because it passes through A, meets d and from $d'\alpha\, d'AB$. If $d \cap \alpha = \emptyset$ there is no solution.

If $d \subset \alpha$, then any line determined by A and a point of d represents solution to the problem, so there are infinite solutions.

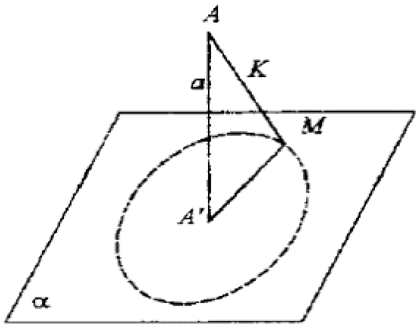
Solution to Problem 233.



$$\alpha \cap \beta = d \Rightarrow \begin{matrix} d \subset \alpha \\ d \subset \beta \end{matrix}.$$

$$\left.\begin{matrix} MM_1 \perp \alpha \Rightarrow MM_1 \perp d \\ MM_2 \perp \beta \Rightarrow MM_2 \perp d \end{matrix}\right\} \Rightarrow d \perp (MM_1M_2)$$

Solution to Problem 234.



Let M be a point such that $||AM|| = k$.

We draw $AA' \perp \alpha \Rightarrow A'$ fixed point and $AA' \perp A'M$.

We write $||AA'|| = a$.

Then

$$\left.\begin{matrix} \|AM\| = \sqrt{k^2 - a^2} = ct \\ A^2 - \text{ fix} \end{matrix}\right\} \Rightarrow$$

$M \in$ a circle centered at A' and of radius $\sqrt{k^2 - a^2}$, for $k > a$.

For $k = a$ we obtain 1 point.

For $k < a$ empty set.

Vice-versa, let M be a point on this circle $\Longrightarrow$

$$\left.\begin{matrix} \|A'M\| = \sqrt{k^2 - a^2} \\ \|AA'\| = a \end{matrix}\right\} \Rightarrow \|AM\| =$$

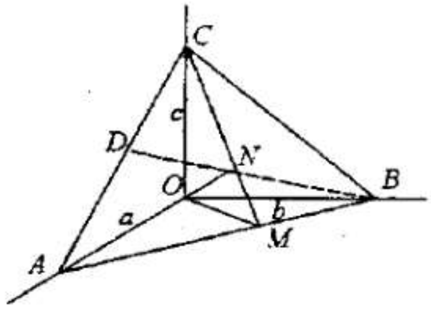
$$= \sqrt{k^2 - a^2 + a^2} = k$$

so M has the property from the statement.

Solution to Problem 235.

1) $\|AB\| = \sqrt{a^2 + b^2}$; $\|BC\| = \sqrt{b^2 + c^2}$, $\|CA\| = \sqrt{a^2 + c^2}$

2) $$\left.\begin{matrix} OM \perp AB \\ CO \perp (OAB) \end{matrix}\right\} \Rightarrow CM \perp AB$$



In ΔOCM:

$$\|CM\| = \sqrt{c^2 + \frac{a^2b^2}{a^2+b^2}} = \sqrt{\frac{c^2a^2 + c^2b^2 + a^2b^2}{a^2+b^2}}$$

$$\sigma[ABC] = \frac{\|AB\| \cdot \|CM\|}{2} = \sqrt{\frac{a^2+b^2}{2}} \times$$

$$\times \sqrt{\frac{a^2c^2 + c^2b^2 + a^2b^2}{a^2+b^2}} \Rightarrow$$

$$\Rightarrow \sigma^2[ABC] = \frac{a^2b^2 + a^2c^2 + b^2c^2}{4}.$$

But

$$\sigma[OAB] = \frac{ab}{2}$$

$$\alpha[BOC] = \frac{bc}{2};\ \sigma[COA] = \frac{ac}{2}$$

$$\sigma^2[ABC] = \frac{a^2b^2}{4} + \frac{a^2c^2}{4} + \frac{b^2c^2}{4} = \sigma^2[AOB] + \sigma^2[DOC] + \sigma^2[COA].$$

3. Let H be the projection of O lcp. plane ABC, so

$$OH \perp (ABC) \Rightarrow OH \perp AC$$

$$\left.\begin{array}{l} OC \perp DA \\ OC \perp OB \end{array}\right\} \Rightarrow OC \perp (OAB) \Rightarrow\Rightarrow OC \perp AB$$

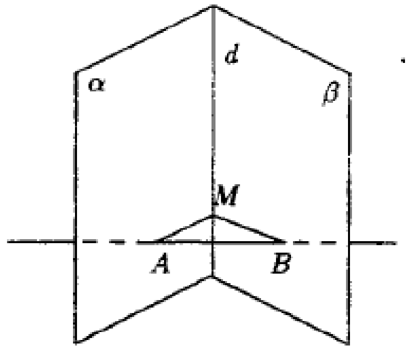
$$\Rightarrow AB \perp (OHC) \Rightarrow AB \perp CH \Rightarrow$$

$H \in$ corresponding heights of side AB. We show in the same way that $AC \perp BH$ and thus H is the point of intersection of the heights, thus orthocenter.

