On Neutrosophic Ideals of Neutrosophic BCI-Algebras

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Abstract

The objective of this paper is to introduce and study neutrosophic ideals of neutrosophic BCI-algebras. Elementary properties of neutrosophic ideals of neutrosophic BCI-algebras are presented.

Keywords

BCI/BCK-algebra, Neutrosophic set, Neutrosophic BCI/BCK- algebra, Neutrosophic ideal.

1 Introduction

BCI/BCK-algebras are generalizations of the concepts of set-theoretic difference and propositional calculi. These two classes of logical algebras were introduced by Imai and Iséki [8, 9] in 1966. It is well known that the class of MV-algebras introduced by Chang in [4] is a proper subclass of the class of BCK- algebras which in turn is a proper subclass of the class of BCI-algebras. Since the introduction of BCI/BCK-algebras, a great deal of literature has been produced, for example see [5, 9, 10, 11, 14]. For the general develop- ment of BCI/BCK-algebras, the ideal theory plays an important role. Hence much research emphasis has been on the ideal theory of BCI/BCK-algebras, see [3, 6, 7, 15].

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms, for all $x, y, z \in X$,

(1) $((x*y)*(x*z))*(z*y)=0$, (2) $(x*(x*y))*y=0$,

(3) $x*x=0$,

(4) $x*y=0$ and $y*x=0$ imply $x=y$.

Example 1.

(1) Every abelian group is a BCI-algebra, with group substraction and 0 the group identity.

(2) Consider $X = \{0, a, b\}$. Then, X with the following Cayley table is a BCI-algebra.

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y=0$.

If a BCI-algebra X satisfies $0*x=0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$,

(1) $(x*y)*z=(x*z)*y$,

(2) $((x*z)*(y*z))*(x*y)=0$,

(3) $x*0=x$,

(4) $x*y=0 \Rightarrow (x*z)*(y*z)=0, (z*y)*(z*x)=0$.

Example 2.

(1) The subsets of a set form a BCK-algebra, where $A*B$ is the difference $A \setminus B$.

(2) A Boolean algebra is a BCK-algebra, if $A*B$ is defined to be $A \wedge \neg B$ (A does not imply B).

A subset A of a BCI/BCK-algebra $(X, *, 0)$ is called a subalgebra of X if $x*y \in A$ for all $x, y * A$.

Let $(X, *, 0)$ be a BCI-algebra. A subset A of X is called an ideal of X if the following conditions hold:

(1) $0 \in A$.

(2) For all $x, y \in X$, $x*y \in A$ and $y*A$ implies that $x \in A$.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by

Smarandache as generalizations of fuzzy set and respectively intuitionistic fuzzy logic. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$, see [16, 17, 18]. Neutrosophic logic has wide applications in science, engineering, Information Technology, law, politics, economics, finance, econometrics, operations research, optimization theory, game theory and simulation etc.

The notion of neutrosophic algebraic structures was introduced by Kandasamy and Smarandache in 2006, see [12, 13]. Since then, several researchers have studied the concepts and a great deal of literature has been produced. For example, Agboola et al. in [1] continued the study of some types of neutrosophic algebraic structures. Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras in [2].

Let X be a nonempty set and let I be an indeterminate.

The set $X(I) = < X, I > = \{(x, yI) : x, y \in X\}$ is called a neutrosophic set generated by X and I. If + and . are ordinary addition and multiplication, I has the following properties:

1) $I + I + \cdots + I = nI$.
2) $I + (-I) = 0$.
3) $I.I. \cdots .I = I^{n} = I$ for all positive integer n.
4) $0.I = 0$.
5) I^{-1} is undefined and therefore does not exist.

If $*: X(I) \times X(I) \to X(I)$ is a binary operation defined on $X(I)$, then the couple $(X(I), *)$ is called a neutrosophic algebraic structure and it is named according the axioms satisfied by $*$. If $(X(I), *)$ and $(Y(I), *')$ are two neutrosophic algebraic structures, the mapping $\varphi : (X(I), *) \to (Y(I), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi((w,xI) * (y,zI)) = \varphi((w,xI)) *' \varphi((y,zI))$.

(2) $\varphi(I) = I \ \forall (w,xI), (y,zI) \in X(I)$.

We recall the definition of a neutrosophic group.

Definition 1.1.

