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Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies

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Abstract. In this paper, we make a short history about: the neutrosophic set, neutrosophic numerical components and neutrosophic literal components, neutrosophic numbers, and elementary neutrosophic algebraic structures. Afterwards, their generalizations to refined neutrosophic set, respectively refined neutrosophic numerical and literal components, then refined neutrosophic numbers and refined neutrosophic algebraic structures. The aim of this paper is to construct examples of splitting the literal indeterminacy (I) into literal sub-indeterminacies $(I_1, I_2, \dots I_r)$, and to define a multiplication law of these literal sub-indeterminacies in order to be able to build refined I-neutrosophic algebraic structures. Also, examples of splitting the numerical indeterminacy (i) into numerical sub-indeterminacies, and examples of splitting neutrosophic numerical components into neutrosophic numerical sub-components are given.

Keywords: neutrosophic set, elementary neutrosophic algebraic structures, neutrosophic numerical components, neutrosophic literal components, neutrosophic numbers, refined neutrosophic set, refined elementary neutrosophic algebraic structures, refined neutrosophic numerical components, refined neutrosophic literal components, refined neutrosophic numbers, literal indeterminacy, literal sub-indeterminacies, I-neutrosophic algebraic structures.

1 Introduction

Neutrosophic Set was introduced in 1995 by Florentin Smarandache, who coined the words „neutrosophy" and its derivative „neutrosophic". The first published work on neutrosophics was [1].

There exist two types of neutrosophic components: numerical and literal.

1.1 Neutrosophic Numerical Components

Of course, the *neutrosophic numerical components* (t, i, f) are numbers, intervals, or in general subsets of the unitary standard or nonstandard unit interval.

Let $\mathcal{U}$ be a universe of discourse, and M a set included in $\mathcal{U}$. A generic element x from $\mathcal{U}$ belongs to the set M in the following way: $x(t, i, f) \in M$, meaning that x's degree of membership/truth with respect to the set M is t, x's degree of indeterminacy with respect to the set M is i, and x's degree of non-membership/falsehood with respect to the set M is f, where t, i, f are independent standard subsets of the interval $[0, 1]$, or non-standard subsets of the non-standard interval $]^-0, 1^+[$, in the case when one needs to make distinctions between absolute and relative truth, indeterminacy, or falsehood.

Many papers and books have been published for the cases when t, i, f were single values, or t, i, f were intervals.

1.2 Neutrosophic Literal Components

In 2003, W.B. Vasantha Kandasamy and Florentin Smarandache [4] introduced the *literal indeterminacy* "I", such that $I^2 = I$ (whence $I^n = I$ for $n \geq 1$, n integer). They extended this to *neutrosophic numbers* of the form: $a + bI$, where a, b are real or complex numbers, and

$$(a_1 + b_1 I) + (a_2 + b_2 I) = (a_1 + a_2) + (b_1 + b_2)I \quad (1)$$

and developed many I neutrosophic algebraic structures based on sets formed of neutrosophic numbers.

The neutrosophic number $N = a + bI$ can be interpreted as: "a" represents the determinate part of numbers N, while "bI" the indeterminate part of number N.

1.3 Notations

In order to make distinctions between the numerical and literal neutrosophic components, we start denoting the *numerical indeterminacy* by lower case letter "i" (whence consequently similar notations for *numerical truth* "t", and for *numerical falsehood* "f"), and *literal indeterminacy* by upper case letter "I" (whence consequently similar notations for *literal truth* "T", and for *literal falsehood* "F").

2 Refined Neutrosophic Components

In 2013, F. Smarandache [3] introduced the refined neutrosophic components in the following way: the neutrosophic numerical components t, i, f can be refined (split) into respectively the following refined neutrosophic numerical components:

$$\langle t_1, t_2, \dots t_p;\ i_1, i_2, \dots i_r;\ f_1, f_2, \dots f_s; \rangle, \qquad (2)$$

where p, r, s are integers ≥ 1 and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and t_j represents types of numeral truths, i_k represents types of numeral indeterminacies, and f_l represents types of numeral falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

t_j, i_k, f_l are called numerical subcomponents, or respectively numerical sub-truths, numerical sub-indeterminacies, and numerical sub-falsehoods.

Similarly, the neutrosophic literal components T, I, F can be refined (split) into respectively the following neutrosophic literal components:

$$\langle T_1, T_2, \dots T_p;\ I_1, I_2, \dots I_r;\ F_1, F_2, \dots F_s; \rangle, \qquad (3)$$

where p, r, s are integers ≥ 1 too, and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and similarly t_j represents types of literal truths, i_k represents types of literal indeterminacies, and f_l represents types of literal falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

t_j, i_k, f_l are called literal subcomponents, or respectively literal sub-truths, literal sub-indeterminacies, and literal sub-falsehoods.

