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Paul P. Wang

John N. Mordeson

Mark J. Wierman

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Center for Mathematics of Uncertainty

Creighton University

Paul P. Wang
Department of Electrical and Computer Engineering
Pratt School of Engineering
Duke University
Durham, NC 27708-0271
ppw@ee.duke.edu

John N. Mordeson
Department of Mathematics
Creighton University
Omaha, Nebraska 68178
mordes@creighton.edu

Mark J. Wierman
Department of Journalism, Media & Computing
Creighton University
Omaha, Nebraska 68178
wierman@creighton.edu

From the Editors

We devote part of this issue of Critical Review to Lotfi Zadeh.

Contents

From Unbiased Numerical Estimates to Unbiased Interval Estimates

Baokun Li[1], Gang Xiang[2],
Vladik Kreinovich[3], and Panagios Moscopoulos[4]

[1]Southwestern University of Finance and Economics
Chengdu, 610074, China, bali@swufe.edu.cn

[2]Applied Biomathematics, 100 North Country Rd.
Setauket, NY 11733, USA, gxiang@sigmaxi.net

[3]Department of Computer Science
[4]Department of Mathematical Sciences
University of Texas at El Paso
El Paso, Texas 79968, USA, vladik@utep.edu

Abstract

One of the main objectives of statistics is to estimate the parameters of a probability distribution based on a sample taken from this distribution. Of course, since the sample is finite, the estimate $\widehat{\theta}$ is, in general, different from the actual value θ of the corresponding parameter. What we can require is that the corresponding estimate is *unbiased*, i.e., that the mean value of the difference $\widehat{\theta} - \theta$ is equal to 0: $E\left[\widehat{\theta}\right] = \theta$. In some problems, unbiased estimates are not possible. We show that in some such problems, it is possible to have *interval* unbiased estimates, i.e., interval-valued estimates $\left[\widehat{\underline{\theta}}, \widehat{\overline{\theta}}\right]$ for which $\theta \in E\left[\widehat{\underline{\theta}}, \widehat{\overline{\theta}}\right] \stackrel{\text{def}}{=} \left[E\left[\widehat{\underline{\theta}}\right], E\left[\widehat{\overline{\theta}}\right]\right]$. In some such cases, it is possible to have *asymptotically sharp* estimates, for which the interval $\left[E\left[\widehat{\underline{\theta}}\right], E\left[\widehat{\overline{\theta}}\right]\right]$ is the narrowest possible.

Keywords: statistics, interval uncertainty, unbiased numerical estimates, unbiased interval estimates

1 Traditional Unbiased Estimates: A Brief Reminder

Estimating parameters of a probability distribution: a practical problem. Many real-life phenomena are random. This randomness often come from diversity: e.g., different plants in a field of wheat are, in general, of somewhat different heights. In practice, we observe a *sample* $x_1, \ldots, x_n$ of the corresponding values – e.g., we measure the heights of several plants, or we perform several temperature measurements. Based on this sample, we want to estimate the original probability distribution.

Let us formulate this problem in precise terms.

Estimating parameters of a probability distribution: towards a precise formulation of the problem. We want to estimate a probability distribution F that describes the actual values corresponding to possible samples $x = (x_1, \ldots, x_n)$. In other words, we need to estimate a probability distribution on the set $\mathbb{R}^n$ of all n-tuples of real numbers.

In statistics, it is usually assumed that we know the class $\mathcal{D}$ of possible distributions. For example, we may know that the distribution is normal, in which case $\mathcal{D}$ is the class of all normal distributions.

Usually, a distribution is characterized by several numerical characteristics – usually known as its *parameters*. For example, a normal distribution $N(\mu, \sigma^2)$ can be uniquely characterized by its mean μ and variance σ^2. In general, to describe a parameter θ means to describe, for each probability distribution F from the class $\mathcal{F}$, the numerical value $\theta(F)$ of this parameter for the distribution F. For example, when $\mathcal{D}$ is a family of all normal distributions $N(\mu, \sigma^2)$, then the parameter θ describing the mean assigns, to each distribution $F = N(\mu, \sigma^2)$ from the class $\mathcal{F}$, the value $\theta(F) = \mu$. Alternatively, we can have a parameter θ for which $\theta(N(\mu, \sigma^2)) = \sigma^2$, or a parameter for which $\theta(N(\mu, \sigma^2)) = \mu + 2\sigma$.

In general, a parameter can be defined as a mapping from the class $\mathcal{F}$ to real numbers. In these terms, to estimate a distribution means to estimate all relevant parameters.

In some cases, we are interested in learning the values of *all* possible parameters. In other situations, we are only interested in the values of *some* parameters. For example, when we analyze the possible effect of cold weather on the crops, we may be only interested in the lowest temperature. On the other hand, when we are interested in long-term effects, we may be only interested in the average temperature.

We need to estimate the value of this parameter based on the observations. Due to the random character of the sample $x_1, \ldots, x_n$, the resulting estimate $f(x_1, \ldots, x_n)$ is, in general different from the desired parameter $\theta(F)$. In principle, it is possible to have estimates that tend to overestimate $\theta(F)$ and estimates that tend to underestimate $\theta(F)$. It is reasonable to consider *unbiased* estimates, i.e., estimates for which the mean value $E_F\left[\hat{\theta}(x_1, \ldots, x_n)\right]$ coincides with $\theta(F)$.

Thus, we arrive at the following definition.

Definition 1. *Let $n > 0$ be a positive integer, and let $\mathcal{F}$ be a class of probability distributions on $\mathbb{R}^n$.*

- *By a* parameter*, we mean a mapping $\theta : \mathcal{F} \to \mathbb{R}$.*
- *For each parameter θ, by its* unbiased estimate*, we mean a function $\widehat{\theta} : \mathbb{R}^n \to \mathbb{R}$ for which, for every $F \in \mathcal{F}$, we have*

$$E_F\left[\widehat{\theta}(x_1, \ldots, x_n)\right] = \theta(F).$$

Examples. One can easily check that when each distribution from the class $\mathcal{F}$ corresponds to n independent, identically distributed random variables, then the arithmetic average $\widehat{\mu}(x_1, \ldots, x_n) = \dfrac{x_1 + \ldots + x_n}{n}$ is an unbiased estimate for the mean μ of the individual distribution. When, in addition, the individual distributions are normal, the sample variance

$$\widehat{V}(x_1, \ldots, x_n) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \widehat{\mu})^2$$

is an unbiased estimate for the variance V of the corresponding distribution.

2 What If We Take Measurement Uncertainty into Account

Need to take measurement uncertainty into account. In the traditional approach, we assume that we know the exact sample values $x_1, \ldots, x_n$. In practice, measurements are never absolutely accurate: due to measurement imprecision, the observed values $\widetilde{x}_i$ are, in general, different from the actual values x_i of the corresponding quantities.

Since we do not know the exact values $x_1, \ldots, x_n$, we need to estimate the desired parameter $\theta(F)$ based on the observed values $\widetilde{x}_1, \ldots, \widetilde{x}_n$.

Towards a precise formulation of the problem. In addition to the probability distribution of possible values x_i, we also have, for each x_i, a probability distribution of possible values of the difference $\widetilde{x}_i - x_i$. In other words, we have a *joint* distribution J on the set of all possible tuples $(x_1, \ldots, x_n, \widetilde{x}_1, \ldots, \widetilde{x}_n)$.

The meaning of this joint distribution is straightforward:

- first, we use the distribution on the set of all tuples x to generate a random tuple $x \in \mathbb{R}^n$;
- second, for this tuple x, we use the corresponding probability distribution of measurement errors to generate the corresponding values $\widetilde{x}_i - x_i$, and thus, the values $\widetilde{x}_1, \ldots, \widetilde{x}_n$.

Similarly to the previous case, we usually have some partial information about the joint distribution – i.e., we know that the distribution J belongs to a known class $\mathcal{D}$ of distributions.

We are interested in the parameter $\theta(F)$ corresponding to the distribution F of all possible tuples $x = (x_1, \ldots, x_n)$. In statistical terms, F is a marginal distribution of J corresponding to x (i.e., obtained from J by averaging over $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$): $F = J_x$. Thus, we arrive at the following definition.

Definition 2. *Let $n > 0$ be a positive integer, and let $\mathcal{D}$ be a class of probability distributions on the set $(\mathbb{R}^n)^2$ of all pairs $(x, \widetilde{x})$ of n-dimensional tuples. For each distribution $J \in \mathcal{D}$, we will denote the marginal distribution corresponding to x by J_x. The class of all such marginal distributions is denoted by $\mathcal{D}_x$.*

- *By a* parameter, *we mean a mapping $\theta : \mathcal{D}_x \to \mathbb{R}$.*
- *For each parameter θ, by its* unbiased estimate, *we mean a function $\widehat{\theta} : \mathbb{R}^n \to \mathbb{R}$ for which, for every $J \in \mathcal{D}$, we have*

$$E_J\left[\widehat{\theta}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right] = \theta(J_x).$$

Example. When the sample values are independent, identically distributed random variables, and the measurement errors have 0 mean, (i.e., $E[\widetilde{x}_i] = x_i$ for each i), then the arithmetic average $\widehat{\mu}$ is still an unbiased estimate for the mean.

What we show in this paper. In this paper, we show that in some real-life situations, it is not possible to have number-valued unbiased estimates, but we can have *interval-valued* estimates which are unbiased in some reasonable sense.

3 A Realistic Example In Which Unbiased Numerical Estimates Are Impossible

Description of an example. Let us assume that the actual values $x_1, \ldots, x_n$ are independent identically distributed (i.i.d.) normal variables $N(\mu, \sigma^2)$ for some unknown values μ and $\sigma^2 \geq 0$, and that the only information that we have about the measurement errors $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is that each of these differences is bounded by a known bound $\Delta_i > 0$: $|\Delta x_i| \leq \Delta_i$. The situation in which we only know the upper bound on the measurement errors (and we do not have any other information about the probabilities) is reasonably frequent in real life; see, e.g., [3].

In this case, $\mathcal{D}$ is the class of all probability distributions for which the marginal J_x corresponds to i.i.d. normal distributions, and $|\widetilde{x}_i - x_i| \leq \Delta_i$ for all i with probability 1. In other words, the variables $x_1, \ldots, x_n$ are i.i.d. normal, and $\widetilde{x}_i = x_i + \Delta x_i$, where Δx_i can have any distribution for which Δx_i is located on the interval $[-\Delta_i, \Delta_i]$ with probability 1 (the distribution of Δx_i may depend on $x_1, \ldots, x_n$, as long as each difference Δx_i is located within the corresponding interval).

Let us denote the class of all such distributions by $\mathcal{I}$. By definition, the corresponding marginal distributions J_x correspond to i.i.d. normals. As a parameter, let us select the parameter μ of the corresponding normal distribution.

Proposition 1. *For the class $\mathcal{I}$, no unbiased estimate of μ is possible.*

Proof. Let us prove this result by contradiction. Let us assume that there is an unbiased estimate $\hat{\mu}(x_1, \ldots, x_n))$. By definition of the unbiased distribution, we must have $E_J\left[\widehat{\mu}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right] = \mu$ for all possible distributions $J \in \mathcal{I}$.

Let us take two distributions from this class. In both distributions, we take $\sigma^2 = 0$, meaning that all the values x_i coincide with μ with probability 1.

In the first distribution, we assume that each value Δx_i is equal to 0 with probability 1. In this case, all the values $\widetilde{x}_i = x_i + \Delta x_i$ coincide with μ with probability 1. Thus, the estimate $\widehat{\mu}(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ coincides with $\widehat{\mu}(\mu, \ldots, \mu)$ with probability 1. So, its expected value $E_J\left[\widehat{\mu}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right]$ is also equal to $\widehat{\mu}(\mu, \ldots, \mu)$ with probability 1, and thus, the equality that described that this estimate is unbiased takes the form

$$\widehat{\mu}(\mu, \ldots, \mu) = \mu.$$

In other words, for every real number x, we have

$$\widehat{\mu}(x, \ldots, x) = x.$$

In the second distribution, we select a number $\delta = \min\limits_i \Delta_i > 0$, and assume that each value Δx_i is equal to δ with probability 1. In this case, all the values $\widetilde{x}_i = x_i + \Delta x_i$ coincide with $\mu + \delta$ with probability 1. Thus, the estimate $\widehat{\mu}(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ coincides with $\widehat{\mu}(\mu + \delta, \ldots, \mu + \delta)$ with probability 1. So, its expected value $E_J\left[\widehat{\mu}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right]$ is also equal to $\widehat{\mu}(\mu + \delta, \ldots, \mu + \delta)$ with probability 1, and thus, the equality that described that this estimate is unbiased takes the form

$$\widehat{\mu}(\mu + \delta, \ldots, \mu + \delta) = \mu.$$

However, from $\widehat{\mu}(x, \ldots, x) = x$, we conclude that

$$\widehat{\mu}(\mu + \delta, \ldots, \mu + \delta) = \mu + \delta \neq \mu.$$

This contradiction proves that an unbiased estimate for μ is not possible.

4 Unbiased Interval Estimates: From Idea to Definition

Analysis of the problem. In the above example, the reason why we did not have an unbiased estimate is that the estimate $\widehat{\theta}$ depends only on the distribution of the values $\widetilde{x}_1, \ldots, \widetilde{x}_n$, i.e., only on the marginal distribution $J_{\tilde{x}}$. On the other hand, what we try to reconstruct is the characteristic of the marginal distribution J_x. In the above example, even if we know $J_{\tilde{x}}$, we cannot uniquely determine J_x, because there exists another distribution J' for which $J'_{\tilde{x}} = J_{\tilde{x}}$ but for which $J'_x \neq J_x$ and, moreover, $\theta(J'_x) \neq \theta(J_x)$. In this case, we cannot uniquely reconstruct $\theta(J_x)$ from the sample $\widetilde{x}_1, \ldots, \widetilde{x}_n$ distributed according to the distribution $J_{\tilde{x}}$.

From numerical to interval-valued estimates. While we cannot uniquely reconstruct the value $\theta(J_x)$ – because we may have distributions J' with the same marginal $J'_{\tilde{x}} = J_{\tilde{x}}$ for which the value $\theta(J'_x)$ is different – we can try to reconstruct the set of all possible values $\theta(J'_x)$ corresponding to such distributions J'.

Often, for every distribution J, the class $\mathcal{C}$ of all distributions J' for which $J'_{\tilde{x}} = J_{\tilde{x}}$ is connected, and the function that maps a distribution J' into a parameter $\theta(J'_x)$ is continuous. In this case, the resulting set $\{\theta(J'_x) : J' \in \mathcal{C}\}$ is also connected, and is, thus, an interval (finite or infinite). In such cases, it is reasonable to consider interval-valued estimates, i.e., estimates $\widehat{\theta}$ that map each sample $\widetilde{x}$ into an interval $\widehat{\theta}(\widetilde{x}) = \left[\underline{\widehat{\theta}}(\widetilde{x}), \overline{\widehat{\theta}}(\widetilde{x})\right]$.

How to define expected value of an interval estimate. On the set of all intervals, addition is naturally defined as

$$\mathbf{a} + \mathbf{b} \stackrel{\text{def}}{=} \{a + b : a \in \mathbf{a}, b \in \mathbf{b}\},$$

which leads to component-wise addition $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$. Similarly, we can define an arithmetic mean of several intervals $[\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n]$, and it will be equal to the interval $[\underline{a}_{\text{av}}, \overline{a}_{\text{av}}]$, where $\underline{a}_{\text{av}} \stackrel{\text{def}}{=} \dfrac{\underline{a}_1 + \ldots + \underline{a}_n}{n}$ and $\overline{a}_{\text{av}} \stackrel{\text{def}}{=} \dfrac{\overline{a}_1 + \ldots + \overline{a}_n}{n}$. Thus, it is natural to define the expected value $E[\mathbf{a}]$ of an interval-valued random variable $\mathbf{a} = [\underline{a}, \overline{a}]$ component-wise, i.e., as an interval formed by the corresponding expected values $E[\mathbf{a}] \stackrel{\text{def}}{=} [E[\underline{a}], E[\overline{a}]]$.

When is an interval-valued estimate unbiased? Main idea. It is natural to say that an interval-valued estimate $\widehat{\theta}(\widetilde{x})$ is unbiased if the actual value of the parameter $\theta(J_x)$ is contained in the interval $E\left[\widehat{\theta}(\widetilde{x})\right]$.

Let us take into account that the expected value is not always defined. The above idea seems a reasonable definition, but it may be a good idea to make this definition even more general, by also considering situations when, e.g., the expected value $E[\underline{a}]$ is not defined – i.e., when the function $\underline{a}$ is not integrable. In this case, instead of the exactly defined integral $E[\underline{a}]$, we have a *lower integral* $\underline{E}[\underline{a}]$ and an *upper integral* $\overline{E}[\underline{a}]$. Let us remind what these notions mean.

Lower and upper integrals: a brief reminder. These notions are known in calculus, where we often first define an integral of simple functions $s(x)$ (e.g., piece-wise constant ones).

To define the integral of a general function, we can then use the fact that if $s(x) \leq f(x)$ for all x, then $\int s(x)\,dx \leq \int f(x)\,dx$. Thus, the desired integral

$\int f(x)\,dx$ is larger than or equal to the integrals of all simple functions $s(x)$ for which $s(x) \le f(x)$. Hence, the desired integral is larger than or equal to the supremum of all such integrals $\int s(x)\,dx$.

Similarly, if $f(x) \le s(x)$ for all x, then $\int f(x)\,dx \le \int s(x)\,dx$. So, the integral $\int f(x)\,dx$ is smaller than or equal to the integrals of all simple functions $s(x)$ for which $s(x) \le f(x)$. Thus, the desired integral is smaller than or equal to the infimum of all such integrals $\int s(x)\,dx$.

For well-behaving functions, both the supremum of the values $\int s(x)\,dx$ for all $s(x) \le f(x)$ and the infimum of the values $\int s(x)\,dx$ for all $s(x) \ge f(x)$ coincide – and are equal to the integral. For some functions, however, these supremum and infimum are different. The supremum – which is known to be smaller than or equal to the desired integral $\int f(x)\,dx$ – is called the *lower integral*, and the infimum – which is known to be larger than or equal to the desired integral $\int f(x)\,dx$ – is called the *upper integral*.

For the expected value $E[a] \stackrel{\text{def}}{=} \int x \cdot \rho(x)\,dx$, the corresponding lower and upper integrals are called lower and upper expected values, and denoted by $\underline{E}[a]$ and $\overline{E}[a]$.

Towards the final definition. In the case of an integrable estimate, we would like to require that $E\left[\widehat{\underline{\theta}}\right] \le \theta(J_x)$ and that $\theta(J_x) \le E\left[\widehat{\overline{\theta}}\right]$. When the estimate $\widehat{\underline{\theta}}$ is not integrable, this means, crudely speaking, that we do not know the expected value $E\left[\widehat{\underline{\theta}}\right]$, we only know the lower and upper bounds $\underline{E}\left[\widehat{\underline{\theta}}\right]$ and $\overline{E}\left[\widehat{\underline{\theta}}\right]$ for this mean value. When we know that $E\left[\widehat{\underline{\theta}}\right] \le \theta(J_x)$, we cannot conclude anything about the upper bound, but we can conclude that $\underline{E}\left[\widehat{\underline{\theta}}\right] \le \theta(J_x)$.

Similarly, crudely speaking, we do not know the expected value $E\left[\widehat{\overline{\theta}}\right]$, we only know the lower and upper bounds $\underline{E}\left[\widehat{\overline{\theta}}\right]$ and $\overline{E}\left[\widehat{\overline{\theta}}\right]$ for this mean value. When we know that $\theta(J_x) \le E\left[\widehat{\overline{\theta}}\right]$, we cannot conclude anything about the lower bound, but we can conclude that $\theta(J_x) \le \overline{E}\left[\widehat{\overline{\theta}}\right]$.

Thus, we conclude that $\underline{E}\left[\widehat{\underline{\theta}}\right] \le \theta(J_x) \le \overline{E}\left[\widehat{\overline{\theta}}\right]$, i.e., that

$$\theta(J_x) \in \left[\underline{E}\left[\widehat{\underline{\theta}}\right], \overline{E}\left[\widehat{\overline{\theta}}\right]\right].$$

So, we arrive at the following definition:

Definition 3. *Let $n > 0$ be a positive integer, and let $\mathcal{D}$ be a class of probability distributions on the set $(\mathbb{R}^n)^2$ of all pairs $(x, \widetilde{x})$ of n-dimensional tuples. For each distribution $J \in \mathcal{D}$, we will denote:*

- *the marginal distribution corresponding to x by J_x, and*
- *the marginal distribution corresponding to $\widetilde{x}$ by $J_{\widetilde{x}}$.*

The classes of all such marginal distributions are denoted by $\mathcal{D}_x$ *and* $\mathcal{D}_{\widetilde{x}}$.

- *By a* parameter, *we mean a mapping* $\theta : \mathcal{D}_x \to \mathbb{R}$.
- *For each parameter* θ, *by its* unbiased interval estimate, *we mean a function* $\widehat{\theta} : \mathbb{R}^n \to \mathbb{I}$ *that maps* $\mathbb{R}^n$ *into the set* $\mathbb{I}$ *of all intervals for which, for every* $J \in \mathcal{D}$, *we have*

$$\theta(J_x) \in \left[\underline{E}_J\left[\underline{\widehat{\theta}}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right], \overline{E}_J\left[\overline{\widehat{\theta}}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right]\right].$$

Comment. When the interval-values estimate $\widehat{\theta}(\widetilde{x}) = \left[\underline{\widehat{\theta}}(\widetilde{x}), \underline{\widehat{\theta}}(\widetilde{x})\right]$ is integrable, and its expected value is well-defined, the above requirement takes a simpler form

$$\theta(J_x) \in E_J\left[\widehat{\theta}(\widetilde{x}_1, \ldots, \widetilde{x}_n)\right].$$

5 Unbiased Interval Estimates are Often Possible when Unbiased Numerical Estimates are Not Possible

Let us show that for examples similar to the one presented above – for which unbiased *numerical* estimates are not possible – it is possible to have unbiased *interval* estimates.