4) $\|OH\| \cdot \|CM\| = \|OC\| \cdot \|OM\| \Rightarrow OH \cdot \sqrt{\dfrac{a^2c^2 + c^2b^2 + b^2a^2}{a^2+b^2}} = \dfrac{ab}{\sqrt{a^2+b^2}} \Rightarrow$

$$\Rightarrow \|OH\| = \frac{\dfrac{cab}{\sqrt{a^2+b^2}}}{\dfrac{\sqrt{a^2c^2 + c^2b^2 + b^2a^2}}{\sqrt{a^2+b^2}}} = \frac{abc}{\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}.$$

Solution to Problem 236.



First we prove that if a line is ⊥ to two concurrent planes ⟹ the planes coincide.

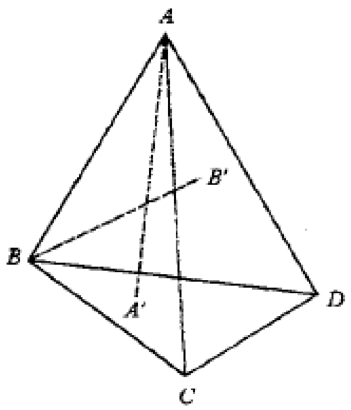
$$\left.\begin{array}{r} a\perp\alpha \\ a\perp\beta \\ \alpha\cap\beta=\alpha \end{array}\right\} \Rightarrow \alpha=\beta.$$

Let

$A = a\cap\alpha$, $B = a\cap\beta$, $M \in d$.

$$\left.\begin{array}{l} a\perp\alpha \Rightarrow a\perp AM \\ a\perp\beta \Rightarrow a\perp BM \end{array}\right\} \Rightarrow \triangle ABM$$

ΔABM has two right angles. False. We return to the given problem.



$$\left.\begin{array}{l} AA'\perp(BCD) \Rightarrow AA'\perp CD \\ BB'\perp(ACD) \Rightarrow BB'\perp CD \end{array}\right\} \Rightarrow CD\perp(AA', BB')$$

being concurrent, they determine a plane ⟹ $CD \perp AB$.

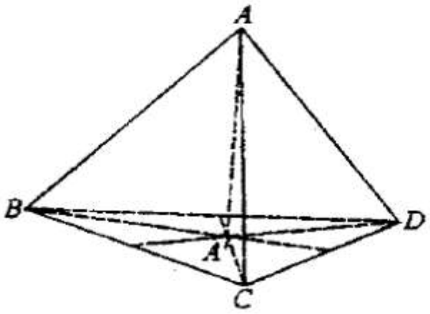
$CC'\perp(ABD) \Rightarrow CC'\perp AB$

$DD'\perp(ABC) \Rightarrow DD'\perp AB$

$$\left.\begin{array}{l} \left.\begin{array}{l} AB\perp CD \\ AB\perp DD' \end{array}\right\} \Rightarrow AB\perp(CDD') \\ \qquad\qquad (CDC')\cap(CDD') = CD \\ \left.\begin{array}{l} AB\perp CD \\ AB\perp CC' \end{array}\right\} \Rightarrow AB'\perp(CDC') \end{array}\right\}$$

C, D, C', D' are coplanar ⟹ CC' and DD' are coplanar.

Solution to Problem 237.



We draw

$$AA' \perp (BCD)$$

$$\left.\begin{array}{l} AA' \perp BC \\ AB \perp DC \end{array}\right\} \Rightarrow DC \perp (ABA') \Rightarrow DC \perp BA'$$

$\Longrightarrow BA'$ height in ΔBCD (1)

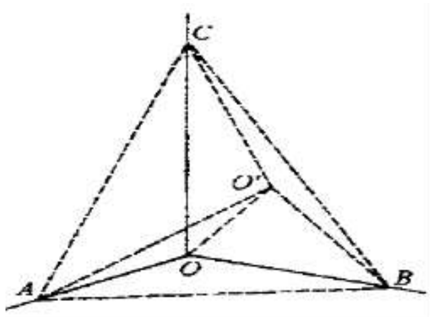
$$\left.\begin{array}{l} AA' \perp BD \\ AC \perp BD \end{array}\right\} \Rightarrow BD \perp (AA'C) \Rightarrow BD \perp A'C$$

$\Longrightarrow A'C$ height in ΔABC (2)

From (1) and (2) $\Longrightarrow A'$ is the orthocenter $\Delta ABC \Longrightarrow OA' \perp BC$.

$$\left.\begin{array}{l} DA' \perp BC \\ AA' \perp BC \end{array}\right\} \Rightarrow BC \perp (DAA') \Rightarrow BC \perp AD$$

Solution to Problem 238.



Let

$$\left.\begin{array}{r} OO' \perp (ABC) \Rightarrow OO' \perp AO' \\ OO' \perp BO' \\ OO' \perp CO' \end{array}\right\} \Rightarrow \Delta AOO',$$

$\Delta BOO'$ and $\Delta COO'$ are right at O'.