Let consider a *simple example*.

Suppose that a country C is composed of two districts D_1 and D_2, and a candidate John Doe competes for the position of president of country C. Per whole country, NL (Joe Doe) $= (0.6, 0.1, 0.3)$, meaning that 60% of people voted for him, 10% of people were indeterminate or neutral – i.e. didn't vote, or gave a black vote, or a blank vote –, and 30% of people voted against him, where NL means the neutrosophic logic values.

But a political analyst does some research to find out what happened to each district separately. So, he does a refinement and he gets:

$$\begin{pmatrix} 0.40 & 0.20 \\ t_1 & t_2 \end{pmatrix}; \begin{pmatrix} 0.08 & 0.02 \\ i_1 & i_2 \end{pmatrix}; \begin{pmatrix} 0.05 & 0.25 \\ f_1 & f_2 \end{pmatrix}$$

which means that 40% of people that voted for Joe Doe were from district D_1, and 20% of people that voted for Joe Doe were from district D_2; similarly, 8% from D_1 and 2% from D_2 were indeterminate (neutral), and 5% from D_1 and 25% from D_2 were against Joe Doe.

It is possible, in the same example, to refine (split) it in a different way, considering another criterion, namely: what percentage of people did not vote (i_1), what percentage of people gave a blank vote – cutting all candidates on the ballot – (i_2), and what percentage of people gave a blank vote – not selecting any candidate on the ballot (i_3). Thus, the numerical indeterminacy (i) is refined into i_1, i_2, and i_3:

$$\begin{pmatrix} 0.60 \\ t \end{pmatrix}; \begin{pmatrix} 0.05 & 0.04 & 0.01 \\ i_1 & i_2 & i_3 \end{pmatrix}; \begin{pmatrix} 0.30 \\ f \end{pmatrix}$$

3 Refined Neutrosophic Numbers

In 2015, F. Smarandache [6] introduced the refined literal indeterminacy (I) refined as $I_1, I_2, \dots, I_r$, with $r \geq 2$, where I_k, for $k = 1, 2, \dots, r$, represents types of literal indeterminacies. A refined neutrosophic number has the general form:

$$N_r = a + b_1 I_1 + b_2 I_2 + \cdots + b_r I_r, \qquad (4)$$

where $a, b_1, b_2, \dots, b_r$ are real numbers, and in this case N_r is called a *refined neutrosophic real number*; if at least one of $a, b_1, b_2, \dots, b_r$ is a complex number (of the form $\alpha + \beta\sqrt{-1}$ with $\beta \neq 0$), then N_r is called a *refined neutrosophic complex number*.

Then F. Smarandache [6] defined the *refined I-neutrosophic algebraic structures* as algebraic structures based on sets of refined neutrosophic numbers.

Soon after this definition, Dr. Adesina Agboola wrote a paper on refined neutrosophic algebraic structures [7].

They were called "I-neutrosophic" because the refinement is done with respect to the literal indeterminacy (I), in order to distinguish them from the refined (t, i, f)-neutrosophic algebraic structures, where "(t, i, f)-neutrosophic" is referred to as refinement of the neutrosophic numerical components t, i, f.

Said Broumi and F. Smarandache published a paper [8] on refined neutrosophic numerical components.

4 Neutrosophic Graphs

We now introduce for the first time the general definition of a *neutrosophic graph*, which is a (direct or not direct) graph that has some indeterminacy with respect to its edges, or with respect to its vertexes, or with respect to both (edges and vertexes simultaneously). We have four main categories of neutrosophic graphs:

1) The (t, i, f)-*Edge Neutrosophic Graph.*

In such a graph, the connection between two vertexes A and B, represented by edge AB:

A------------------------------B

has the neutroosphic value of (t, i, f).

2) I-*Edge Neutrosophic Graph.*

This one was introduced in 2003 in the book "Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps", by Dr. Vasantha Kandasamy and F. Smarandache, that used a different approach for the edge:

A------------------------------B

which can be just I = literal indeterminacy of the edge, with $I^2 = I$ (as in I-Neutrosophic algebraic structures). Therefore, simply we say that the connection between vertex A and vertex B is indeterminate.

3) *I-Vertex Neutrosophic Graph.*

Or a literal indeterminate vertex, meaning we do not know what this vertex represents.

4) *(t, i, f)-Vertex Neutrosophic Graph.*

We can also have neutrosophic vertex, for example vertex A only partially belongs to the graph (t), indeter-minate appurtenance to the graph (i), does not partially belong to the graph (f), we can say $A(t, i, f)$.

