The Geometry of Polychronous Wavefront Computation

Abstract

This paper consolidates all the salient geometrical aspects of the principle of Polychronous Wavefront Computation. A novel set of simple and closed planar curves are constructed based on this principle, using MATLAB. The algebraic and geometric properties of these curves are then elucidated as theorems, propositions and conjectures.

Keywords

Polychronous wavefront computation, hyperbola, Fermat-Torricelli point, Jordan curve

List of Abbreviations

PWC – Polychronous Wavefront Computation
ISI – Inter-Source stimulation Interval
FT – Fermat-Torricelli point

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To the Glory of God - my Heavenly Father,

To the Glory of Christ - my Risen Savior,

To the Glory of the Holy Spirit - my Guide, my Light and my Counsellor

Contents

1. Introduction
2. Preliminaries: Definitions, Elements and Principles
   - 2.1 Definitions
   - 2.2 Elements
   - 2.3 Principles
3. Two Theorems, One Proposition and Two Special Cases
   - 3.1 Dynamic Hyperbola Theorem
   - 3.2 Theorem
   - 3.3 Proposition
   - 3.4 Special Cases
     - 3.4.1 Source Configuration – At the Vertices of an Equilateral Triangle
     - 3.4.2 Source Configuration – At the Vertices of a Right Isosceles Triangle
4. Stimulation of a Source lying in the Medium
5. Calculation of ISIs when the Stimulating Wavefront is Linear
   - 5.1 Summary of the Localization Algorithm for a Linear Stimulating Wavefront
6. Calculation of ISIs when the Stimulating Wavefront is Circular
   - 6.1 For an Equilateral Triangle Configuration of Sources
   - 6.2 Summary of the Localization Algorithm for a Circular Stimulating Wavefront
   - 6.3 For a Scalene Triangle Configuration of Sources
   - 6.4 Summary of the Localization Algorithm for a Circular Stimulating Wavefront
7. Results: Numerical-Graphical Simulation using MATLAB
   - 7.1 For a Linear Stimulating Wavefront and Scalene Configuration of Sources
   - 7.2 For a Linear Stimulating Wavefront and Right Isosceles Configuration of Sources
   - 7.3 For a Linear Stimulating Wavefront and Equilateral Configuration of Source
   - 7.4 For a Circular Stimulating Wavefront and Scalene Configuration of Source
   - 7.5 For a Circular Stimulating Wavefront and Right Isosceles Configuration of Source
   - 7.6 For a Circular Stimulating Wavefront and Equilateral Configuration of Source
8. Discussion
   - 8.1 Nomenclature and Classification of PWC Curves
   - 8.2 On General Curve Morphology
   - 8.3 Some Final Theorems, Propositions and Conjectures on PWC Curves
     - 8.3.1 Theorem
     - 8.3.2 Corollary
     - 8.3.3 Theorem
     - 8.3.4 Theorem
     - 8.3.5 Proposition
     - 8.3.6 Proposition
     - 8.3.7 Proposition
     - 8.3.8 Proposition
     - 8.3.9 Proposition
     - 8.3.10 Proposition
     - 8.3.11 Proposition
     - 8.3.12 Proposition
     - 8.3.13 Proposition
     - 8.3.14 Proposition
     - 8.3.15 Proposition
     - 8.3.16 Conjecture
     - 8.3.17 Conjecture
   - 8.4 Prior Results
   - 8.5 Future Directions

References

Supplementary Material (MATLAB Code)

1. Introduction

Polychronous Wavefront Computation is the name given to a recently propounded principle in Theoretical Neuroscience [1]. It can be easily understood with the help of an example. Imagine the quiet surface of a pond on which a stone has just been dropped. Consequent to surface impact, a series of concentric circular ripples of disturbance are generated, which radiate outwards from the point of contact. These water waves spread out uniformly in all directions and with uniform speed.

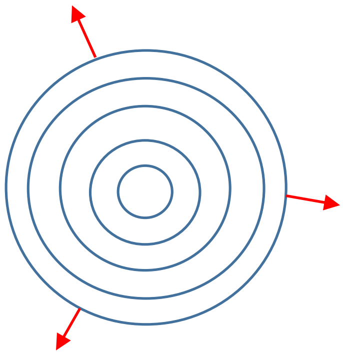


Fig. 1.1: *Concentric circular ripples emanating from a single contact point*

Now consider the case, where two stones have been dropped over neighboring points on the pond surface, at slightly different time instants. Consequent to surface impact, a pair of concentric circular ripples are generated.

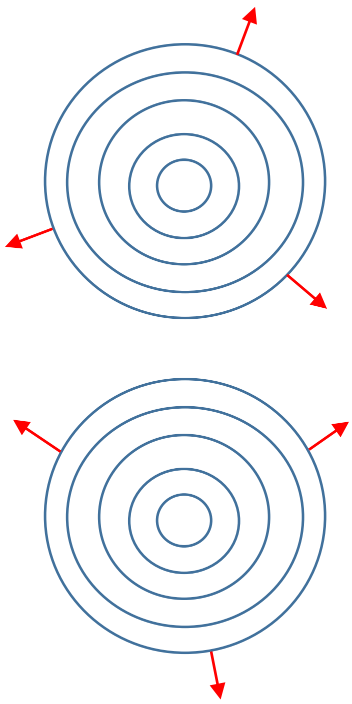


Fig. 1.2: *A pair of concentric ripples emanating from two neighboring contact points*

These ripples of disturbance then grow to intersect each other, first at a single point V along the line joining their centers and immediately thereafter, at a pair of points P and P'.

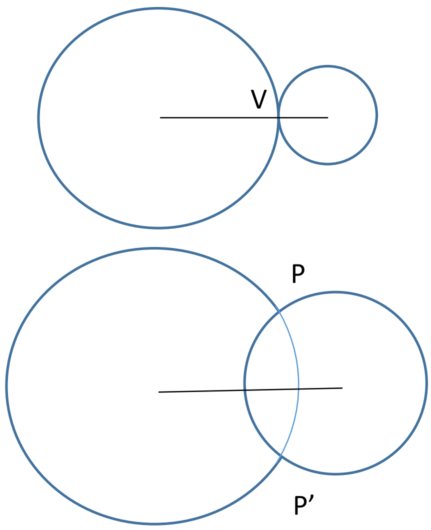


Fig. 1.3: *Circular ripples grow to touch each other first at a single point V, following which they intersect at a pair of points P and P'*

A trace of the intersection points over time yields the branch of a hyperbola. A mathematical proof of this statement in the form of a theorem, called the Dynamic Hyperbola Theorem (DHT), has been previously forwarded by the author [2]. DHT states that two dynamic circles intersect in a branch of a dynamic hyperbola. The term *dynamic*, emphasizes the temporal aspect of the generation of the hyperbolic branch from the intersection points of two expanding circles with centers located at, say, points A and B.

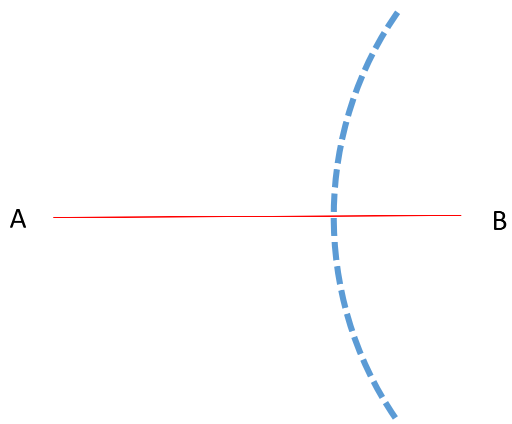


Fig. 1.4: *Hyperbolic branch formed from tracing the points of intersection between two dynamic circles ($r_A > r_B$)*

When the instantaneous radius of the dynamic circle emanating from point A is greater than that from point B, the mouth of the traced hyperbolic branch is open towards point B. Similarly, when the instantaneous radius of the dynamic circle emanating from point B is greater than that from point A, the mouth of the traced hyperbolic branch is open towards point A.

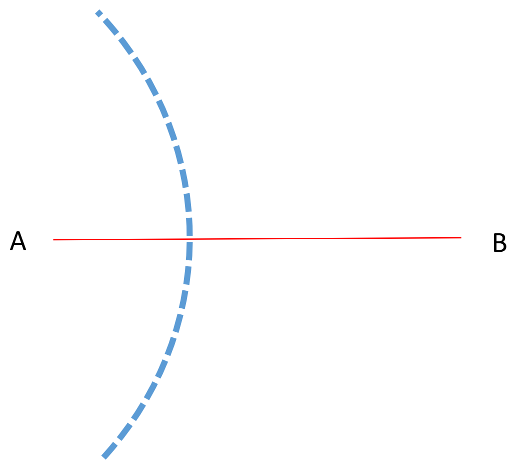


Fig. 1.5: *Hyperbolic branch formed from tracing the points of intersection between two dynamic circles ($r_B > r_A$)*

In the context of a computational process, it can be said that the hyperbolic branch spatially encodes the magnitude of the time interval between the instants at which the two dynamic circles begin their expansion, from zero radius. Say that t_A and t_B represent the instants at which the dynamic circles begin their expansion from zero radius. Then, it follows that the time interval $t_B - t_A$ is spatially encoded in that hyperbolic branch which has its mouth open towards point B. Similarly, the time interval $t_A - t_B$ is spatially encoded in that hyperbolic branch which has its mouth open towards point A.

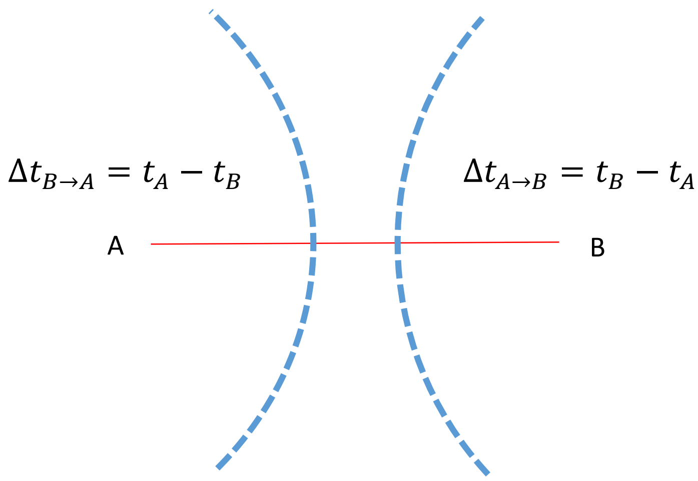


Fig. 1.6: *The time interval $\Delta t_{B\to A}$ is spatially encoded in the left hyperbolic branch and the time interval $\Delta t_{A\to B}$ is spatially encoded in the right hyperbolic branch*

The meaning of each word in the so called principle - *polychronous wavefront computation (PWC)*, may now be fully substantiated with clarity. The words '*poly', 'chronous'* and '*wavefront'*, refers to circular ripples of disturbance emanating from different point sources lying in a plane, at different time instants. The word '*computation'* refers to the notion that the traced points of intersection of those wavefronts, spatially encodes the difference in the time instants at which the circular ripples began their expansion from zero radius.

This paper consolidates all the salient geometrical aspects of the principle of Polychronous Wavefront Computation. A novel set of simple and closed planar curves (which we here, refer to as *PWC curves of the Jordan kind*) are constructed based on this principle, using MATLAB. The algebraic and geometric properties of these curves are then finally elucidated as theorems, propositions and conjectures.

2. Preliminaries: Definitions, Elements and Principles

2.1 Definitions

1. Dynamic Circle

A dynamic circle is one whose radius is a function of time.

2. Dynamic Hyperbola

A dynamic hyperbola is one that is formed from the locus of the intersection points of two dynamic circles.

2.2 Elements

1. Medium

A medium is a two-dimensional, homogenous, geometrical plane in which, circular ripples or wavefronts of disturbance can propagate.

2. Source

A source is a geometrical point in the medium, which upon stimulation, emanates circular ripples that expand radially outwards. The source lies at the center of this expansion.

2.3 Principles

1. Isotropic Propagation

The circular ripples that emanate from a stimulated source, spreads out uniformly in all directions with uniform speed.

2. Elastic Wave Collision

Wavefronts emanating from any pair of stimulated sources separated by a finite distance in the medium, pass through each other intact and unaltered in character (i.e. in terms of phase, amplitude, frequency).

3. Superposition of Waves

When the crest (or trough) of one wave falls on the crest (or trough) of another wave, the resultant amplitude is equal to the sum of the individual amplitudes. And when the crest (or trough) one wave falls on the trough (or crest) of another wave, the resultant amplitude is equal to the difference of the individual amplitudes.

4. Polychronous Wavefront Computation (PWC)

Circular ripples that emanate from a given pair of stimulated sources, spread out uniformly in all directions with equal speeds through the medium, to intersect in a particular member branch of a family of confocal hyperbolas. Each hyperbolic branch spatially encodes two equivalent quantities:

i. The time interval spanning the individual stimulations of the source pair, called the Inter-Source stimulation time Interval (ISI),
ii. The magnitude of the difference in the times of arrival of the circular ripples at any point on a particular member hyperbolic branch (TDOA).