Proposition 2. *Let* $\mathcal{D}_0$ *be a class of probability distributions on* $\mathbb{R}^n$, *let* θ *be a parameter, let* $\widehat{\theta}(x_1, \ldots, x_n)$ *be a continuous function which is an unbiased numerical estimate for* θ, *and let* $\Delta_1, \ldots, \Delta_n$ *be positive real numbers. Let* $\mathcal{D}$ *denote the class of all distributions* J *on* $(x, \widetilde{x})$ *for which the marginal* J_x *belongs to* $\mathcal{D}_0$ *and for which, for all* i, *we have* $|x_i - \widetilde{x}_i| \le \Delta_i$ *with probability 1. Then, the following interval-values function is an unbiased interval estimate for* θ*:*

$$\widehat{\theta}_r(\widetilde{x}_1, \ldots, \widetilde{x}_n) \stackrel{\text{def}}{=}$$

$$\left\{\widehat{\theta}(x_1, \ldots, x_n) : x_1 \in [\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1], \ldots, x_n \in [\widetilde{x}_n - \Delta_n, \widetilde{x}_n + \Delta_n]\right\}.$$

Comment. Since the function $\widehat{\theta}(x_1, \ldots, x_n)$ is continuous, its range $\widehat{\theta}_r(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ on the box $[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] \times \ldots \times [\widetilde{x}_n - \Delta_n, \widetilde{x}_n + \Delta_n]$ is an interval. Method of estimating this intervals are known as methods of *interval computations*; see, e.g., [1, 2].

Proof. For every tuple $x = (x_1, \ldots, x_n)$, since $|x_i - \widetilde{x}_i| \le \Delta_i$, we have $x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. Thus,

$$\theta(x_1, \ldots, x_n) \in$$

$$\left\{\widehat{\theta}(x_1, \ldots, x_n) : x_1 \in [\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1], \ldots, x_n \in [\widetilde{x}_n - \Delta_n, \widetilde{x}_n + \Delta_n]\right\} =$$

$$\widehat{\theta}_r(\widetilde{x}) = \left[\underline{\widehat{\theta}_r}(\widetilde{x}), \overline{\widehat{\theta}_r}(\widetilde{x})\right]$$

and thus,

$$\underline{\widehat{\theta}_r}(\widetilde{x}) \le \theta(x) \le \overline{\widehat{\theta}_r}(\widetilde{x}).$$

It is known that if $f(x) \le g(x)$, then $\underline{E}(f) \le \underline{E}[g]$ and $\overline{E}[f] \le \overline{E}[g]$. Thus, we get

$$\underline{E}\left[\underline{\widehat{\theta}_r}(\widetilde{x})\right] \le \underline{E}[\theta(x)] \text{ and } \overline{E}[\theta(x)] \le \overline{E}\left[\overline{\widehat{\theta}_r}(\widetilde{x})\right].$$

We have assumed that θ is an unbiased estimate; this means that the man $E[\theta(x)]$ is well defined and equal to $\theta(J_x)$. Since the mean is well-defined, this means that $\underline{E}[\theta(x)] = \overline{E}[\theta(x)] = E[\theta(x)] = \theta(J_x)$. Thus, the above two inequalities take the form

$$\underline{E}\left[\underline{\widehat{\theta}_r}(\widetilde{x})\right] \le \theta(J_x) \le \overline{E}\left[\overline{\widehat{\theta}_r}(\widetilde{x})\right].$$

This is exactly the inclusion that we want to prove. The proposition is thus proven.

6 Case When We Can Have Sharp Unbiased Interval Estimates

Need for sharp unbiased interval estimates. All we required in our definition of an unbiased interval estimate (Definition 3) is that the the actual value θ of the desired parameter is contained in the interval obtained as an expected value of the interval-valued estimates $\widehat{\theta}_r(x)$.

So, if, instead of the original interval-valued estimate $\widehat{\theta}_r(x) = \left[\underline{\widehat{\theta}_r}(x), \overline{\widehat{\theta}_r}(x)\right]$, we take a wider enclosing interval, e.g., an interval $\left[\underline{\widehat{\theta}_r}(x) - 1, \overline{\widehat{\theta}_r}(x) + 1\right]$, this wider interval estimate will also satisfy our definition.

It is therefore desirable to come up with the *narrowest possible* ("sharpest") unbiased interval estimates.

A realistic example where sharp unbiased interval estimates are possible. Let us give a realistic example in which a sharp unbiased interval estimate is possible. This example will be a (slight) generalization of the example on which we showed that an unbiased numerical estimate is not always possible.

Specifically, let us assume that the actual values $x_1, \ldots, x_n$ have a joint normal distribution $N(\mu, \Sigma)$ for some unknown means $\mu_1, \ldots, \mu_n$ and an unknown covariance matrix Σ, and that the only information that we have about the measurement errors $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is that each of these differences is bounded by a known bound $\Delta_i > 0$: $|\Delta x_i| \le \Delta_i$. As we have mentioned earlier, the situation in which we only know the upper bound on the measurement errors (and we do not have any other information about the probabilities) is reasonably frequent in real life.

In this case, $\mathcal{D}$ is the class of all probability distributions for which the marginal distribution J_x is normal, and $|\widetilde{x}_i - x_i| \le \Delta_i$ for all i with probability 1. In other words, the tuple $(x_1, \ldots, x_n)$ is normally distributed, and $\widetilde{x}_i = x_i + \Delta x_i$, where Δx_i can have any distribution for which Δx_i is located on the interval $[-\Delta_i, \Delta_i]$ with probability 1 (the distribution of Δx_i may depend on $x_1, \ldots, x_n$, as long as each Δx_i is located within the corresponding interval).

Let us denote the class of all such distributions by $\mathcal{I}'$. By definition, the corresponding marginal distributions J_x correspond to n-dimensional normal distribution. As a parameter, let us select the average

$$\beta \stackrel{\text{def}}{=} \frac{\mu_1 + \ldots + \mu_n}{n}.$$

For the class of all marginal distributions J_x, there is an unbiased numerical estimate: namely, we can take $\widehat{\beta}(x_1, \ldots, x_n) = \dfrac{x_1 + \ldots + x_n}{n}$. Indeed, one can easily check that since the expected value of each variable x_i is equal to μ_i, the expected value of the estimate $\widehat{\beta}(x)$ is indeed equal to β. Due to Proposition 2, we can conclude that the range

$$\widehat{\beta}_r(\widetilde{x}_1, \ldots, \widetilde{x}_n) \stackrel{\text{def}}{=}$$

$$\left\{\widehat{\beta}(x_1, \ldots, x_n) : x_1 \in [\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1], \ldots, x_n \in [\widetilde{x}_n - \Delta_n, \widetilde{x}_n + \Delta_n]\right\} =$$

$$\left\{\frac{x_1 + \ldots + x_n}{n} : x_1 \in [\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1], \ldots, x_n \in [\widetilde{x}_n - \Delta_n, \widetilde{x}_n + \Delta_n]\right\}$$

is an unbiased interval estimate for the parameter β.

This range can be easily computed if we take into account that the function $\widehat{\beta}(x_1, \ldots, x_n) = \dfrac{x_1 + \ldots + x_n}{n}$ is an increasing function of all its variables. Thus:

- the smallest value of this function is attained when each of the variables x_i attains its smallest possible value $x_i = \widetilde{x}_i - \Delta_i$, and
- the largest value of this function is attained when each of the variables x_i attains its largest possible value $x_i = \widetilde{x}_i + \Delta_i$.

So, the range has the form

$$\widehat{\beta}_r(\widetilde{x}_1, \ldots, \widetilde{x}_n) =$$

$$\left[\frac{(\widetilde{x}_1 - \Delta_1) + \ldots + (\widetilde{x}_n - \Delta_n)}{n}, \frac{(\widetilde{x}_1 + \Delta_1) + \ldots + (\widetilde{x}_n + \Delta_n)}{n}\right].$$

Let us show that this unbiased interval estimate is indeed sharp.

Proposition 3. *For the class $\mathcal{I}'$, if $\widehat{\beta'}_r(\widetilde{x})$ is an unbiased interval estimate for β, then for every tuple $\widetilde{x}$, we have $\widehat{\beta}_r(\widetilde{x}) \subseteq \widehat{\beta'}_r(\widetilde{x})$.*

Comment. So, the above interval estimate $\widehat{\beta}_r(\widetilde{x})$ is indeed the narrowest possible.

Proof. Let us pick an arbitrary tuple $\widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$, and let us show that for this tuple, the interval $\widehat{\beta}_r(\widetilde{y})$ is contained in the interval $\widehat{\beta'}_r(\widetilde{y})$. To prove this, it is sufficient to prove that both endpoints of the interval $\widehat{\beta}_r(\widetilde{y})$ are contained in the interval $\widehat{\beta'}_r(\widetilde{y})$. Without losing generality, let us consider the left endpoint $\dfrac{(\widetilde{y}_1 - \Delta_1) + \ldots + (\widetilde{y}_n - \Delta_n)}{n}$ of the interval $\widehat{\beta}_r(\widetilde{y})$; for the right endpoint $\dfrac{(\widetilde{y}_1 + \Delta_1) + \ldots + (\widetilde{y}_n + \Delta_n)}{n}$ of this interval, the proof is similar.

To prove that $\dfrac{(\widetilde{y}_1 - \Delta_1) + \ldots + (\widetilde{y}_n - \Delta_n)}{n} \in \widehat{\beta'}_r(\widetilde{y})$, we will use the fact that the function $\widehat{\beta'}_r(\widetilde{x})$ is an unbiased interval estimate. Let us consider a distribution $J \in \mathcal{I}'$ for which each value x_i is equal to $\mu_i = \widetilde{y}_i - \Delta_i$ with probability 1, and each value $\widetilde{x}_i$ is equal to $\widetilde{y}_i$ with probability 1. One can easily see that here, $|\widetilde{x}_i - x_i| \le \Delta_i$ and therefore, this distribution indeed belongs to the desired class $\mathcal{I}'$.

For this distribution, since $\mu_i = \widetilde{y}_i - \Delta_i$, the actual value

$$\beta(J_x) = \frac{\mu_1 + \ldots + \mu_n}{n}$$

is equal to

$$\beta(J_x) = \frac{(\widetilde{y}_1 - \Delta_1) + \ldots + (\widetilde{y}_n - \Delta_n)}{n}.$$

On the other hand, since $\widetilde{x}_i = \widetilde{y}_i$ with probability 1, we have $\widehat{\beta'}_r(\widetilde{x}) = \widehat{\beta'}_r(\widetilde{y})$ with probability 1, and thus, the expected value of $\widehat{\beta'}_r(\widetilde{x})$ also coincides with the interval $\widehat{\beta'}_r(\widetilde{y})$: $E_J\left[\widehat{\beta'}_r(\widetilde{x})\right] = \widehat{\beta'}_r(\widetilde{y})$.

So, from the condition that $\beta(J_x) \in E_J\left[\widehat{\beta'}_r(\widetilde{x})\right]$, we conclude that

$$\frac{(\widetilde{y}_1 - \Delta_1) + \ldots + (\widetilde{y}_n - \Delta_n)}{n} \in \widehat{\beta'}_r(\widetilde{y}),$$

i.e., that the left endpoint of the interval $\widetilde{\beta}_r(\widetilde{y})$ indeed belongs to the interval $\widehat{\beta'}_r(\widetilde{y})$. We can similarly prove that the right endpoint of the interval $\widetilde{\beta}_r(\widetilde{y})$ belongs to the interval $\widehat{\beta'}_r(\widetilde{y})$. Thus, the whole interval $\widetilde{\beta}_r(\widetilde{y})$ is contained in the interval $\widehat{\beta'}_r(\widetilde{y})$. The proposition is proven.

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References

[1] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer-Verlag, London, 2001.

[2] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylvania, 2009.

[3] S. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer-Verlag, New York, 2005.

New Distance and Similarity Measures of Interval Neutrosophic Sets

Said Broumi[1], Hassan II University Mohammedia-Casablanca, Hay El Baraka Ben M'sik Casablanca B.P. 7951, Hassan II University Mohammedia-Casablanca, Morocco
Broumisaid78@gmail.com

Florentin Smarandache[2], Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA
fsmarandache@gmail.com

Abstract: In this paper we proposed a new distance and several similarity measures between interval neutrosophic sets.

Keywords: Neutrosophic set, Interval neutrosohic set, Similarity measure.

I. INTRODUCTION

The neutrsophic set, founded by F.Smarandache [1], has capability to deal with uncertainty, imprecise, incomplete and inconsistent information which exist in the real world. Neutrosophic set theory is a powerful tool in the formal framework, which generalizes the concepts of the classic set, fuzzy set [2], interval-valued fuzzy set [3], intuitionistic fuzzy set [4], interval-valued intuitionistic fuzzy set [5], and so on.

After the pioneering work of Smarandache, in 2005 Wang [6] introduced the notion of interval neutrosophic set (INS for short) which is a particular case of the neutrosophic set. INS can be described by a membership interval, a non-membership interval, and the indeterminate interval. Thus the interval value neutrosophic set has the virtue of being more flexible and practical than single value neutrosophic set. And the Interval Neutrosophic Set provides a more reasonable mathematical framework to deal with indeterminate and inconsistent information.

Many papers about neutrosophic set theory have been done by various researchers [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

A similarity measure for neutrosophic set (NS) is used for estimating the degree of similarity between two neutrosophic sets. Several researchers proposed some similarity measures between NSs, such as S. Broumi and F. Smarandache [26], Jun Ye [11, 12], P. Majumdar and S.K.Smanta [23].

In the literature, there are few researchers who studied the distance and similarity measure of IVNS.

In 2013, Jun Ye [12] proposed similarity measures between interval neutrosophic set based on the Hamming and Euclidean distance, and developed a multicriteria decision–making method based on the similarity degree. S. Broumi and F. Smarandache [10] proposed a new similarity measure, called "cosine similarity measure of interval valued neutrosophic sets". On the basis of numerical computations, S. Broumi and F. Smarandache found out that their similarity measures are stronger and more robust than Ye's measures.

We all know that there are various distance measures in mathematics. So, in this paper, we will extend the generalized distance of single valued neutrosophic set proposed by Ye [12] to the case of interval neutrosophic set and we'll study some new similarity measures.

This paper is organized as follows. In section 2, we review some notions of neutrosophic set andinterval valued neutrosophic set. In section 3, some new similarity measures of interval valued neutrosophic sets and their proofs are introduced. Finally, the conclusions are stated in section 4.

II. PRELIMIAIRIES

This section gives a brief overview of the concepts of neutrosophic set, and interval valued neutrosophic set.

A. *Neutrosophic Sets*

1) *Definition [1]*

Let X be a universe of discourse, with a generic element in X denoted by x, then a neutrosophic set A is an object having the form:

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F : X→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ X to the set A with the condition:

$$^-0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3^+. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. Therefore, instead of $]^-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NSs, A_{NS}= {<x, $T_A(x)$, $I_A(x)$, $F_A(x)$>|x ∈ X} **(2)**

and B_{NS} ={ <x, $T_B(x)$, $I_B(x)$, $F_B(x)$> | x ∈ X } the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$.

(2) $A_{NS} = B_{NS}$ if and only if , $T_A(x)=T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$.

B. *Interval Valued Neutrosophic Sets*

In actual applications, sometimes, it is not easy to express the truth-membership, indeterminacy-membership and falsity-membership by crisp value, and they may be easier to be expressed by interval numbers. Wang et al. [6] further defined interval neutrosophic sets (INS) shows as follows:

1) Definition [6]

Let X be a universe of discourse, with generic element in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function$T_A(x)$, indeteminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x), I_A(x), F_A(x) \in [0,1]$.

For two IVNS, A_{IVNS}={<x,[$T_A^L(x)$,$T_A^U(x)$], [$I_A^L(x), I_A^U(x)$], [$F_A^L(x), F_A^U(x)$]> | x ∈ X } **(3)**

and B_{IVNS}= {<x, [$T_B^L(x)$,$T_B^U(x)$], [$I_B^L(x), I_B^U(x)$], [$F_B^L(x), F_B^U(x)$] > | x ∈ X } the two relations are defined as follows:

(1) $A_{IVNS} \subseteq B_{IVNS}$ if and only if $T_A^L(x) \leq T_B^L(x), T_A^U(x) \leq T_B^U(x), I_A^L(x) \geq I_B^L(x), I_A^U(x) \geq I_B^U(x), F_A^L(x) \geq F_B^L(x), F_A^U(x) \geq F_B^U(x)$.

(2) $A_{IVNS} = B_{IVNS}$ if and only if $T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$, $I_A^L(x_i) = I_B^L(x_i)$,$I_A^U(x_i) = I_B^U(x_i)$, $F_A^L(x_i) = F_B^L(x_i)$ and $F_A^U(x_i) = F_B^U(x_i)$ for any x ∈ X.

C. *Defintion*

Let A and B be two interval valued neutrosophic sets, then

- **i. $0 \leq S(A,B) \leq 1$.**
- ii. $S(A,B) = S(B,A)$.
- iii. $S(A,B) = 1$ if A= B, i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$,
$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and
$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$, for i = 1, 2,…., n.

- iv . A⊂ B ⊂ C ⇒ S(A,B) ≤ min (S(A,B), S(B,C).

III. New Distance Measure of Interval Valued Neutrosophic Sets

Let A and B be two single neutrosophic sets, then J. Ye [11] proposed a generalized single valued neutrosophic weighted distance measure between A and B as follows:

$$d_\lambda(A,B) = \left\{\frac{1}{3}\sum_{i=1}^{n} w_i\left[|T_A(x_i) - T_B(x_i)|^\lambda + |I_A(x_i) - I_B(x_i)|^\lambda + |F_A(x_i) - F_B(x_i)|^\lambda\right]\right\}^{\frac{1}{\lambda}} \quad (4)$$

where

$\lambda > 0$ and $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i) \in [0, 1]$.

Based on the geometrical distance model and using the interval neutrosophic sets, we extended the distance (4) as follows:

$$d_\lambda(A,B) = \left\{\frac{1}{6}\sum_{i=1}^{n} w_i\left[|T_A^L(x_i) - T_B^L(x_i)|^\lambda + |T_A^U(x_i) - T_B^U(x_i)|^\lambda + |I_A^L(x_i) - I_B^L(x_i)|^\lambda + |I_A^U(x_i) - I_B^U(x_i)|^\lambda + |F_A^L(x_i) - F_B^L(x_i)|^\lambda + |F_A^U(x_i) - F_B^U(x_i)|^\lambda\right]\right\}^{\frac{1}{\lambda}}. \quad (5)$$

The normalized generalized interval neutrosophic distance is

$$d_\lambda(A,B) = \left\{\frac{1}{6n}\sum_{i=1}^{n} w_i\left[|T_A^L(x_i) - T_B^L(x_i)|^\lambda + |T_A^U(x_i) - T_B^U(x_i)|^\lambda + |I_A^L(x_i) - I_B^L(x_i)|^\lambda + |I_A^U(x_i) - I_B^U(x_i)|^\lambda + |F_A^L(x_i) - F_B^L(x_i)|^\lambda + |F_A^U(x_i) - F_B^U(x_i)|^\lambda\right]\right\}^{\frac{1}{\lambda}}. \quad (6)$$

If w={$\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}$}, the distance (6) is reduced to the following distances:

$$d_\lambda(A,B) = \left\{\frac{1}{6}\sum_{i=1}^{n}\left[|T_A^L(x_i) - T_B^L(x_i)|^\lambda + |T_A^U(x_i) - T_B^U(x_i)|^\lambda + |I_A^L(x_i) - I_B^L(x_i)|^\lambda + |I_A^U(x_i) - I_B^U(x_i)|^\lambda + |F_A^L(x_i) - F_B^L(x_i)|^\lambda + |F_A^U(x_i) - F_B^U(x_i)|^\lambda\right]\right\}^{\frac{1}{\lambda}}. \quad (7)$$

$$d_\lambda(A,B) = \left\{\frac{1}{6n}\sum_{i=1}^{n}\left[|T_A^L(x_i) - T_B^L(x_i)|^\lambda + |T_A^U(x_i) - T_B^U(x_i)|^\lambda + |I_A^L(x_i) - I_B^L(x_i)|^\lambda + |I_A^U(x_i) - I_B^U(x_i)|^\lambda + |F_A^L(x_i) - F_B^L(x_i)|^\lambda + |F_A^U(x_i) - F_B^U(x_i)|^\lambda\right]\right\}^{\frac{1}{\lambda}}. \quad (8)$$

Particular case.

(i) If $\lambda = 1$ then the distances (7) and (8) are reduced to the following Hamming distance and respectively normalized Hamming distance defined by Ye Jun [11]:

$$d_H(A, B) = \left\{\frac{1}{6}\sum_{i=1}^{n}[|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|]\right\}, \quad \textbf{(9)}$$

$$d_{NH}(A, B) = \left\{\frac{1}{6n}\sum_{i=1}^{n}[|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|]\right\}. \quad \textbf{(10)}$$

(ii) If $\lambda = 2$ then the distances (7) and (8) are reduced to the following Euclidean distance and respectively normalized Euclidean distance defined by Ye Jun [12]:

$$d_E(A, B) = \left\{\frac{1}{6}\sum_{i=1}^{n}[|T_A^L(x_i) - T_B^L(x_i)|^2 + |T_A^U(x_i) - T_B^U(x_i)|^2 + |I_A^L(x_i) - I_B^L(x_i)|^2 + |I_A^U(x_i) - I_B^U(x_i)|^2 + |F_A^L(x_i) - F_B^L(x_i)|^2 + |F_A^U(x_i) - F_B^U(x_i)|^2]\right\}^{\frac{1}{2}}, \quad \textbf{(11)}$$

$$d_{NE}(A, B) = \left\{\frac{1}{6n}\sum_{i=1}^{n}[|T_A^L(x_i) - T_B^L(x_i)|^2 + |T_A^U(x_i) - T_B^U(x_i)|^2 + |I_A^L(x_i) - I_B^L(x_i)|^2 + |I_A^U(x_i) - I_B^U(x_i)|^2 + |F_A^L(x_i) - F_B^L(x_i)|^2 + |F_A^U(x_i) - F_B^U(x_i)|^2]\right\}^{\frac{1}{2}}. \quad \textbf{(12)}$$

IV. New Similarity Measures of Interval Valued Neutrosophic Set

A. Similarity measure based on the geometric distance model

Based on distance (4), we define the similarity measure between the interval valued neutrosophic sets A and B as follows:

$$S_{DM}(A, B) = 1 - \left\{\frac{1}{6n}\sum_{i=1}^{n}\left[\left|T_A^L(x_i) - T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) - T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) - I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) - I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) - F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) - F_B^U(x_i)\right|^\lambda\right]\right\}^{\frac{1}{\lambda}}, \quad \textbf{(13)}$$

where $\lambda > 0$ and $S_{DM}(A, B)$ is the degree of similarity of A and B .