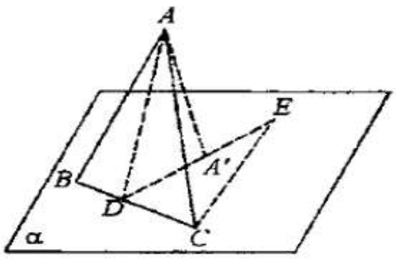
As $\left.\begin{array}{l}|OA| \equiv |OB| \equiv |OC| \\ |OO'| \text{ common side}\end{array}\right\}$

$\Rightarrow \triangle AOO' \equiv \triangle BOO' \equiv \triangle COO'$

$\Rightarrow \|OA\| = \|BO'\| = \|CO'\| \Rightarrow O'$

is the center of the circumscribed circle ΔABC.

Solution to Problem 239.



Let D be the midpoint of $[BC]$ and $E \in |DA'$ such that $||DE|| = ||DA||$.

AD is median in the Δ isosceles

$\left.\begin{array}{l}\Rightarrow AD \perp BC \\ AA' \perp \alpha\end{array}\right\} \Rightarrow A'D \perp BC$

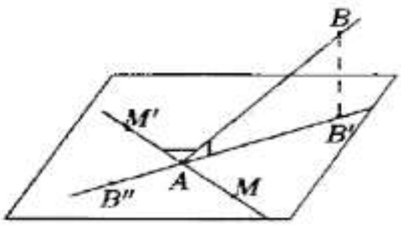
$\left.\begin{array}{l}|AD| \equiv |DE| \\ |DC| \quad \text{common}\end{array}\right\} \Rightarrow \Delta ADC \equiv \Delta DCE \Rightarrow \widehat{DAC} \equiv \widehat{DEC}$

$\widehat{DA'C} > \widehat{DEC}$

being external for

$\Delta CA'E \Rightarrow \widehat{DAC} > \widehat{BAC} \Rightarrow 2\widehat{DA'C} > 2\widehat{DAC} \Rightarrow \widehat{BA'C} > \widehat{BAC}.$

Solution to Problem 240.



Let M be a point in the plane and $|AM'$ the opposite ray to AM.

According to theorem 1

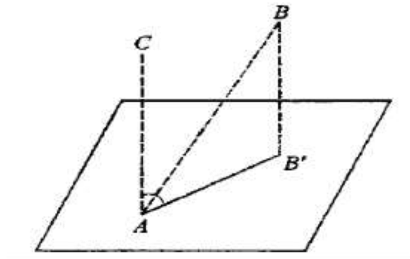
$\Rightarrow \widehat{B'AB} < \widehat{MAB} \Rightarrow$

$\Rightarrow m(\widehat{B'AB}) < \widehat{M'AB} \Rightarrow m(\widehat{B'AB}) < m(\widehat{M'AB}) \Rightarrow$

$\Rightarrow -m(\widehat{B'AB}) > -m(\widehat{M'AB}) \Rightarrow$

$\Rightarrow 180° - m(\widehat{B'AB}) > 180° - m(\widehat{M'AB}) \Rightarrow m(\widehat{B''AB}) > m(\widehat{MAB}) \Rightarrow \widehat{B''AB} > \widehat{MAB}.$

Solution to Problem 241.

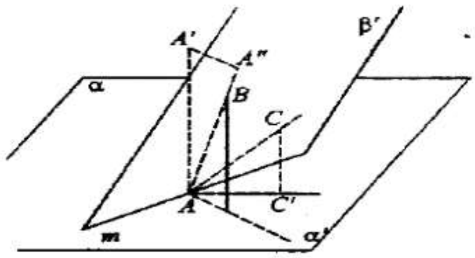


We construct B on the plane

$$\left.\begin{array}{r} \Rightarrow BB'\perp\alpha \\ AC\perp\alpha \end{array}\right\} \Rightarrow$$

$\Longrightarrow AC$ and BB' determine a plane $\beta = (AC, BB') \Longrightarrow AB \subset \beta$ and on this plane $M(\widehat{CAB}) = 90^0 - m(\widehat{BAB'})$.

Solution to Problem 242.



Let ray $|AB \subset \beta'$ such that $AB \perp m$. Let $|AC$ another ray such that $|AC \subset \beta'$. We draw $BB' \perp \alpha$ and $CC' \perp \alpha$ to obtain the angle of the 2 rays with α, namely $\widehat{BAB'} > \widehat{CAC'}$. We draw line $|AA'$ such that $AA' \perp \alpha$ and is on the same side of plane α as well as half-plane β'.

$$\left.\begin{array}{r} AA'\perp\alpha \;\Rightarrow\; AA'\perp m \\ AB\perp m \\ A'A''\perp AB \end{array}\right\} \Longrightarrow A'A''\perp\beta' \Rightarrow$$

$[AB$ is the projection of ray $[AA'$ on plane β

$\beta \overset{t.1}{\Longrightarrow} \widehat{A'AB} < A,\widehat{AC} \Rightarrow m(\widehat{A'AB}) < m(\widehat{A'AC}) \Rightarrow -m(\widehat{A'AB}) > -m(\widehat{A'AC}) \Rightarrow 90° -$
$-m(\widehat{A'AB}) > 90° - m(\widehat{A'AC}) \Rightarrow \widehat{BAB'} > \widehat{CAC'}$

Solution to Problem 243.