Let $(G, *)$ be a group. Then, the neutrosophic group is generated by I and G under $*$ defined by $(< G, I >, *)$. The present paper is concerned with the introduction of the concept of neutrosophic of ideals of neutrosophic BCI-algebras. Some elementary properties of neutrosophic ideals of neutrosophic BCI-algebras are presented. First, we recall some basic concepts from [2].

Definition 1.2.

Let $(X, *, 0)$ be any BCI/BCK-algebra and let $X(I) = < X, I >$ be a set generated by X and I. The triple $(X(I), *, (0, 0))$ is called a neutrosophic BCI/BCK-algebra. If (a, bI) and (c, dI) are any two elements of $X(I)$ with $a, b, c, d \in X$, we define

$$(a,bI) * (c,dI) = (a * c, (a * d \wedge b * c \wedge b * d)I) \quad (1)$$

An element $x \in X$ is represented by $(x, 0) \in X(I)$ and $(0, 0)$ represents the constant element in $X(I)$. For all $(x, 0), (y, 0) \in X$, we define

$$(x, 0) * (y, 0) = (x * y, 0) = (x \wedge \neg y, 0), \quad (2)$$

where $\neg y$ is the negation of y in X.

Definition 1.3.

Let $(X, *, 0)$ be any BCI/BCK-algebra and let $X(I) =< X, I >$ be a set generated by X and I. The triple $(X(I), *, (0,))$ is called a neutrosophic BCI/BCK-algebra. If (a, bI) and (c, dI) are any two elements of $X(I)$ with $a, b, c, d \in X$, we define

$$(a,bI) * (c,dI) = (a * c, (a * d \wedge b * c \wedge b * d)I) \quad (3)$$

An element $x \in X$ is represented by $(x, 0) \in X(I)$ and $(0, 0)$ represents the constant element in $X(I)$. For all $(x, 0), (y, 0) \in X$, we define

$$(x, 0) * (y, 0) = (x * y, 0) = (x \wedge \neg y, 0) \quad (4)$$

where $\neg y$ is the negation of y in X.

Example 3.

Let $(X(I), +)$ be any commutative neutrosophic group.

For all $(a, bI), (c, dI) \in X(I)$ define

$$(a, bI) * (c, dI) = (a, bI) - (c, dI) = (a - c, (b - d)I). \quad (5)$$

Then, $(X(I), *, (0,0))$ is a neutrosophic BCI-algebra.

Theorem 1.4.

(1) Every neutrosophic BCK-algebra $(X(I), *, (0,0))$ is a neutrosophic BCI-algebra.

(2) Every neutrosophic BCK-algebra $(X(I), *, (0,0))$ is a BCI-algebra and not the converse.

(3) Let $(X(I), *, (0,0))$ be a neutrosophic BCK-algebra. Then, $(a, bI) * (0,0) = (a, bI)$ if and only if $a = b$.

Definition 1.5.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI/BCK-algebra. A non-empty subset $A(I)$ is called a neutrosophic subalgebra of $X(I)$ if the following conditions hold:

1) $(0,0) \in A(I)$.

2) $(a, bI) * (c, dI) \in A(I)$ for all $(a, bI), (c, dI) \in A(I)$.

3) $A(I)$ contains a proper subset which is a BCI/BCK-algebra.

If $A(I)$ does not contain a proper subset which is a BCI/BCK-algebra, then $A(I)$ is called a pseudo neutrosophic subalgebra of $X(I)$.

2 Main Results

Theorem 2.1.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI-algebra and let $X_\omega(I)$ be a subset of $X(I)$ defined by

$$X_\omega(I) = \{(x, xI) : x \in X\}. \tag{6}$$

Then, $X_\omega(I)$ is a neutrosophic subalgebra of $X(I)$.

Proof.

Obviously, $(0,0) \in X_\omega(I)$. Let $(x, xI), (y, yI) \in X_\omega(I)$ be arbitrary. Then, we have

$$(x, xI) * (y, yI) = (x * y, (x * y)I) = (x \wedge \neg y, (x \wedge \neg y)I) \in X_\omega(I).$$

Remark 1.

Since $(X_\omega(I), *, (0,0))$ is a neutrosophic subalgebra, then $X_\omega(I)$ is a neutrosophic BCI-algebra in its own right.

Example 4.