If we take the weight of each element $x_i \in X$ into account, then

$$S_{DM}^w(A, B) = 1 - \left\{\frac{1}{6}\sum_{i=1}^{n} w_i\left[\left|T_A^L(x_i) - T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) - T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) - I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) - I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) - F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) - F_B^U(x_i)\right|^\lambda\right]\right\}^{\frac{1}{\lambda}}. \quad \textbf{(14)}$$

If each elements has the same importance, i.e. $w = \{\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\}$, then similarity (14) reduces to (13).

By (definition C) it can easily be known that $S_{DM}(A, B)$ satisfies all the properties of the definition..

Similarly, we define another similarity measure of A and B, as:

$$S(A, B) = 1 - \left[\frac{\sum_{i=1}^{n}\left(\left|T_A^L(x_i) - T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) - T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) - I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) - I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) - F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) - F_B^U(x_i)\right|^\lambda\right)}{\sum_{i=1}^{n}\left(\left|T_A^L(x_i) + T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) + T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) + I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) + I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) + F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) + F_B^U(x_i)\right|^\lambda\right)}\right]^{\frac{1}{\lambda}}. \quad \textbf{(15)}$$

If we take the weight of each element $x_i \in X$ into account, then

$$S(A, B) = 1 - \left[\frac{\sum_{i=1}^{n} w_i\left(\left|T_A^L(x_i) - T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) - T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) - I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) - I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) - F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) - F_B^U(x_i)\right|^\lambda\right)}{\sum_{i=1}^{n} w_i\left(\left|T_A^L(x_i) + T_B^L(x_i)\right|^\lambda + \left|T_A^U(x_i) + T_B^U(x_i)\right|^\lambda + \left|I_A^L(x_i) + I_B^L(x_i)\right|^\lambda + \left|I_A^U(x_i) + I_B^U(x_i)\right|^\lambda + \left|F_A^L(x_i) + F_B^L(x_i)\right|^\lambda + \left|F_A^U(x_i) + F_B^U(x_i)\right|^\lambda\right)}\right]^{\frac{1}{\lambda}}. \quad \textbf{(16)}$$

It also has been proved that all conditions of the definition are satisfied. If each elements has the same importance, and then the similarity (16) reduces to (15).

B. Similarity measure based on the interval valued neutrosophic theoretic approach:

In this section, following the similarity measure between two neutrosophic sets defined by P. Majumdar in [24], we extend Majumdar's definition to interval valued neutrosophic sets.

Let A and B be two interval valued neutrosophic sets, then we define a similarity measure between A and B as follows:

$$S_{TA}(A,B)=\frac{\sum_{i=1}^{n}\{\min\{T_A^L(x_i),T_B^L(x_i)\}+\min\{T_A^U(x_i),T_B^U(x_i)\}+\min\{I_A^L(x_i),I_B^L(x_i)\}+\min\{I_A^U(x_i),I_B^U(x_i)\}+\min\{F_A^L(x_i),F_B^L(x_i)\}+\min\{F_A^U(x_i),F_B^U(x_i)\}}{\sum_{i=1}^{n}\{\max\{T_A^L(x_i),T_B^L(x_i)\}+\max\{T_A^U(x_i),T_B^U(x_i)\}+\max\{I_A^L(x_i),I_B^L(x_i)\}+\max\{I_A^U(x_i),I_B^U(x_i)\}+\max\{F_A^L(x_i),F_B^L(x_i)\}+\max\{F_A^U(x_i),F_B^U(x_i)\}} \quad \textbf{(17)}$$

1) Proposition

Let A and B be two interval valued neutrosophic sets, then

iv. $\mathbf{0} \le S_{TA}(A,B) \le \mathbf{1}$.

v. $S_{TA}(A,B) = S_{TA}(A,B)$.

vi. $S(A,B) = 1$ if A = B i.e.

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$, $I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and $F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i = 1, 2, …., n.

iv. $A \subset B \subset C \Rightarrow S_{TA}(A,B) \le \min(S_{TA}(A,B), S_{TA}(B,C))$.

Proof. Properties (i) and (ii) follow from the definition.

(iii) It is clearly that if A = B ⇒ $S_{TA}(A,B)$ =1

$\Rightarrow \sum_{i=1}^{n}\{\min\{T_A^L(x_i),T_B^L(x_i)\} + \min\{T_A^U(x_i),T_B^U(x_i)\} + \min\{I_A^L(x_i),I_B^L(x_i)\} + \min\{I_A^U(x_i),I_B^U(x_i)\} + \min\{F_A^L(x_i),F_B^L(x_i)\} + \min\{F_A^U(x_i),F_B^U(x_i)\}$

$=\sum_{i=1}^{n}\{max\{T_A^L(x_i),T_B^L(x_i)\} + max\{T_A^U(x_i),T_B^U(x_i)\} + max\{I_A^L(x_i),I_B^L(x_i)\} + max\{I_A^U(x_i),I_B^U(x_i)\} + max\{F_A^L(x_i),F_B^L(x_i)\} + max\{F_A^U(x_i),F_B^U(x_i)\}$

$\Rightarrow \sum_{i=1}^{n}\{[\min\{T_A^L(x_i),T_B^L(x_i)\} - \max\{T_A^L(x_i),T_B^L(x_i)\}] + [\min\{T_A^U(x_i),T_B^U(x_i)\} - \max\{T_A^U(x_i),T_B^U(x_i)\}] + [\min\{I_A^L(x_i),I_B^L(x_i)\} - \max\{I_A^L(x_i),I_B^L(x_i)\}] + [\min\{I_A^U(x_i),I_B^U(x_i)\} - \max\{I_A^U(x_i),I_B^U(x_i)\}] + [\min\{F_A^L(x_i),F_B^L(x_i)\} - \max\{F_A^L(x_i),F_B^L(x_i)\}] + [\min\{F_A^U(x_i),F_B^U(x_i)\} - \max\{F_A^U(x_i),F_B^U(x_i)\}]\} = 0$.

Thus for each x, one has that

$[\min\{T_A^L(x_i),T_B^L(x_i)\} - \max\{T_A^L(x_i),T_B^L(x_i)\}] = 0$

$[\min\{T_A^U(x_i),T_B^U(x_i)\} - \max\{T_A^U(x_i),T_B^U(x_i)\}] = 0$

$[\min\{I_A^L(x_i),I_B^L(x_i)\} - \max\{I_A^L(x_i),I_B^L(x_i)\}] = 0$

$[\min\{I_A^U(x_i),I_B^U(x_i)\} - \max\{I_A^U(x_i),I_B^U(x_i)\}] = 0$

$[\min\{F_A^L(x_i),F_B^L(x_i)\} - \max\{F_A^L(x_i),F_B^L(x_i)\}] = 0$

$[\min\{F_A^U(x_i),F_B^U(x_i)\} - \max\{F_A^U(x_i),F_B^U(x_i)\}] = 0$

hold.

Thus $T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$, $I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$, $F_A^L(x_i) = F_B^L(x_i)$ and $F_A^U(x_i) = F_B^U(x_i) \Rightarrow$ A=B

(iv) Now we prove the last result.

Let $A \subset B \subset C$, then we have

$T_A^L(x) \le T_B^L(x) \le T_C^L(x)$, $T_A^U(x) \le T_B^U(x) \le T_C^L(x)$, $I_A^L(x) \ge I_B^L(x) \ge I_C^L(x)$, $I_A^U(x) \ge I_B^U(x) \ge I_C^U(x)$, $F_A^L(x) \ge F_B^L(x) \ge F_C^L(x)$, $F_A^U(x) \ge F_B^U(x) \ge F_C^U(x)$ for all x ∈ X.

Now

$T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_B^L(x) + F_B^U(x) \ge T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_C^L(x) + F_C^U(x)$

and

$T_B^L(x) + T_B^U(x) + I_B^L(x) + I_B^U(x) + F_A^L(x) + F_A^U(x) \ge T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_A^L(x) + F_A^U(x)$.

$$S(A,B) = \frac{T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_B^L(x) + F_B^U(x)}{T_B^L(x) + T_B^U(x) + I_B^L(x) + I_B^U(x) + F_A^L(x) + F_A^U(x)} \ge \frac{T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_C^L(x) + F_C^U(x)}{T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_A^L(x) + F_A^U(x)} = S(A,C).$$

Again, similarly we have

$T_B^L(x) + T_B^U(x) + I_B^L(x) + I_B^U(x) + F_C^L(x) + F_C^U(x) \ge T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_C^L(x) + F_C^U(x)$

$T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_A^L(x) + F_A^U(x) \ge T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_B^L(x) + F_B^U(x)$

$$S(B,C) = \frac{T_B^L(x) + T_B^U(x) + I_B^L(x) + I_B^U(x) + F_C^L(x) + F_C^U(x)}{T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_B^L(x) + F_B^U(x)} \ge \frac{T_A^L(x) + T_A^U(x) + I_A^L(x) + I_A^U(x) + F_C^L(x) + F_C^U(x)}{T_C^L(x) + T_C^U(x) + I_C^L(x) + I_C^U(x) + F_A^L(x) + F_A^U(x)} = S(A,C)$$

$\Rightarrow S_{TA}(A,B) \le \min(S_{TA}(A,B), S_{TA}(B,C))$.

Hence the proof of this proposition.

If we take the weight of each element $x_i \in X$ into account, then

$$S(A,B)=\frac{\sum_{i=1}^{n} w_i\{\min\{T_A^L(x_i),T_B^L(x_i)\}+\min\{T_A^U(x_i),T_B^U(x_i)\}+\min\{I_A^L(x_i),I_B^L(x_i)\}+\min\{I_A^U(x_i),I_B^U(x_i)\}+\min\{F_A^L(x_i),F_B^L(x_i)\}+\min\{F_A^U(x_i),F_B^U(x_i)\}}{\sum_{i=1}^{n} w_i\{\max\{T_A^L(x_i),T_B^L(x_i)\}+\max\{T_A^U(x_i),T_B^U(x_i)\}+\max\{I_A^L(x_i),I_B^L(x_i)\}+\max\{I_A^U(x_i),I_B^U(x_i)\}+\max\{F_A^L(x_i),F_B^L(x_i)\}+\max\{F_A^U(x_i),F_B^U(x_i)\}} \quad \textbf{(18)}$$

Particularly, if each element has the same importance, then (18) is reduced to (17), clearly this also satisfies all the properties of the definition.

C. Similarity measure based on matching function by using interval neutrosophic sets:

Chen [24] and Chen et al. [25] introduced a matching function to calculate the degree of similarity between fuzzy sets. In the following, we extend the matching function to deal with the similarity measure of interval valued neutrosophic sets.

Let A and B be two interval valued neutrosophic sets, then we define a similarity measure between A and B as follows:

S_{MF}(A,B) =

$$\frac{\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\Big(\big(T_A^L(x_i)\cdot\ T_B^L(x_i)\big)+\big(T_A^U(x_i)\cdot\ T_B^U(x_i)\big)+\big(I_A^L(x_i)\cdot\ I_B^L(x_i)\big)+\big(I_A^U(x_i)\cdot\ I_B^U(x_i)\big)+\big(F_A^L(x_i)\cdot\ F_B^L(x_i)\big)+\big(F_A^U(x_i)\cdot\ F_B^U(x_i)\big)\Big)}{\max\big(\sum_{i=}^{n}(T_A^L(x_i)^2+T_A^U(x_i)^2+I_A^L(x_i)^2+I_A^U(x_i)^2+F_A^L(x_i)^2+F_A^U(x_i)^2),\ \sum_{i=}^{n}(T_B^L(x_i)^2+T_B^U(x_i)^2+I_B^L(x_i)^2+I_B^U(x_i)^2+F_B^L(x_i)^2+F_B^U(x_i)^2)\big)} \quad \textbf{(19)}$$

Proof.

i. $\mathbf{0} \le S_{MF}$(A,B) $\le \mathbf{1}$**.**

The inequality S_{MF}(A,B) ≥ 0 is obvious. Thus, we only prove the inequality S(A, B) ≤ 1.

S_{MF}(A,B)= $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\Big(\big(T_A^L(x_i)\cdot\ T_B^L(x_i)\big)+\big(T_A^U(x_i)\cdot\ T_B^U(x_i)\big)+\big(I_A^L(x_i)\cdot\ I_B^L(x_i)\big)+\big(I_A^U(x_i)\cdot\ I_B^U(x_i)\big)+\big(F_A^L(x_i)\cdot\ F_B^L(x_i)\big)+\big(F_A^U(x_i)\cdot\ F_B^U(x_i)\big)\Big)$

= $T_A^L(x_1)\cdot\ T_B^L(x_1)$+$T_A^L(x_2)\cdot\ T_B^L(x_2)$+…+$T_A^L(x_n)\cdot\ T_B^L(x_n)$+$T_A^U(x_1)\cdot\ T_B^U(x_1)$+$T_A^U(x_2)\cdot\ T_B^U(x_2)$+…+$T_A^U(x_n)\cdot\ T_B^U(x_n)$+ $I_A^L(x_1)\cdot\ I_B^L(x_1)$+$I_A^L(x_2)\cdot\ I_B^L(x_2)$+…+$I_A^L(x_n)\cdot\ I_B^L(x_n)$+$I_A^U(x_1)\cdot\ I_B^U(x_1)$+$I_A^U(x_2)\cdot\ I_B^U(x_2)$+…+$I_A^U(x_n)\cdot\ I_B^U(x_n)$+ $F_A^L(x_1)\cdot\ F_B^L(x_1)$+$F_A^L(x_2)\cdot\ F_B^L(x_2)$+…+$F_A^L(x_n)\cdot\ F_B^L(x_n)$+$F_A^U(x_1)\cdot\ T_B^U(x_1)$+$F_A^U(x_2)\cdot\ F_B^U(x_2)$+…+$F_A^U(x_n)\cdot\ F_B^U(x_n)$.

According to the Cauchy–Schwarz inequality:

$(x_1\cdot y_1+x_2\cdot y_2+\cdots+x_n\cdot y_n)^2 \le ({x_1}^2+{x_2}^2+\cdots+{x_n}^2)\ \cdot({y_1}^2+{y_2}^2+\cdots+{y_n}^2)$

where $(x_1, x_2, \dots, x_n) \in\ R^n$ and $(y_1, y_2, \dots, y_n)\ \in R^n$

we can obtain

$[S_{MF}(\mathrm{A,B})]^2 \le \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\big(T_A^L(x_i)^2+T_A^U(x_i)^2+I_A^L(x_i)^2+I_A^U(x_i)^2+F_A^L(x_i)^2+F_A^U(x_i)^2\big)\cdot$ $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\big(T_B^L(x_i)^2+T_B^U(x_i)^2+I_B^L(x_i)^2+I_B^U(x_i)^2+F_B^L(x_i)^2+F_B^U(x_i)^2\big)$= S(A, A) · S(B, B)

Thus S_{MF}(A,B)≤ $[S(A,A)]^{\frac{1}{2}}\ \cdot[S(B,B)]^{\frac{1}{2}}$.

Then S_{MF}(A,B)≤ max{S(A,A), S(B,B)].

Therefore S_{MF}(A, B) ≤ 1.

If we take the weight of each element x_i ∈ X into account, then

$$S_{MF}^{w}(\mathrm{A,B})=\frac{\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} w_i\Big(\big(T_A^L(x_i)\cdot T_B^L(x_i)\big)+\big(T_A^U(x_i)\cdot T_B^U(x_i)\big)+\big(I_A^L(x_i)\cdot I_B^L(x_i)\big)+\big(I_A^U(x_i)\cdot I_B^U(x_i)\big)+\big(F_A^L(x_i)\cdot F_B^L(x_i)\big)+\big(F_A^U(x_i)\cdot F_B^U(x_i)\big)\Big)}{\max\big(\sum_{i=}^{n} w_i\,(T_A^L(x_i)^2+T_A^U(x_i)^2+I_A^L(x_i)^2+I_A^U(x_i)^2+F_A^L(x_i)^2+F_A^U(x_i)^2),\ \sum_{i=}^{n} w_i\,(T_B^L(x_i)^2+T_B^U(x_i)^2+I_B^L(x_i)^2+I_B^U(x_i)^2+F_B^L(x_i)^2+F_B^U(x_i)^2)\big)} \quad \textbf{(20)}$$

Particularly, if each element has the same importance, then the similarity (20) is reduced to (19). Clearly this also satisfies all the properties of definition.

The larger the value of S(A,B), the more the similarity between A and B.

V. Comparison of New Similarity Measure of IVNS With The Existing Measures.

Let A and B be two interval valued neutrosophic sets in the universe of discourse X = {x_1, x_2 ,.., x_n}.The new similarity $S_{TA}(A,B)$ of IVNS and the existing similarity measures of interval valued neutrosophic sets (examples 1 and 2) introduced in [10, 12, 23] are listed as follows:

***Pinaki similarity I*:**

this similarity measure was proposed as concept of association coefficient of the neutrosophic sets as follows

$$S_{PI}=\frac{\sum_{i=1}^{n}\{\min\{T_A(x_i),T_B(x_i)\}+\min\{I_A(x_i),I_B(x_i)\}+\min\{F_A(x_i),F_B(x_i)\}\}}{\sum_{i=1}^{n}\{\max\{T_A(x_i),T_B(x_i)\}+\max\{I_A(x_i),I_B(x_i)\}+\max\{F_A(x_i),F_B(x_i)\}\}} \quad \textbf{(21)}$$

Broumi and Smarandache cosine similarity:

$$C_N(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A^L(x_i)+T_A^U(x_i))\,(T_B^L(x_i)+T_B^U(x_i))+(I_A^L(x_i)+I_A^U(x_i))\,(I_B^L(x_i)+I_B^U(x_i))+(F_A^L(x_i)+F_A^U(x_i))\,(F_B^L(x_i)+F_B^U(x_i))}{\sqrt{(T_A^L(x_i)+T_A^U(x_i))^2+(I_A^L(x_i)+I_A^U(x_i))^2+(F_A^L(x_i)+F_A^U(x_i))^2}\sqrt{(T_B^L(x_i)+T_B^U(x_i))^2+(I_B^L(x_i)+I_B^U(x_i))^2+(F_B^L(x_i)+F_B^U(x_i))^2}} \quad \textbf{(22)}$$

Ye similarity

$$S_{ye}(A,B)=1-\frac{1}{6}\sum_{i=1}^{n}\left[|\inf T_A(x_i)-\inf T_B(x_i)|+|\sup T_A(x_i)-\sup T_B(x_i)|+|\inf I_A(x_i)-\inf I_B(x_i)|+|\sup I_A(x_i)-\sup I_B(x_i)|+|\inf F_A(x_i)-\inf F_B(x_i)|+|\sup F_A(x_i)-\sup F_B(x_i)|\right]. \quad \textbf{(23)}$$

Example 1

Let A = {<x, (a, 0.2 , 0.6 , 0.6), (b, 0.5, 0.3 , 0.3), (c, 0.6 , 0.9 , 0.5)>}

and B = {<x, (a, 0.5 , 0.3 , 0.8), (b, 0.6 , 0.2 , 0.5), (c, 0.6 , 0.4 , 0.4)>}.

Pinaki similarity I = 0.6.

S_{ye}(A, B) = 0.38 (With w_i =1).

Cosine similarity $\mathbf{C_N(A,B)}$ = 0.95.

$S_{TA}(A,B)$ = **0.8.**

Example 2:

Let A= {<x, (a, [0.2 , 0.3] ,[0.2, 0.6], [0.6 , 0.8]), (b, [0.5 , 0.7], [0.3, 0.5], [0.3 , 0.6]), (c, [0.6 , 0.9] ,[0.3, 0.9], [0.3, 0.5])>} and

B={<x, (a, [0.5 , 0.3] ,[0.3, 0.6], [0.6 , 0.8]), (b, [0.6, 0.8] ,[0.2, 0.4], [0.5 , 0.6]), (c, [0.6 , 0.9] ,[0.3, 0.4], [0.4 , 0.6])>}.

Pinaki similarity I = NA.

S_{ye}(A, B) = 0.7 (With w_i =1).

Cosine similarity $\mathbf{C_N(A,B)}$ = 0.92.

$S_{TA}(A,B)$ = **0.9.**

On the basis of computational study Jun Ye [12] has shown that their measure is more effective and reasonable. A similar kind of study with the help of the proposed new measure based on theoretic approach, it has been done and it is found that the obtained results are more refined and accurate. It may be observed from the above examples that the values of similarity measures are closer to 1 with $S_{TA}(A,B)$ which is this proposed similarity measure.

VI. Conclusions

Few distance and similarity measures have been proposed in literature for measuring the distance and the degree of similarity between interval neutrosophic sets. In this paper, we proposed a new method for distance and similarity measure for measuring the degree of similarity between two weighted interval valued neutrosophic sets, and we have extended the work of Pinaki, Majumdar and S. K. Samant and Chen. The results of the proposed similarity measure and existing similarity measure are compared.

In the future, we will use the similarity measures which are proposed in this paper in group decision making

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VIII. References

[1] F. Smarandache, “A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic”. Rehoboth: American Research Press, (1998).

[2] L. A. Zadeh, "Fuzzy Sets". Information and Control, 8, (1965), pp. 338-353.

[3] Turksen, “Interval valued fuzzy sets based on normal forms”.Fuzzy Sets and Systems, 20, (1968), pp.191–210.

[4] Atanassov, K. “Intuitionistic fuzzy sets”. Fuzzy Sets and Systems, 20, (1986), pp. 87–96.

[5] Atanassov, K., & Gargov, G “Interval valued intuitionistic fuzzy sets”. Fuzzy Sets and Systems. International Journal of General Systems 393, 31,(1998), pp. 343–349

[6] Wang, H., Smarandache, F., Zhang, Y.-Q. and Sunderraman, R., ”Interval Neutrosophic Sets and Logic: Theory and Applications in Computing”, Hexis, Phoenix, AZ, (2005).

[7] A. Kharal, “A Neutrosophic Multicriteria Decision Making Method”,New Mathematics & Natural Computation, to appear in Nov. 2013.

[8] S. Broumi and F. Smarandache, “Intuitionistic Neutrosophic Soft Set”, Journal of Information and Computing Science, England, UK, ISSN 1746-7659,Vol. 8, No. 2, (2013), pp. 130-140.

[9] S.Broumi, “Generalized Neutrosophic Soft Set”, International Journal of Computer Science, Engineering and Information Technology (IJCSEIT), ISSN: 2231-3605, E-ISSN: 2231-3117, Vol.3, No.2, (2013), pp. 17-30.

[10] S. Broumi and F. Smarandache,” Cosine similarity measure of interval valued neutrosophic sets”, 2014 (submitted).

[11] J. Ye.” Single valued netrosophiqc minimum spanning tree and its clustering method” De Gruyter journal of intelligent system, (2013), pp. 1-24.