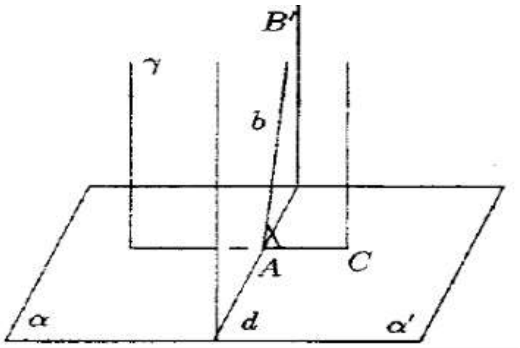
Let d be the border of α' and $A \in d$. We draw a plane $\perp$ on d in A, which we mark as γ.

$$\left.\begin{array}{r}\gamma\cap c=c\\ d\perp\gamma\end{array}\right\}\Rightarrow d\perp c$$

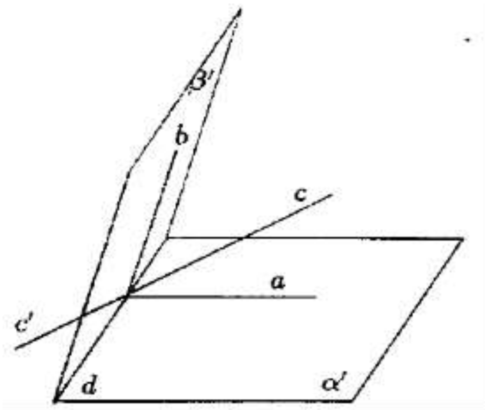
In this plane, there is only one ray b, with its origin in A, such that $m(\widehat{c,b})=a$.

The desired half-plane is determined by d and ray b, because from

$$\left.\begin{array}{r}d\perp\gamma\Rightarrow d\perp b\\ d\perp c\end{array}\right\}\Rightarrow m(\alpha'\beta')=m(\widehat{b,c})=a.$$



Solution to Problem 244.



Let d be the edge of the dihedral angle and $A\in d$. We draw $a\perp d, a\subset\alpha'$ and $b\perp d$, $b\subset\beta'$ two rays with origin in A. It results $d\perp(ab)$. We draw on plane (a,b) ray c such that $m(\widehat{ac})=m(\widehat{cb})$ (1).

As $d\perp(ab)\Longrightarrow d\perp c$.

Half-plane $\gamma'=(d,c)$ is the desired one, because

$$\left.\begin{array}{l}m(\widehat{\alpha'\gamma'})=m(\widehat{a,c})\\ m(\widehat{\gamma'\beta'})=m(\widehat{c,b})\end{array}\right\}\Rightarrow(\alpha'\gamma')=m(\gamma'\beta').$$

If we consider the opposite ray to c,c', half-plane $\gamma''=(d,c')$ also forms concurrent angles with the two half-planes, being supplementary to the others.

Solution to Problem 245.

Let M be a point in space equally distant from the half-planes $\alpha', \beta' \Longrightarrow ||MA|| =$ $||MB||$.

$$\left.\begin{array}{l} MA \perp \alpha \Rightarrow MA \perp d \\ MB \perp \beta \Rightarrow MB \perp d \end{array}\right\} \Rightarrow d \perp (MAB),$$

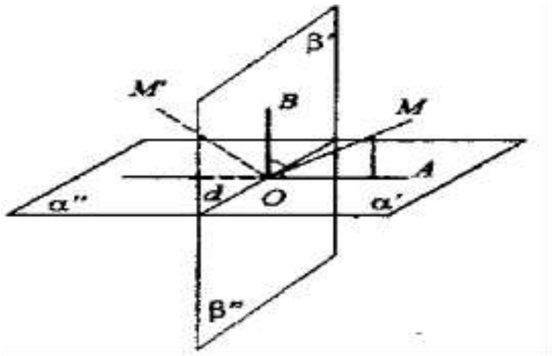
where $d = \alpha \cap \beta$.

Let

$d \cap (MAB) = \{O\} \Rightarrow$

$$\left.\begin{array}{l} d \perp OA \\ d \perp OB \end{array}\right\} \Rightarrow m(\alpha'\beta') =$$

$= m(\widehat{AOB})$.



$$\left.\begin{array}{r} |MA| = |MB| \\ |OM| \text{ common side right triangle} \end{array}\right\} \Longrightarrow$$

$\Rightarrow \triangle MOA \equiv \triangle MOB \Rightarrow \widehat{MOA} \equiv \widehat{MOB} \Rightarrow$

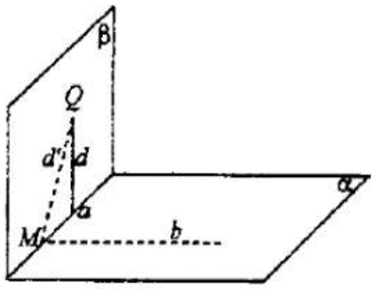
$\Longrightarrow M \in$ bisector of the angle.