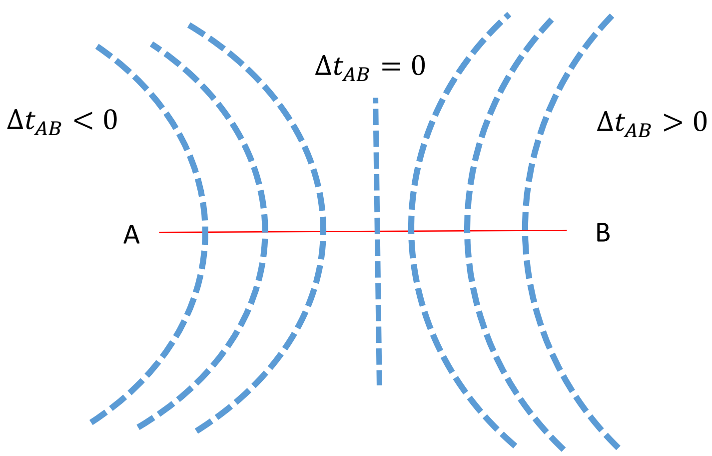


Fig. 2.3: *Different members of a family of confocal hyperbolas, each encoding a unique value of ISI, or equivalently TDOA*

3. Two Theorems, One Proposition and Two Special Cases

3.1 Dynamic Hyperbola Theorem [2], [3]

Two dynamic circles with equal expansion rates, non-coincident source centers and distinct instants of emergence, come to intersect each other in a branch of a hyperbola. That is, the locus of the intersection points of two such dynamic circles is a dynamic hyperbola.

The analytical equation of the latter curve, in the XY-plane is:

$$\frac{x^2}{\left(\frac{u.\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{a^2 - \left(\frac{u.\Delta t_{AB}}{2}\right)^2} = 1$$

Where $(-a, 0)$ and $(a, 0)$ are the point locations of sources A and B respectively, Δt_{AB} is the Inter-Source Interval and u is the uniform rate of expansion of the dynamic circles. The source centers A and B, thus behave as the common foci of the family of dynamic hyperbolas with each of its members corresponding to a particular Δt_{AB} value.

Fig. 3.1: *The locus of the intersection points is depicted with red broken lines for both cases, when source A is stimulated before source B and vice versa.*

3.2 Theorem*

Three dynamic circles with equal expansion rates, non-collinear source centers and distinct instants of emergence, come to meet at a common point in the plane of the medium, if and only if the three dynamic hyperbolic branches, generated from the pair wise intersections of the dynamic circles, share a point of concurrence.

Let $A(-a, 0)$, $B(b, 0)$ and $C(0, c)$ be the coordinate positions of three non-collinear sources in the medium, which emanate circular ripples upon stimulation at time instants t_A, t_B and t_C, respectively. Also, let u be the expansion rate of the dynamic circles and Δt_{AB}, Δt_{AC} and Δt_{CB} be the magnitudes of the time intervals spanning the instants at which pair wise source stimulations occur. (**N.B.** a, b and c are non-negative numbers).

*Proof given in § 8.3.3

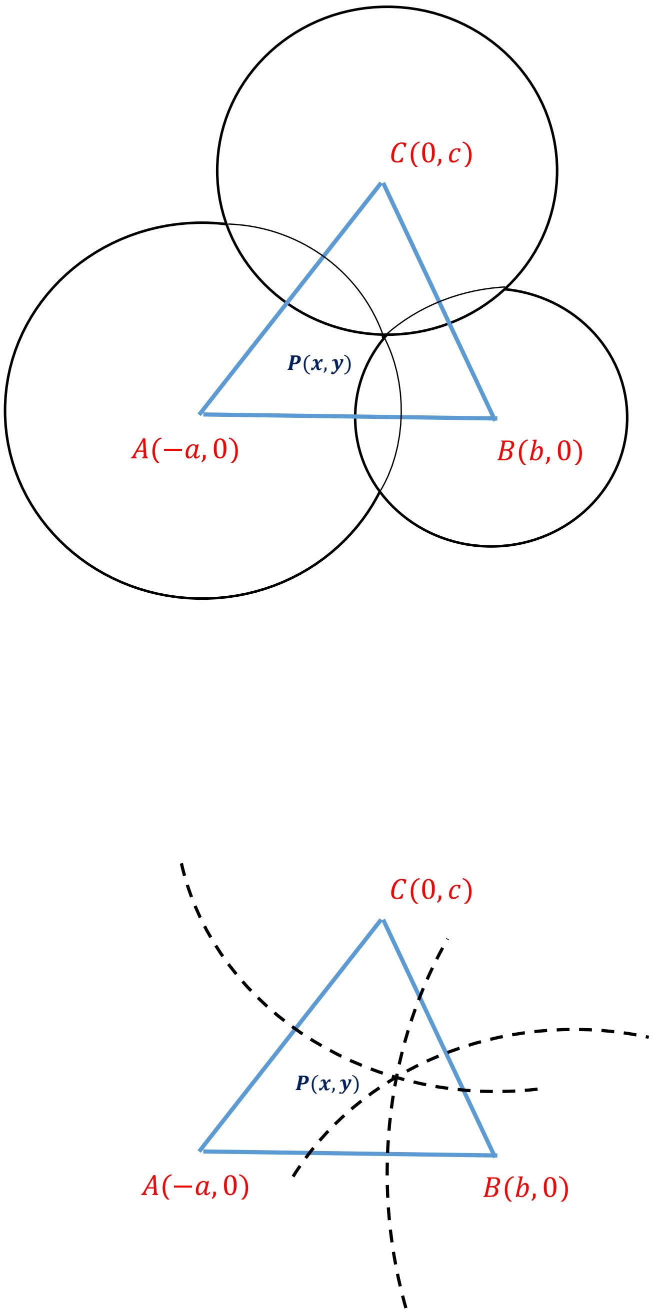


Fig. 3.2: *The dynamic circles are depicted with black solid lines; the dynamic hyperbolas are depicted with black dotted lines; $P(x, y)$ is the instantaneous point of concurrence of three dynamic circles, or equivalently the point of concurrence of three dynamic hyperbolas. The sequence of source stimulations in the diagram is A → C → B.*

The analytical equations of three dynamic hyperbolas generated from the pair wise intersections of three dynamic circles are given below [3],[4]. The point of concurrence $P(x, y)$ can be determined by either solving these equations using numerical methods or by graphically plotting the curves that they represent, using MATLAB.

(i) Equation of Dynamic Hyperbola with side AB as transverse axis,

$$y = \pm\sqrt{\left(\left(\frac{a+b}{2}\right)^2 - J^2{}_1\right)}\sqrt{\left(\frac{\left(x-\frac{b-a}{2}\right)^2}{J^2{}_1} - 1\right)}$$

(ii) Equation of Dynamic Hyperbola with side CB as transverse axis,

$$y = \frac{-4[-2bcx+c(b^2-c^2)+4cJ_2^2] \pm \sqrt{64J_2^2(b^2+c^2-4J_2^2)(4x^2-4bx+b^2+c^2-4J_2^2)}}{8(c^2-4J_2^2)}$$

(iii) Equation of Dynamic Hyperbola with side AC as transverse axis,

$$y = \frac{-4[2acx+c(a^2-c^2)+4cJ_3^2] \pm \sqrt{64J_3^2(a^2+c^2-4J_3^2)(4x^2+4ax+a^2+c^2-4J_3^2)}}{8(c^2-4J_3^2)}$$

Where, $J_1 = \frac{u|\Delta t_{AB}|}{2}$, $J_2 = \frac{u|\Delta t_{CB}|}{2}$ and $J_3 = \frac{u|\Delta t_{AC}|}{2}$

3.3 Proposition

A single dynamic hyperbola spatially encodes a singlet set time interval say, {Δt_{AB}}, whereas a point of concurrence of three dynamic hyperbolas spatially encodes a triplet set of time intervals say, {Δt_{AB}, Δt_{AC}, Δt_{CB}}.

3.4 Special Cases

When considering the spatial arrangement of the sources in the medium, there are two configurations that are worth special mention. The first, is when the three sources lie at the vertices of an equilateral triangle and the second, is when they lie at the vertices of a right isosceles triangle. The triplet set of Dynamic Hyperbola Equations in each of these cases are listed below.

3.4.1 Source Configuration – At the vertices of an Equilateral Triangle

The co-ordinates of the vertices of an equilateral ΔABC of side length $2a$, are obtained by placing $b = a$ and $c = \sqrt{3}a$. Hence, the source positions in the medium are $A(-a, 0)$, $B(a, 0)$ and $C(0, \sqrt{3}a)$. The triplet set of dynamic hyperbola equations of § 3.2 become:

(i) Equation of Dynamic Hyperbola with side AB as transverse axis,

$$y = \pm\sqrt{(a^2 - J^2{}_1)}\sqrt{\left(\frac{x^2}{J^2{}_1} - 1\right)}$$

(ii) Equation of Dynamic Hyperbola with side CB as transverse axis,

$$y = \frac{\sqrt{3}.a(ax + a^2 - 2J_2{}^2) \pm 4J_2\sqrt{(a^2 - J_2{}^2)(x^2 - ax + a^2 - J_2{}^2)}}{(3a^2 - 4J_{CB}{}^2)}$$

(iii) Equation of Dynamic Hyperbola with side AC as transverse axis,

$$y = \frac{-\sqrt{3}.a(ax - a^2 + 2J_3{}^2) \pm 4J_3\sqrt{(a^2 - J_3{}^2)(x^2 + ax + a^2 - J_3{}^2)}}{(3a^2 - 4J_3{}^2)}$$

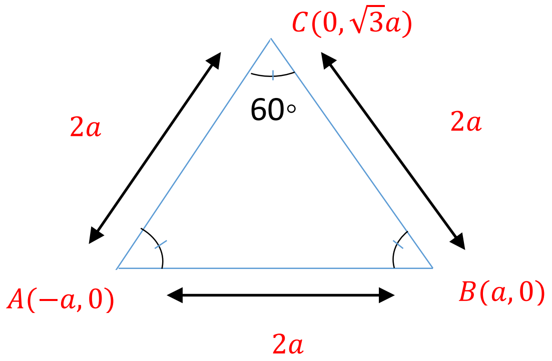


Fig. 3.4.1: *Equilateral Source Configuration*

3.4.2 Source Configuration – At the vertices of a Right Isosceles Triangle

The co-ordinates of the vertices of a right isosceles ΔABC with hypotenuse length $2a$ and adjacent side lengths $\sqrt{2}a$, are obtained by placing $b = a$ and $c = a$. Hence, the source positions in the medium are $A(-a, 0)$, $B(a, 0)$ and $C(0, a)$. The triplet set of dynamic hyperbola equations of § 3.2 become:

(i) Equation of Dynamic Hyperbola with side AB as transverse axis,

$$y = \pm\sqrt{(a^2 - J^2{}_1)}\sqrt{\left(\frac{x^2}{J^2{}_1} - 1\right)}$$

(ii) Equation of Dynamic Hyperbola with side CB as transverse axis,

$$y = \frac{a(ax - 2{J_2}^2) \pm \sqrt{4\,{J_2}^2(a^2 - 2{J_2}^2)(2x^2 - 2ax + a^2 - 2{J_2}^2)}}{(a^2 - 4{J_2}^2)}$$

(iii) Equation of Dynamic Hyperbola with side AC as transverse axis,

$$y = \frac{-a(ax + 2{J_3}^2) \pm \sqrt{4{J_3}^2(a^2 - 2{J_3}^2)(2x^2 + 2ax + a^2 - 2{J_3}^2)}}{(a^2 - 4{J_3}^2)}$$

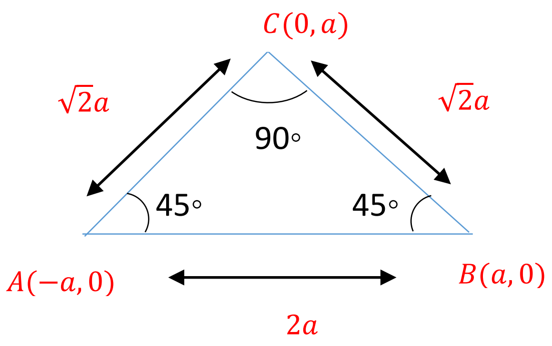


Fig. 3.4.2: *Right Isosceles Source Configuration*

4. Stimulation of a Source lying in the Medium

A source lying in the medium is said to be stimulated when a wavefront, either linear or circular in shape, passes through it. Following stimulation, a dynamic circle emanates from the source, which expands radially outwards in all directions, undiminished.

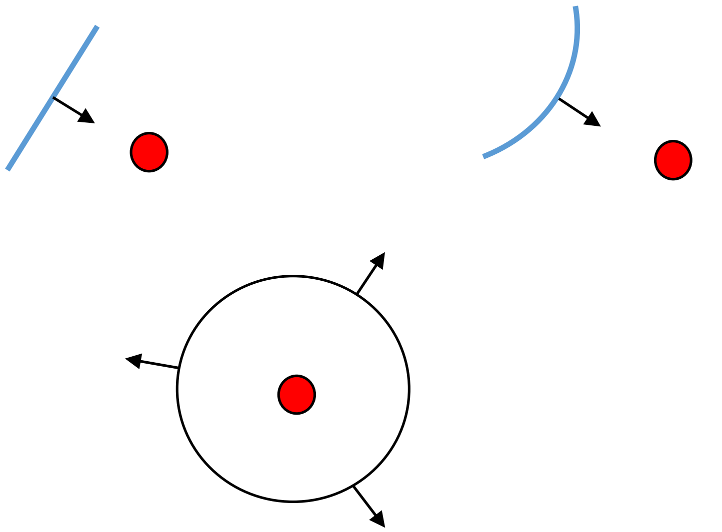


Fig. 4.1: *Source lying in the medium is shown in red; the stimulating wavefronts, both linear and circular, are shown in blue; the dynamic circle emanating from the source consequent to stimulation, is shown in black*

For the purpose of illustration, depicted below is the case of a linear stimulating wavefront, passing through three non-collinear sources in rapid succession. Following stimulation, three dynamic circles emanate from the sources.