[12] J. Ye, ”Similarity measures between interval neutrosophic sets and their multicriteria decision-making method “Journal of Intelligent & Fuzzy Systems, DOI: 10.3233/IFS-120724 ,(2013),.

[13] M. Arora, R. Biswas, U. S. Pandy, “Neutrosophic Relational Database Decomposition”, International Journal of Advanced Computer Science and Applications, Vol. 2, No. 8, (2011) 121-125.

[14] M. Arora and R. Biswas,” Deployment of Neutrosophic technology to retrieve answers for queries posed in natural language”, in 3rdInternational Conference on Computer Science and Information Technology ICCSIT, IEEE catalog Number CFP1057E-art, Vol.3, ISBN: 978-1-4244-5540-9,(2010) 435-439.

[15] Ansari, Biswas, Aggarwal,”Proposal for Applicability of Neutrosophic Set Theory in Medical AI”, International Journal of Computer Applications (0975 – 8887), Vol. 27– No.5, (2011) 5-11.

[16] F. G Lupiáñez, "On neutrosophic topology", Kybernetes, Vol. 37, Iss: 6, (2008), pp. 797 - 800, Doi: 10.1108/03684920810876990.

[17] S. Aggarwal, R. Biswas, A. Q. Ansari, ”Neutrosophic Modeling and Control”,978-1-4244-9034-/10 IEEE,(2010), pp. 718-723.

[18] H. D. Cheng,& Y Guo. “A new neutrosophic approach to image thresholding”. New Mathematics and Natural Computation, 4(3), (2008), pp.291–308.

[19] Y. Guo, &, H. D. Cheng “New neutrosophic approach to image segmentation”. Pattern Recognition, 42, (2009), pp.587–595.

[20] M. Zhang, L. Zhang, and H. D. Cheng. “A neutrosophic approach to image segmentation based on watershed method”. Signal Processing 5, 90, (2010), pp. 1510-1517.

[21] Wang, H., Smarandache, F., Zhang, Y. Q., Sunderraman, R, ”Single valued neutrosophic sets”. Multispace and Multistructure, 4 (2010), pp. 410–413.

[22] S. Broumi, F. Smarandache , “Correlation Coefficient of Interval Neutrosophic set”, Periodical of Applied Mechanics and Materials, Vol. 436, 2013, with the title Engineering Decisions and Scientific Research in Aerospace, Robotics, Biomechanics, Mechanical Engineering and Manufacturing; Proceedings of the International Conference ICMERA, Bucharest, October 2013.

[23] P. Majumdar, S.K. Samant,” On similarity and entropy of neutrosophic sets”, Journal of Intelligent and Fuzzy Systems,1064-1246(Print)-1875-8967(Online), (2013), DOI:10.3233/IFS-130810, IOS Press.

[24] S.M. Chen, S. M. Yeh and P.H. Hsiao, “A comparison of similarity measure of fuzzy values” Fuzzy sets and systems, 72, 1995. 79-89.

[25] S.M. Chen, “A new approach to handling fuzzy decision making problem”, IEEE transactions on Systems, man and cybernetics, 18, 1988. , pp. 1012-1016.

[26] Said Broumi, and Florentin Smarandache, Several Similarity Measures of Neutrosophic Sets”, Neutrosophic Sets and Systems, Vol. 1 (2013), pp. 54-62.

Lower and Upper Soft Interval Valued Neutrosophic Rough Approximations of An IVNSS-Relation

Said Broumi[1], Florentin Smarandache[2]
[1] Faculty of Arts and Humanities, Hay El Baraka Ben M'sik Casablanca B.P. 7951, Hassan II University Mohammedia-Casablanca , Morocco

broumisaid78@gmail.com

[2]Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA

fsmarandache@gmail.com

Abstract: In this paper, we extend the lower and upper soft interval valued intuitionstic fuzzy rough approximations of IVIFS –relations proposed by Anjan et al. to the case of interval valued neutrosophic soft set relation(IVNSS-relation for short)

Keywords: Interval valued neutrosophic soft , Interval valued neutrosophic soft set relation

0. Introduction

This paper is an attempt to extend the concept of interval valued intuitionistic fuzzy soft relation (IVIFSS-relations) introduced by A. Mukherjee et al [45]to IVNSS relation .

The organization of this paper is as follow: In section 2, we briefly present some basic definitions and preliminary results are given which will be used in the rest of the paper. In section 3, relation interval neutrosophic soft relation is presented. In section 4 various type of interval valued neutrosophic soft relations. In section 5, we concludes the paper

1. Preliminaries

Throughout this paper, let U be a universal set and E be the set of all possible parameters under consideration with respect to U, usually, parameters are attributes, characteristics, or properties of objects in U. We now recall some basic notions of neutrosophic set, interval neutrosophic set, soft set, neutrosophic soft set and interval neutrosophic soft set.

Definition 2.1.

Let U be an universe of discourse then the neutrosophic set A is an object having the form $A= \{< x: \mu_{A(x)}, \nu_{A(x)}, \omega_{A(x)} >, x \in U\}$, where the functions $\mu, \nu, \omega : U \to]^{-}0,1^{+}[$ define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$^{-}0 \le \mu_{A(x)} + \nu_{A(x)} + \omega_{A(x)} \le 3^{+}.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0,1^{+}[$.so instead of $]^{-}0,1^{+}[$ we need to take the interval [0,1] for technical applications, because $]^{-}0,1^{+}[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2. A neutrosophic set A is contained in another neutrosophic set B i.e. A ⊆ B if ∀x ∈ U, $\mu_A(x) \le \mu_B(x)$, $\nu_A(x) \ge \nu_B(x)$, $\omega_A(x) \ge \omega_B(x)$.

Definition 2.3. Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $\boldsymbol{\mu_A(x)}$, indeteminacy-membership function $\boldsymbol{\nu_A(x)}$ and falsity-membership function $\boldsymbol{\omega_A(x)}$. For each point **x in X, we have that $\boldsymbol{\mu_A(x)}$, $\boldsymbol{\nu_A(x)}$, $\boldsymbol{\omega_A(x) \in [0,1]}$.**
For two IVNS , A_{IVNS} ={ <x , $[\mu_A^L(x), \mu_A^U(x)]$, $[\nu_A^L(x), \nu_A^U(x)]$, $[\omega_A^L(x), \omega_A^U(x)]$ > | x ∈ X }
And B_{IVNS} ={ <x , $[\mu_B^L(x), \mu_B^U(x)]$, $[\nu_B^L(x), \nu_B^U(x)]$, $[\omega_B^L(x), \omega_B^U(x)]$> | x ∈ X } the two relations are defined as follows:
(1) $A_{\text{IVNS}} \subseteq B_{\text{IVNS}}$ if and only if $\mu_A^L(x) \le \mu_B^L(x)$,$\mu_A^U(x) \le \mu_B^U(x)$,$\nu_A^L(x) \ge \nu_B^L(x)$, $\omega_A^U(x) \ge \omega_B^U(x)$, $\omega_A^L(x) \ge \omega_B^L(x)$, $\omega_A^U(x) \ge \omega_B^U(x)$
(2) $A_{\text{IVNS}} = B_{\text{IVNS}}$ if and only if , $\mu_A(x) = \mu_B(x)$, $\nu_A(x) = \nu_B(x)$, $\omega_A(x) = \omega_B(x)$ for any x ∈ X
As an illustration ,let us consider the following example.
Example 2.4. Assume that the universe of discourse U={x_1,x_2,x_3},where x_1 characterizes the capability, x2 characterizes the trustworthiness and x3 indicates the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in [0,1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components viz. the degree of goodness,
the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is an interval neutrosophic set (INS) of U, such that,
A = {< x_1,[0.3 0.4],[0.5 0.6],[0.4 0.5] >,< x_2, ,[0.1 0.2],[0.3 0.4],[0.6 0.7]>,< x_3, [0.2 0.4],[0.4 0.5],[0.4 0.6] >}, where the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.4 etc.

Definition 2.5.
Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).
As an illustration, let us consider the following example.
Example 2.6 .
Suppose that U is the set of houses under consideration, say U = {h_1, h_2, . . ., h_5}. Let E be the set of some attributes of such houses, say E = {e_1, e_2, . . ., e_8}, where e_1, e_2, . . ., e_8 stand for the attributes "beautiful", "costly", "in the green surroundings'", "moderate", respectively.
In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K,A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:
A={e_1,e_2,e_3,e_4,e_5};
K(e_1) = {h_2, h_3, h_5}, K(e_2) = {h_2, h_4}, K(e_3) = {h_1}, K(e_4) = U, K(e_5) = {h_3, h_5}.

Definition 2.7 .
Let U be an initial universe set and A ⊂ E be a set of parameters. Let IVNS(U) denotes the

set of all interval neutrosophic subsets of U. The collection (K,A) is termed to be the soft interval neutrosophic set over U, where F is a mapping given by K : A → IVNS(U).
The interval neutrosophic soft set defined over an universe is denoted by INSS.
To illustrate let us consider the following example:
Let U be the set of houses under consideration and E is the set of parameters (or qualities). Each parameter is a interval neutrosophic word or sentence involving interval neutrosophic words. Consider E = { beautiful, costly, in the green surroundings, moderate, expensive }. In this case, to define a interval neutrosophic soft set means to point out beautiful houses, costly houses, and so on. Suppose that, there are five houses in the universe U given by, U = $\{h_1,h_2,h_3,h_4,h_5\}$ and the set of parameters A = $\{e_1,e_2,e_3,e_4\}$, where each e_i is a specific criterion for houses:

e_1 stands for 'beautiful',
e_2 stands for 'costly',
e_3 stands for 'in the green surroundings',
e_4 stands for 'moderate',

Suppose that,

K(beautiful)={< h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,< h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, < h_3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,< h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,< h_5,[0.8, 0.4] ,[0.2 ,0.6],[0.3, 0.4] >}.K(costly)={< b_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,< h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, < h_3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,< h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,< h_5,[0.8, 0.4] ,[0.2 ,0.6],[0.3, 0.4] >}.
K(in the green surroundings)= {< h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,< b_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, < h_3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,< h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,< h_5,[0.8, 0.4] ,[0.2 ,0.6],[0.3, 0.4] >}.K(moderate)={< h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,< h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, < h_3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,< h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >,< h_5,[0.8, 0.4] ,[0.2 ,0.6],[0.3, 0.4] >}.

Definition 2.8.

Let U be an initial universe and (F,A) and (G,B) be two interval valued neutrosophic soft set . Then a relation between them is defined as a pair (H, AxB), where H is mapping given by H: AxB→IVNS(U). This is called an interval valued neutrosophic soft sets relation (IVNSS-relation for short).the collection of relations on interval valued neutrosophic soft sets on Ax Bover U is denoted by $\sigma_U(A\text{x } B)$.

Defintion 2.9. Let P, Q ∈ $\sigma_U(Ax\ B)$ and the ordre of their relational matrices are same. Then P ⊆ Q if H $(e_{j,}e_j)$ ⊆ J $(e_{j,}e_j)$ for $(e_{j,}e_j)$ ∈ A x B where P=(H, A x B) and Q = (J, A x B)
Example:
P

U	$(e_1\ ,e_2)$	$(e_1\ ,e_4)$	$(e_3\ ,e_2)$	$(e_3\ ,e_4)$
h_1	([0.2, 0.3],[0.2, 0.3],[0.4, 0.5])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])
h_2	([0.6, 0.8],[0.3, 0.4],[0.1,0.7])	([1, 1],[0, 0],[0, 0])	([0.1, 0.5],[0.4, 0.7],[0.5,0.6])	([0.1, 0.5],[0.4, 0.7],[0.5,0.6])
h_3	([0.3, 0.6],[0.2, 0.7],[0.3,0.4])	([0.4, 0.7],[0.1, 0.3],[0.2,0.4])	([1, 1],[0, 0],[0, 0])	([0.4, 0.7],[0.1, 0.3],[0.2,0.4])
h_4	([0.6, 0.7],[0.3, 0.4],[0.2,0.4])	([0.3, 0.4],[0.7, 0.9],[0.1,0.2])	([0.3, 0.4],[0.7, 0.9],[0.1,0.2])	([1, 1],[0, 0],[0, 0])

Q

U	$(e_1\ ,e_2)$	$(e_1\ ,e_4)$	$(e_3\ ,e_2)$	$(e_3\ ,e_4)$
h_1	([0.3, 0.4],[0, 0],[0, 0])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])	([0.4, 0.6],[0.7, 0.8],[0.1,0.4])
h_2	([0.6, 0.8],[0.3, 0.4],[0.1,0.7])	([1, 1],[0, 0],[0, 0])	([0.1, 0.5],[0.4, 0.7],[0.5,0.6])	([0.1, 0.5],[0.4, 0.7],[0.5,0.6])

h_3	([0.3, 0.6],[0.2, 0.7],[0.3,0.4])	([0.4, 0.7],[0.1, 0.3],[0.2,0.4])	([1, 1],[0, 0],[0, 0])	([0.4, 0.7],[0.1, 0.3],[0.2,0.4])
h_4	([0.6, 0.7],[0.3, 0.4],[0.2,0.4])	([0.3, 0.4],[0.7, 0.9],[0.1,0.2])	([0.3, 0.4],[0.7, 0.9],[0.1,0.2])	([1, 1],[0, 0],[0, 0])

Definition 2.10**.**
Let U be an initial universe and (F, A) and (G, B) be two interval valued neutrosophic soft sets. Then a null relation between them is denoted by 0_U and is defined as 0_U =(H_O , A xB) where H_O (e_i, e_j)={<h_k , [0, 0],[1, 1],[1, 1]>; h_k ∈ U} for (e_i, e_j) ∈ A xB.

Example. Consider the interval valued neutrosophic soft sets (F, A) and (G, B). Then a null relation between them is given by

U	(e_1, e_2)	(e_1, e_4)	(e_3, e_2)	(e_3, e_4)
h_1	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])
h_2	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])
h_3	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])
h_4	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])	([0, 0],[1, 1],[1, 1])

Remark. It can be easily seen that P ∪ 0_U =P and P ∩ 0_U =0_U for any P ∈ $\sigma_U(Ax\ B)$

Definition 2.11.
Let U be an initial universe and (F, A) and (G, B) be two interval valued neutrosophic soft sets. Then an absolute relation between them is denoted by I_U and is defined as I_U =(H_I , A xB) where H_I (e_i, e_j)={<h_k , [1, 1],[0, 0],[0, 0]>; h_k ∈ U} for (e_i, e_j) ∈ A xB.

U	(e_1, e_2)	(e_1, e_4)	(e_3, e_2)	(e_3, e_4)
h_1	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])
h_2	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])
h_3	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])
h_4	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])	([1, 1],[0, 0],[0, 0])

Defintion.2.12 Let P ∈ $\sigma_U(Ax\ B)$, P= (H, AxB) ,Q = (J, AxB) and the order of their relational matrices are same.Then we define

(i) P ⋃ Q= (H ∘J, AxB) where H∘ J :AxB →IVNS(U) is defined as (H ∘J)(e_i, e_j)= H(e_i, e_j) ∨ J(e_j, e_j) for (e_i, e_j) ∈ A x B, where ∨ denotes the interval valued neutrosophic union.

(ii) P ∩ Q= (H ■J, AxB) where H■J :AxB →IVNS(U) is defined as (H■J)(e_i, e_j)= H(e_i, e_j) ∧ J(e_i, e_j) for (e_j, e_j) ∈ A x B, where ∧ denotes the interval valued neutrosophic intersection

(iii) P^c= (∼H, AxB) , where ∼H :AxB →IVNS(U) is defined as ∼H(e_i, e_j)=[H(e_i, e_j)] ^{c} for (e_i, e_j) ∈ A x B, where c denotes the interval valued neutrosophic complement.

Defintion.2.13.
Let R be an equivalence relation on the universal set U. Then the pair (U, R) is called a Pawlak approximation space. An equivalence class of R containing x will be denoted by $[x]_R$. Now for X ⊆ U, the lower and upper approximation of X with respect to (U, R) are denoted by respectively R $_*$X and R* X and are defined by

R*X={x∈U: $[x]_R$ ⊆X},
R*X={ x∈U: $[x]_R \cap X \neq$}.
Now if R *X = R* X, then X is called definable; otherwise X is called a rough set.

3-Lower and upper soft interval valued neutrosophic rough approximations of an IVNSS-relation

Defntion 3.1 .Let R $\in \sigma_U(A\text{x} A)$ and R=(H, Ax A). Let Θ=(f,B) be an interval valued neutrosophic soft set over U and S= (U, Θ) be the soft interval valued neutrosophic approximation space. Then the lower and upper soft interval valued neutrosophic rough approximations of R with respect to S are denoted by $\mathrm{Lwr_S}$(R)and $\mathrm{Upr_S}$(R) respectively, which are IVNSS- relations over AxB in U given by:
$\mathrm{Lwr_S}$(R)= (J, A xB) and $\mathrm{Upr_S}$(R) =(K, A xB)
J($\boldsymbol{e_i}$ **,** $\boldsymbol{e_k}$**) ={<x, [**$\wedge_{e_j\in A}(\inf \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\inf \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x))$],
[$\wedge_{e_j\in A}(\inf \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x))$] :x ∈ U}.

K($\boldsymbol{e_i}$ **,** $\boldsymbol{e_k}$**) ={<x, [**$\wedge_{e_i\in A}(\inf \mu_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \mu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \mu_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \mu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\inf \nu_{\mathbf{H}(e_i,e_j)}(x) \wedge \inf \nu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \nu_{\mathbf{H}(e_i,e_j)}(x) \wedge \sup \nu_{\mathbf{f}(e_k)}(x))$],
[$\wedge_{e_j\in A}(\inf \omega_{\mathbf{H}(e_i,e_j)}(x) \wedge \inf \omega_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \omega_{\mathbf{H}(e_i,e_j)}(x) \wedge \sup \omega_{\mathbf{f}(e_k)}(x))$] :x ∈ U}.

For $e_i \in$ A , $e_K \in$ B

Theorem 3.2. Let be an interval valued neutrosophic soft over U and S = (U,Θ) be the soft approximation space. Let R_1 , $R_2 \in \sigma_U(A\text{x} A)$ and R_1=(G,Ax A) and R_2=(H,Ax A).Then

(i) $\mathrm{Lwr_S}(0_U)= 0_U$
(ii) $\mathrm{Lwr_S}(1_U)= 1_U$
(iii) $\boldsymbol{R_1} \subseteq \boldsymbol{R_2} \Longrightarrow \mathrm{Lwr_S}\ (\boldsymbol{R_1}) \subseteq \mathrm{Lwr_S}\ (\boldsymbol{R_2})$
(iv) $\boldsymbol{R_1} \subseteq \boldsymbol{R_2} \Longrightarrow \mathrm{Upr_S}\ (\boldsymbol{R_1}) \subseteq \mathrm{Upr_S}\ (\boldsymbol{R_2}$
(v) $\mathrm{Lwr_S}\ (\boldsymbol{R_1} \cap \boldsymbol{R_2}) \subseteq \mathrm{Lwr_S}\ (\boldsymbol{R_1}) \cap \mathrm{Lwr_S}\ (\boldsymbol{R_2})$
(vi) $\mathrm{Upr_S}\ (\boldsymbol{R_1} \cap \boldsymbol{R_2}) \subseteq \mathrm{Upr_S}\ (\boldsymbol{R_1}) \cap \mathrm{Upr_S}\ (\boldsymbol{R_2})$
(vii) $\mathrm{Lwr_S}\ (\boldsymbol{R_1}) \cup \mathrm{Lwr_S}\ (\boldsymbol{R_2}) \subseteq \mathrm{Lwr_S}\ (\boldsymbol{R_1} \cup \boldsymbol{R_2})$
(viii) $\mathrm{Upr_S}\ (\boldsymbol{R_1}) \cup \mathrm{Upr_S}\ (\boldsymbol{R_2}) \subseteq \mathrm{Upr_S}\ (\boldsymbol{R_1} \cup \boldsymbol{R_2})$

Proof. (i) –(iv) are straight forward.
Let Lwrs($R_1 \cap R_2$) =(S, Ax B).Then for ($\boldsymbol{e_i}$, $\boldsymbol{e_k}$) ∈ A xB , we have

S($\boldsymbol{e_i}$, $\boldsymbol{e_k}$) ={<x, **[**$\wedge_{e_j\in A}(\inf \mu_{\mathbf{G\circ H}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \mu_{\mathbf{G\circ H}(e_i,e_j)}(x)$
$\wedge \sup \mu_{\mathbf{f}(e_k)}(x))$],
[$\wedge_{e_j\in A}(\inf \nu_{\mathbf{G\circ H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \nu_{\mathbf{G\circ H}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\inf \omega_{\mathbf{G\circ H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \omega_{\mathbf{G\circ H}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x))$] :x ∈ U}

$=\{<x, [\wedge_{e_j\in A}(\min(\inf \mu_{G(e_i,e_j)}(x), \inf \mu_{H(e_i,e_j)}(x)) \wedge \inf \mu_{f(e_k)}(x))$
$, \wedge_{e_j\in A}(\min(\sup \mu_{G(e_i,e_j)}(x), \sup \mu_{H(e_i,e_j)}(x)) \wedge \sup \mu_{f(e_k)}(x))],$

$[\wedge_{e_j\in A}(\max(\inf \nu_{G(e_i,e_j)}(x), \inf \nu_{H(e_i,e_j)}(x)) \vee \inf \nu_{f(e_k)}(x))$
$, \wedge_{e_j\in A}(\max(\sup \nu_{G(e_i,e_j)}(x), \sup \nu_{H(e_i,e_j)}(x)) \vee \sup \nu_{f(e_k)}(x))],$

$[\wedge_{e_j\in A}(\max(\inf \omega_{G(e_i,e_j)}(x), \inf \omega_{H(e_i,e_j)}(x)) \vee \inf \omega_{f(e_k)}(x))$
$, \wedge_{e_j\in A}(\max(\sup \omega_{G(e_i,e_j)}(x), \sup \omega_{H(e_i,e_j)}(x)) \vee \sup \omega_{f(e_k)}(x))] : x \in U\}$

Also for $\mathrm{Lwr}_S(\boldsymbol{R_1}) \cap \mathrm{Lwr}_S(\boldsymbol{R_2})$ =(Z,A x B) and $(\boldsymbol{e_i}, \boldsymbol{e_K}) \in$ A xB ,we have ,
$Z(\boldsymbol{e_i}, \boldsymbol{e_K}) = \{<x, [\,\mathrm{Min}\,(\wedge_{e_j\in A}(\inf \mu_{G(e_i,e_j)}(x) \wedge \inf \mu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\inf \mu_{H(e_i,e_j)}(x) \wedge \inf \mu_{f(e_k)}(x))),\ \mathrm{Min}(\wedge_{e_j\in A}(\sup \mu_{G(e_i,e_j)}(x) \wedge \sup \mu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\sup \mu_{H(e_i,e_j)}(x) \wedge \sup \mu_{f(e_k)}(x)))],$
$[\mathrm{Max}\,(\wedge_{e_j\in A}(\inf \nu_{G(e_i,e_j)}(x) \vee \inf \nu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\inf \nu_{H(e_i,e_j)}(x) \vee \inf \nu_{f(e_k)}(x))),$
$\mathrm{Max}(\wedge_{e_j\in A}(\sup \nu_{G(e_i,e_j)}(x) \vee \sup \nu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\sup \nu_{H(e_i,e_j)}(x) \vee \sup \nu_{f(e_k)}(x)))],$

$[\mathrm{Max}\,(\wedge_{e_j\in A}(\inf \omega_{G(e_i,e_j)}(x) \vee \inf \omega_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\inf \omega_{H(e_i,e_j)}(x) \vee \inf \omega_{f(e_k)}(x))),$
$\mathrm{Max}(\wedge_{e_j\in A}(\sup \omega_{G(e_i,e_j)}(x) \vee \sup \omega_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\sup \omega_{H(e_i,e_j)}(x) \vee \sup \omega_{f(e_k)}(x)))] : x \in U\}$

Now since $\min(\inf \mu_{G(e_i,e_j)}, \inf \mu_{H(e_i,e_j)}(x)) \le \inf \mu_{G(e_i,e_j)}(x)$ and
$\min(\inf \mu_{G(e_i,e_j)}, \inf \mu_{H(e_i,e_j)}(x)) \le \inf \mu_{H(e_i,e_j)}(x)$ we have
$\wedge_{e_j\in A}(\min(\inf \mu_{G(e_i,e_j)}(x), \inf \mu_{H(e_i,e_j)}(x)) \wedge \inf \mu_{f(e_k)}(x)) \le \mathrm{Min}\,(\wedge_{e_j\in A}(\inf \mu_{G(e_i,e_j)}(x) \wedge \inf \mu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\inf \mu_{H(e_i,e_j)}(x) \wedge \inf \mu_{f(e_k)}(x)))$.
Similarly we can get
$\wedge_{e_j\in A}(\min(\sup \mu_{G(e_i,e_j)}(x), \sup \mu_{H(e_i,e_j)}(x)) \wedge \sup \mu_{f(e_k)}(x)) \le \mathrm{Min}\,(\wedge_{e_j\in A}(\sup \mu_{G(e_i,e_j)}(x) \wedge \sup \mu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\sup \mu_{H(e_i,e_j)}(x) \wedge \sup \mu_{f(e_k)}(x)))$.