Let $X_\omega(I) = \{(0,0), (a, aI), (b, bI), (c, cI)\}$ be a set and let $*$ be a binary operation defined on $X_\omega(I)$ as shown in the Cayley table below:

$*$	$(0,0)$	(a, aI)	(b, bI)	(c, cI)
$(0,0)$	$(0,0)$	$(0,0)$	(c, cI)	(b, bI)
(a, aI)	(a, aI)	$(0,0)$	(c, cI)	(b, bI)
(b, bI)	(b, bI)	(b, bI)	$(0,0)$	(c, cI)
(c, cI)	(c, cI)	(c, cI)	(b, bI)	$(0, 0I)$

Then, $(X_\omega(I), *, (0,0))$ is a neutrosophic BCI-algebra.

Definition 2.2.

Let $(X(I), *, (0, 0))$ be a neutrosophic BCI-algebra. A subset $A(I)$ is called a neutrosophic ideal of $X(I)$ if the following conditions hold:

(1) $(0,0) \in A(I)$.

(2) For all $(a, bI), (c, dI) \in X(I)$, $(a, bI)*(c, dI) \in A(I)$ and $(c, dI) \in A(I)$ implies that $(a, bI) \in A(I)$.

Definition 2.3.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI-algebra and let $A(I)$ be a neutrosophic ideal of $X(I)$.

1) $A(I)$ is called a closed neutrosophic ideal of $X(I)$ if $A(I)$ is also a neutrosophic subalgebra of $X(I)$.
2) $A(I)$ is called a closed pseudo neutrosophic ideal of $X(I)$ if $A(I)$ is also a pseudo neutrosophic subalgebra of $X(I)$.

Lemma 2.4.

Let A(I) be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0, 0))$. Then,

1) $A(I) * A(I) = A(I)$.
2) $(a, bI) * A(I) = A(I)$ if and only if $(a, bI) \in A(I)$.

Definition 2.5.

Let $A_\omega(I)$ be a non-empty subset of $X_\omega(I)$.

1) $A_\omega(I)$ is called a neutrosophic α-ideal of $X_\omega(I)$ if the following conditions hold:
   a. $(0,0) \in A_\omega(I)$.

b. For all (x, xI), (y, yI), $(z, zI) \in X_\omega(I)$, $((x, xI) * (z, zI)) * ((y, yI) * (z, zI)) \in A_\omega(I)$ and $(y, yI) \in A_\omega(I)$ imply that $(x, xI) \in A_\omega(I)$.

2) $A_\omega(I)$ is called a neutrosophic β-ideal of $X_\omega(I)$ if the following conditions hold:

a. $(0, 0) \in A_\omega(I)$.

b. For all (x, xI), (y, yI), $(z, zI) \in X_\omega(I)$, $(x, xI) * ((y, yI) * (z, zI)) \in A_\omega(I)$ and $(y, yI) \in A_\omega(I)$ imply that $(x, xI) * (z, zI) \in A_\omega(I)$.

Theorem 2.6.

Every neutrosophic α-ideal of $X_\omega(I)$ is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Putting $(z, zI) = (0, 0)$ in Definition 2.5 (1-b), the result follows.

Theorem 2.7.

Every neutrosophic β-ideal of $X_\omega(I)$ is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Follows easily by putting $(z, zI) = (0, 0)$ in Definition 2.5 (2-b).

Theorem 2.8.

Let $A_\omega(I)$ and $B_\omega(I)$ be neutrosophic α-ideal and neutrosophic β-ideal of $X_\omega(I)$, respectively. Then,

$$A_\omega(I) * B_\omega(I) = \{(a, aI) * (b, bI) : (a, aI) \in A_\omega(I), (b, bI) \in B_\omega(I)\} \quad (7)$$

is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Follows easily from Theorems 2.6 and 2.7.

Theorem 2.9.

Let $(X(I), *, (0, 0))$ be a neutrosophic BCI-algebra and let $A(I)$ be a neutrosophic ideal of $X(I)$. For all (a, bI), $(c, dI) \in X(I)$, let τ be a relation defined on $X(I)$ by

$$(a, bI)\tau(c, dI) \iff (a, bI) * (c, dI), (c, dI) * (a, bI) \in A(I).$$

Then, τ is a congruence relation on $X(I)$.

Proof.