$\widehat{ADB} \Longrightarrow M \in$ bisector half-plane of the angle of half-planes α', β'.

If M' is equally distant from half-planes β' and α'' we will show in the same way that $M' \in$ bisector half-plane of these half-planes. We assume that M and M' are on this plane $\perp$ to d, we remark that $m(\widehat{MOM'}) = 90^0$, so the two half-planes are $\perp$. Considering the two other dihedral angles, we obtain 2 perpendicular planes, the 2 bisector planes.

Vice-versa: we can easily show that a point on these planes is equally distant form planes α and β.

Solution to Problem 246.



$$\left.\begin{array}{l} \alpha \perp \beta \\ Q \in \beta \\ d \perp \alpha \\ Q \in d \end{array}\right\} \Rightarrow d \subset \beta.$$

Let $\alpha \cap \beta = \alpha$. In plane β we draw

$d' \perp a,\ Q \in d'$.

As

$$\left.\begin{array}{r} \alpha \perp \beta \Rightarrow m(\widehat{d'b}) = 90^\circ \Rightarrow d' \perp b \\ d' \perp a \end{array}\right\}$$

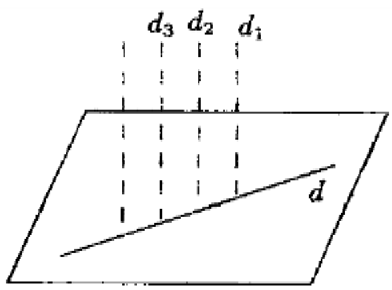
but

$$\left.\begin{array}{l} d' \perp \alpha \\ d \perp \alpha \end{array}\right\} \Rightarrow d = d'$$

so from a point it can be drawn only one perpendicular line to a plane,

$d' \subset \beta \Rightarrow d \subset \beta$.

Solution to Problem 247.

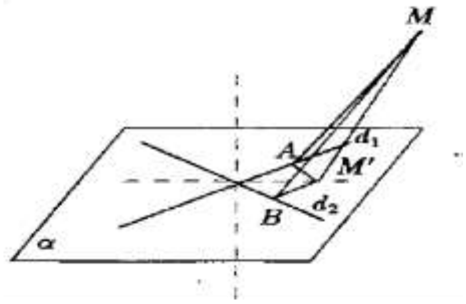


Let

$$\left.\begin{array}{l} d_1 \perp \alpha \\ d_2 \perp \alpha \\ d_3 \perp \alpha \end{array}\right\} \Rightarrow d_1 \parallel d_2 \parallel d_3 \Rightarrow$$

the line with the same direction. We know that the union of the lines with the same direction and are based on a given line is a plane. As this plane contains a perpendicular line to α, it is perpendicular to α.

Solution to Problem 248.



Let $\alpha = (d_1, d_2)$ the plane of the two concurrent lines and M is a point with the property $d(M, d_1) = d(M, d_2)$. We draw

$MA \perp d_1, \quad MB \perp d_2 \ \Rightarrow \ \|MA\| \ = \ \|MB\|.$

Let

$M' \ = \ \mathrm{pr}_\alpha M \ \Rightarrow \ \triangle MAM' \equiv \triangle MBM' \Rightarrow \|M'A\| \ =$

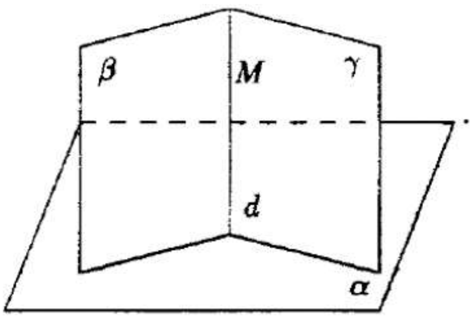
$\|M'B\| \Rightarrow M' \in$

a bisector of the angle formed by the two lines, and M is on a line α which meets a bisector $\Rightarrow M \in$ a plane $\perp$ α and which intersects α after a bisector. Thus the locus will be formed by two planes ⊥ α and which intersects α after the two bisectors of the angle formed by d_1, d_2. The two planes are ⊥.

$$\Rightarrow \left.\begin{array}{c} \|M'A\| = \|M'B\| \\ \|MM'\| \text{ common side} \end{array}\right\} \Longrightarrow MA \perp d_1$$

And in the same way $MB \perp d_2 \Rightarrow M$ has the property from the statement.

Solution to Problem 249.

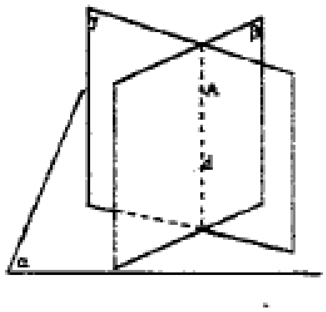


Let $\beta \cap \gamma = d$ and $M \in d \Longrightarrow M \in \beta, M \in \gamma$. We draw $\perp$ from M to α, line d'.