Again as $\max(\inf \nu_{G(e_i,e_j)}, \inf \nu_{H(e_i,e_j)}(x)) \ge \inf \nu_{G(e_i,e_j)}(x)$,and
$\max(\inf \nu_{G(e_i,e_j)}, \inf \nu_{H(e_i,e_j)}(x)) \ge \inf \nu_{H(e_i,e_j)}(x)$

we have

$\wedge_{e_j\in A}(\max(\inf \nu_{G(e_i,e_j)}(x), \inf \nu_{H(e_i,e_j)}(x)) \vee \inf \nu_{f(e_k)}(x)) \ge \mathrm{Max}\,(\wedge_{e_j\in A}(\inf \nu_{G(e_i,e_j)}(x) \vee \inf \nu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\inf \nu_{H(e_i,e_j)}(x) \vee \inf \nu_{f(e_k)}(x)))$.
Similarly we can get

$\wedge_{e_j\in A}(\max(\sup \nu_{G(e_i,e_j)}(x), \sup \nu_{H(e_i,e_j)}(x)) \vee \sup \nu_{f(e_k)}(x)) \ge \mathrm{Max}\,(\wedge_{e_j\in A}(\sup \nu_{G(e_i,e_j)}(x) \vee \sup \nu_{f(e_k)}(x)),\ \wedge_{e_j\in A}(\sup \nu_{H(e_i,e_j)}(x) \vee \sup \nu_{f(e_k)}(x)))$.
Again as $\max(\inf \omega_{G(e_i,e_j)}, \inf \omega_{H(e_i,e_j)}(x)) \ge \inf \omega_{G(e_i,e_j)}(x)$,and
$\max(\inf \omega_{G(e_i,e_j)}, \inf \omega_{H(e_i,e_j)}(x)) \ge \inf \omega_{H(e_i,e_j)}(x)$

we have

$\wedge_{e_j\in A}(\max(\inf \omega_{\mathbf{G}(e_i,e_j)}(x), \inf \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \inf \omega_{\mathbf{f}(e_k)}(x)) \geq$ Max $(\wedge_{e_j\in A}(\inf \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\inf \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x)))$.

Similarly we can get

$\wedge_{e_j\in A}(\max(\sup \omega_{\mathbf{G}(e_i,e_j)}(x), \sup \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \sup \omega_{\mathbf{f}(e_k)}(x)) \geq$ Max $(\wedge_{e_j\in A}(\sup \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x)))$.

Consequently, $\mathrm{Lwr}_S\,(\boldsymbol{R_1} \cap \boldsymbol{R_2}) \subseteq \mathrm{Lwr}_S\,(\boldsymbol{R_1}) \cap \mathrm{Lwr}_S\,(\boldsymbol{R_2})$

(vi) Proof is similar to (v)

(vii) Let $\mathrm{Lwr}_S\,(\boldsymbol{R_1} \cup \boldsymbol{R_2})$ **=(S, A xB).Then for** $(\boldsymbol{e_i}, \boldsymbol{e_k}) \in$ A xB , we have

S$(\boldsymbol{e_i}, \boldsymbol{e_k})$={<x, **[**$\wedge_{e_j\in A}(\inf \mu_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\sup \mu_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\inf \nu_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\inf \nu_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\inf \omega_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\inf \omega_{\mathbf{G}\blacksquare\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$] :x ∈ U}

={<x, **[**$\wedge_{e_j\in A}(\max(\inf \mu_{\mathbf{G}(e_i,e_j)}(x), \inf \mu_{\mathbf{H}(e_i,e_j)}(x)) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\max(\sup \mu_{\mathbf{G}(e_i,e_j)}(x), \sup \mu_{\mathbf{H}(e_i,e_j)}(x)) \wedge \sup \mu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\min(\inf \nu_{\mathbf{G}(e_i,e_j)}(x), \inf \nu_{\mathbf{H}(e_i,e_j)}(x)) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\min(\sup \nu_{\mathbf{G}(e_i,e_j)}(x), \sup \nu_{\mathbf{H}(e_i,e_j)}(x)) \vee \sup \nu_{\mathbf{f}(e_k)}(x))$],

[$\wedge_{e_j\in A}(\min(\inf \omega_{\mathbf{G}(e_i,e_j)}(x), \inf \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$,$\wedge_{e_j\in A}(\min(\sup \omega_{\mathbf{G}(e_i,e_j)}(x), \sup \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \sup \omega_{\mathbf{f}(e_k)}(x))$] :x ∈ U}

Also for $\mathrm{Lwrs}(\boldsymbol{R_1}) \cup \mathrm{Lwrs}(\boldsymbol{R_2})$ = **(Z, AxB) and** $(\boldsymbol{e_i}, \boldsymbol{e_k}) \in$ A xB ,we have ,

Z $(\boldsymbol{e_i}, \boldsymbol{e_K})$= {<x, [Max $(\wedge_{e_j\in A}(\inf \mu_{\mathbf{G}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\inf \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x)))$, Max$(\wedge_{e_j\in A}(\sup \mu_{\mathbf{G}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x)))$] ,
[Min $(\wedge_{e_j\in A}(\inf \nu_{\mathbf{G}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\inf \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x)))$, Min$(\wedge_{e_j\in A}(\sup \nu_{\mathbf{G}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x)))$] ,

[Min $(\wedge_{e_j\in A}(\inf \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\inf \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x)))$, Min$(\wedge_{e_j\in A}(\sup \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x)))$] :x ∈ U}

Now since $\max(\inf \mu_{\mathbf{G}(e_i,e_j)}, \inf \mu_{\mathbf{H}(e_i,e_j)}(x)) \geq \inf \mu_{\mathbf{G}(e_i,e_j)}(x)$ and $\max(\inf \mu_{\mathbf{G}(e_i,e_j)}, \inf \mu_{\mathbf{H}(e_i,e_j)}(x)) \geq \inf \mu_{\mathbf{H}(e_i,e_j)}(x)$ we have

$\wedge_{e_j\in A}(\max(\inf \mu_{\mathbf{G}(e_i,e_j)}(x), \inf \mu_{\mathbf{H}(e_i,e_j)}(x)) \wedge \inf \mu_{\mathbf{f}(e_k)}(x)) \geq$ max $(\wedge_{e_j\in A}(\inf \mu_{\mathbf{G}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\inf \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \inf \mu_{\mathbf{f}(e_k)}(x)))$.

Similarly we can get

$\wedge_{e_j\in A}(\max(\sup \mu_{\mathbf{G}(e_i,e_j)}(x), \sup \mu_{\mathbf{H}(e_i,e_j)}(x)) \wedge \sup \mu_{\mathbf{f}(e_k)}(x)) \geq$ max$(\wedge_{e_j\in A}(\sup \mu_{\mathbf{G}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x))$, $\wedge_{e_j\in A}(\sup \mu_{\mathbf{H}(e_i,e_j)}(x) \wedge \sup \mu_{\mathbf{f}(e_k)}(x)))$.

Again as $\min(\inf \nu_{\mathbf{G}(e_i,e_j)}, \inf \nu_{\mathbf{H}(e_i,e_j)}(x)) \leq \inf \nu_{\mathbf{G}(e_i,e_j)}(x)$,and
$\min(\inf \nu_{\mathbf{G}(e_i,e_j)}, \inf \nu_{\mathbf{H}(e_i,e_j)}(x)) \leq \inf \nu_{\mathbf{H}(e_i,e_j)}(x)$

we have

$\wedge_{e_j \in A}(\min(\inf \nu_{\mathbf{G}(e_i,e_j)}(x), \inf \nu_{\mathbf{H}(e_i,e_j)}(x)) \vee \inf \nu_{\mathbf{f}(e_k)}(x)) \leq \text{Min}\,(\wedge_{e_j \in A}(\inf \nu_{\mathbf{G}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x)), \wedge_{e_j \in A}(\inf \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \nu_{\mathbf{f}(e_k)}(x)))$.
Similarly we can get

$\wedge_{e_j \in A}(\min(\sup \nu_{\mathbf{G}(e_i,e_j)}(x), \sup \nu_{\mathbf{H}(e_i,e_j)}(x)) \vee \sup \nu_{\mathbf{f}(e_k)}(x)) \leq \text{Min}\,(\wedge_{e_j \in A}(\sup \nu_{\mathbf{G}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x)), \wedge_{e_j \in A}(\sup \nu_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \nu_{\mathbf{f}(e_k)}(x)))$.

Again as $\min(\inf \omega_{\mathbf{G}(e_i,e_j)}, \inf \omega_{\mathbf{H}(e_i,e_j)}(x)) \leq \inf \omega_{\mathbf{G}(e_i,e_j)}(x)$,and
$\min(\inf \omega_{\mathbf{G}(e_i,e_j)}, \inf \omega_{\mathbf{H}(e_i,e_j)}(x)) \leq \inf \omega_{\mathbf{H}(e_i,e_j)}(x)$

we have

$\wedge_{e_j \in A}(\min(\inf \omega_{\mathbf{G}(e_i,e_j)}(x), \inf \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \inf \omega_{\mathbf{f}(e_k)}(x)) \leq \text{Min}\,(\wedge_{e_j \in A}(\inf \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x)), \wedge_{e_j \in A}(\inf \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \inf \omega_{\mathbf{f}(e_k)}(x)))$.
Similarly we can get

$\wedge_{e_j \in A}(\min(\sup \omega_{\mathbf{G}(e_i,e_j)}(x), \sup \omega_{\mathbf{H}(e_i,e_j)}(x)) \vee \sup \omega_{\mathbf{f}(e_k)}(x)) \leq \text{Min}\,(\wedge_{e_j \in A}(\sup \omega_{\mathbf{G}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x)), \wedge_{e_j \in A}(\sup \omega_{\mathbf{H}(e_i,e_j)}(x) \vee \sup \omega_{\mathbf{f}(e_k)}(x)))$.

Consequently $\text{Lwr}_S(\boldsymbol{R_1}) \cup \text{Lwr}_S(\boldsymbol{R_2}) \subseteq \text{Lwrs}(\boldsymbol{R_1} \cap \boldsymbol{R_2})$
(vii) Proof is similar to (vii).

Conclusion

In the present paper we extend the concept of Lower and upper soft interval valued intuitionstic fuzzy rough approximations of an IVIFSS-relation to the case IVNSS and investigated some of their properties.

Reference

[1] A. Mukherjee and S. B. Chakraborty , Relations on intuitionistic fuzzy soft sets and their applications, Proceedings of International Conference on Modeling and Simulation. C.I.T., Coimbator, 27-29 August, 2007, 671-677.
[2] D. Molodtsov, Soft set theory- results, Computers and Mathematics with Applications. 37 (1999) 19-31.
[3] Z. Pawlak , Rough Sets , Int. J. Comput. Inform. Sci. 11 (1982), 341-356.
[4] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965), 338-353

Cosine Similarity Measure of Interval Valued Neutrosophic Sets

Said Broumi, [1]Faculty of Arts and Humanities, Hay El Baraka Ben M'sik Casablanca B.P. 7951, Hassan II University Mohammedia-Casablanca , Morocco.
broumisaid78@gmail.com

Florentin Smarandache, [2]Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, Gallup , USA
fsmarandache@gmail.com

Abstract. In this paper, we define a new cosine similarity between two interval valued neutrosophic sets based on Bhattacharya's distance [19]. The notions of interval valued neutrosophic sets (IVNS, for short) will be used as vector representations in 3D-vector space. Based on the comparative analysis of the existing similarity measures for IVNS, we find that our proposed similarity measure is better and more robust. An illustrative example of the pattern recognition shows that the proposed method is simple and effective.

Keywords: Cosine Similarity Measure; Interval Valued Neutrosophic Sets .

I. INTRODUCTION

The neutrsophic sets (**NS**), pioneered by F. Smarandache [1], has been studied and applied in different fields, including decision making problems [2, 3, 4 , 5, 23], databases [6-7], medical diagnosis problems [8] , topology [9], control theory [10], Image processing [11,12,13] and so on. The character of NSs is that the values of its membership function, non-membership function and indeterminacy function are subsets. The concept of neutrosophic sets generalizes the following concepts: the classic set, fuzzy set, interval valued fuzzy set, Intuitionistic fuzzy set, and interval valued intuitionistic fuzzy set and so on, from a philosophical point of view. Therefore, Wang et al [14] introduced an instance of neutrosophic sets known as single valued neutrosophic sets (SVNS), which were motivated from the practical point of view and that can be used in real scientific and engineering application, and provide the set theoretic operators and various properties of SVNSs. However, in many applications, due to lack of knowledge or data about the problem domains, the decision information may be provided with intervals, instead of real numbers. Thus, interval valued neutrosophic sets (IVNS), as a useful generation of NS, was introduced by Wang et al [15], which is characterized by a membership function, non-membership function and an indeterminacy function, whose values are intervals rather than real numbers. Also, the interval valued neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exist in the real world. As an important extension of NS, IVNS has many applications in real life [16, 17].

Many methods have been proposed for measuring the degree of similarity between neutrosophic set, S.Broumi and F. Smarandache [22] proposed several definitions of similarity measure between NS. P.Majumdar and S.K.Samanta [21] suggested some new methods for measuring the similarity between neutrosophic set. However, there is a little investigation on the similarity measure of IVNS, although some method on measure of similarity between intervals valued neutrosophic sets have been presented in [5] recently.

Pattern recognition has been one of the fastest growing areas during the last two decades because of its usefulness and fascination. In pattern recognition, on the basis of the knowledge of known pattern, our aim is to classify the unknown pattern. Because of the complex and uncertain nature of the problems. The problem pattern recognition is given in the form of interval valued neutrosophic sets.

In this paper, motivated by the cosine similarity measure based on Bhattacharya's distance [19], we propose a new method called "cosine similarity measure for interval valued neutrosophic sets. Also the proposed and existing similarity measures are compared to show that the proposed similarity measure is more reasonable than some similarity measures. The proposed similarity measure is applied to pattern recognition

This paper is organized as follow: In section 2 some basic definitions of neutrosophic set, single valued neutrosophic set, interval valued neutrosophic set and cosine similarity measure are presented briefly. In section 3, cosine similarity measure of interval valued neutrosophic sets and their proofs are introduced. In section 4, results of the proposed similarity measure and existing similarity measures are compared .In section 5, the proposed similarity measure is applied to deal with the problem related to medical diagnosis. Finally we conclude the paper.

II. PRELIMINARIE

This section gives a brief overview of the concepts of neutrosophic set, single valued neutrosophic set, interval valued neutrosophic set and cosine similarity measure.

A. *Neutrosophic Sets*

1) *Definition [1]*

Let U be an universe of discourse then the neutrosophic set A is an object having the form

$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in U\}$, where the functions T, I, F : U→]−0, 1+[define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ U to the set A with the condition.

$$^{-}0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3^{+}. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. So instead of $]-0, 1^{+}[$ we need to take the interval [0, 1] for technical applications, because $]^{-}0, 1^{+}[$will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS, A_{NS} = {<x,$T_A(x)$, $I_A(x)$, $F_A(x)$> | x ∈ X}

And B_{NS} = {<x, $T_B(x)$, $I_B(x)$, $F_B(x)$> | x ∈ X} the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ If and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$

(2) $A_{NS} = B_{NS}$ if and only if, $T_A(x)=T_B(x)$, $I_A(x)=I_B(x)$, $F_A(x)=F_B(x)$

B. Single Valued Neutrosophic Sets

1) Definition [14]

Let X be a space of points (objects) with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, for each point x in X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.

When X is continuous, an SVNS A can be written as

$$A=\int_X \frac{<T_A(x), I_A(x), F_A(x),>}{x}, x \in X. \quad \textbf{(2)}$$

When X is discrete, an SVNS A can be written as

$$A= \sum_1^n \frac{<T_A(x_i), I_A(x_i), F_A(x_i),>}{x_i}, x_i \in X \quad \textbf{(3)}$$

For two SVNS, A_{SVNS} = {<x,$T_A(x)$, $I_A(x)$, $F_A(x)$> | x ∈ X}

And B_{SVNS} ={ <x, $T_B(x)$, $I_B(x)$, $F_B(x)$> | x ∈ X} the two relations are defined as follows:

(1) $A_{SVNS} \subseteq B_{SVNS}$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$

(2) $A_{SVNS} = B_{SVNS}$ if and only if , $T_A(x)=T_B(x)$, $I_A(x)=I_B(x)$, $F_A(x)=F_B(x)$ for any x ∈ X.

C. Interval Valued Neutrosophic Sets

1) Definition [15]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that$T_A(x), I_A(x), F_A(x) \in [0, 1]$.

For two IVNS, A_{IVNS} ={<x, $[T_A^L(x), T_A^U(x)]$, $[F_A^L(x), F_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$ > | x ∈ X}

And B_{IVNS}= {<x, $[T_B^L(x), T_B^U(x)]$, $[F_B^L(x), F_B^U(x)]$, $[I_B^L(x), I_B^U(x)]$>| x ∈ X} the two relations are defined as follows:

(1) $A_{IVNS} \subseteq B_{IVNS}$ if and only if $T_A^L(x) \leq T_B^L(x)$,$T_A^U(x) \leq T_B^U(x)$, $I_A^L(x) \geq I_B^L(x)$, $I_A^U(x) \geq I_B^U(x)$, $F_A^L(x) \geq F_B^L(x)$, $F_A^U(x) \geq F_B^U(x)$.

(2) $A_{IVNS} = B_{IVNS}$ if and only if , $T_A^L(x)=T_B^L(x)$, $T_A^U(x)=T_B^U(x)$, $I_A^L(x)=I_B^L(x)$, $I_A^U(x)=I_B^U(x)$, $F_A^L(x)=F_B^L(x)$, $F_A^U(x)=F_B^U(x)$ for any x ∈ X.

D. Cosine Similarity

1) Definition

Cosine similarity is a fundamental angle-based measure of similarity between two vectors of n dimensions using the cosine of the angle between them Candan and Sapino [20]. It measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them. Given two vectors of attributes, X = $(x_1, x_2, \ldots, x_n)$ and Y= $(y_1, y_2, \ldots, y_n)$, the cosine similarity, cosθ, is represented using a dot product and magnitude as

$$\mathbf{Cos\theta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \quad \textbf{(4)}$$

In vector space, a cosine similarity measure based on Bhattacharya's distance [19] between two fuzzy set $\mu_A(x_i)$ and $\mu_B(x_i)$ defined as follows:

$$C_F(A,B) = \frac{\sum_{i=1}^{n} \mu_A(x_i)\ \mu_B(x_i)}{\sqrt{\sum_{i=1}^{n} \mu_A(x_i)^2}\sqrt{\sum_{i=1}^{n} \mu_B(x_i)^2}} \quad \textbf{(5)}$$

The cosine of the angle between the vectors is within the values between 0 and 1.

In 2-D vector space, J. Ye [18] defines cosine similarity measure between IFS as follows:

$$C_{IFS}(A,B) = \frac{\sum_{i=1}^{n} \mu_A(x_i)\ \mu_B(x_i)+\nu_A(x_i)\nu_B(x_i)}{\sqrt{\sum_{i=1}^{n} \mu_A(x_i)^2+\nu_A(x_i)^2}\sqrt{\sum_{i=1}^{n} \mu_B(x_i)^2+\nu_B(x_i)^2}} \quad \textbf{(6)}$$

III . Cosine Similarity Measure for Interval Valued Neutrosophic Sets.

The existing cosine similarity measure is defined as the inner product of these two vectors divided by the product of their lengths. The cosine similarity measure is the cosine of the angle between the vector representations of the two fuzzy sets. The cosine similarity measure is a classic measure used in information retrieval and is the most widely reported measures of vector similarity [19]. However, to the best of our Knowledge, the existing cosine similarity measures does not deal with interval valued neutrosophic sets. Therefore, to overcome this limitation in this section, a new cosine similarity measure between interval valued neutrosophic sets is proposed in 3-D vector space.