It is clear that τ is an equivalence relation on $X(I)$. For τ to be a congruence relation on $X(I)$, we must show that for all $(x, yI) \neq (0, 0)$ in $X(I)$, $(a, bI)\tau$

(c, dI) implies that $(a, bI) * (x, yI) \tau (c, dI) * (x, yI)$ and $(x, yI) * (a, bI) \tau (x, yI) * (c, dI)$. To this end, if $(a, bI) \tau (c, dI)$, then $(a, bI) * (c, dI), (c, dI) * (a, bI) \in A(I)$ that is $(a * c, (a * d \wedge b * c \wedge b * d)I), (c * a, (c * b \wedge d * a \wedge d * b)I) \in A(I)$ so that $(a \wedge \neg c, (a \wedge \neg d \wedge b \neg c)I), (c \wedge \neg a, (c \wedge \neg b \wedge d \wedge \neg a)I) \in A(I)$ and thus $a \wedge \neg c, a \wedge \neg d \wedge b \neg c, c \wedge \neg a, c \wedge \neg b \wedge d \wedge \neg a \in A$.

Now, let $(p, qI) = (a, bI) * (x, yI)$ and $(u, vI) = (c, dI) * (x, yI)$.

Then,

$$(p, qI) = (a * x, (a * y \wedge b * x \wedge b * y)I)$$

$$= (a \wedge \neg x, (a \wedge \neg y \wedge b \wedge \neg x)I).$$

$$(u, vI) = (c * x, (c * y \wedge d * x \wedge d * y)I)$$

$$= (c \wedge \neg x, (c \wedge \neg y \wedge d \wedge \neg x)I).$$

Now, we have

$$(p, qI) * (u, vI) = (p * u, (p * v \wedge q * u \wedge q * v)I)$$

$$= (p \wedge \neg u, (p \wedge \neg v \wedge q \wedge \neg u)I) \equiv (m, kI),$$

where

$$m = p \wedge \neg u = a \wedge \neg x \wedge (\neg c \vee x) = a \wedge \neg x \wedge \neg c \in A. \; k = p \wedge \neg v \wedge q \wedge \neg u$$

$$= a \wedge \neg x \wedge (\neg c \vee x) \wedge a \neg y \wedge b \wedge \neg x \wedge (\neg c \vee y \vee \neg d \vee x)$$

$$= a \wedge \neg x \wedge \neg c \wedge \neg y \wedge b \wedge (\neg c \vee y \vee \neg d \vee x)$$

$$= (a \wedge \neg c \wedge b \wedge \neg x \wedge \neg y) \vee (a \wedge \neg d \wedge b \wedge \neg c \wedge \neg x \wedge \neg y) \in A.$$

These show that $(m, kI) \in A(I)$ that is $((a, bI) * (x, yI)) * ((c, dI) * (x, yI)) \in A(I)$. Similarly, it can be shown that $(((x, yI) * (a, bI)) * ((x, yI) * (c, dI)) \in A(I)$. Thus, $(a, bI) * (x, yI) \tau (c, dI) * (x, yI)$ and $(x, yI) * (a, bI) \tau (x, yI) * (c, dI)$. Hence, τ is a congruence relation on $X(I)$.

For all $(a, bI) \in X(I)$, let $[(a, bI)]$ denote the congruence class containing (a, bI) and let $X(I)/A(I)$ denote the set of all congruence classes. For all $[(a, bI)], [(c, dI)] \in X(I)/A(I)$, we define

$$[(a, bI)] = (a, bI) * A(I), \tag{8}$$

$$[(0, 0)] = \{(0, 0) * (x, yI) : (x, yI) \in A(I)\}. \tag{9}$$

Theorem 2.10.

Let $A(I)$ be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0, 0))$. Then, $(X(I)/A(I), *, [(0, 0)])$ is a neutrosophic BCI-algebra.

Definition 2.11.

Let $(X(I), *, (0, 0))$ and $(X'(I), \circ, (0', 0'))$ be two neutrosophic BCI-algebras. A mapping $\varphi : X(I) \to X'(I)$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi((a,bI) * (c,dI)) = \varphi((a,bI)) \circ \varphi((c,dI)), \forall (a,bI),(c,dI) \in X(I)$.

(2) $\varphi((0,I)) = (0,I)$.

If in addition:

1) φ is injective, then φ is called a neutrosophic monomorphism.

2) φ is surjective, then φ is called a neutrosophic epimorphism.

3) φ is a bijection, then φ is called a neutrosophic isomorphism. A bijective neutrosophic homomorphism from $X(I)$ onto $X(I)$ is called a neutrosophic automorphism.

Definition 2.12.

Let $\varphi : X(I) \to Y(I)$ be a neutrosophic homomorphism of neutrosophic BCI-algebras.

(1) $Ker\varphi = \{(a,bI) \in X(I) : \varphi((a,bI)) = (0,0)\}$.