And combinations of any two, three, or four of the above four possibilities of neutrosophic graphs.

If (t, i, f) or the literal I are refined, we can get corresponding refined neurosophic graphs.

7 Example of Refined Indeterminacy and Multiplication Law of Sub-Indeterminacies

Discussing the development of Refined I-Neutrosophic Structures with Dr. W.B. Vasantha Kandasamy, Dr. A.A.A. Agboola, Mumtaz Ali, and Said Broumi, a question has arised: if I is refined into $I_1, I_2, \ldots, I_r$, with $r \geq 2$, how to define (or compute) $I_j * I_k$, for $j \neq k$?

We need to design a Sub-Indeterminacy $*$ Law Table.

Of course, this depends on the way one defines the algebraic binary multiplication law $*$ on the set:

$$\{N_r = a + b_1 I_1 + b_2 I_2 + \cdots + b_r I_r | a, b_1, b_2, \ldots, b_r \in M\},$$

where M can be $\mathbb{R}$ (the set of real numbers), or $\mathbb{C}$ (the set of complex numbers).

We present the below example.

But, first, let's present several (possible) interconnections between logic, set, and algebra.

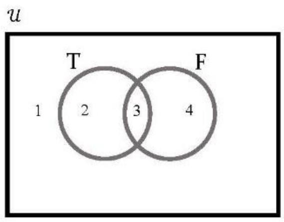
	Logic	*Set*	*Algebra*
operators	Disjunction (or) $\vee$	Union $\cup$	Addition $+$
	Conjunction (and) $\wedge$	Intersection $\cap$	Multiplication $\cdot$
	Negation $\neg$	Complement $\complement$	Subtraction $-$
	Implication $\rightarrow$	Inclusion $\subseteq$	Subtraction, Addition $-, +$
	Equivalence $\leftrightarrow$	Identity $\equiv$	Equality $=$

Table 1: Interconnections between logic, set, and algebra.

In general, if a Venn Diagram has n sets, with $n \geq 1$, the number of disjoint parts formed is 2^n. Then, if one combines the 2^n parts either by none, or by one, or by 2, ..., or by 2^n, one gets:

$$C_{2^n}^0 + C'_{2^n} + C_{2^n}^2 + \cdots + C_{2^n}^{2^n} = (1+1)^{2^n} = 2^{2^n}. \quad (5)$$

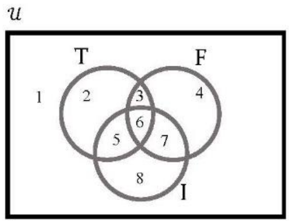
Hence, for $n = 2$, the Venn diagram, with literal truth (T), and literal falsehood (F), will make $2^2 = 4$ disjoint parts, where the whole rectangle represents the whole universe of discourse $(\mathcal{U})$.



Then, combining the four disjoint parts by none, by one, by two, by three, by four, one gets

$$C_4^0 + C_4^1 + C_4^2 + C_4^3 + C_4^4 = (1+1)^4 = 2^4 = 16 = 2^{2^2}.$$

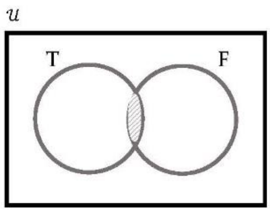
For $n = 3$, one has $2^3 = 8$ disjoint parts,



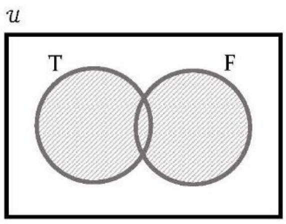
and combining them by none, by one, by two, and so on, by eight, one gets $2^8 = 256$, or $2^2 = 256$.

For the case when $n = 2 = \{T, F\}$ one can make up to 16 sub-indeterminacies, such as:

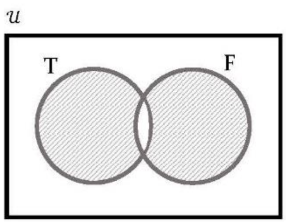
$$I_1 = C = \textbf{contradiction} = \text{True and False} = T \wedge F$$



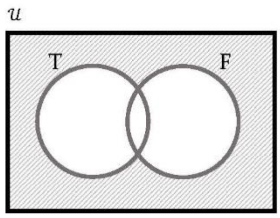
$$I_2 = Y = \textbf{uncertainty} = \text{True or False} = T \vee F$$