Let A be an interval valued neutrosophic sets in a universe of discourse X ={x}, the interval valued neutrosophic sets is characterized by the interval of membership $[T_A^L(x), T_A^U(x)]$,the interval degree of non-membership $[F_A^L(x), F_A^U(x)]$ and the interval degree of indeterminacy $[I_A^L(x), I_A^U(x)]$ which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure for interval neutrosophic sets is proposed in an analogous manner to the cosine similarity measure proposed by J. Ye [18].

E. Definition

Assume that there are two interval neutrosophic sets A and B in X $=\{x_1, x_2, \ldots, x_n\}$ Based on the extension measure for fuzzy sets, a cosine similarity measure between interval valued neutrosophic sets A and B is proposed as follows

$$C_N(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A^L(x_i)+T_A^U(x_i))(T_B^L(x_i)+T_B^U(x_i))+(I_A^L(x_i)+I_A^U(x_i))(I_B^L(x_i)+I_B^U(x_i))+(F_A^L(x_i)+F_A^U(x_i))(F_B^L(x_i)+F_B^U(x_i))}{\sqrt{(T_A^L(x_i)+T_A^U(x_i))^2+(I_A^L(x_i)+I_A^U(x_i))^2+(F_A^L(x_i)+F_A^U(x_i))^2}\sqrt{(T_B^L(x_i)+T_B^U(x_i))^2+(I_B^L(x_i)+I_B^U(x_i))^2+(F_B^L(x_i)+F_B^U(x_i))^2}} \quad \textbf{(7)}$$

F. Proposition

Let A and B be interval valued neutrosophic sets then

- **i.** $\mathbf{0} \le C_N(A,B) \le \mathbf{1}$
- ii. $C_N(A,B) = C_N(B,A)$
- iii. $C_N(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$,
$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and
$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n

Proof : (i) it is obvious that the proposition is true according to the cosine valued

(ii) it is obvious that the proposition is true.

(iii) when A =B, there are
$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$,
$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and
$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…, n , So there is $C_N(A,B) = 1$

If we consider the weights of each element x_i, a weighted cosine similarity measure between IVNSs A and B is given as follows:

$$C_{WN}(A,B)=\frac{1}{n}\sum_{i=1}^{n} w_i\frac{(T_A^L(x_i)+T_A^U(x_i))(T_B^L(x_i)+T_B^U(x_i))+(I_A^L(x_i)+I_A^U(x_i))(I_B^L(x_i)+I_B^U(x_i))+(F_A^L(x_i)+F_A^U(x_i))(F_B^L(x_i)+F_B^U(x_i))}{\sqrt{(T_A^L(x_i)+T_A^U(x_i))^2+(I_A^L(x_i)+I_A^U(x_i))^2+(F_A^L(x_i)+F_A^U(x_i))^2}\sqrt{(T_B^L(x_i)+T_B^U(x_i))^2+(I_B^L(x_i)+I_B^U(x_i))^2+(F_B^L(x_i)+F_B^U(x_i))^2}} \quad \textbf{(8)}$$

Where $w_i \in [0.1]$,i =1,2,…,n ,and $\sum_{i=1}^{n} w_i =1$.

If we take $w_i = \frac{1}{n}$, i =1,2,…,n , then there is $C_{WN}(A,B) = C_N(A,B)$.

The weighted cosine similarity measure between two IVNSs A and B also satisfies the following properties:

- **i.** $\mathbf{0} \le C_{WN}(A,B) \le \mathbf{1}$
- ii. $C_{WN}(A,B) = C_{WN}(B,A)$
- iii. $C_{WN}(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$,
$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and
$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…, n

G. Proposition

Let the distance measure of the angle as **d(A,B)= arcos ($\boldsymbol{C_N(A,B)}$)** then it satisfies the following properties.

- **i.** d(A, B) ≥ 0, if $\mathbf{0} \le C_s(A,B) \le \mathbf{1}$
- **ii.** d(A, B) = **arcos(1) = 0, if** $\boldsymbol{C_N(A,B) = 1}$
- **iii.** d(A, B) = d(B, A) if $C_N(A,B) = C_N(B,A)$
- **iv.** d(A, C) ≤ d(A, B) + d(B, C) if A ⊆ B ⊆ C for any interval valued neutrosophic sets C.

Proof : obviously, d(A,B) satisfies the (i) – (iii). In the following , d(A,B) will be proved to satisfy the (iv).

For any C = { x_i}, A ⊆ B ⊆ C since Eq (7) is the sum of terms. Let us consider the distance measure of the angle between vectors:

d_i(A(x_i), B(x_i)) = **arcos($\boldsymbol{C_N(A(x_i), B(x_i))}$),**
d_i(B(x_i), C(x_i)) = **arcos($\boldsymbol{C_N(B(x_i), C(x_i))}$) and**
d_i(A(x_i), C(x_i)) = **arcos($\boldsymbol{C_N(A(x_i), C(x_i))}$),** for i=1, 2, .., n, where

$$C_N(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A^L(x_i)+T_A^U(x_i))(T_B^L(x_i)+T_B^U(x_i))+(I_A^L(x_i)+I_A^U(x_i))(I_B^L(x_i)+I_B^U(x_i))+(F_A^L(x_i)+F_A^U(x_i))(F_B^L(x_i)+F_B^U(x_i))}{\sqrt{(T_A^L(x_i)+T_A^U(x_i))^2+(I_A^L(x_i)+I_A^U(x_i))^2+(F_A^L(x_i)+F_A^U(x_i))^2}\sqrt{(T_B^L(x_i)+T_B^U(x_i))^2+(I_B^L(x_i)+I_B^U(x_i))^2+(F_B^L(x_i)+F_B^U(x_i))^2}} \quad \textbf{(9)}$$

$$C_N(B,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_B^L(x_i)+T_B^U(x_i))(T_C^L(x_i)+T_C^U(x_i))+(I_B^L(x_i)+I_B^U(x_i))(I_C^L(x_i)+I_C^U(x_i))+(F_B^L(x_i)+F_B^U(x_i))(F_C^L(x_i)+F_C^U(x_i))}{\sqrt{(T_B^L(x_i)+T_B^U(x_i))^2+(I_B^L(x_i)+I_B^U(x_i))^2+(F_B^L(x_i)+F_B^U(x_i))^2}\sqrt{(T_C^L(x_i)+T_C^U(x_i))^2+(I_C^L(x_i)+I_C^U(x_i))^2+(F_C^L(x_i)+F_C^U(x_i))^2}} \quad \textbf{(10)}$$

$$C_N(A,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A^L(x_i)+T_A^U(x_i))(T_C^L(x_i)+T_C^U(x_i))+(I_A^L(x_i)+I_A^U(x_i))(I_C^L(x_i)+I_C^U(x_i))+(F_A^L(x_i)+F_A^U(x_i))(F_C^L(x_i)+F_C^U(x_i))}{\sqrt{(T_A^L(x_i)+T_A^U(x_i))^2+(I_A^L(x_i)+I_A^U(x_i))^2+(F_A^L(x_i)+F_A^U(x_i))^2}\sqrt{(T_C^L(x_i)+T_C^U(x_i))^2+(I_C^L(x_i)+I_C^U(x_i))^2+(F_C^L(x_i)+F_C^U(x_i))^2}} \quad \textbf{(11)}$$

For three vectors

A(x_i) = <x_i, [$T_A^L(x_i)$, $T_A^U(x_i)$], [$I_A^L(x_i)$, $I_A^U(x_i)$], [$F_A^L(x_i)$, $F_A^U(x_i)$] >

B(x_i) = < x_i, [$T_B^L(x_i)$, $T_B^U(x_i)$],[$I_B^L(x_i)$, $I_B^U(x_i)$], [$F_B^L(x_i)$, $F_B^U(x_i)$] >

C(x_i) = <x_i, [$T_C^L(x_i)$, $T_C^U(x_i)$], [$I_C^L(x_i)$, $I_C^U(x_i)$] , [$F_C^L(x_i)$, $F_C^U(x_i)$] >, in a plane , If A (x_i) ⊆ B (x_i) ⊆ C (x_i) (I =1, 2,…, n), then it is obvious that d(A(x_i), C(x_i)) ≤ d(A(x_i), B(x_i)) + d(B(x_i), C(x_i)), According to the triangle inequality. Combining the inequality with Eq (7), we can obtain d(A, C) ≤ d(A, B) + d(B, C)

Thus, d(A,B) satisfies the property (iv). So we have finished the proof.

IV. Comparison of New Similarity Measure with the Existing Measures.

Let A and B be two interval neutrosophic set in the universe of discourse X={x_1, x_2,.,x_n}. For the cosine similarity and the existing similarity measures of interval valued neutrosophic sets introduced in [5, 21], they are listed as follows:

Pinaki's similarity I [21]

$$S_{PI}=\frac{\sum_{i=1}^{n}\{\min\{T_A(x_i),T_B(x_i)\}+\min\{I_A(x_i),I_B(x_i)\}+\min\{F_A(x_i),F_B(x_i)\}\}}{\sum_{i=1}^{n}\{\max\{T_A(x_i),T_B(x_i)\}+\max\{I_A(x_i),I_B(x_i)\}+\max\{F_A(x_i),F_B(x_i)\}\}} \quad \textbf{(12)}$$

Also ,P. Majumdar [21] proposed weighted similarity measure for neutrosophic set as follows:

$$S_{PII}=\frac{\sum_{i=1}^{n} w_i\,(T_A(x_i)\cdot T_B(x_i)+(I_A(x_i)\cdot I_B(x_i)+(F_A(x_i)\cdot F_B(x_i))}{\max(w_i\sqrt{T_A(x_i)^2+I_A(x_i)^2+F_A(x_i)^2}\,,\,w_i\sqrt{T_B(x_i)^2+I_B(x_i)^2+F_B(x_i)^2}))} \quad \textbf{(13)}$$

Where, S_{PI}, S_{PII} denotes Pinaki's similarity I and Pinaki's similarity II

Ye's similarity [5] is defined as the following:

$$S_{ye}(A,B) = 1-\frac{1}{6}\sum_{i=1}^{n} w_i\,[|\inf T_A(x_i)-\inf T_B(x_i)| + |\sup T_A(x_i)-\sup T_B(x_i)| + |\inf I_A(x_i)-\inf I_B(x_i)| + |\sup I_A(x_i)-\sup I_B(x_i)| + |\inf F_A(x_i)-\inf F_B(x_i)| + |\sup F_A(x_i)-\sup F_B(x_i)|] \quad (14)$$

Example 1:

Let A = {<x, (0.2, 0.2 0.3)>} and B= {<x, (0.5, 0.2 0.5)>}

Pinaki similarity I = 0.58

Pinaki similarity II (with w_i =1) = 0.29

Ye similarity (**w**ith w_i =1) = 0.83

Cosine similarity $\mathbf{C_N(A,B)}$ = 0.95

Example 2:

Let A= {<x, ([0.2, 0.3], [0.5, 0.6] ,[0.3, 0.5])>} and B{<x, ([0.5, 0.6], [0.3, 0.6] ,[0.5, 0.6])>}

Pinaki similarty I = NA

Pinaki similarty II(With w_i =1) = NA

Ye similarity (With w_i =1) =0.81

Cosine similarity $\mathbf{C_N(A,B)}$ = 0.92

On the basis of computational study. J.Ye [5] have shown that their measure is more effective and reasonable .A similar kind of study with the help of the proposed new measure based on the cosine similarity, has been done and it is found that the obtained results are more refined and accurate. It may be observed from the example 1 and 2 that the values of similarity measures are more closer to 1 with $\mathbf{C_N(A,B)}$,the proposed similarity measure. This implies that we may be more deterministic for correct diagnosis and proper treatment.

V. Application of Cosine Similarity Measure for Interval Valued Neutrosophic Numbers to Pattern Recognition

In order to demonstrate the application of the proposed cosine similarity measure for interval valued neutrosophic numbers to pattern recognition, we discuss the medical diagnosis problem as follows:

For example the patient reported temperature claiming that the patient has temperature between 0.5 and 0.7 severity /certainty, some how it is between 0.2 and 0.4 indeterminable if temperature is cause or the effect of his current disease. And it between 0.1 and 0.2 sure that temperature has no relation with his main disease. This piece of information about one patient and one symptom may be written as:

(patient , Temperature) = <[0.5, 0.7], [0.2 ,0.4], [0.1, 0.2]>

(patient , Headache) = < [0.2, 0.3], [0.3 ,0.5], [0.3, 0.6]>

(patient , Cough) = <[0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>

Then, P= {< x_1, [0.5, 0.7], [0.2 ,0.4], [0.1, 0.2] >, < x_2, [0.2, 0.3], [0.3, 0.5], [0.3, 0.6] > ,< x_3, [0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>}

And each diagnosis A_i (i=1, 2, 3) can also be represented by interval valued neutrosophic numbers with respect to all the symptoms as follows:

A_1= {< x_1, [0.5, 0.6], [0.2 ,0.3], [0.4, 0.5] >, < x_2 , [0.2 , 0.6], [0.3 ,0.4], [0.6 , 0.7]>,< x_3, [0.1, 0.2], [0.3 ,0.6], [0.7, 0.8]>}

A_2= {< x_1, [0.4, 0.5], [0.3, 0.4], [0.5, 0.6] >, < x_2 , [0.3, 0.5], [0.4 ,0.6], [0.2, 0.4]> ,< x_3 , [0.3, 0.6], [0.1, 0.2], [0.5, 0.6]>}

A_3= {< x_1, [0.6, 0.8], [0.4 ,0.5], [0.3, 0.4]>, <x_2 , [0.3, 0.7], [0.2, 0.3], [0.4, 0.7]> ,< x_3, [0.3, 0.5], [0.4, 0.7], [0.2, 0.6]>}

Our aim is to classify the pattern P in one of the classes A_1, A_2, A_3 .According to the recognition principle of maximum degree of similarity measure between interval valued neutrosophic numbers, the process of diagnosis A_k to patient P is derived according to

k = arg Max{ $\boldsymbol{C_N}$(A_i , $\boldsymbol{P}$))}

from the previous formula (7) , we can compute the cosine similarity between A_i (i=1, 2, 3) and P as follows;

$\boldsymbol{C_N}$(A_1 , $\boldsymbol{P}$)**=0.8988,** $\boldsymbol{C_N}$(A_2 , $\boldsymbol{P}$)**=0.8560,** $\boldsymbol{C_N}$(A_3 , $\boldsymbol{P}$) **=0.9654**

Then, we can assign the patient to diagnosis A_3 (Typoid) according to recognition of principal.

VI. Conclusions.

In this paper a cosine similarity measure between two and weighted interval valued neutrosophic sets is proposed. The results of the proposed similarity measure and existing similarity measure are compared. Finally, the proposed cosine similarity measure is applied to pattern recognition.

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[1] F. Smarandache, “A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic”. Rehoboth: American Research Press,(1998).

[2] A. Kharal, “A Neutrosophic Multicriteria Decision Making Method”,New Mathematics & Natural Computation, to appear in Nov 2013

[3] S. Broumi and F. Smarandache, “Intuitionistic Neutrosophic Soft Set”, Journal of Information and Computing Science, England, UK ,ISSN 1746-7659,Vol. 8, No. 2, (2013) 130-140.

[4] S. Broumi, “Generalized Neutrosophic Soft Set”, International Journal of Computer Science, Engineering and Information Technology (IJCSEIT), ISSN: 2231-3605, E-ISSN : 2231-3117, Vol.3, No.2, (2013) 17-30.

[5] J. Ye, ”Similarity measures between interval neutrosophic sets and their multicriteria decision-making method “Journal of Intelligent & Fuzzy Systems, DOI: 10.3233/IFS-120724 ,(2013),pp.

[6] M. Arora, R. Biswas, U.S.Pandy, “Neutrosophic Relational Database Decomposition”, International Journal of Advanced Computer Science and Applications, Vol. 2, No. 8, (2011) 121-125.

[7] M. Arora and R. Biswas,” Deployment of Neutrosophic technology to retrieve answers for queries posed in natural language”, in 3rdInternational Conference on Computer Science and Information Technology ICCSIT, IEEE catalog Number CFP1057E-art,Vol.3, ISBN: 978-1-4244-5540-9,(2010) 435-439.

[8] Ansari, Biswas, Aggarwal,”Proposal for Applicability of Neutrosophic Set Theory in Medical AI”, International Journal of Computer Applications (0975 – 8887),Vo 27– No.5, (2011) 5-11.

[9] F.G Lupiáñez, "On neutrosophic topology", Kybernetes, Vol. 37 Iss: 6,(2008), pp.797 - 800 ,Doi:10.1108/03684920810876990.

[10] S. Aggarwal, R. Biswas, A.Q.Ansari, ”Neutrosophic Modeling and Control”,978-1-4244-9034-/10 IEEE,(2010) 718-723.

[11] H. D. Cheng, Y Guo. “A new neutrosophic approach to image thresholding”. New Mathematics and Natural Computation, 4(3), (2008) 291–308.

[12] Y. Guo, H. D. Cheng “New neutrosophic approach to image segmentation”.Pattern Recognition, 42, (2009) 587–595.

[13] M.Zhang, L.Zhang, and H.D.Cheng. “A neutrosophic approach to image segmentation based on watershed method”. Signal Processing 5, 90 , (2010) 1510-1517.

[14] Wang, H, Smarandache, F, Zhang, Y. Q, Sunderraman,R, ”Single valued neutrosophic”,sets.Multispace and Multistructure, 4,(2010) 410–413.

[15] Wang, H, Smarandache, F, Zhang, Y.-Q. and Sunderraman, R, ”Interval Neutrosophic Sets and Logic: Theory and Applications in Computing”, Hexis, Phoenix, AZ, (2005).

[16] S. Broumi, F. Smarandache , “Correlation Coefficient of Interval Neutrosophic set”, Periodical of Applied Mechanics and Materials, Vol. 436, 2013, with the title Engineering Decisions and Scientific Research in Aerospace, Robotics, Biomechanics, Mechanical Engineering and Manufacturing; Proceedings of the International Conference ICMERA, Bucharest, October 2013.

[17] L. Peide, “Some power generalized aggregation operators based on the interval neutrosophic numbers and their application to decision making”, IEEE Transactions on Cybernetics,2013,12 page .

[18] J. Ye.“Cosine Similarity Measures for Intuitionistic Fuzzy Sets and Their Applications.”Mathematical and Computer Modelling 53, (2011) 91–97

[19] A. Bhattacharya, ” On a measure of divergence of two multinomial population”. Sanakhya Ser A 7 ,(1946) 401-406

[20] Candan, K. S. and M. L. Sapino, “Data management for multimedia retrieval”, Cambridge University Press,(2010).

[21] P. Majumdar, S.K. Samant,” On similarity and entropy of neutrosophic sets”, Journal of Intelligent and Fuzzy Systems,1064-1246(Print)-1875-8967(Online),(2013),DOI:10.3233/IFS-130810, IOSPress.

[22]S, Broumi and F, Smarandache, ” Several Similarity Measures of Neutrosophic Sets”, Neutrosophic Sets and Systems, An International Journal in Information Science and Engineering, December (2013).

[23] P. K. Maji, ” Neutrosophic Soft Set”, Annals of Fuzzy Mathematics and Informatics,Vol 5, No. 1,ISSN: 2093-9310 , ISSN:2287-623.

Reliability and Importance Discounting of Neutrosophic Masses

Florentin Smarandache, University of New Mexico, Gallup, NM 87301, USA

Abstract. In this paper, we introduce for the first time the discounting of a neutrosophic mass in terms of reliability and respectively the importance of the source.

We show that reliability and importance discounts commute when dealing with classical masses.

1. **Introduction.** Let $\Phi = \{\Phi_1, \Phi_2, \dots, \Phi_\mathrm{n}\}$ be the frame of discernment, where $n \geq 2$, and the set of **focal elements**:

$$F = \{A_1, A_2, \dots, A_m\}, \text{ for } m \geq 1, F \subset G^{\Phi}. \ (1)$$

Let $G^{\Phi} = (\Phi, \cup, \cap, \mathcal{C})$ be the **fusion space**.

A **neutrosophic mass** is defined as follows:

$$m_n: G \to [0,1]^3$$

for any $x \in G$, $m_n(x) = (t(x), i(x), f(x))$, (2)

where $t(x)$ = believe that x will occur (truth);

$i(x)$ = indeterminacy about occurence;

and $f(x)$ = believe that x will not occur (falsity).

Simply, we say in neutrosophic logic:

$t(x)$ = believe in x;

$i(x)$ = believe in neut(x)
[the neutral of x, i.e. neither x nor anti(x)];

and $f(x)$ = believe in anti(x) [the opposite of x].

Of course, $t(x)$, $i(x)$, $f(x) \in [0,1]$, and

$$\sum_{x \in G}[t(x) + i(x) + f(x)] = 1, (3)$$

while

$$m_n(\phi) = (0, 0, 0). \ (4)$$

It is possible that according to some parameters (or data) a source is able to predict the believe in a hypothesis x to occur, while according to other parameters (or other data) the same source may be able to find the believe in x not occuring, and upon a third category of parameters (or data) the source may find some indeterminacy (ambiguity) about hypothesis occurence.

An element $x \in G$ is called **focal** if

$$n_m(x) \neq (0, 0, 0), (5)$$

i.e. $t(x) > 0$ or $i(x) > 0$ or $f(x) > 0$.

Any **classical mass**:

$$m : G^{\phi} \rightarrow [0, 1] \ (6)$$

can be simply written as a neutrosophic mass as:

$$m(A) = (m(A), 0, 0). \ (7)$$

2. Discounting a Neutrosophic Mass due to Reliability of the Source.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be the reliability coefficient of the source, $\alpha \in [0,1]^3$.