According to a previous problem

$$\Rightarrow \left.\begin{array}{l} d' \subset \beta \\ d' \subset \alpha \end{array}\right\} \Rightarrow d \perp \alpha.$$

Solution to Problem 250.



Let β and γ be such planes, that is

$A \in \beta,\ \beta \perp \alpha$

$A \in \gamma,\ \gamma \perp \alpha$

From

$$\left.\begin{array}{l} A \in \beta \\ A \in \gamma \end{array}\right\} \Rightarrow \beta \cap \gamma \neq \emptyset \Rightarrow$$

are secant planes and $\perp$ to α.

$\alpha \overset{pr.T}{\Longrightarrow} \alpha \perp (\beta \cap \gamma) = d,\ A \in d.$

So their intersection is $\perp$ through A to plane α.

Solution to Problem 251.

We construct the point on the two planes and the desired plane is determined by the two perpendicular lines.

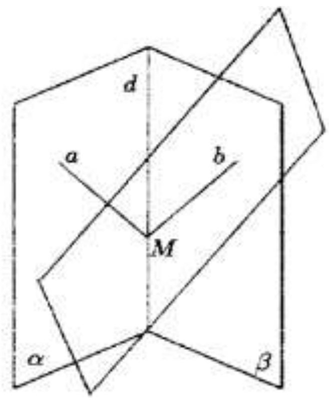
Solution to Problem 252.

Let $\alpha \cap \beta = d$ and $M \in d$. We consider a ray originating in M, $a \in \alpha$ and we construct a $\perp$ plane to a in M, plane γ.

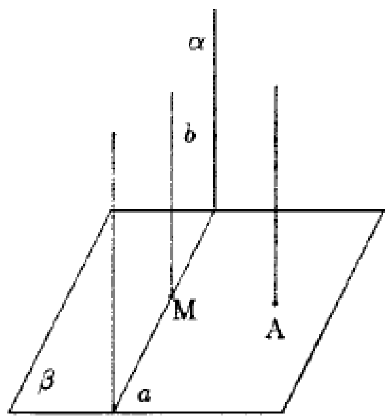
Because

$$\left.\begin{array}{l} M \in \beta \\ M \in \gamma \end{array}\right\} \Rightarrow \beta \cap \gamma \neq \emptyset$$

and let a ray originating in M, $b \subset \beta \cap \gamma \Rightarrow b \subset \beta$, $b \subset \gamma$ As $a \perp \gamma$ $\Longrightarrow a \perp b$ and the desired plane is that determined by rays (a, b).



Solution to Problem 253.



Let $\alpha \cap \beta = a$ and $d \cap \beta = \{A\}$.

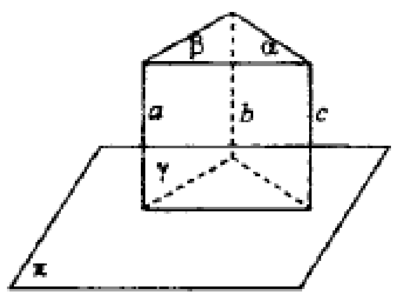
We suppose that $A \notin a$. Let $M \in a$, we build $b \perp \beta$, $M \in b \Longrightarrow b \subset \alpha$.

$b \perp \beta,\ d \perp \beta \Rightarrow d \parallel b \Rightarrow d \parallel \alpha.$

If

$$\left.\begin{array}{l} A \in a \\ d \perp \beta \end{array}\right\} \Rightarrow d \subset \alpha.$$

Solution to Problem 254.



$\alpha \cap \gamma = b$

$\alpha \cap \beta = c$

$\gamma \cap \beta = a$

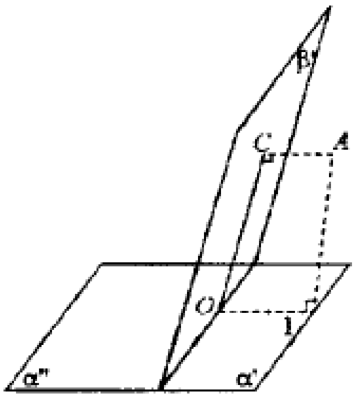
$$\left.\begin{array}{l}\pi \perp \alpha \\ \pi \perp \beta\end{array}\right\} \Rightarrow \pi \perp c \qquad (1)$$

$$\left.\begin{array}{l}\pi \perp \alpha \\ \pi \perp \gamma\end{array}\right\} \Rightarrow \pi \perp b \qquad (2)$$

$$\left.\begin{array}{l}\pi \perp \beta \\ \pi \perp \gamma\end{array}\right\} \Rightarrow \pi \perp a \qquad (3)$$

From (1), (2), (3) $\Longrightarrow a \parallel b \parallel c$.

Solution to Problem 255.