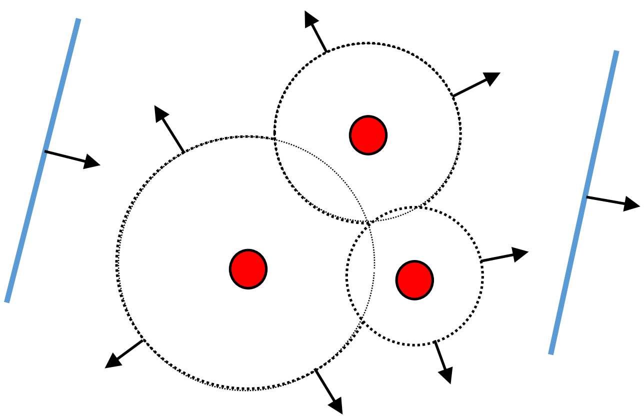


Fig. 4.2: *Three Sources (red) stimulated in succession, by a linear stimulating wavefront (blue) and consequently, three dynamic circles emanate from them (black)*

In the course of radial expansion, the three dynamic circles may eventually come to meet at some unique spatial point in the plane of the medium, for an instant of time. Such an instantaneous point of confluence can happen only if the three dynamic hyperbolas generated from the pair wise intersections of the dynamic circles, share a common point of intersection (Theorem 3.2).

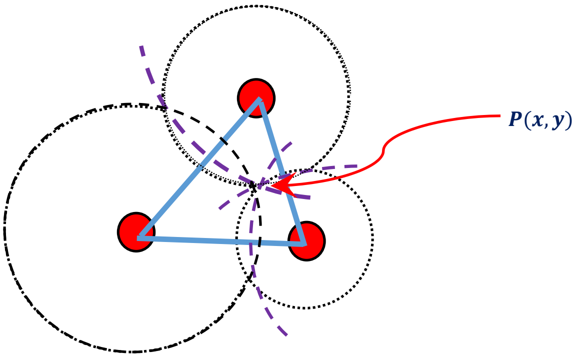


Fig. 4.3: *Dynamic circles and dynamic hyperbolas share a point of concurrence* $P(x,y)$

If this singular point does indeed exist, then it can be said to encode the triplet set of time intervals spanning successive source stimulations (Proposition 3.3). Let us next compute what the magnitude of these intervals are, for both linear and circular stimulating wavefronts.

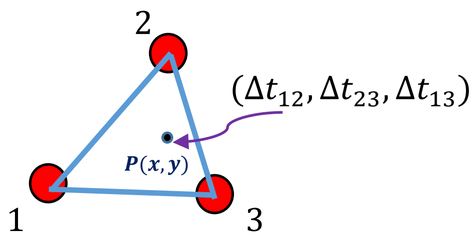


Fig. 4.4: *Point of concurrence -* $P(x,y)$*, if it exists, spatially encodes* $\Delta t_{12}, \Delta t_{23}, \Delta t_{13}$*. Here the sources are labelled 1, 2 and 3. The subscripts 12, 23 and 13 indicates the sequence of source stimulations 1→2, 2→3 and 1→3, respectively.*

5. Calculation of ISIs when the Stimulating Wavefront is Linear

A linear wavefront is represented by a straight line *l* that is inclined at an angle β with the base of a *ΔABC*. Three sources are located at its vertices A, B and C. Let α be the angle made by the linear stimulus *l* with the first side of the triangle that it passes across, θ be the direction of its motion with respect to the base and *v* be the speed of its sweep across the sources. Finally, let t_A, t_B and t_C be the instants at which the sources A, B and C get stimulated by the linear wavefront.

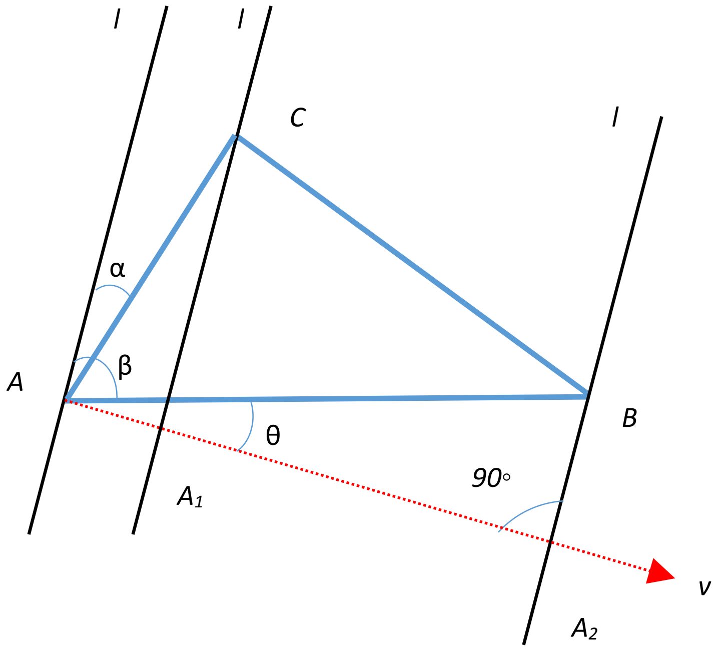


Fig. 5.1: *The linear stimulating wavefront contacts sources A, C and B in succession*

From Fig. 5.1, it is clear that the sources A, C and B can be stimulated in temporal succession, provided that the angular condition $A \leq \beta \leq A + C$ holds. (**N.B.** The angles of the triangle are denoted by their corresponding vertices).

$$\beta + \theta = 90°$$

$$A = \beta - \alpha$$

From Right ΔAA_1C,

$$
\begin{aligned}
AA_1 &= AC.Cos(\beta - \alpha + \theta) \\
&= AC.Cos(\beta - \alpha + \theta) \\
&= AC.Cos(A + 90° - \beta) \\
&= -AC.Sin(A - \beta) \\
&= AC.Sin(\beta - A)
\end{aligned}
$$

From Right ΔAA_2B,

$$
\begin{aligned}
AA_2 &= AB.Cos\theta \\
&= AB.Cos\theta \\
&= AB.Cos(90° - \beta) \\
&= AB.Sin\beta
\end{aligned}
$$

$$
\begin{aligned}
A_1A_2 &= AA_2 - AA_1 \\
&= AB.Sin\beta - AC.Sin(\beta - A)
\end{aligned}
$$

$$\Delta t_{AC} = t_C - t_A = \frac{AA_1}{v} = \frac{AC.Sin(\beta - A)}{v}$$

$$\Delta t_{AB} = t_B - t_A = \frac{AA_2}{v} = \frac{AB.Sin\beta}{v}$$

$$\Delta t_{CB} = t_B - t_C = \frac{A_1A_2}{v} = \frac{AB.Sin\beta - AC.Sin(\beta - A)}{v}$$

The above time intervals that span successive source stimulations may now be expressed more generally, by adopting the schematic shown below. The sides of the triangle here, are labelled as Side-1, Base and Side-2 according to the sequence of source stimulation by the linear wavefront *l*.

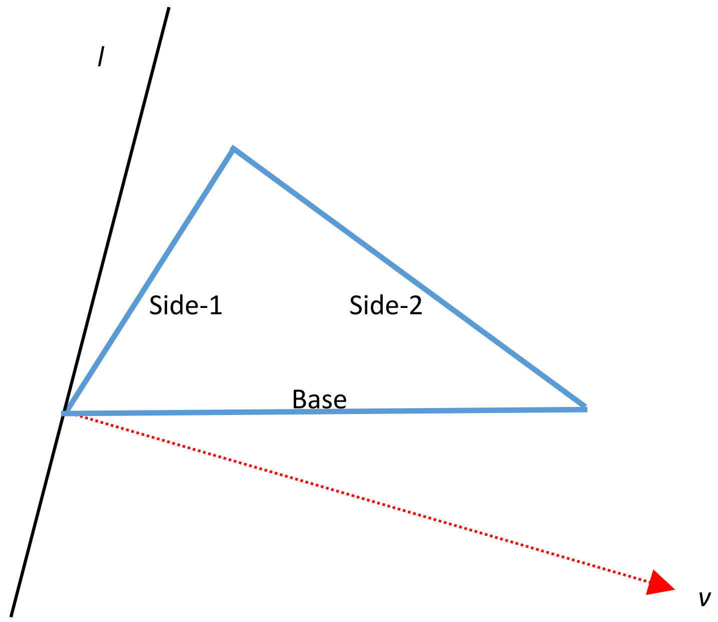


Fig. 5.2

The previous triplet set of ISI equations can therefore, be rewritten as follows:

$$\Delta t_{Side1} = \frac{(Side1).Sin(\beta - Angle\ between\ Side1\ \&\ Base)}{v}$$

$$\Delta t_{Base} = \frac{(Base).Sin\beta}{v}$$

$$\Delta t_{Side2} = \frac{(Base).Sin\beta - (Side1).Sin(\beta - Angle\ between\ Side1\ \&\ Base)}{v}$$

There are in total, six possible sequences of source stimulations that are listed in Table 5.3 below. The sides of the triangle over which the linear stimulus sweeps across, are ordered in correspondence to these sequences, as Side-1, Base and Side-2. The last column indicates the angular range (β) over which the specific sequence of source stimulations is valid.

Sequence of Source stimulations	**Side-1**	**Base**	**Side-2**	**Angular Range of Validity for the Specific Sequence of Source Stimulations β**
$A \rightarrow C \rightarrow B$	*AC*	*AB*	*CB*	$A \leq \beta \leq A + C$
$A \rightarrow B \rightarrow C$	*AB*	*AC*	*BC*	$A \leq \beta \leq A + B$
$B \rightarrow A \rightarrow C$	*BA*	*BC*	*AC*	$B \leq \beta \leq B + A$
$B \rightarrow C \rightarrow A$	*BC*	*BA*	*CA*	$B \leq \beta \leq B + C$
$C \rightarrow A \rightarrow B$	*CA*	*CB*	*AB*	$C \leq \beta \leq C + A$
$C \rightarrow B \rightarrow A$	*CB*	*CA*	*BA*	$C \leq \beta \leq C + B$

Table 5.3

5.1 Summary of the Localization Algorithm for a Linear Stimulating Wavefront

The general algorithm for the determination of the point of concurrence of three dynamic circles, when the stimulating wavefront is linear and the three sources are distributed at the vertices of a scalene triangle, is summarized below:

1. Choose an arbitrary sequence of source stimulations, by varying the angle of inclination β with the base of the triangle that has three sources lying at its vertices. (The inclination β, should lie within the range spanning the magnitude of the angle that corresponds to the first vertex of contact, upto the magnitude of the sum of the two angles that correspond to the first and second vertices of contact).
2. Compute $|\Delta t_{AB}|$, $|\Delta t_{CB}|$ and $|\Delta t_{AC}|$ using the ISI-Equations.
3. Compute the J-parameters corresponding to each of the ISIs.
4. Obtain the graphical plot of the triplet set of Dynamic Hyperbola Equations. Each hyperbola has one side of the triangle as its transverse axis.
5. Locate the point of concurrence of the three Dynamic Circles, by zooming into the point of intersection of the three Dynamic Hyperbolic branches.

6. Calculation of ISIs when the Stimulating Wavefront is Circular

6.1 For an Equilateral Triangle Configuration of Sources

For the sake of simplicity, let us first consider the case where the three sources lie at the vertices of an equilateral triangle. Say, that the external source P, from which the circular stimulating wavefront emanates, lies at a distance R from the center O of the equilateral triangle. (**N.B.** The five significant centers of a triangle, namely the centroid, the incenter, the circumcenter, the orthocenter and the Fermat-Torricelli point all coincide with each other, if all the sides of that triangle are equal. This unique point of concurrence is therefore, simply referred to here as the center of the equilateral triangle).

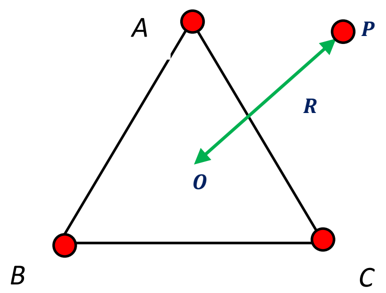


Fig. 6.1: *Sources are located at A, B, C and P. The Center of the triangle is O and* $OP = R$.