Then, for any $x \in G^{\theta} \setminus \{\theta, I_t\}$,

where $\theta =$ the empty set

and $I_t =$ total ignorance,

$$m_n(x)_a = \big(\alpha_1 t(x), \alpha_2 i(x), \alpha_3 f(x)\big), \ (8)$$

and

$$m_n(I_t)_\alpha = \left(t(I_t) + (1-\alpha_1) \sum_{x\in G^\theta \setminus \{\phi, I_t\}} t(x)\,, \right.$$

$$\left. i(I_t) + (1-\alpha_2) \sum_{x\in G^\theta \setminus \{\phi, I_t\}} i(x), f(I_t) + (1-\alpha_3) \sum_{x\in G^\theta \setminus \{\phi, I_t\}} f(x) \right) \quad (9),$$

and, of course,

$$m_n(\phi)_\alpha = (0, 0, 0).$$

The missing mass of each element x, for $x \neq \phi, x \neq I_t$, is transferred to the mass of the total ignorance in the following way:

$t(x) - \alpha_1 t(x) = (1-\alpha_1) \cdot t(x)$ is transferred to $t(I_t)$, (10)

$i(x) - \alpha_2 i(x) = (1-\alpha_2) \cdot i(x)$ is transferred to $i(I_t)$, (11)

and $f(x) - \alpha_3 f(x) = (1-\alpha_3) \cdot f(x)$ is transferred to $f(I_t)$. (12)

3. Discounting a Neutrosophic Mass due to the Importance of the Source.

Let $\beta \in [0, 1]$ be the importance coefficient of the source. This discounting can be done in several ways.

a. For any $x \in G^\theta \setminus \{\phi\}$,

$$m_n(x)_{\beta_1} = \big(\beta \cdot t(x), i(x), f(x) + (1-\beta) \cdot t(x)\big), (13)$$

which means that $t(x)$, the believe in x, is diminished to $\beta \cdot t(x)$, and the missing mass, $t(x) - \beta \cdot t(x) = (1-\beta) \cdot t(x)$, is transferred to the believe in $anti(x)$.

b. Another way:

For any $x \in G^\theta \setminus \{\phi\}$,

$$m_n(x)_{\beta_2} = \big(\beta \cdot t(x), i(x) + (1-\beta) \cdot t(x), f(x)\big), (14)$$

which means that $t(x)$, the believe in x, is similarly diminished to $\beta \cdot t(x)$, and the missing mass $(1-\beta) \cdot t(x)$ is now transferred to the believe in $neut(x)$.

c. The third way is the most general, putting together the first and second ways.

For any $x \in G^{\theta} \setminus \{\phi\}$,

$$m_n(x)_{\beta_3} = \big(\beta \cdot t(x), i(x) + (1-\beta) \cdot t(x) \cdot \gamma, f(x) + (1-\beta) \cdot t(x) \cdot (1-\gamma)\big), (15)$$

where $\gamma \in [0,1]$ is a parameter that splits the missing mass $(1-\beta) \cdot t(x)$ a part to $i(x)$ and the other part to $f(x)$.

For $\gamma = 0$, one gets the first way of distribution, and when $\gamma = 1$, one gets the second way of distribution.

4. Discounting of Reliability and Importance of Sources in General Do Not Commute.

a. Reliability first, Importance second.

For any $x \in G^{\theta} \setminus \{\phi, I_t\}$, one has after reliability α discounting, where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3):$$

$$m_n(x)_{\alpha} = \big(\alpha_1 \cdot t(x), \alpha_2 \cdot t(x), \alpha_3 \cdot f(x)\big), (16)$$

and

$$m_n(I_t)_{\alpha} = \left(t(I_t) + (1-\alpha_1) \cdot \sum_{x \in G^{\theta} \setminus \{\phi, I_t\}} t(x), i(I_t) + (1-\alpha_2) \cdot \sum_{x \in G^{\theta} \setminus \{\phi, I_t\}} i(x), f(I_t) + (1-\alpha_3) \cdot \sum_{x \in G^{\theta} \setminus \{\phi, I_t\}} f(x) \right) \stackrel{\text{def}}{=} \left(T_{I_t}, I_{I_t}, F_{I_t}\right). \quad (17)$$

Now we do the importance β discounting method, the third importance discounting way which is the most general:

$$m_n(x)_{\alpha\beta_3} = \big(\beta\alpha_1 t(x), \alpha_2 i(x) + (1-\beta)\alpha_1 t(x)\gamma, \alpha_3 f(x) + (1-\beta)\alpha_1 t(x)(1-\gamma)\big) \quad (18)$$

and

$$m_n(I_t)_{\alpha\beta_3} = \left(\beta \cdot T_{I_t}, I_{I_t} + (1-\beta)T_{I_t} \cdot \gamma, F_{I_t} + (1-\beta)T_{I_t}(1-\gamma)\right). \ (19)$$

b. Importance first, Reliability second.

For any $x \in G^\theta \setminus \{\phi, I_t\}$, one has after importance β discounting (third way):

$$m_n(x)_{\beta_3} = \left(\beta \cdot t(x), i(x) + (1-\beta)t(x)\gamma, f(x) + (1-\beta)t(x)(1-\gamma)\right) \ (20)$$

and

$$m_n(I_t)_{\beta_3} = \left(\beta \cdot t(I_{I_t}), i(I_{I_t}) + (1-\beta)t(I_t)\gamma,\ f(I_t) + (1-\beta)t(I_t)(1-\gamma)\right). \ (21)$$

Now we do the reliability $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ discounting, and one gets:

$$m_n(x)_{\beta_3\alpha} = \left(\alpha_1 \cdot \beta \cdot t(x), \alpha_2 \cdot i(x) + \alpha_2(1-\beta)t(x)\gamma, \alpha_3 \cdot f(x) + \alpha_3 \cdot (1-\beta)t(x)(1-\gamma)\right) \ (22)$$

and

$$m_n(I_t)_{\beta_3\alpha} = \left(\alpha_1 \cdot \beta \cdot t(I_t), \alpha_2 \cdot i(I_t) + \alpha_2(1-\beta)t(I_t)\gamma, \alpha_3 \cdot f(I_t) + \alpha_3(1-\beta)t(I_t)(1-\gamma)\right). \ (23)$$

Remark.

We see that (a) and (b) are in general different, so reliability of sources does not commute with the importance of sources.

5. Particular Case when Reliability and Importance Discounting of Masses Commute.

Let's consider a classical mass

$$m: G^\theta \to [0,1] \ (24)$$

and the focal set $F \subset G^\theta$,

$$F = \{A_1, A_2, \dots, A_m\}, m \geq 1, \ (25)$$

and of course $m(A_i) > 0$, for $1 \leq i \leq m$.

Suppose $m(A_i) = a_i \in (0,1]$. (26)

a. Reliability first, Importance second.

Let $\alpha \in [0, 1]$ be the reliability coefficient of $m\ (\cdot)$.

For $x \in G^{\theta} \setminus \{\phi, I_t\}$, one has

$$m(x)_{\alpha} = \alpha \cdot m(x), (27)$$

$$\text{and } m(I_t) = \alpha \cdot m(I_t) + 1 - \alpha. (28)$$

Let $\beta \in [0, 1]$ be the importance coefficient of $m\ (\cdot)$.

Then, for $x \in G^{\theta} \setminus \{\phi, I_t\}$,

$$m(x)_{\alpha\beta} = \big(\beta\alpha m(x), \alpha m(x) - \beta\alpha m(x)\big) = \alpha \cdot m(x) \cdot (\beta, 1 - \beta), (29)$$

considering only two components: believe that x occurs and, respectively, believe that x does not occur.

Further on,

$$m(I_t)_{\alpha\beta} = (\beta\alpha m(I_t) + \beta - \beta\alpha, \alpha m(I_t) + 1 - \alpha - \beta\alpha m(I_t) - \beta + \beta\alpha) = [\alpha m(I_t) + 1 - \alpha] \cdot (\beta, 1 - \beta). (30)$$

b. Importance first, Reliability second.

For $x \in G^{\theta} \setminus \{\phi, I_t\}$, one has

$$m(x)_{\beta} = \big(\beta \cdot m(x), m(x) - \beta \cdot m(x)\big) = m(x) \cdot (\beta, 1 - \beta), (31)$$

$$\text{and } m(I_t)_{\beta} = \big(\beta m(I_t), m(I_t) - \beta m(I_t)\big) = m(I_t) \cdot (\beta, 1 - \beta). (32)$$

Then, for the reliability discounting scaler α one has:

$$m(x)_{\beta\alpha} = \alpha m(x)(\beta, 1 - \beta) = \big(\alpha m(x)\beta, \alpha m(x) - \alpha\beta m(m)\big)\ (33)$$

and $m(I_t)_{\beta\alpha} = \alpha \cdot m(I_t)(\beta, 1 - \beta) + (1 - \alpha)(\beta, 1 - \beta) = [\alpha m(I_t) + 1 - \alpha] \cdot (\beta, 1 - \beta) = (\alpha m(I_t)\beta, \alpha m(I_t) - \alpha m(I_t)\beta) + (\beta - \alpha\beta, 1 - \alpha - \beta + \alpha\beta) = (\alpha\beta m(I_t) + \beta - \alpha\beta, \alpha m(I_t) - \alpha\beta m(I_t) + 1 - \alpha - \beta - \alpha\beta)$. (34)

Hence (a) and (b) are equal in this case.

6. Examples.

1. Classical mass.

The following classical is given on $\theta = \{A, B\}$:

	A	B	AUB
m	0.4	0.5	0.1

(35)

Let $\alpha = 0.8$ be the reliability coefficient and $\beta = 0.7$ be the importance coefficient.

a. Reliability first, Importance second.

	A	B	AUB
m_α	0.32	0.40	0.28
$m_{\alpha\beta}$	(0.224, 0.096)	(0.280, 0.120)	(0.196, 0.084)

(36)

We have computed in the following way:

$$m_\alpha(A) = 0.8m(A) = 0.8(0.4) = 0.32, \quad (37)$$

$$m_\alpha(B) = 0.8m(B) = 0.8(0.5) = 0.40, \quad (38)$$

$$m_\alpha(AUB) = 0.8(\text{AUB}) + 1 - 0.8 = 0.8(0.1) + 0.2 = 0.28, \quad (39)$$

and

$$m_{\alpha\beta}(B) = \big(0.7m_\alpha(A), m_\alpha(A) - 0.7m_\alpha(A)\big) = \big(0.7(0.32), 0.32 - 0.7(0.32)\big) = (0.224, 0.096), \quad (40)$$

$$m_{\alpha\beta}(B) = \big(0.7m_\alpha(B), m_\alpha(B) - 0.7m_\alpha(B)\big) = \big(0.7(0.40), 0.40 - 0.7(0.40)\big) = (0.280, 0.120), \quad (41)$$

$$m_{\alpha\beta}(AUB) = \big(0.7m_\alpha(AUB), m_\alpha(AUB) - 0.7m_\alpha(AUB)\big) = \big(0.7(0.28), 0.28 - 0.7(0.28)\big) = (0.196, 0.084). \quad (42)$$

b. Importance first, Reliability second.

	A	B	AUB
m	0.4	0.5	0.1
m_β	(0.28, 0.12)	(0.35, 0.15)	(0.07, 0.03)
$m_{\beta\alpha}$	(0.224, 0.096	(0.280, 0.120)	(0.196, 0.084)

(43)

We computed in the following way:

$$m_\beta(A) = \big(\beta m(A), (1-\beta)m(A)\big) = \big(0.7(0.4), (1-0.7)(0.4)\big) = (0.280, 0.120), \quad (44)$$

$$m_\beta(B) = \big(\beta m(B), (1-\beta)m(B)\big) = \big(0.7(0.5), (1-0.7)(0.5)\big) = (0.35, 0.15), \quad (45)$$

$$m_\beta(AUB) = \big(\beta m(AUB), (1-\beta)m(AUB)\big) = \big(0.7(0.1), (1-0.1)(0.1)\big) = (0.07, 0.03), \quad (46)$$

and $m_{\beta\alpha}(A) = \alpha m_\beta(A) = 0.8(0.28, 0.12) = \big(0.8(0.28), 0.8(0.12)\big) = (0.224, 0.096), \quad (47)$

$$m_{\beta\alpha}(B) = \alpha m_\beta(B) = 0.8(0.35, 0.15) = \big(0.8(0.35), 0.8(0.15)\big) = (0.280, 0.120), \quad (48)$$

$$m_{\beta\alpha}(AUB) = \alpha m(AUB)(\beta, 1-\beta) + (1-\alpha)(\beta, 1-\beta) = 0.8(0.1)(0.7, 1-0.7) + (1-0.8)(0.7, 1-0.7) = 0.08(0.7, 0.3) + 0.2(0.7, 0.3) = (0.056, 0.024) + (0.140, 0.060) = (0.056 + 0.140, 0.024 + 0.060) = (0.196, 0.084). \quad (49)$$

Therefore reliability discount commutes with importance discount of sources when one has classical masses.

The result is interpreted this way: believe in A is 0.224 and believe in $nonA$ is 0.096, believe in B is 0.280 and believe in $nonB$ is 0.120, and believe in total ignorance AUB is 0.196, and believe in non-ignorance is 0.084.

7. Same Example with Different Redistribution of Masses Related to Importance of Sources.

Let's consider the third way of redistribution of masses related to importance coefficient of sources. $\beta = 0.7$, but $\gamma = 0.4$, which means that 40% of β is redistributed to $i(x)$ and 60% of β is redistributed to $f(x)$ for each $x \in G^{\theta} \setminus \{\phi\}$; and $\alpha = 0.8$.

a. Reliability first, Importance second.

	A	B	AUB
m	0.4	0.5	0.1
m_{α}	0.32	0.40	0.28
$m_{\alpha\beta}$	(0.2240, 0.0384, 0.0576)	(0.2800, 0.0480, 0.0720)	(0.1960, 0.0336, 0.0504).

(50)

We computed m_{α} in the same way.

But:

$$m_{\alpha\beta}(A) = \big(\beta \cdot m_{\alpha}(A), i_{\alpha}(A) + (1-\beta)m_{\alpha}(A)\cdot\gamma, f_{\alpha}(A) + (1-\beta)m_{\alpha}(A)(1-\gamma)\big) = \big(0.7(0.32), 0 + (1-0.7)(0.32)(0.4), 0 + (1-0.7)(0.32)(1-0.4)\big) = (0.2240, 0.0384, 0.0576). \quad (51)$$

Similarly for $m_{\alpha\beta}(B)$ and $m_{\alpha\beta}(AUB)$.

b. Importance first, Reliability second.

	A	B	AUB
m	0.4	0.5	0.1
m_{β}	(0.280, 0.048, 0.072)	(0.350, 0.060, 0.090)	(0.070, 0.012, 0.018)
$m_{\beta}\alpha$	(0.2240, 0.0384, 0.0576)	(0.2800, 0.0480, 0.0720)	(0.1960, 0.0336, 0.0504).

(52)

We computed $m_{\beta}(\cdot)$ in the following way:

$$m_{\beta}(A) = \big(\beta \cdot t(A), i(A) + (1-\beta)t(A)\cdot\gamma, f(A) + (1-\beta)t(A)(1-\gamma)\big) = \big(0.7(0.4), 0 + (1-0.7)(0.4)(0.4), 0 + (1-0.7)0.4(1-0.4)\big) = (0.280, 0.048, 0.072). \quad (53)$$

Similarly for $m_\beta(B)$ and $m_\beta(AUB)$.

To compute $m_{\beta\alpha}(\cdot)$, we take $\alpha_1 = \alpha_2 = \alpha_3 = 0.8$, (54)

in formulas (8) and (9).

$$m_{\beta\alpha}(A) = \alpha \cdot m_\beta(A) = 0.8(0.280, 0.048, 0.072)$$
$$= \big(0.8(0.280), 0.8(0.048), 0.8(0.072)\big)$$
$$= (0.2240, 0.0384, 0.0576). \ (55)$$

Similarly
$m_{\beta\alpha}(B) = 0.8(0.350, 0.060, 0.090) = (0.2800, 0.0480, 0.0720)$. (56)

For $m_{\beta\alpha}(AUB)$ we use formula (9):

$$m_{\beta\alpha}(AUB) = \big(t_\beta(AUB) + (1-\alpha)\big[t_\beta(A) + t_\beta(B)\big],\ i_\beta(AUB)$$
$$+ (1-\alpha)\big[i_\beta(A) + i_\beta(B)\big],$$
$$f_\beta(AUB) + (1-\alpha)\big[f_\beta(A) + f_\beta(B)\big]\big)$$
$$= (0.070 + (1-0.8)[0.280 + 0.350], 0.012$$
$$+ (1-0.8)[0.048 + 0.060], 0.018 + (1-0.8)[0.072 + 0.090])$$
$$= (0.1960, 0.0336, 0.0504).$$

Again, the reliability discount and importance discount commute.

8. Conclusion.

In this paper we have defined a new way of discounting a classical and neutrosophic mass with respect to its importance. We have also defined the discounting of a neutrosophic source with respect to its reliability.

In general, the reliability discount and importance discount do not commute. But if one uses classical masses, they commute (as in Examples 1 and 2).

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References.

1. F. Smarandache, J. Dezert, J.-M. Tacnet, *Fusion of Sources of Evidence with Different Importances and Reliabilities*, Fusion 2010 International Conference, Edinburgh, Scotland, 26-29 July, 2010.

2. Florentin Smarandache, *Neutrosophic Masses & Indeterminate Models. Applications to Information Fusion*, Proceedings of the International Conference on Advanced Mechatronic Systems [ICAMechS 2012], Tokyo, Japan, 18-21 September 2012.

Neutrosophic Code

Mumtaz ali[1], Florentin Smarandache[2*], Munazza Naz[3], and Muhammad Shabir[4]

[1,4]Department of Mathematics, Quaid-i-Azam University, Islamabad, 44000,Pakistan. E-mail: mumtazali770@yahoo.com, mshabirbhatti@yahoo.co.uk

[2]University of New Mexico, 705 Gurley Ave., Gallup, New Mexico 87301, USA E-mail: fsmarandache@gmail.com

[3]Department of Mathematical Sciences, Fatima Jinnah Women University, The Mall, Rawalpindi, 46000, Pakistan. E-mail: munazzanaz@yahoo.com

Abstract: The idea of neutrosophic code came into my mind at that time when i was reading the literature about linear codes and i saw that, if there is data transfremation between a sender and a reciever. They want to send 11 and 00 as codewords. They suppose 11 for true and 00 for false. When the sender sends the these two codewords and the error occures. As a result the reciever recieved 01 or 10 instead of 11 and 00. This story give a way to the neutrosophic codes and thus we introduced neutrosophic codes over finite field in this paper.

Introduction

Florentin Smarandache for the first time intorduced the concept of neutrosophy in 1995 which is basically a new branch of philosophy which actually studies the origion, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interal valued fuzzy set. Neutrosophic logic is used to overcome the problems of imperciseness, indeterminate, and inconsistentness of data etc. The theory of neutrosophy is so applicable to every field of agebra.W.B Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings, neutrosophic vectorspaces, neutrosophic groups, neutrosophic bigroups and neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrsosophic N-semigroups, neutrosophic loops, nuetrosophic biloops, and neutrosophic N-loops, and so on. Mumtaz ali $et\ al$. introduced nuetrosophic LA-semigoups.

Algebriac codes are used for data compression, cryptography, error correction and and for network coding. The theory of codes was first focused by Claude Shanon in 1948 and then gradually developed by time to time. There are many types of codes which is important to its algebriac structures such as Linear block codes, Hamming codes, BCH codes etc. The most common type of code is a linear code over the field F_q. There are also linear codes which are define over the finite rings. The linear codes over finite ring are initiated by Blake in a series of papers $[2]$, $[3]$, Spiegel $[4]$, $[5]$ and Forney et al. $[6]$. Klaus Huber defined codes over Gaussian integers.

In the further section, we intorduced the concept of neutrosophic code and establish the basic results of codes. We also developed the decoding procedures for neutrosophic codes and illustrate it with examples.

Basic concepts

Definition 1 Let A be a finite set of q symbols where $(q>1)$ and let $V=A^n$ be the set of n-tuples of elements of A where n is some positive integer greater than 1. In fact V is a vecotr space over A. Now let C be a non empty subset of V. Then C is called a q-ary code of length n over A.

Definition 2 Let F^n be a vector space over the field F, and $x, y \in F^n$ where $x = x_1x_2...x_n$, $y = y_1y_2...y_n$. The Hamming distance between the vectors x and y is denoted by $d(x, y)$, and is defined as $d(x,y) = |i : x_i \neq y_i|$.

Definition 3 The minimum distance of a code C is the smallest distance between any two distinct codewords in C which is denoted by $d(C)$, that is $d(C) = \min\{d(x,y) : x, y \in C, x \neq y\}$.

Definition 4 Let F be a finite field and n be a positive integer. Let C be a subspace of the vector space $V = F^n$. Then C is called a linear code over F.

Definition 5 The linear code C is called linear $[n,k]$-code if $\dim(C) = k$.

Definition 6 Let C be a linear $[n,k]$-code. Let G be a $k \times n$ matrix whose rows form basis of C. Then G is called generator matrix of the code C.

Definition 7 Let C be an $[n,k]$-code over F. Then the dual code of C is defined to be

$$C^{\perp} = \{y \in F^n : x \cdot y = 0, \forall x \in C\}$$

Definition 8 Let C be an $[n,k]$-code and let H be the generator matrix of the dual code $C^{\perp}$. Then H is called a parity-check matrix of the code C.

Definition 9 A code C is called self-orthogonal code if $C \subset C^{\perp}$.

Definition 10 Let C be a code over the field F and for every $x \in F^n$, the coset of C is defined to be

$$C = \{x + c : c \in C\}$$

Definition 11 Let C be a linear code over F. The coset leader of a given coset is defined to be the vector with least weight in that coset.

Definition 12 If a codeword x is transmitted and the vector y is received, then $e = y - x$ is called error vector. Therefore a coset leader is the error vector for each vector y lying in that coset.

Nuetrosophic code

Definition 13 Let C be a q-ary code of length n over the field F. Then the neutrosophic code is denoted by $N(C)$, and is defined to be

$$N(C) = \langle C \cup nI \rangle$$

where I is indeterminacy and called neutrosophic element with the property $I + I = 2I, I^2 = I$. For an integer n, $n + I$, nI are neutrosophic elements and $0.I = 0$. I^{-1}, the inverse of I is not defined and hence does not exist.

Example 1 Let $C = \{000,111\}$ be a binary code of length 3 over the field $F = Z_2$. Then the corresponding neutrosophic binary code is $N(C)$, where

$$N(C) = \langle C \cup III \rangle = \{000,111, III, I'I'I'\}$$

where $I' = (1 + I)$ which is called dual bit or partially determinate bit. In $1 + I$, 1 is determinate bit and I is indeterminate bit or multibit. This multibit is sometimes 0 and sometimes 1.

Theorem 1 A neutrosophic code $N(C)$ is a neutrosophic vector space over the field F.

Definition 14 Let $F^n(I)$ be a neutrosophic vector space over the field F and $x, y \in F^n(I)$, where

$x = x_1 x_2 ... x_n$, $y = y_1 y_2 ... y_n$. The Haming neutrosophic distance between the neutrosophic vectors x and y is denoted by $d_N(x, y)$, and is defined as $d_N(x, y) = \left| j : x_j \neq y_j \right|$.