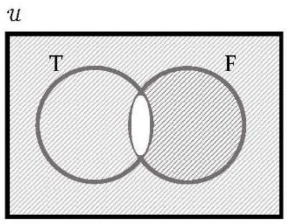
$$I_3 = S = \textbf{unsureness} = \text{either True or False} = T \veebar F$$



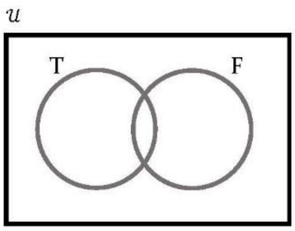
$I_4 = H =$ **nihilness** = neither True nor False
$= \neg T \wedge \neg F$



$I_5 = V =$ **vagueness** = not True or not False
$= \neg T \vee \neg F$



$I_6 = E =$ **emptiness** = neither True nor not True
$= \neg T \wedge \neg(\neg T) = \neg T \wedge T$



Let's consider the literal indeterminacy (I) refined into only six literal sub-indeterminacies as above.

The binary multiplication law $*$:

$$\{I_1, I_2, I_3, I_4, I_5, I_6\}^2 \rightarrow \{I_1, I_2, I_3, I_4, I_5, I_6\} \qquad (6)$$

defined as:
$I_j * I_k$ = intersections of their Venn diagram representations;
or $I_j * I_k$ = application of $\wedge$ operator, i.e. $I_j \wedge I_k$.

We make the following *Sub-Indeterminacies Multiplication Law Table*:

$*$	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_1	I_6	I_6	I_6	I_6
I_2	I_1	I_2	I_3	I_6	I_3	I_6
I_3	I_6	I_3	I_3	I_6	I_3	I_6
I_4	I_6	I_6	I_6	I_4	I_4	I_6
I_5	I_6	I_3	I_3	I_4	I_5	I_6
I_6	I_6	I_6	I_6	I_6	I_6	I_6

Remark

One can construct in various ways the diagrams that represent the subindeterminacies and similarly one can define in many ways the $*$ algebraic multiplication law, $I_j * I_k$, depending on the problem or application to solve.

What we constructed above is just an example, not a general procedure.

Let's see several calculations, so the reader gets familiar:

$I_1 * I_2 =$ (shaded area of I_1) $\cap$ (shaded area of I_2) = shaded area of I,
or $I_1 * I_2 = (T \wedge F) \wedge (T \vee F) = T \wedge F = I_1$.
$I_3 * I_4 =$ (shaded area of I_3) $\cap$ (shaded area of I_4) = empty set $= I_6$,
or $I_3 * I_4 = (T \underline{\vee} F) \wedge (\neg T \wedge \neg F) = [T \wedge (\neg T \wedge \neg F)] \underline{\vee} [F \wedge (\neg T \wedge \neg F)] = (T \wedge \neg T \wedge \neg F) \underline{\vee} (F \wedge \neg T \wedge \neg F) =$ (impossible) $\underline{\vee}$ (impossible)
because of $T \wedge \neg T$ in the first pair of parentheses and because of $F \wedge \neg F$ in the second pair of parentheses
= (impossible) $= I_6$.
$I_5 * I_5 =$ (shaded area of I_5) $\cap$ (shaded area of I_5)
= (shaded area of I_5)
or $I_5 * I_5 = (\neg T \vee \neg F) \wedge (\neg T \vee \neg F) = \neg T \vee \neg F = I_5$.

Now we are able to build refined I-neutrosophic algebraic structures on the set

$$S_6 = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6 \text{ for } a_0, a_1, a_2, \ldots a_6 \in \mathbb{R}\},$$

by defining the addition of refined neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) + (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2 + \cdots + (a_6 + b_6) I_6 \in S_6.$$

and the multiplication of refined neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) \cdot (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = a_0 b_0 + (a_0 b_1 + a_1 b_0) I_1 + (a_0 b_2 + a_2 b_0) I_2 + \cdots + (a_0 b_6 + a_6 b_0) I_6 +$$
$$+ \sum_{j,k=1}^{6} a_j b_k (I_j * I_k) = a_0 b_0 + \sum_{k=1}^{6} (a_0 b_k + a_k b_0) I_k + \sum_{j,k=1}^{6} a_j b_k (I_j * I_k) \in S_6.$$

where the coefficients (scalars) $a_m \cdot b_n$, for $m = 0, 1, 2, \ldots 6$ and $n = 0, 1, 2, \ldots 6$, are multiplied as any real numbers, while $I_j * I_k$ are calculated according to the previous Sub-Indeterminacies Multiplication Law Table.

Clearly, both operators (addition and multiplication of refined neutrosophic numbers) are well-defined on the set S_6.

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