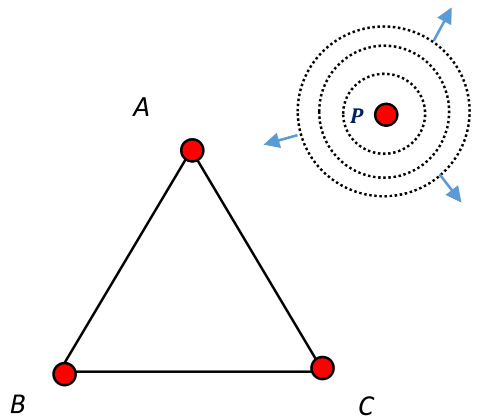


Fig. 6.2: *The circular stimulating wavefront emanates from external source P.*

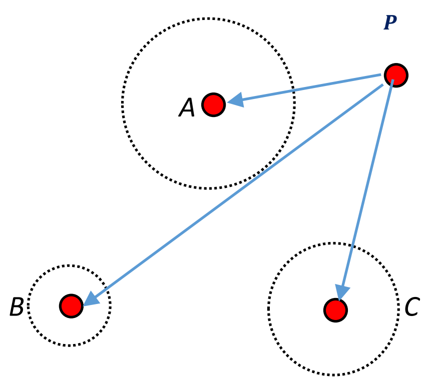


Fig. 6.3: *After traversing distances PA, PB and PC, the sources A, B and C get stimulated and emanate dynamic circles.*

Consider an equilateral ΔABC, with sides $AB = BC = CA = \rho$ and center O. Say that the external source *P* lies on a circle centered at *O* with radius $OP = R$. Extend OA, OB and OC so that they meet the circle at M_1, M_2 and M_3, respectively. Recall that *O* is also the centroid of the equilateral triangle. So, if *h* be the length of its median, then AO must be equal to $\frac{2}{3}h$, according to the *Median 2:1 Intersection Theorem*.
Also, the altitude h of an equilateral triangle (or equivalently, its median) is related to its side length ρ by the expression $h = \frac{\sqrt{3}}{2}\rho$. Finally, let α, β and γ be the central angles made by OP with OA, OB and OC, respectively.

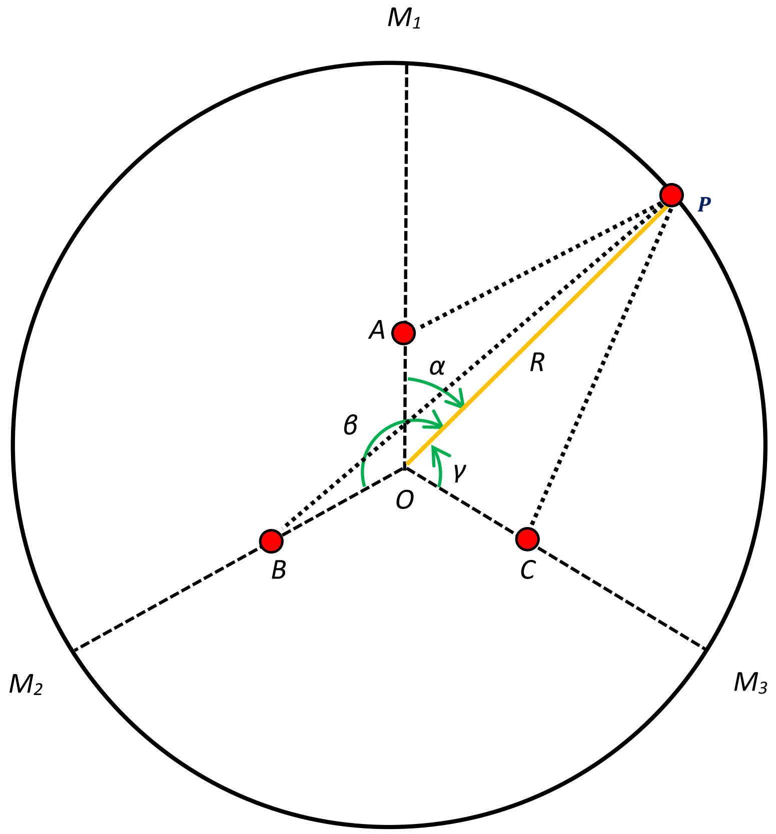


Fig. 6.4: *O is the center of both the equilateral ΔABC and the circle of radius R. The sources are located at A, B, C and P. The distance OP is equal to the radius R. (**N.B.** OA, OB and OC are the reference lines for the angular measurements* α, β *and* γ*, respectively).*

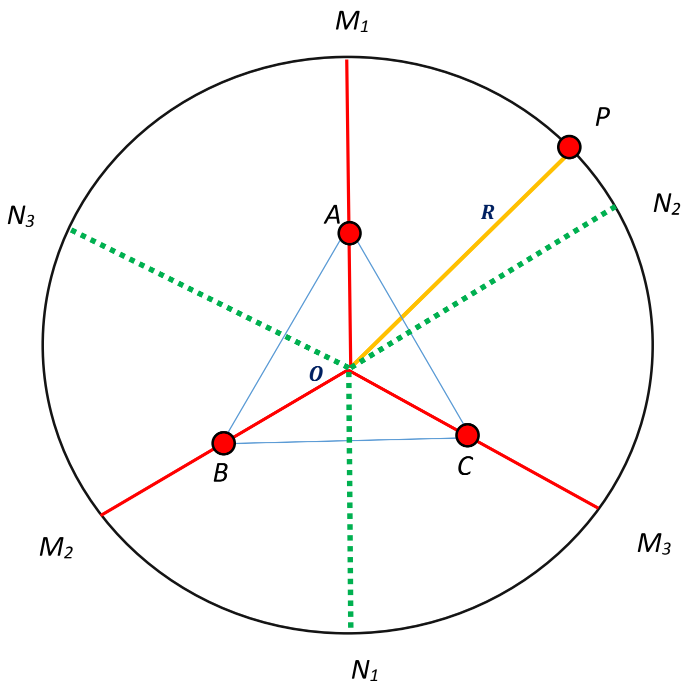


Fig. 6.5: *The dotted green lines trisect the circle into three equal sectors $N_1O\ N_2$, $N_2O\ N_3$ and $N_3O\ N_1$. Each of these sectors are further bisected by the red lines into a total of six equal sectors $M_1O\ N_2$, $N_2O\ M_3$, $M_3O\ N_1$, $N_1O\ M_2$, $M_2O\ N_3$* and *$N_3O\ M_1$.*

To summarize,

$$AB = BC = CA = \rho$$

$$h = \frac{\sqrt{3}}{2}\rho$$

$$OA = OB = OC = \frac{2}{3}h = \frac{\rho}{\sqrt{3}}$$

$$OP = R$$

In ΔAOP (Fig. 6.4),

$$cos\alpha = \frac{OA^2 + OP^2 - AP^2}{2.OA.OP}$$

$$AP^2 = OA^2 + OP^2 - 2.OA.OPcos\alpha$$

$$AP^2 = \frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho Rcos\alpha$$

Similarly, in ΔBOP (Fig. 6.4),

$$BP^2 = \frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\beta$$

And in ΔCOP (Fig. 6.4),

$$CP^2 = \frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\gamma$$

The above results for source position with respect to the center O, is summarized below:

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\alpha}$$

$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\beta}$$

$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\gamma}$$

By making the following substitutions, the central angles α, β and γ with respect to the reference lines OA, OB and OC, respectively can be related to each other:

$$\alpha = \theta$$
$$\beta = 120° + \theta$$
$$\gamma = 120° - \theta$$

So, provided that $0° \leq \theta \leq 60°$, the above Equations for Source-Position become:

$$AP = \sqrt{\tfrac{\rho^2}{3} + R^2 - \tfrac{2}{\sqrt{3}}\rho R cos\theta}$$

$$BP = \sqrt{\tfrac{\rho^2}{3} + R^2 - \tfrac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$

$$CP = \sqrt{\tfrac{\rho^2}{3} + R^2 - \tfrac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$

Depending on which sector the external source P lies in, the sequence of stimulation of sources A, B and C will differ. This may be geometrically intuited with the help of Fig. 6.6. The sequence of source stimulations corresponding to each of the arc sectors where the source P may be located, is given in Table 6.7.

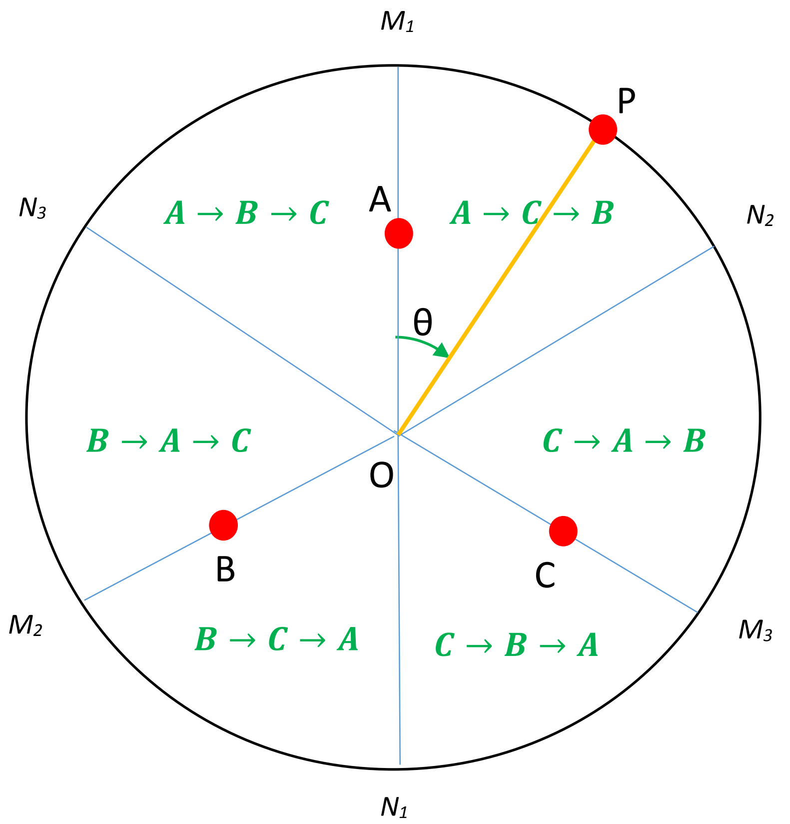


Fig. 6.6: *Each of the six arc sectors subtends an angle of* $60°$ *at the center O. The external source P is located at distances AP, BP and CP from the sources A, B and C, respectively. The radial line OP makes a central angle θ with the vertical* M_1N_1.

Arc Sector No.	Sequence of Source stimulation	Source Position Relations	Reference line for central angle θ measurement
Sector-1: $M_1O\ N_2$	$A \to C \to B$	$AP \leq CP \leq BP$	OA
Sector-2: $N_2O\ M_3$	$C \to A \to B$	$CP \leq AP \leq BP$	OC
Sector-3: $M_3O\ N_1$	$C \to B \to A$	$CP \leq BP \leq AP$	OC
Sector-4: $N_1O\ M_2$	$B \to C \to A$	$BP \leq CP \leq AP$	OB
Sector-5: $M_2O\ N_3$	$B \to A \to C$	$BP \leq AP \leq CP$	OB
Sector-6: $N_3O\ M_1$	$A \to B \to C$	$AP \leq BP \leq CP$	OA

Table 6.7

By restricting the range of the central angle θ to between 0° and 60°, and permuting the reference lines for its measurement amongst the lines OA, OB and OC, the triplet Source Position Equations for each of the six sectors can be appropriately written.

CASE-1: When External Source P lies in Sector-1 (OA is reference line & Seq. A→C→B)

$$\alpha = \theta$$
$$\beta = 120° + \theta$$
$$\gamma = 120° - \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$

CASE-2: When External Source P lies in Sector-2 (OC is reference line & Seq. C→A→B)

$$\alpha = 120° - \theta$$
$$\beta = 120° + \theta$$
$$\gamma = \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$

CASE-3: When External Source P lies in Sector-3 (OC is reference line & Seq. C→B→A)

$$\alpha = 120° + \theta$$
$$\beta = 120° - \theta$$
$$\gamma = \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$

CASE-4: When External Source P lies in Sector-4 (OB is reference line & Seq. B→C→A)

$$\alpha = 120° + \theta$$
$$\beta = \theta$$
$$\gamma = 120° - \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$

CASE-5: When External Source P lies in Sector-5 (OB is reference line & Seq. B→A→C)

$$\alpha = 120° - \theta$$
$$\beta = \theta$$
$$\gamma = 120° + \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$

CASE-6: When External Source P lies in Sector-6 (OA is reference line & Seq. A→B→C)

$$\alpha = \theta$$
$$\beta = 120° - \theta$$
$$\gamma = 120° + \theta$$

$$AP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos\theta}$$
$$BP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° - \theta)}$$
$$CP = \sqrt{\frac{\rho^2}{3} + R^2 - \frac{2}{\sqrt{3}}\rho R cos(120° + \theta)}$$

The purpose of calculating the source positions *AP, BP* and *CP* in this sectorial manner, is to determine the magnitude of the ISIs for each source pair, i.e. $|\Delta t_{AC}|, |\Delta t_{AB}|$ and $|\Delta t_{CB}|$.