Example 2 Let the neutrosophic code $N(C)$ be as in above example. Then the Hamming neutrosophic distance $d_N(x, y) = 3$, for all $x, y \in N(C)$.

Remark 1 The Hamming neutrosophic distance satisfies the three conditions of a distance function:

1) $d_N(x, y) = 0$ if and only if $x = y$.
2) $d_N(x, y) = d_N(y, x)$ for all $x, y \in F^n(I)$.
3) $d_N(x, z) \le d_N(x, y) + d_N(y, z)$ for all $x, y, z \in F^n(I)$.

Definition 15 The minimum neutrosophic distance of a neutrosophic code $N(C)$ is the smallest distance between any two distinct neutrosophic codewords in $N(C)$. We denote the minimum neutrosophic distance by $d_N(N(C))$. Equivalently $d_N(N(C)) = \min\{d_N(x, y) : x, y \in N(C), x \neq y\}$.

Example 3 Let $N(C)$ be a neutrosophic code over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \langle C \cup III \rangle = \{000, 111, III, I'I'I'\}$$

with $I' = (1 + I)$. Then the minimum neutrosophic distance $d_N(N(C)) = 3$.

Definition 16 Let C be a linear code of lenght n over the field F. Then $N(C) = \langle C \cup nI \rangle$ is called neutrosophic linear code over F.

Example 4 In example (3), $C = \{000, 111\}$ is a linear binary code over the field $F = Z_2 = \{0,1\}$, and the required neutrosophic linear code is

$$N(C) = \langle C \cup III \rangle = \{000, 111, III, I'I'I'\}$$

where $I' = (1 + I)$.

Theorem 2 Let C be a linear code and $N(C)$ be a neutrosohic linear code. If $\dim(C) = k$, then $\dim(N(C)) = k + 1$.

Definition 17 The linear neutrosophic code $N(C)$ is called linear $[n, k+1]$-neutrosophic code if $\dim(N(C)) = k + 1$.

Theorem 3 The linear $[n, k+1]$-neutrosophic code $N(C)$ contains the linear $[n, k]$-code C.

Theorem 4 The linear code C is a subspace of the neutrosophic code $N(C)$ over the field F.

Theorem 5 The linear code C is a sub-neutrosophic code of the neutrosophic code $N(C)$ over the field F.

Theorem 6 Let B be a basis of a linear code C of lenght n over the field F. Then $B \cup nI$ is the neutrosophic basis of the neutrosophic code $N(C)$, where I is the neutrosophic element.

Theorem 7 If the linear code C has a code rate $\frac{k}{n}$, then the neutrosophic linear code $N(C)$ has code rate

$\frac{k+1}{n}$.

Theorem 8 If the linear code C has redundancy $n-k$, then the neureosophic code has redundancy $n-(k+1)$.

Definition 18 Let $N(C)$ be a linear $[n,k+1]$-neutrosophic code. Let $N(G)$ be a $(k+1)\times n$ matrix whose rows form basis of $N(C)$. Then $N(G)$ is called neutrosophic generator matrix of the neutrosophic code $N(C)$.

Example 5 Let $N(C)$ be the linear neutrosophic code of length 3 over the field F, where

$$N(C)=\langle C\cup III\rangle=\{000,111,III,I^{'}I^{'}I^{'}\}$$

with $I^{'}=1+I$.

Let $N(G)$ be a $(k+1)\times n$ neutrosophic matrixe where

$$N(G)=\begin{bmatrix}1 & 1 & 1\\ I & I & I\end{bmatrix}_{2\times 3}$$

Then clearly $N(G)$ is a neutrosophic generator matrix of the neutrosophic code $N(C)$ because the rows of $N(G)$ generates the linear neutrosophic code $N(C)$. In fact the rows of $N(G)$ form a basis of $N(G)$.

Remark 2 The neutrosophic generator matrix of a neutrosophic code $N(C)$ is not unique.

We take the folowing example to prove the remark.

Example 6 Let $N(C)$ be a linear neutrosophic code of length 3 over the field F, where

$$N(C)=\langle C\cup III\rangle=\{000,111,III,I^{'}I^{'}I^{'}\}$$

with $I^{'}=1+I$. Then clearly $N(C)$ has three neutrosophic generator matrices which are follows.

$$N(G_1)=\begin{bmatrix}1 & 1 & 1\\ I & I & I\end{bmatrix}_{2\times 3}, N(G_2)=\begin{bmatrix}1 & 1 & 1\\ I^{'} & I^{'} & I^{'}\end{bmatrix}_{2\times 3},$$

$$N(G_3)=\begin{bmatrix}I & I & I\\ I^{'} & I^{'} & I^{'}\end{bmatrix}$$

Theorem 9 Let G be a generator matrix of a linear code C and $N(G)$ be the neutrosophic generator matrix of the neutrosophic linear code $N(C)$, then G is always contained in $N(G)$.

Definition 19 Let $N(C)$ be an $[n,k+1]$-neutrosophic code over F. Then the neutrosophic dual code of the neutrosophic code $N(C)$ is defined to be

$$N(C)^{\perp}=\{y\in F^{n}(I): xgy=0\forall x\in N(C)\}$$

Example 7 Let $N(C)$ be a linear neutrosophic code of length 2 over the neutrosophic field $F=Z_2$, where

$$N(C)=\{00,11,II,I'I'\}$$

with $I'=1+I$.
Since

$$F_2^{\,2}(I)=\begin{Bmatrix}00,01,0I,0I',10,11,1I,1I', \\ I0,I1,II,II',I'0,I'1,I'I,I'I'\end{Bmatrix}$$

where $I'=1+I$. Then the neutrosophic dual code $N(C)^{\perp}$ of the neutrosophic code $N(C)$ is given as follows,

$$N(C)^{\perp}=\{00,11,II,I'I'\}.$$

Theorem 10 If the neutrosophic code $N(C)$ has dimension $k+1$, then the neutrosophic dual code $N(C)^{\perp}$ has dimension $2n-(k+1)$.

Theorem 11 If $C^{\perp}$ is a dual code of the code C over F, then $N(C)^{\perp}$ is the neutrosophic dual code of the neutrosophic code $N(C)$ over the field F, where $N(C)^{\perp}=\langle C^{\perp}\cup nI\rangle$.

Definition 20 A neutrosophic code $N(C)$ is called self neutrosophic dual code if $N(C)=N(C)^{\perp}$.

Example 8 In example (7), the neutrosophic code $N(C)$ is self neutrosophic dual code because $N(C)=N(C)^{\perp}$.

Definition 21 Let $N(C)$ be an $[n,k+1]$-neutrosophic code and let $N(H)$ be the neutrosophic generator matrix of the neutrosophic dual code $N(C)^{\perp}$. Then $N(H)$ is called a neutrosophic parity-check matrix of the neutrosophi code $N(C)$.

Example 9 Let $N(C)$ be the linear neutrosophic code of length 3 over the field F, where

$$N(C)=\{000,111,III,I'I'I'\}$$

with $I'=1+I$. The neutrosophic generator matrix is

$$N(G)=\begin{bmatrix}1 & 1 & 1\\ I & I & I\end{bmatrix}_{2\times 3}$$

The neutrosophic dual code $N(C)^{\perp}$ of the above neutrosophic code $N(C)$ is as following,

$$N(C)^{\perp}=\begin{Bmatrix}000,011,101,110,1II',1I'I, \\ 0II,I0I,I1I',II0,I'0I',I'1I, \\ I'I1,I'I'0,II'1,0I'I'\end{Bmatrix}$$

The corresponding neutrosophic parity check matrix is given as follows,

$$N\left(H\right)=\begin{bmatrix}1 & 1 & 0\\ I & I^{'} & I\\ 0 & I & I\\ I^{'} & 0 & I\end{bmatrix}$$

Theorem 12 Let $N\left(C\right)$ be an $\left[n,k+1\right]$-neutrosophic code. Let $N\left(G\right)$ and $N\left(H\right)$ be neutrosophic generator matrix and neutrosophic parity check matrix of $N\left(C\right)$ respectively. Then

$$N\left(G\right)N\left(H\right)^{T}=0=N\left(H\right)N\left(G\right)^{T}$$

Remark 3 The neutrosophic parity check matrix $N\left(H\right)$ of a neutrosophic code $N\left(C\right)$ is not unique. To see the proof of this remark, we consider the following example.

Example 10 Let the neutrosophic code $N\left(C\right)$ be as in above example. The neutrosophic parity check matrices of $N\left(C\right)$ are given as follows.

$$N\left(H_1\right)=\begin{bmatrix}1 & 1 & 0\\ I & I^{'} & 1\\ 0 & I & I\\ I^{'} & 0 & I\end{bmatrix},N\left(H_2\right)=\begin{bmatrix}1 & 0 & 1\\ I & I^{'} & 1\\ I & 0 & I\\ I^{'} & 0 & I\end{bmatrix}$$

and so on.

Definition 22 A neutrosophic code $N\left(C\right)$ is called self-orthogonal neutrosophic code if $N\left(C\right)\subset N\left(C\right)^{\perp}$.

Example 11 Let $N\left(C\right)$ be the linear $\left[4,2\right]$-neutrosophic code of length 4 over the neutrosophic field $F=Z_2$, where

$$N\left(C\right)=\left\{0000,1111,IIII,I^{'}I^{'}I^{'}I^{'}\right\}$$

with $I^{'}=1+I$. The neutrosophic dual code $N\left(C\right)^{\perp}$ of $N\left(C\right)$ is following;

$$N\left(C\right)^{\perp}=\left\{\begin{matrix}0000,1100,1010,1001,0110,0101,0011,1111,IIII,\\ I^{'}I^{'}I^{'}I^{'},I^{'}I^{'}00,I^{'}0I^{'}0,I^{'}00I^{'},0I^{'}I^{'}0,0I^{'}0I^{'},00I^{'}I^{'},\ldots\end{matrix}\right\}$$

Then clearly $N\left(C\right)\subset N\left(C\right)^{\perp}$. Hence $N\left(C\right)$ is self-orthogonal neutrosophic code.

Theorem 13 If C is self-orthognal code then $N\left(C\right)$ is self-orthognal neutrosophic code.

Pseudo Neutrosophic Code

Definition 23 A linear $\left[n,k+1\right]$-neutrosophic code $N\left(C\right)$ is called pseudo linear $\left[n,k+1\right]$-neutrosophic code if it does not contain a proper subset of S which is a linear $\left[n,k\right]$-code.

Example 12 Let $N(C)$ be the linear neutrosophic code of length 3 over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \{000, 111, III, I'I'I'\}$$

with $I' = 1 + I$. Then clearly $N(C)$ is a pseudo linear $[3,2]$-neutrosophic code because it does not contain a proper subset of C which is a linear $[3,1]$-code.

Theorem 14 Every pseudo linear $[n,k+1]$-neutrosophic code $N(C)$ is a trivially a linear $[n,k+1]$-neutrosophic code but the converse is not true.
We prove the converse by taking the following example.

Example 13 Let $C = \{00, 01, 10, 11\}$ be a linear $[2,2]$-code and $N(C)$ be the corresponding linear $[2,3]$-neutrosophic code of length 2 over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \{00, 01, 10, 11, II, II', I'I, I'I'\}$$

with $I' = 1 + I$.
Then clearly $N(C)$ is not a pseudo linear $[2,3]$-neutrosophic code because $\{00,11\}$ is a proper subspace of C which is a code.

Strong or Pure Neutrosophic Code

Definition 24 A neutrosophic code $N(C)$ is called strong or pure neutrosophic code if $0 \neq y$ is neutrosophic codeword for all $y \in N(C)$.

Example 14 Let $N(C)$ be the linear neutrosophic code of length 3 over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \{000, III\}$$

Then clearly $N(C)$ is a strong or pure neutrosophic code over the field F.

Theorem 15 Every strong or pure neutrosophic code is trivially a neutrosophic code but the converse is not true.
For converse, let us see the following example.

Example 15 Let $N(C)$ be a neutrosophic code of length 3 over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \{000, 111, III, I'I'I'\}$$

with $I' = 1 + I$. Then clearly $N(C)$ is not a strong or pure neutrosophic code.

Theorem 16 There is one to one correspondence between the codes and strong or pure neutrosophic codes.

Theorem 17 A neutrosophic vector space have codes, neutrosophic codes, and strong or pure neutrosophic codes.

Decoding Algorithem

Definition 25 Let $N(C)$ be a neutrosophic code over the field F and for every $x \in F^n(I)$, the neutrosophic coset of $N(C)$ is defined to be

$$N(C)_C = \{x + c : c \in N(C)\}$$

Theorem 18 Let $N(C)$ be a linear neutrosophic code over the field F and let $y \in F^n(I)$. Then the

neutrosophic codeword x nearest to y is given by $x = y - e$, where e is the neutosophic vector of the least weight in the neutrosophic coset containing y.
if the neutrosophic coset containing y has more than one neutrosophic vector of least weight, then there are more than one neutosophic codewords nearest to y.

Definition 26 Let $N(C)$ be a linear neutrosophic code over the field F. The neutrosophic coset leader of a given neutrosophic coset $N(C)_C$ is defined to be the neutrosophic vector with least weight in that neutrosophic coset.

Theorem 19 Let F be a field and $F(I)$ be the corresponding neutrosophic vector space. If $|F| = q$, then $|F(I)| = q^2$.

Proof It is obvious.

Algorithem

Let $N(C)$ be an $[n, k+1]$-neutrosophic code over the field F_q with $|F_q(I)| = q^2$. As $F_q^{\,n}(I)$ has q^{2n} elements and so there are q^{k+1} elements in the coset of $N(C)$. Therefore the number of distinct cosets of $N(C)$ are $q^{2n-(k+1)}$. Let the coset leaders be denoted by $e_1, e_2, \ldots, e_N$, where $N = q^{2n-(k+1)}$. We also consider that the neutrosophic coset leaders are arranged in ascending order of weight; i.e $w(e_i) \le w(e_{i+1})$ for all i and consequently $e_1 = 0$ is the coset leader of $N(C) = 0 + N(C)$. Let $N(C) = \{c_1, c_2, \ldots, c_M\}$, where $M = q^{k+1}$ and $c_1 = 0$. The q^{2n} vectors can be arranged in an $N \times M$ table, which is given below. In this table the (i, j)-entry is the neutrosophic vector $e_i + c_j$. The elements of the coset $e_i + N(C)$ are in ith row with the coset leader e_i as the first entry. The neutrosophic code $N(C)$ will be placed on the top row. The corresponding table is termed as the standard neutrosophic array for the neutrosophic code $N(C)$.

$e_1 = 0 = c_1$	c_2	$\cdots$	c_j	$\cdots$	c_M
e_2	$e_2 + c_2$	$\cdots$	$e_2 + c_j$	$\cdots$	$e_2 + c_M$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
e_i	$e_i + c_2$	$\cdots$	$e_i + c_j$	$\cdots$	$e_i + c_M$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
e_N	$e_N + c_2$	$\cdots$	$e_N + c_J$	$\cdots$	$e_N + c_M$

Table 1.

For decoding, the standard neutrosophic array can be used as following:
Let us suppose that a neutrosophic vector $y \in F_q^{\,n}(I)$ is recieved and then look at the position of y in the table. In the table if y is the (i, j)-entry, then $y = e_i + c_j$ and also e_i is the neutrosophic vector of least weight in the neutrosophic coset and by theoem it follows that $x = y - e_i = c_j$. Hence the recieved neutrosophic

vector y is decoded as the neutrosophic codeword at the top of the column in which y appears.

Definition 27 If a neutrosophic codeword x is transmitted and the neutosophic vector y is received, then $e = y - x$ is called neutrosophic error vector. Therefore a neutrosophic coset leader is the neutrosophic error vector for each neutrosophic vector y lying in that neutrosophic coset.

Example 16 Let $F_2^2(I)$ be a neutrosophic vector space over the field $F = Z_2 = \{0,1\}$, where

$$F^2{}_2(I) = \left\{\begin{matrix} 00, 01, 0I, 0I', 10, 1I, 1I', \\ I0, I1, II, II', I'0, I'1, I'I, I'I' \end{matrix}\right\}$$

with $I' = 1 + I$. Let $N(C)$ be a neutrosophic code over the field $F = Z_2 = \{0,1\}$, where

$$N(C) = \{00, 11, II, I'I'\}$$

The following are the neutrosophic cosets of $N(C)$:

$$00 + N(C) = N(C),$$
$$01 + N(C) = \{01, 10, II', I'I\} = 10 + N(C),$$
$$0I + N(C) = \{0I, 1I', I0, I'1\} = I0 + N(C),$$
$$0I' + N(C) = \{0I', 1I, I1, I'0\} = I'0 + N(C)$$

The standard neutrosophic array table is given as under;

00	11	II	$I'I'$
01	10	II'	$I'I$
0I	$1I'$	I0	$I'1$
$0I'$	1I	I1	$I'0$

Table 2.

We want to decode the neutrosophic vector $1I'$. Since $1I'$ occures in the second coloumn and the top entry in that column is 11. Hence $1I'$ is decoded as the neutrosophic codeword 11.

Sydrome Decoding

Definition 28 Let $N(C)$ be an $[n, k+1]$-neutrosophic code over the field F with neutrosophic parity-check matrix $N(H)$. For any neutrosophic vector $y \in F_q^n(I)$, the syndrome of y is denoted by $S(y)$ and is defined to be

$$S(y) = yN(H)^T$$

Definition 29 A table with two columns showing the coset leaders e_i and the corresponding syndromes $S(e_i)$ is called syndrome table.

To decode a recieved neutrosophic vector y, compute the syndrome $S(y)$ and then find the neutrosophic coset leader e in the table for which $S(e)=S(y)$. Then y is decoded as $x = y-e$. This algorithem is known as syndrome decoding.

Example 17 Let $F_2^2(I)$ be a neutrosophic vector space over the field $F = Z_2 = \{0,1\}$, where

$$F^2{}_2(I)=\left\{\begin{matrix}00,01,0I,0I^{'},10,1I,1I^{'}, \\ I0,I1,II,II^{'},I^{'}0,I^{'}1,I^{'}I,I^{'}I^{'}\end{matrix}\right\}$$

with $I^{'}=1+I$.

Let $N(C)$ be a neutrosophic code over the field $F = Z_2 = \{0,1\}$, where

$$N(C)=\{00,11,II,I^{'}I^{'}\}$$

The neutosophic parity-check matrix of $N(C)$ is

$$N(H)=\begin{bmatrix}1 & 1\\ I & I\end{bmatrix}$$

First we find the neutrosophic cosets of $N(C)$:

$$00+N(C)=N(C),$$
$$01+N(C)=\{01,10,II^{'},I^{'}I\}=10+N(C),$$
$$0I+N(C)=\{0I,1I^{'},I0,I^{'}1\}=I0+N(C),$$
$$0I^{'}+N(C)=\{0I^{'},1I,I1,I^{'}0\}=I^{'}0+N(C)$$

These are the neutrosophic cosets of $N(C)$. After computing the syndrome $eN(H)^T$ for every coset leader, we get the following syndrome table.

Coset leaders	Syndrome
00	00
01	$1I$
$0I$	II
$0I^{'}$	$I^{'}0$

Table 3.

Let $y=10$, and we want do decode it. So

$$S(y) = yN(H)^T = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & I \\ 1 & I\end{bmatrix}$$
$$= \begin{bmatrix}1 & I\end{bmatrix}$$

Hence $S(10) = S(01)$ and thus y is decoded as the neutrosophic codeword

$$x = y - e = 10 - 01$$
$$x = 11$$

Thus we can find all the decoding neutrosophic codewords by this way.

Advantages and Betterness of Neutrosophic code

1) The code rate of a neutrosophic code is better than the ordinary code. Since the code rate of a neutrosophic code is $\frac{k+1}{n}$, while the code rate of ordinary code is $\frac{k}{n}$.
2) The redundancy is decrease in neutrosophic code as compared to ordinary codes. The redundancy of neutrosophic code is $n-(k+1)$, while the redundancy of ordinary code is $n-k$.
3) The number of neutrosophic codewords in neutrosophic code is more than the number of codewords in ordinary code.
4) The minimum distance remains same for both of neutrosophic codes as well as ordinary codes.

Conclusion

In this paper we initiated the concept of neutrosophic codes which are better codes than other type of codes. We first construct linear neutrosophic codes and gave illustrative examples. This neutrosophic algebriac structure is more rich for codes and also we found the containement of corresponding code in neutrosophic code. We also found new types of codes and these are pseudo neutrosophic codes and strong or pure neutrosophic codes. By the help of examples, we illustrated in a simple way. We established the basic results for neutosophic codes. At the end, we developed the decoding proceedures for neutrosophic codes.

1) Huber K., "Codes Over Gaussian Integers" IEEE Trans. Inform.Theory, vol. 40, pp. 207-216, jan. 1994.
2) Huber K., "Codes Over Eisenstein-Jacobi Integers," AMS, Contemp. Math., vol.158, pp. 165-179, 1994.
3) Huber K., "The MacWilliams theorem for two-dimensional modulo metrics" AAECC Springer Verlag, vol. 8, pp. 41-48, 1997.
4) Neto T.P. da N., "Lattice Constellations and Codes From Quadratic Number Fields"IEEE Trans. Inform. Theory, vol. 47, pp. 1514-1527, May 2001.
5) Fan Y. and Gao Y., "Codes Over Algebraic Integer Rings of Cyclotomic Fields"IEEE Trans. Inform. Theory, vol. 50, No. 1 jan. 2004.
6) Dresden G. and Dymacek W.M., "Finding Factors of Factor Rings Over The Gaussian Integers" The Mathematical Association of America, Monthly Aug-Sep.2005.
7) E R Berlekamp. Algebraic Coding Theory, Laguna Hills, CA: Aegan Park, 1984.
8) C. Martinez, R. Beivide and E. Gabidulin, Perfect Codes from Cayley Graphs over Lipschitz Integers, IEEE Trans. Inf. Theory, Vol. 55, No. 8, Aug. 2009.
9) I.F. Blake, Codes over certain rings, Information and Control 20 (1972) 396-404.
10) I.F. Blake, Codes over integer residue rings, Information and Control 29 (1975) 295-300.
11) J.H. Conway, N.J.A. Sloane, Self-dual codes over integers modulo 4, J. Combin. Theory 62 (1993) 30-45.
12) F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes (North-Holland, Amsterdam, 1978).
13) E. Spiegel, Codes over Zm, Information and Control 35 (1977) 48-51.
14) E. Spiegel, Codes over 7/m, revisited, Information and Control 37 (1978) 100-104.

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