Let $A \in \text{int.}\left(\widehat{\alpha'\beta'}\right), \alpha \cap \beta = d.$

$$\left.\begin{array}{l}AB \perp \alpha \Rightarrow AB \perp d \\ AC \perp \beta \Rightarrow AC \perp d\end{array}\right\} \Rightarrow d \perp (ABC)$$

$$\Rightarrow \left\{\begin{array}{l}d \cap (ABC) = \{O\} \\ d \perp OC \\ d \perp OB\end{array}\right\} \Rightarrow m(\widehat{\alpha'\beta'}) = m(\widehat{BOC})$$

$$\left.\begin{array}{l}m(\widehat{ABO}) = 90^\circ \\ m(\widehat{ACO}) = 90^\circ\end{array}\right\} \Rightarrow m(\widehat{BAC}) + m(\widehat{\alpha'\beta'}) = 180^\circ.$$

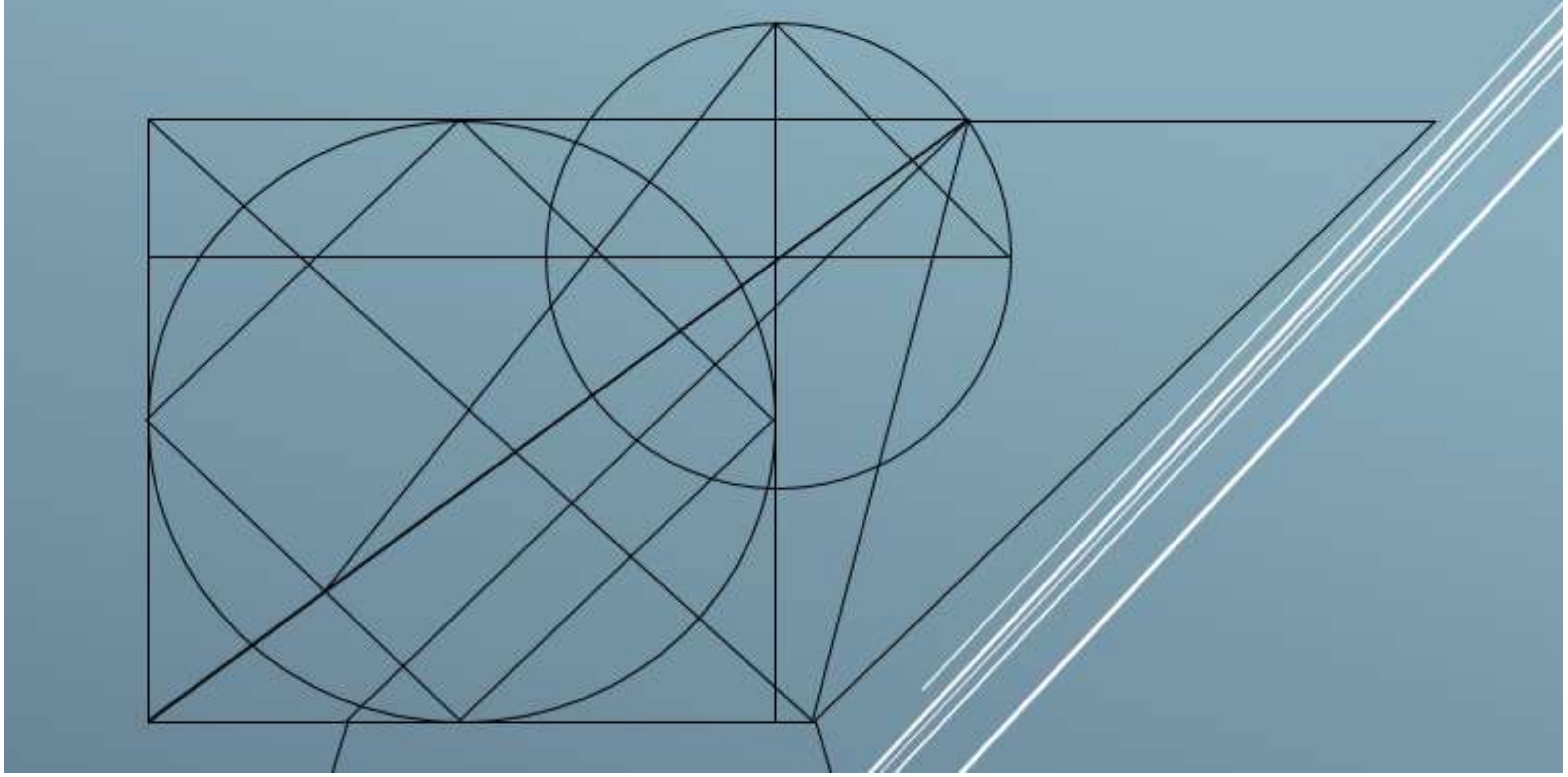
$$\Rightarrow m(\widehat{\alpha'\beta'}) = 180^\circ - m(\widehat{BAC}) \Rightarrow m(\widehat{BAC}) = 180^\circ - m(\widehat{\alpha'\beta'})$$