Recall that if *v* is the speed of the stimulating circular wavefront that emanates from the external source P, then the times of arrival t_A, t_B and t_C of this wavefront at the vertices of *A, B* and *C* of the equilateral *ΔABC,* after having traversed distances *PA, PB* and *PC,* respectively are given by:

$$t_A = \frac{PA}{v}$$

$$t_B = \frac{PB}{v}$$

$$t_C = \frac{PC}{v}$$

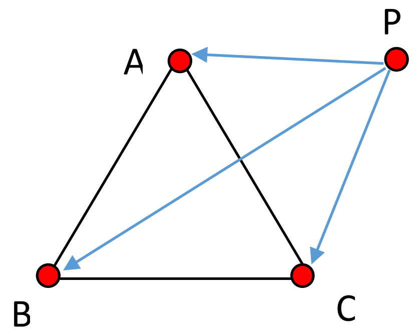


Fig. 6.8

Therefore, the magnitude of the ISIs for each of the successive source pair stimulations are:

$$|\Delta t_{AC}| = |t_C - t_A| = \left|\frac{PC - PA}{v}\right|$$

$$|\Delta t_{AB}| = |t_B - t_A| = \left|\frac{PB - PA}{v}\right|$$

$$|\Delta t_{CB}| = |t_B - t_C| = \left|\frac{PB - PC}{v}\right|$$

The ISIs so calculated above, can then be plugged into the following triplet set of equations for the Dynamic Hyperbolas, in order to obtain the point of concurrence of the three Dynamic Circles in the plane of the medium. Here, the coordinates of the vertices of ΔABC are taken as $A(0, a)$, $B(-b, 0)$ and $C(c, 0)$. (**N.B.** a, b and c are non-negative numbers).

(i) Equation of Dynamic Hyperbola with side CB as transverse axis,

$$y = \pm\sqrt{\left(\left(\frac{b+c}{2}\right)^2 - J^2{}_1\right)}\sqrt{\left(\frac{\left(x-\frac{c-b}{2}\right)^2}{J^2{}_1} - 1\right)}$$

(ii) Equation of Dynamic Hyperbola with side AC as transverse axis,

$$y = \frac{-4[-2cax+a(c^2-a^2)+4aJ_2^2]\pm\sqrt{64J_2^2(c^2+a^2-4J_2^2)(4x^2+4cx+c^2+a^2-4J_2^2)}}{8(a^2-4J_2^2)}$$

(iii) Equation of Dynamic Hyperbola with side AB as transverse axis,

$$y = \frac{-4[2bax+a(b^2-a^2)+4aJ_3^2]\pm\sqrt{64J_3^2(b^2+a^2-4J_3^2)(4x^2-4bx+b^2+a^2-4J_3^2)}}{8(a^2-4J_3^2)}$$

Where, $J_1 = \frac{u|\Delta t_{CB}|}{2}$, $J_2 = \frac{u|\Delta t_{AC}|}{2}$, $J_3 = \frac{u|\Delta t_{AB}|}{2}$ and u is the uniform rate of expansion of the Dynamic Circles. In order to obtain the triplet set of Dynamic Hyperbola equations for an equilateral triangle configuration, substitute $c = b$ and $a = \sqrt{3}b$ in the above.

6.2 Summary of Localization Algorithm for a Circular Stimulating Wavefront

The algorithm used for the determination of the point of concurrence of three dynamic circles, when the stimulating wavefront is circular and the three sources are distributed at the vertices of an equilateral triangle, is summarized below:

1. Choose an arbitrary central angle θ between the radial line OP and one of the reference lines OA, OB or OC. (**N.B.** Range of θ: $0° \leq \theta \leq 60°$; O is the generic center of the equilateral ΔABC; OA, OB, OC are the lines joining the center to the vertices).
2. Write the three central angles α, β and γ as permutations of $\theta, 120° + \theta, 120° - \theta$, based on which of the six sectors, the external source P is located.
3. Compute the Source-Vertex Distances AP, BP and CP using the triplet set of Source-Position Equations.
4. Compute $|\Delta t_{AC}|$, $|\Delta t_{AB}|$ and $|\Delta t_{CB}|$ using the ISI Equations.
5. Compute the J-parameters corresponding to each of the ISIs.
6. Obtain the graphical plot of the triplet set of Dynamic Hyperbola equations. Each hyperbola has one side of the triangle as its transverse axis.
7. Locate the instantaneous point of concurrence of the three Dynamic Circles, by zooming into the point of intersection of the three Dynamic Hyperbolic branches.

6.3 For a Scalene Triangle Configuration of Sources

Let us now consider the more general case, where three sources lie at the vertices *A, B* and *C* of a scalene triangle and all its angles are less than $120°$. The algorithm to be followed for localizing the instantaneous point of concurrence of the three dynamic circles, when the sources are stimulated by a circular stimulating wavefront, is almost exactly the same as that outlined in § 6.2. However, the center chosen for this task is the Fermat-Torricelli point *F* of the triangle [5]. The peculiar property of this geometric point *F*, is that, the lines joining *F* to the vertices A, B and C divides the circle on whose circumference the external source *P* lies, into three equal sectors, each subtending an angle of $120°$ at *F*. Let the external source *P* which emanates the circular stimulating wavefront, lie at a distance *R* from *F.*

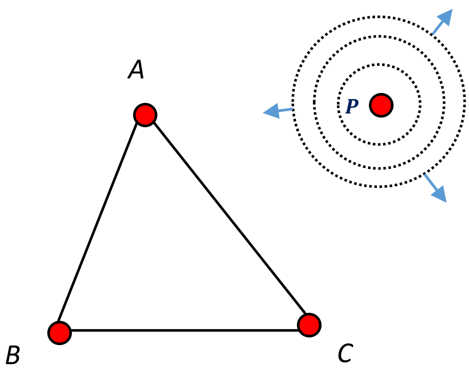


Fig. 6.9: *ΔABC is scalene and angles A, B and C are all less than* $120°$*.*

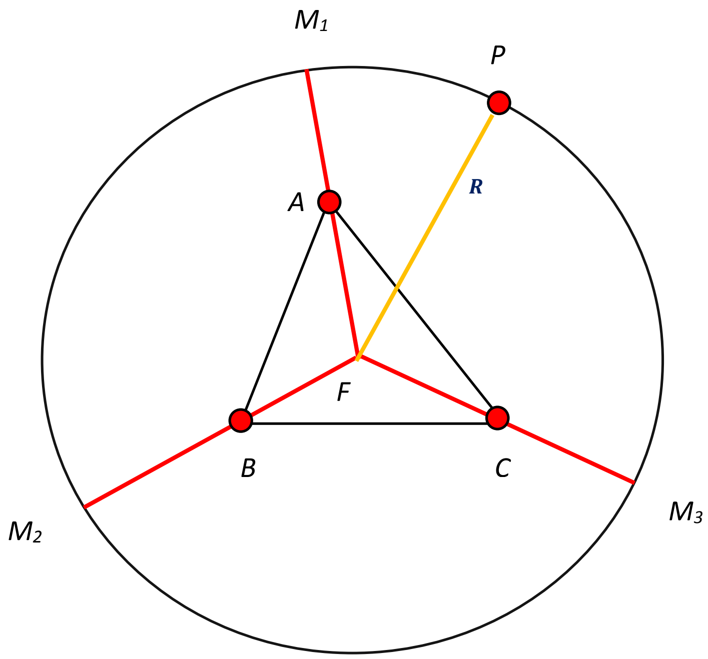


Fig. 6.10: *External source P lies at a distance R from the Fermat-Torricelli point F. AF, BF and CF when extended, meet the circle of radius R and center F at M_1, M_2 and M_3. Lines FM_1, FM_2 and FM_3 divides the circle into three equal sectors, each subtending an angle* $120°$ *at F*

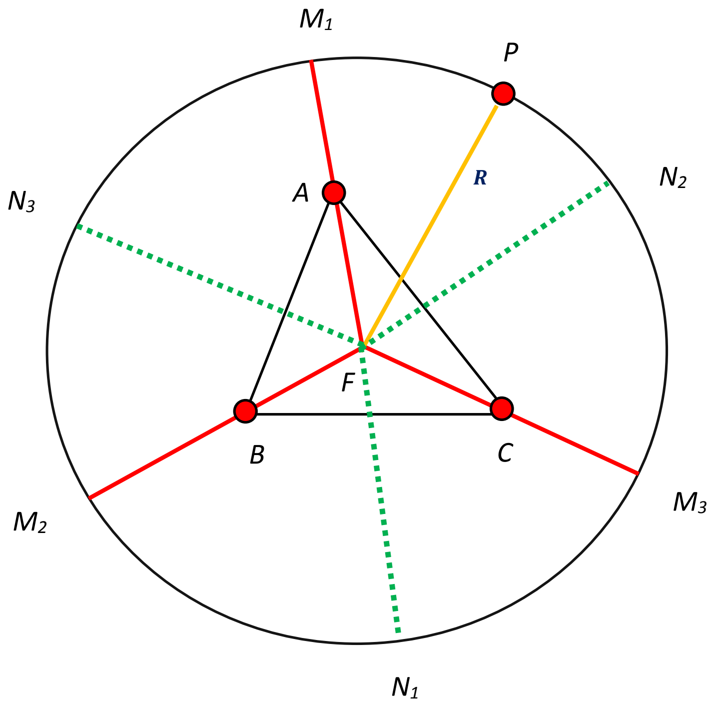


Fig. 6.11: *The dotted (green) lines bisect the three sectors $M_1F\ M_3$, $M_3F\ M_2$ and $M_2F\ M_1$ into a total of six equal sectors M_1FN_2, N_2FM_3, M_3FN_1, N_1FM_2, M_2FN_3* and *N_3FM_1, each subtending an angle of $60°$ at F.*

We begin by determining the coordinate position of the Fermat-Torricelli point, given the coordinates of the vertices A, B and C. The next step is to calculate the lengths of AF, BF and CF using the conventional Cartesian distance formula. If α, β and γ be the angles made by FP with AF, BF and CF respectively, then by applying the Cosine Law of Triangles to ΔAPF, ΔBPF and ΔCPF, the source-vertex distances AP, BP and CP may be computed.

Step-1: Determination of the Coordinates of the Fermat-Torricelli (FT) Point

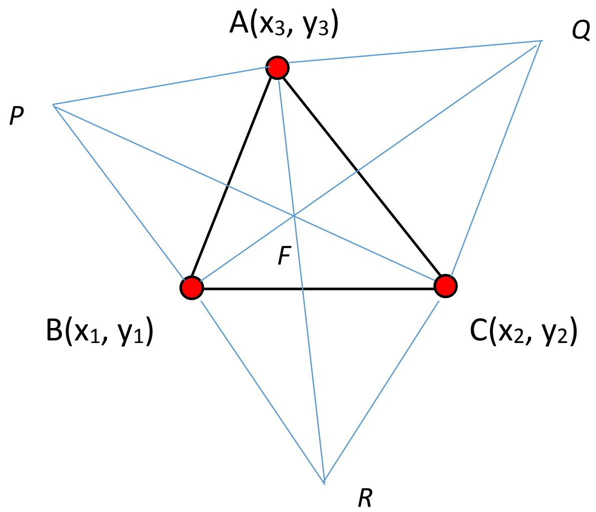


Fig. 6.11: *Three equilateral triangles ΔPAB, ΔRBC and ΔQAC are drawn with sides AB, BC and CA of ΔABC as their respective bases.*

Inorder to find the exact location of the Fermat-Torricelli (FT) point, first draw three equilateral triangles with sides AB, BC and CA of the scalene ΔABC as bases. Then join the vertices P, Q and R of these side triangles to the directly opposite vertices C, B and A respectively, of the main triangle. The lines AR, BQ and CP are concurrent at the FT point [5]. Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the coordinates of the vertices B, C and A respectively.