(2) $Im\varphi = \{\varphi((a,bI)) \in Y(I) : (a,bI) \in X(I)\}$.

Theorem 2.13.

Let $\varphi : X(I) \to Y(I)$ be a neutrosophic homomorphism of neutrosophic BCI-algebras. Then, $Ker\varphi$ is not a neutrosophic ideal of $X(I)$.

Proof.

The proof is straightforward since $(0,I) \in X(I)$ can not be mapped to $(0,0) \in Y(I)$.

Theorem 2.14.

Let $A(I)$ be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0,0))$. Then, the mapping $\varphi : X(I) \to X(I)/A(I)$ defined by

$$\varphi((x,yI)) = [(x,yI)], \quad \forall (x,yI) \in X(I)$$

is not a neutrosophic homomorphism.

Proof.

Straightforward since $\varphi((0,I)) = [(0,I)] \neq (0,I)$.

Theorem 2.15.

Let $\varphi : X_\omega(I) \to Y_\omega(I)$ be a neutrosophic homomorphism. Then, $Ker\varphi$ is a closed neutrosophic ideal of $X_\omega(I)$.

Proof.

Obvious.

Theorem 2.16.

Let $\varphi : X_\omega(I) \to Y_\omega(I)$ be a neutrosophic homomorphism and let $A[I]$ be a neutrosophic ideal of $X_\omega(I)$ such that $Ker\varphi \subseteq A[I]$. Then, $\varphi^{-1}(\varphi(A[I])) = A[I]$.

Proof.

Same as the classical case.

Theorem 2.17.

Let $A[I]$ be a neutrosophic ideal of $X_\omega(I)$. Then, the mapping $\varphi : X_\omega(I) \to X_\omega(I)/A[I]$ defined by

$$\varphi((x, xI)) = [(x, xI)], \quad \forall (x, xI) \in X_\omega(I)$$

is a neutrosophic homomorphism.

Proof.

The proof is straightforward.

Theorem 2.18.

Let $\varphi : X_\omega(I) \to Y_\omega(I)$ be a neutrosophic epimorphism. Then, $X_\omega(I)/Ker\varphi \cong Y_\omega(I)$.

Proof.

Same as the classical case.

3 References

[1] A.A.A. Agboola, Akwu A.O. and Y.T. Oyebo, *Neutrosophic groups and neutrosopic subgroups*, in “Int. J. of Math. Comb.”, 3 (2012), 1-9.

[2] A.A.A. Agboola and B. Davvaz, *Introduction to Neutrosophic BCI/BCK-algebras*, in “Int. J. Math. Math. Sci.”, Article ID 370267, (2015).

[3] M. Aslam, A.B. Thaheem, *On ideals of BCK-algebras*, in “Demonstratio Math.”, 27 (1994), No 3-4, 635-640.

[4] C. C. Chang, *Algebraic analysis of many valued logics*, in "Transactions of the American Mathematical society", 88(2) (1958), 467-490.
[5] Y.S. Huang, *BCI-algebra*, Science Press, Beijing, China, 2006.
[6] C.S. Hoo, *Closed ideals and p-semisimple BCI-algebras*, in "Math. Japon.", (1990), 1103-1112.
[7] C.S. Hoo, *A survey of BCK and BCI-algebras*, Southeast Asian Bull. Math., 12 (1988), 1-9.
[8] Y. Imai and K. Iséki, *On axiom systems of propositional calculi*, XIV, Proc. Japan Academy, 42 (1966), 19-22.
[9] K. Iséki, *An algebra related with a propositional calculus*, Japan Acad., 42 (1966), 26-29.
[10] K. Iséki, *On BCI-algebras*, in "Math. Sem. Notes", 8 (1980), 125-130.
[11] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, in "Math. Japonica", 23(1) (1978), 1-26.
[12] W.B. Kandasamy and F. Smarandache, *Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures*, Hexis, Phoenix, Arizona, 2006.
[13] W.B. Kandasamy and F. Smarandache, *Neutrosophic Rings*, Hexis, Phoenix, Arizona, 2006.
[14] J. Meng, Jie and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Seoul, 1994.
[15] J. Meng, *On ideals in BCK-algebras*, in "Math. Japon.", 40 (1994), 143-154.
[16] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability*, American Research Press, Rehoboth, 2003.
[17] F. Smarandache, *Introduction to Neutrosophic Statistics*, Sitech and Education Publishing, Romania, 2014.
[18] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, Am. Res. Press, Rehoboth, USA, 1998.