Let $A \in \text{int.}\left(\widehat{\alpha''\beta'}\right)$. We show the same way that

$$\left.\begin{array}{l}m\left(\widehat{BAC}\right) = 180^0 - m\left(\widehat{\alpha''\beta'}\right) \\ m\left(\widehat{\alpha''\beta'}\right) = 180^0 - m\left(\widehat{\alpha'\beta'}\right)\end{array}\right\} \Longrightarrow m\left(\widehat{BAC}\right) = 180^0 - 180^0 + m\left(\widehat{\alpha'\beta'}\right) = m\left(\widehat{\alpha'\beta'}\right).$$

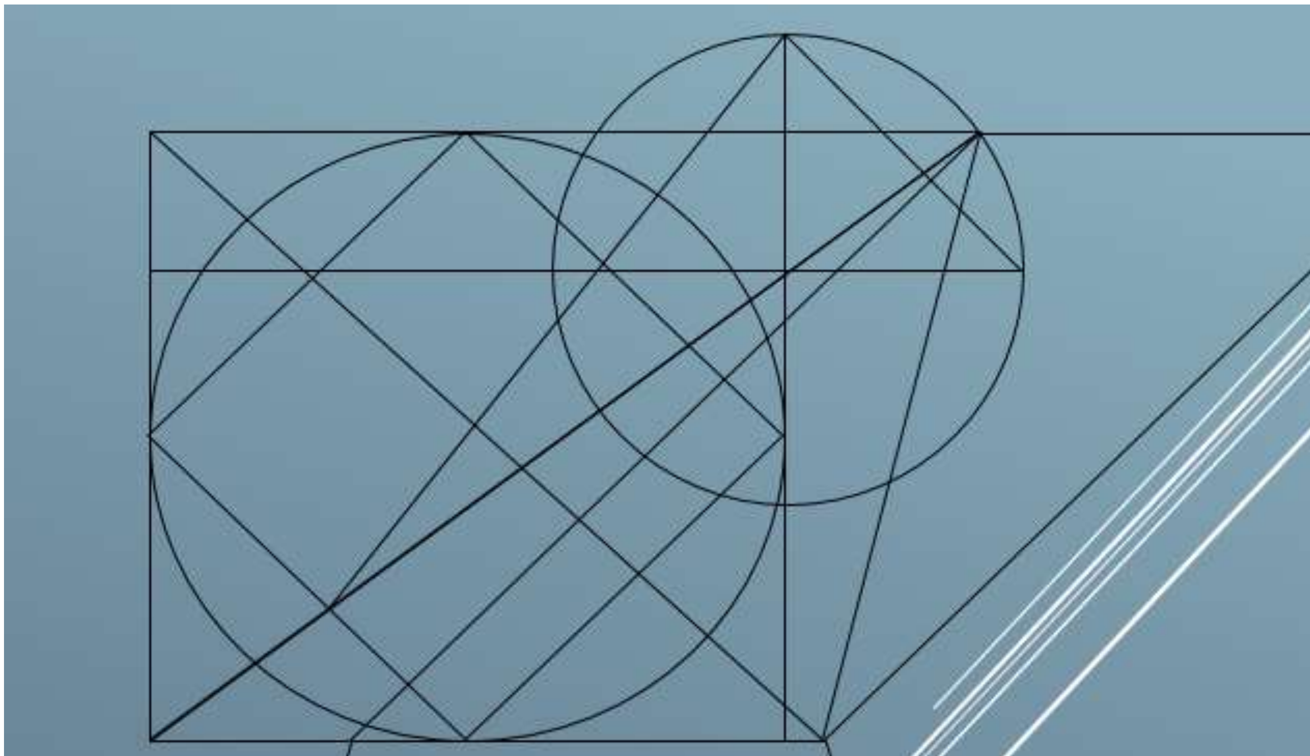
If $A \in \text{int.}\left(\widehat{\alpha''\beta''}\right) \Longrightarrow m\left(\widehat{BAC}\right) = 180^0 - m\left(\widehat{\alpha''\beta'}\right).$

If $A \in \text{int.}\left(\widehat{\alpha'\beta''}\right) \Longrightarrow m\left(\widehat{BAC}\right) = 180^0 - m\left(\widehat{\alpha'\beta'}\right).$



This book is a translation from Romanian of "Probleme Compilate şi Rezolvate de Geometrie şi Trigonometrie" (University of Kishinev Press, Kishinev, 169 p., 1998), and includes problems of 2D and 3D Euclidean geometry plus trigonometry, compiled and solved from the Romanian Textbooks for 9th and 10th grade students, in the period 1981-1988, when I was a professor of mathematics at the "Petrache Poenaru" National College in Balcesti, Valcea (Romania), Lycée Sidi El Hassan Lyoussi in Sefrou (Morroco), then at the "Nicolae Balcescu" National College in Craiova and Dragotesti General School (Romania), but also I did intensive private tutoring for students preparing their university entrance examination. After that, I have escaped in Turkey in September 1988 and lived in a political refugee camp in Istanbul and Ankara, and in March 1990 I immigrated to United States. The degree of difficulties of the problems is from easy and medium to hard. The solutions of the problems are at the end of each chapter. One can navigate back and forth from the text of the problem to its solution using bookmarks. The book is especially a didactical material for the mathematical students and instructors.

The Author



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