Using the Two-Point Formula, the equation of the side AB of ΔABC may be written following some rearrangement of terms, as:

$$y = m_{AB}x - m_{AB}x_1 + y_1$$

Where the slope m_{AB} is,

$$m_{AB} = \frac{(y_3 - y_1)}{(x_3 - x_1)}$$

The line AB makes an angle of 60° with the line PA, since ΔAPC is an equilateral triangle. Therefore, we may write,

$$\tan(60°) = \frac{m_{AB} - m_{PA}}{1 + m_{AB}.m_{PA}}$$

From which we get,

$$m_{PA} = \frac{m_{AB} - \sqrt{3}}{1 + \sqrt{3}.m_{AB}}$$

Similarly, the line AB makes an angle of 60° with the line PB, since ΔAPC is an equilateral triangle. Therefore, we may write,

$$\tan(60°) = \frac{m_{PB} - m_{AB}}{1 + m_{PB}.m_{AB}}$$

From which we get,

$$m_{PB} = \frac{m_{AB} + \sqrt{3}}{1 - \sqrt{3}.m_{AB}}$$

Using the Slope-Point Formula, the equation of line PA is,

$$y = m_{PA}x + c_{PA}$$

Since the point A(x_3, y_3) lies on PA, we may substitute this in the above equation to obtain the Y-intercept c_{PA} :

$$c_{PA} = y_3 - m_{PA}x_3$$

Equation of line PA now becomes,

$$y = m_{PA}x + y_3 - m_{PA}x_3$$

Using the Slope-Point Formula, the equation of line PB is,

$$y = m_{PB}x + c_{PB}$$

Since the point A(x_1, y_1) lies on PB, we may substitute this in the above equation to obtain the Y-intercept c_{PB} :

$$c_{PB} = y_3 - m_{PB}x_3$$

Equation of line PB now becomes,

$$y = m_{PB}x + y_1 - m_{PB}x_3$$

On solving the equations of the lines PA and PB, we get the coordinates of the point P,

$$x_P = \left(\frac{m_{PA}x_3 - m_{PB}x_1 - y_3 + y_1}{m_{PA} - m_{PB}}\right)$$

$$y_P = m_{PA}\left(\frac{m_{PA}x_3 - m_{PB}x_1 - y_3 + y_1}{m_{PA} - m_{PB}}\right) + y_3 - m_{PA}x_3$$

Since the coordinates of the points P and C are now known to us as $\left(\frac{m_{PA}x_3 - m_{PB}x_1 - y_3 + y_1}{m_{PA} - m_{PB}}, m_{PA}\left(\frac{m_{PA}x_3 - m_{PB}x_1 - y_3 + y_1}{m_{PA} - m_{PB}}\right) + y_3 - m_{PA}x_3\right)$ and (x_2, y_2) respectively, we may use the Two-Point Formula to obtain the equation of the line PC.

$$\frac{(y - y_P)}{(y_2 - y_P)} = \frac{(x - x_P)}{(x_2 - x_P)}$$

On rearranging, we get,

$$y = m_{PC}x + y_P - m_{PC}x_P$$

Where the slope m_{PC} is,

$$m_{PC} = \frac{(y_2 - y_P)}{(x_2 - x_P)}$$

Using the Two-Point Formula, the equation of the side AC of ΔABC can be written following some rearrangement of terms, as:

$$y = m_{AC}x - m_{AC}x_2 + y_2$$

Where the slope m_{AB} is,

$$m_{AC} = \frac{(y_3 - y_2)}{(x_3 - x_2)}$$

The line AC makes an angle of $60°$ with the line QA, since ΔAQB is an equilateral triangle. Therefore, we may write,

$$\tan(60°) = \frac{m_{QA} - m_{AC}}{1 + m_{QA}.m_{AC}}$$

From which we get,

$$m_{QA} = \frac{m_{AC} + \sqrt{3}}{1 - \sqrt{3}.m_{AC}}$$

Similarly, the line AC makes an angle of $60°$ with the line QC, since ΔAPC is an equilateral triangle. Therefore, we may write,

$$\tan(60°) = \frac{m_{AC} - m_{QC}}{1 + m_{AC}.m_{QC}}$$

From which we get,

$$m_{QC} = \frac{m_{AC} - \sqrt{3}}{1 + \sqrt{3}.m_{AC}}$$

Using the Slope-Point Formula, the equation of line QA is,

$$y = m_{QA}x + c_{QA}$$

Since the point A(x_3, y_3) lies on QA, we may substitute this in the above equation to obtain the Y-intercept c_{QA} :

$$c_{QA} = y_3 - m_{QA}x_3$$

Equation of line QA now becomes,

$$y = m_{QA}x + y_3 - m_{QA}x_3$$

Using the Slope-Point Formula, the equation of line QC is,

$$y = m_{QC}x + c_{QC}$$

Since the point C(x_2, y_2) lies on QC, we may substitute this in the above equation to obtain the Y-intercept c_{QC} :

$$c_{QC} = y_2 - m_{QC}x_2$$

Equation of line QC now becomes,

$$y = m_{QC}x + y_2 - m_{QC}x_2$$

On solving the equations of the lines QA and QC, we get the coordinates of the point Q,

$$x_Q = \left(\frac{m_{QA}x_3 - m_{QC}x_2 - y_3 + y_2}{m_{QA} - m_{QC}}\right)$$

$$y_Q = m_{QA}\left(\frac{m_{QA}x_3 - m_{QC}x_2 - y_3 + y_2}{m_{QA} - m_{QC}}\right) + y_3 - m_{QA}x_3$$

Since the coordinates of the points Q and B are now known to us as $\left(\frac{m_{QA}x_3 - m_{QC}x_2 - y_3 + y_2}{m_{QA} - m_{QC}}, m_{QA}\left(\frac{m_{QA}x_3 - m_{QC}x_2 - y_3 + y_2}{m_{QA} - m_{QC}}\right) + y_3 - m_{QA}x_3\right)$ and (x_1, y_1) respectively, we may use the Two-Point Formula to obtain the equation of the line QB.

$$\frac{(y - y_Q)}{(y_1 - y_Q)} = \frac{(x - x_Q)}{(x_1 - x_Q)}$$

On rearranging, we get,

$$y = m_{QB}x + y_Q - m_{QB}x_Q$$

Where the slope m_{QB} is,

$$m_{QB} = \frac{(y_1 - y_Q)}{(x_1 - x_Q)}$$

Solving the equations of lines PC and QB, finally yields the coordinates of the FT-point,

$$x_F = \frac{m_{PC}x_P - m_{QB}x_Q - y_P + y_Q}{m_{PC} - m_{QB}}$$

$$y_F = m_{PC}\left(\frac{m_{PC}x_P - m_{QB}x_Q - y_P + y_Q}{m_{PC} - m_{QB}}\right) - m_{PC}x_P + y_P$$

<u>Step-2:</u> Calculation of the Vertex-FT distances AF, BF and CF using the Cartesian distance formula between two points.

$$AF = \sqrt{(x_3 - x_F)^2 + (y_3 - y_F)^2}$$

$$BF = \sqrt{(x_1 - x_F)^2 + (y_1 - y_F)^2}$$

$$CF = \sqrt{(x_2 - x_F)^2 + (y_2 - y_F)^2}$$

<u>Step-3</u>: Calculation of Source-Vertex Distances AP, BP and CP by applying the Cosine Law of Triangles to each of ΔAPF, ΔBPF and ΔCPF, respectively. Let α, β and γ be the angles made by FP with AF, BF and CF, respectively.

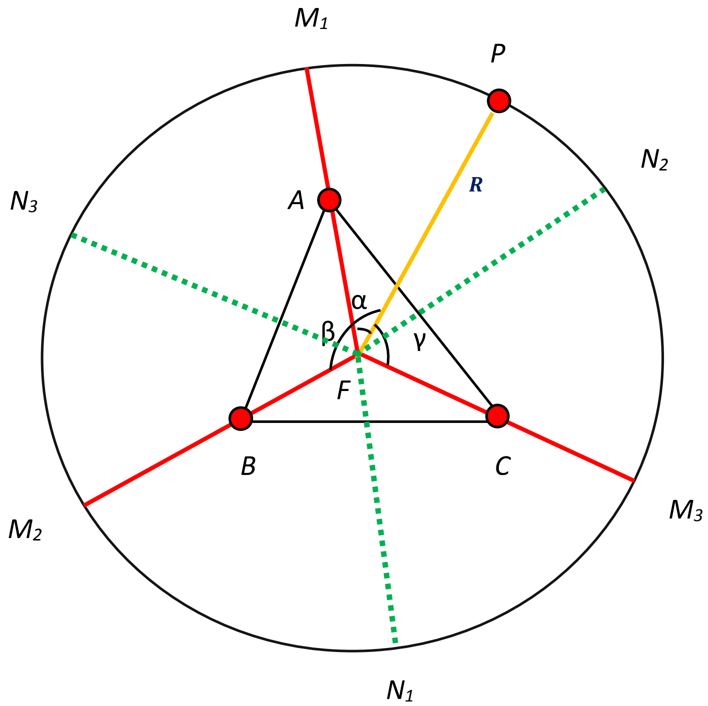


Fig. 6.12: *FP =R. ∠AFP=α. ∠BFP=β. ∠CFP=γ.*

In ΔAFP,

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FP\cos\alpha}$$

In ΔBFP,

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FP\cos\beta}$$

In ΔCFP,

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FP\cos\gamma}$$

By permuting the central angles α, β, γ between θ, $120° + \theta$, $120° - \theta$ and restricting the range of θ within 0° to 60°, the triplet set of Source Position Equations for each of the six sectors (M_1FN_2, N_2FM_3, M_3FN_1, N_1FM_2, M_2FN_3 and N_3FM_1) may be appropriately written.

CASE-1: When External Source P lies in Sector- M_1FN_2 (FA is reference line)

$$\alpha = \theta$$
$$\beta = 120° + \theta$$
$$\gamma = 120° - \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos\theta}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos(120° + \theta)}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos(120° - \theta)}$$

CASE-2: When External Source P lies in Sector- N_2FM_3 (FC is reference line)

$$\alpha = 120° - \theta$$
$$\beta = 120° + \theta$$
$$\gamma = \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos(120° - \theta)}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos(120° + \theta)}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos\theta}$$

CASE-3: When External Source P lies in Sector- M_3FN_1 (FC is reference line)

$$\alpha = 120° + \theta$$
$$\beta = 120° - \theta$$
$$\gamma = \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos(120° + \theta)}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos(120° - \theta)}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos\theta}$$

CASE-4: When External Source P lies in Sector- N_1FM_2 (FB is reference line)

$$\alpha = 120° + \theta$$
$$\beta = \theta$$
$$\gamma = 120° - \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos(120° + \theta)}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos\theta}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos(120° - \theta)}$$

CASE-5: When External Source P lies in Sector- M_2FN_3 (FB is reference line)

$$\alpha = 120° - \theta$$
$$\beta = \theta$$
$$\gamma = 120° + \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos(120° - \theta)}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos\theta}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos(120° + \theta)}$$

CASE-6: When External Source P lies in Sector- N_3FM_1 (FA is reference line)

$$\alpha = \theta$$
$$\beta = 120° - \theta$$
$$\gamma = 120° + \theta$$

$$AP = \sqrt{AF^2 + FP^2 - 2.AF.FPcos\theta}$$

$$BP = \sqrt{BF^2 + FP^2 - 2.BF.FPcos(120° - \theta)}$$

$$CP = \sqrt{CF^2 + FP^2 - 2.CF.FPcos(120° + \theta)}$$

Step-4: Determination of the magnitude of the ISIs for each of the stimulated source pairs, i.e. $|\Delta t_{AC}|, |\Delta t_{AB}|$ and $|\Delta t_{CB}|$. If v be the speed of the circular stimulating wavefront emanating from external source P, then the times of arrival t_A, t_B and t_C of this wavefront at the vertices *A, B* and *C* of scalene *ΔABC,* after having traversed distances *PA, PB* and *PC,* respectively are given by:

$$t_A = \frac{PA}{v}$$

$$t_B = \frac{PB}{v}$$

$$t_C = \frac{PC}{v}$$

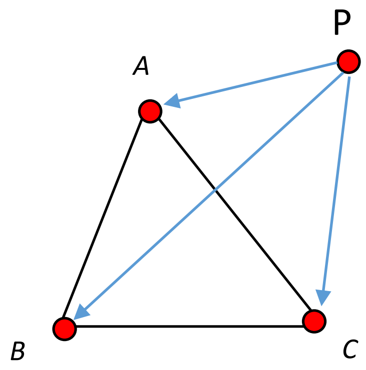


Fig. 6.13

Therefore, the magnitude of the ISIs for each of the successive source pair stimulations are:

$$|\Delta t_{AC}| = |t_C - t_A| = \left|\frac{PC - PA}{v}\right|$$

$$|\Delta t_{AB}| = |t_B - t_A| = \left|\frac{PB - PA}{v}\right|$$

$$|\Delta t_{CB}| = |t_B - t_C| = \left|\frac{PB - PC}{v}\right|$$

<u>**Step-5:**</u> Calculation of the J-parameters corresponding to the ISIs of each source pair.

$$J_1 = \frac{u|\Delta t_{CB}|}{2}, J_2 = \frac{u|\Delta t_{AC}|}{2}, J_3 = \frac{u|\Delta t_{AB}|}{2}$$

<u>**Step-6:**</u> Obtain the graphical plots of the triplet set of Dynamic Hyperbolic Equations. The instantaneous point of concurrence of the three Dynamic Circles can be found by zooming into the point of intersection of the three Dynamic Hyperbolic branches. Please note, that the coordinates of the vertices of ΔABC are $A(0, a)$, $B(-b, 0)$ and $C(c, 0)$ and also, the numbers a, b and c are non-negative.

(i) Equation of Dynamic Hyperbola with side CB as transverse axis,

$$y = \pm\sqrt{\left(\left(\frac{b+c}{2}\right)^2 - J^2{}_1\right)}\sqrt{\left(\frac{\left(x-\frac{c-b}{2}\right)^2}{J^2{}_1} - 1\right)}$$

(ii) Equation of Dynamic Hyperbola with side AC as transverse axis,

$$y = \frac{-4\left[-2cax+a\left(c^2-a^2\right)+4aJ_2^2\right]\pm\sqrt{64J_2^2\left(c^2+a^2-4J_2^2\right)\left(4x^2+4cx+c^2+a^2-4J_2^2\right)}}{8\left(a^2-4J_2^2\right)}$$

(iii) Equation of Dynamic Hyperbola with side AB as transverse axis,

$$y = \frac{-4\left[2bax+a\left(b^2-a^2\right)+4aJ_3^2\right]\pm\sqrt{64J_3^2\left(b^2+a^2-4J_3^2\right)\left(4x^2-4bx+b^2+a^2-4J_3^2\right)}}{8\left(a^2-4J_3^2\right)}$$

Here, u is the uniform rate of expansion of the Dynamic Circles. In order to obtain the triplet set of Dynamic Hyperbola equations for a Right Isosceles triangle configuration, substitute $c = b$ and $a = b$ in the above.

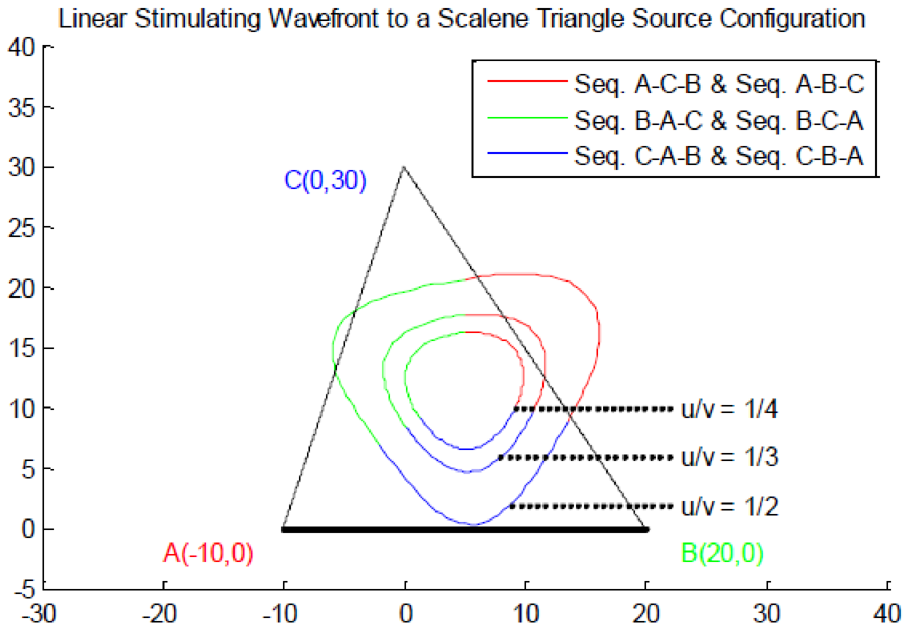
6.4 Summary of the Localization Algorithm for a Circular Stimulating Wavefront

The algorithm used for the determination of the point of concurrence of three dynamic circles, when the stimulating wavefront is circular and the three sources are distributed at the vertices of a scalene triangle, is summarized below:

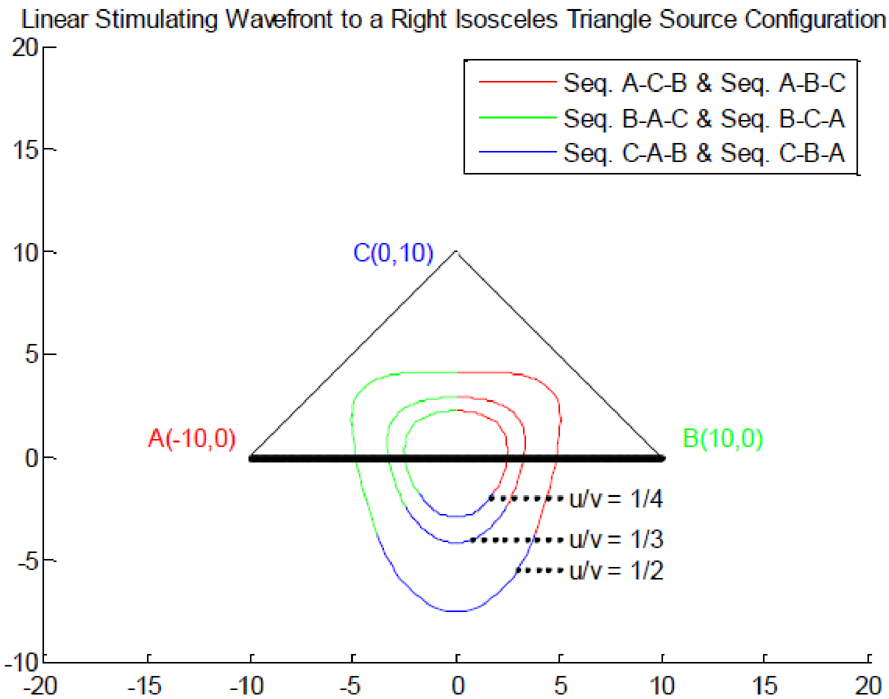
1. Choose an arbitrary central angle θ between the radial line FP and one of the reference lines FA, FB or FC. (**N.B.** Range of θ: $0° \leq \theta \leq 60°$; F is the Fermat-Torricelli point).
2. Write the three central angles α, β, γ as permutations of $\theta, 120° + \theta, 120° - \theta$, based on which of the six sectors, the external source P is located.
3. Compute AP, BP and CP using the triplet set of Source-Position Equations.
4. Compute $|\Delta t_{AC}|$, $|\Delta t_{AB}|$ and $|\Delta t_{CB}|$ using the ISI Equations.
5. Compute the J-parameters corresponding to each of the ISIs.
6. Obtain the graphical plot of the triplet set of Dynamic Hyperbola equations. Each hyperbola has one side of the triangle as its transverse axis.
7. Locate the instantaneous point of concurrence of the three Dynamic Circles, by zooming into the point of intersection of the three Dynamic Hyperbolic branches.

7. Results: Numerical-Graphical Simulation using MATLAB

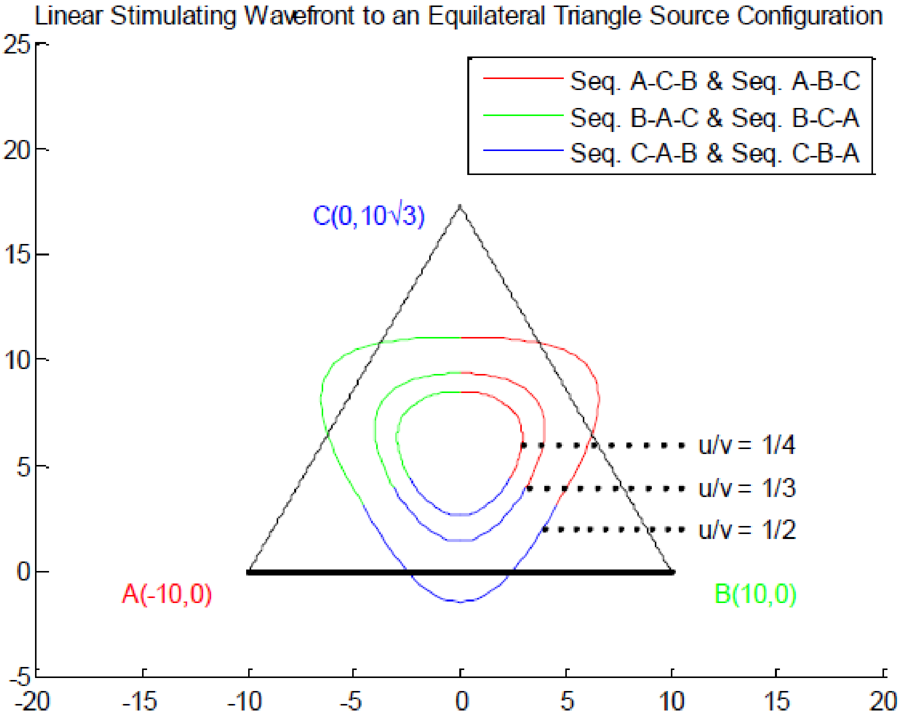
7.1 For a Linear Stimulating Wavefront and Scalene Configuration of Sources



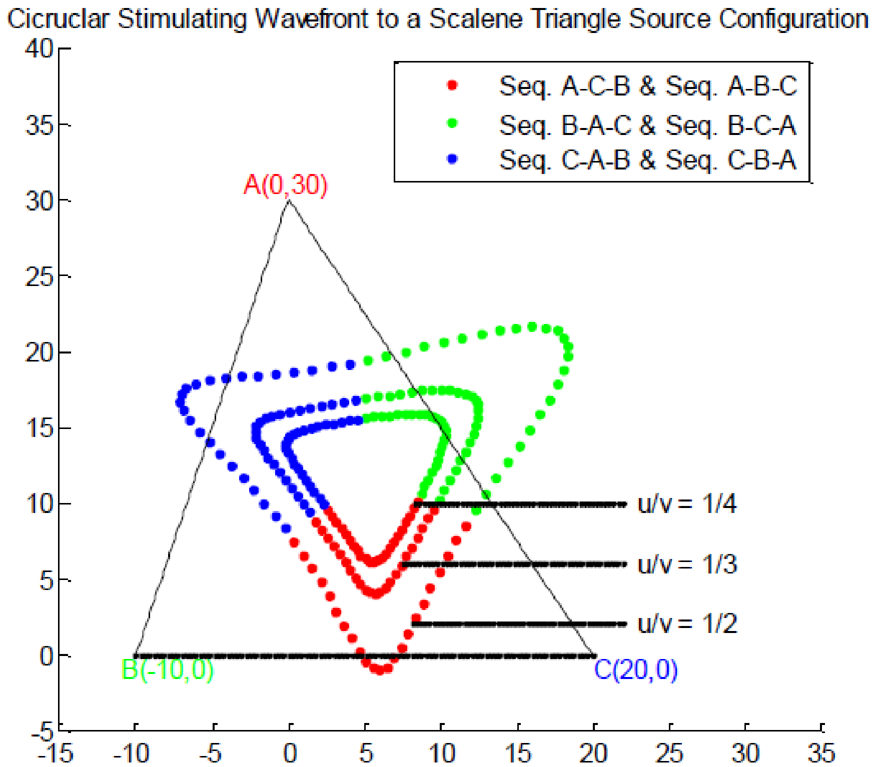
7.2 For a Linear Stimulating Wavefront and Right Isosceles Configuration of Sources



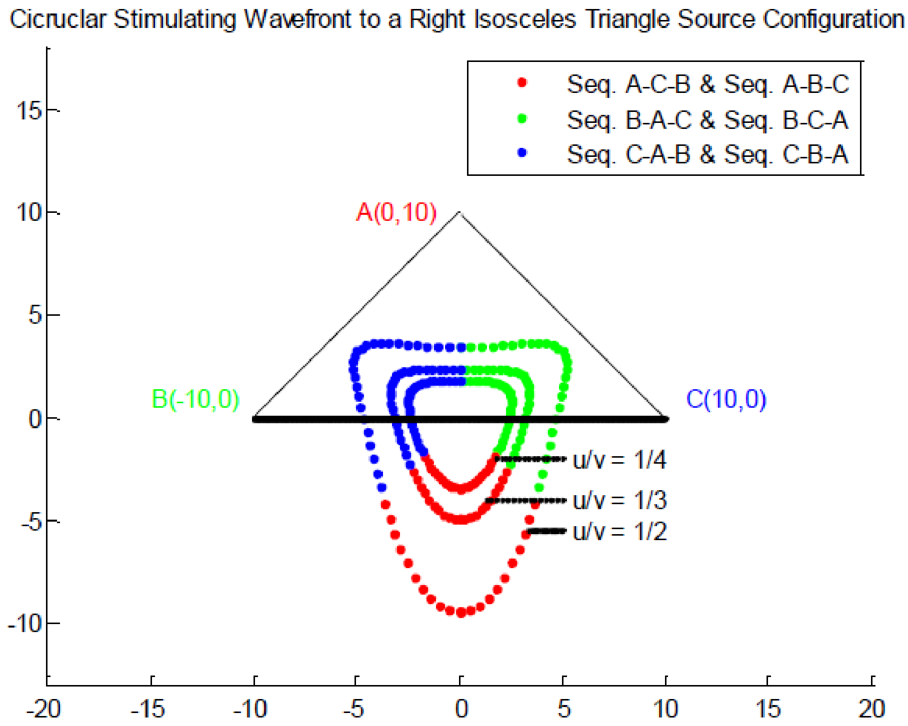
7.3 For a Linear Stimulating Wavefront and Equilateral Configuration of Sources



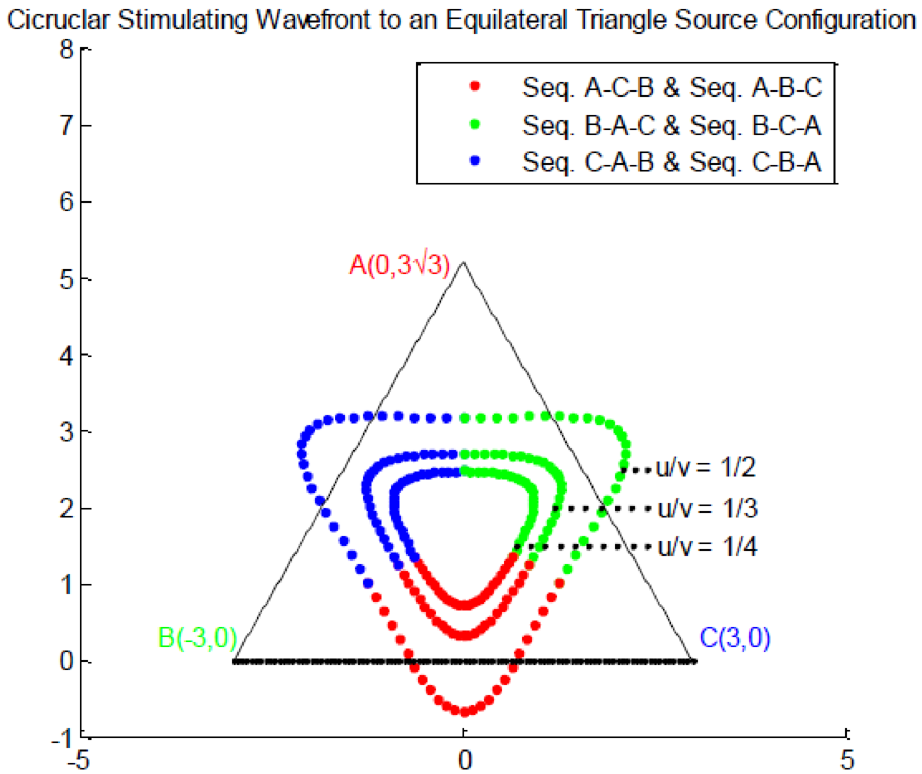
7.4 For a Circular Stimulating Wavefront and Scalene Source Configuration



7.5 For a Circular Stimulating Wavefront and Right Isosceles Source Configuration



7.6 For a Circular Stimulating Wavefront and Equilateral Source Configuration



8. Discussion

8.1 Nomenclature and Classification of PWC Curves

A systematic framework for naming and classifying all the PWC curves obtained in §7, is shown in Table 8.1 below. A PWC curve may be designated as Type-1, when the stimulating wavefront is linear in shape and Type-2, when it is circular. Each PWC curve type, may be further subtyped into a, b and c varieties, corresponding to the different triangular source configurations. These include scalene, right isosceles and equilateral triangle configurations.

PWC Curve	*Shape of Stimulating Wavefront*	*Triangular Configuration of Sources*
Type-1a	Linear	Scalene
Type-1b	Linear	Right Isosceles
Type-1c	Linear	Equilateral
Type-2a	Circular	Scalene
Type-2b	Circular	Right Isosceles
Type-2c	Circular	Equilateral

Table 8.1

8.2 On General Curve Morphology [6]

a. **Simple curve:** A non-self-intersecting curve.

Fig. 8.2.1: *Simple Curve and Non-Simple Curve*

b. **Closed curve:** A curve that has no end-points and encloses a finite area.

Fig. 8.2.2: *Closed Curve and Open Curve*

c. **Jordan curve:** A plane curve that is both simple and closed.

d. **Convex curve:** A plane curve enclosing a finite area that contains all line segments connecting any pair of its points.

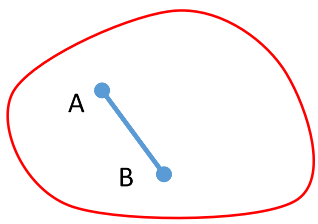


Fig. 8.2.3: *Line segment AB connects arbitrary points A and B*

e. **Concave curve:** A plane curve enclosing a finite area that does not contain all the line segments connecting any pair of its points.

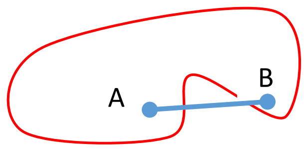


Fig. 8.2.4: *Line segment AB connects arbitrary points A and B*

f. **Reflection Symmetry:** A Jordan curve is said to possess reflection symmetry, if there exists atleast one line that divides it into two halves, each a mirror image of the other. A tri-symmetrical Jordan curve has three lines of reflection symmetry. A mono-symmetrical Jordan curve has one line of reflection symmetry. An asymmetrical Jordan curve has no line of reflection symmetry.

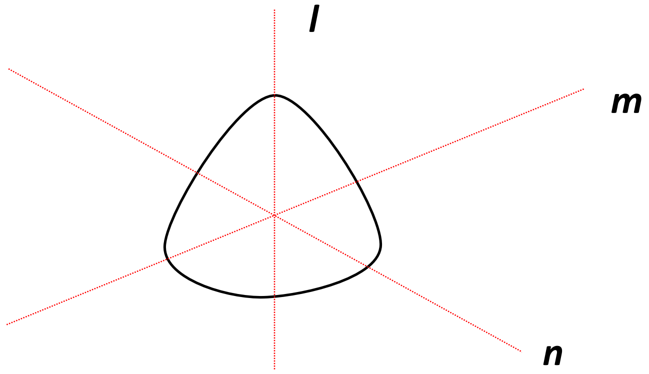


Fig. 8.2.5: *Three lines of reflection symmetry l, m and n*

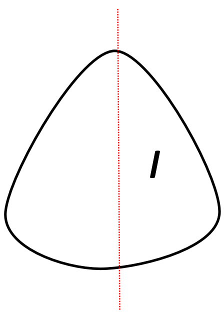


Fig. 8.2.6: *Single line of reflection symmetry l*

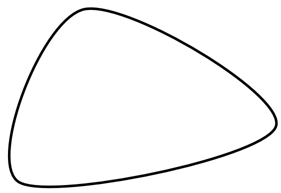


Fig. 8.2.7: *No line of reflection symmetry*

g. **Trifocal ellipse:** The locus of the point, whose sum of distances from three fixed points in the same plane, is constant. The curve so traced is also known by other names - an oval or an egg-lipse or a 3-ellipse. [5],[7],[8],[9],[10]

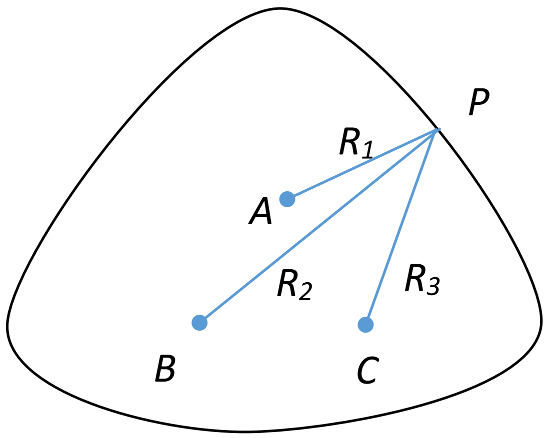


Fig. 8.2.8: *A 3-ellipse is characterized by sum of distances $R_1 + R_2 + R_3$ = constant*

8.3 Some Final Theorems, Propositions and Conjectures on PWC Curves

8.3.1 Theorem

A dynamic hyperbola is generated from the locus of the intersection points of two dynamic circles, if the condition $0 < \left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right) < 1$ *is satisfied. Here, u is the expansion rate of the dynamic circles, v is the speed of propagation of the stimulating wavefront, p is the separation distance between the two sources lying in the medium and f(p) is some function f of p, such that* $0 \leq f(p) \leq p$.

Proof

Let A and B be two point sources lying in the medium, that are separated by a distance, $AB = p$. If Δt_{AB} be the interval between the successive stimulations of sources A and B, and u be the expansion rate of the dynamic circles in the medium, then by the Dynamic Hyperbola Theorem, we may write:

$$\frac{x^2}{\left(\frac{u.\Delta t_{AB}}{2}\right)^2} - \frac{y^2}{\left(\frac{p}{2}\right)^2 - \left(\frac{u.\Delta t_{AB}}{2}\right)^2} = 1$$

Clearly, for the above equation to represent a hyperbola it is necessary that the denominator of the y^2 term, always remain a positive (real) number. That is,

$$\left(\frac{p}{2}\right)^2 - \left(\frac{u.\Delta t_{AB}}{2}\right)^2 > 0$$

$$\Rightarrow \left(\frac{u.\Delta t_{AB}}{p}\right)^2 < 1$$

$$\Rightarrow -1 < \left|\frac{u.\Delta t_{AB}}{p}\right| < 1$$

$$\Rightarrow 0 < \frac{u.|\Delta t_{AB}|}{p} < 1$$

Recall, that the quantity $|\Delta t_{AB}|$ depends on the propagation speed v of the stimulating wavefront and also its shape. For a linear stimulating wavefront, $|\Delta t_{AB}| = \frac{p.Sin\beta}{v}$ where β is the variable inclination of the wavefront with the line AB joining the sources A and B. For a circular stimulating wavefront, $|\Delta t_{AB}| = \frac{|AP-BP|}{v}$ where AP and BP are the distances between the external source P and the sources lying at A and B, respectively. Further, he quantities, AP and BP is expressible in terms of the inter-source separation distance $AB = p$. We are thus, justified in writing $|\Delta t_{AB}|$ in the form $\frac{f(p)}{v}$, where *f* is some function of *p*.

Note that, $f(p) = 0$ for: (i) a linear stimulating wavefront when $\beta = 0$, (ii) a circular stimulating wavefront when $AP = BP$. Also, $f(p) = p$ for: (i) a linear stimulating wavefront when $\beta = 90°$, (ii) a circular stimulating wavefront when $|AP - BP| = p$. Thus, we may write the range of $f(p)$ as $0 \leq f(p) \leq p$. Hence, the condition to be satisfied for the generation of a dynamic hyperbola, from the locus of the intersection points of two dynamic circles is,

$$0 < \left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right) < 1\,;\; 0 \leq f(p) \leq p$$

8.3.2 Corollary

Three dynamic hyperbolas are generated from the locus of the pair wise intersections of three dynamic circles emanating from their respective sources lying in the medium, if the following triplet set of conditions are simultaneously satisfied:

$$0 < \left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right) < 1 \qquad 0 < \left(\frac{u}{v}\right).\left(\frac{g(q)}{q}\right) < 1 \qquad 0 < \left(\frac{u}{v}\right).\left(\frac{h(r)}{r}\right) < 1$$

Here, u is the expansion rate of the dynamic circles, v is the speed of propagation of the stimulating wavefront, p, q and r are the separation distances between each source pair lying in the medium, and f(p), g(q) and h(r) are some functions f, g and h of p, q and r, respectively such that $0 \leq f(p) \leq p$, $0 \leq g(q) \leq q$ *and* $0 \leq h(r) \leq r$.

8.3.3 Theorem

When three non-collinear sources in a medium are successively stimulated by a linear or circular wavefront that propagates through the medium with a speed v, three dynamic circles, each with equal expansion rate u are generated. These dynamic circles come to meet at a particular spatial point in the medium, for a single instant in time, if and only if the three dynamic hyperbolas generated from the pair wise intersections of the three dynamic circles share a point of concurrence.

.

Proof

Case-1:

Let us assume that an instantaneous point of concurrence exists for three mutually intersecting dynamic circles with non-collinear source-centers. That is, when three dynamic circles expand radially outwards from their respective non-collinear source-centers following stimulation, they meet at a single spatial point somewhere in the plane of the medium, at some instant of time. For any two, out of the three mutually intersecting dynamic circles, a single dynamic hyperbolic branch is generated (by Theorem 8.3.1). This implies that, for three mutually intersecting dynamic circles, there are three mutually intersecting dynamic hyperbolic branches generated. Therefore, if an instantaneous point of concurrence of three dynamic circles exists, it implies that a point of concurrence of three dynamic hyperbolic branches also exists.

Case-2:
Let us now assume that a point of concurrence exists for three dynamic hyperbolic branches whose transverse axes together form the three sides a triangle. Say that, three non-collinear sources lie at the vertices of this triangle. For any single dynamic hyperbolic branch with one side of the triangle as transverse axis, there are two dynamic circles that generate it, with source-centers lying at the side end-points. Therefore, for three mutually intersecting dynamic hyperbolic branches, there are three mutually intersecting dynamic circles that generate them and their source-centers lie at the three vertices of the triangle. Therefore, if a point of concurrence of three dynamic hyperbolic branches exists, it implies that an instantaneous point of concurrence of three dynamic circles also exists.

By combining the inferences of case-1 and case-2, we may conclude that three dynamic circles come to meet at a particular spatial point in the medium, for a single instant in time, if and only if the three dynamic hyperbolas generated from the pair wise intersections of the three dynamic circles share a point of concurrence.

8.3.4 Theorem

For a given stimulating wavefront and triangular configuration of sources in the medium, the locus of all the instantaneous points of concurrence obtained when the sequences and intervals of source stimulations are varied, forms a Jordan curve (i.e. a simple and closed curve), if the condition $0 < \frac{u}{v} < 1$ *is satisfied. In other words, a PWC curve is of the Jordan kind, if the condition* $0 < \frac{u}{v} < 1$ *is satisfied.*

Proof

In theorem 8.3.1, we have shown that a dynamic hyperbolic branch is generated from the locus of the intersection points of two dynamic circles emanating from their respective sources, if the condition $0 < \left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right) < 1\,;\; 0 \leq f(p) \leq p$ is satisfied.

By Corollary 8.3.2, for three non-collinear sources A, B and C lying in the medium with inter-source separation distances $AB = p,\ BC = q, CA = r,$ three dynamic hyperbolic branches are generated from the locus of the pair wise intersections of the three dynamic circles, if the following triplet set of conditions are simultaneously satisfied: $0 < \left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right) < 1\ ;\ 0 \leq f(p) \leq p,\ 0 < \left(\frac{u}{v}\right).\left(\frac{g(q)}{q}\right) < 1\ ;\ 0 \leq g(q) \leq q$ and $0 < \left(\frac{u}{v}\right).\left(\frac{h(r)}{r}\right) < 1\ ;\ 0 \leq h(r) \leq r.$

Let us now carefully scrutinize the quantities:$\left\{\left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right), \left(\frac{u}{v}\right).\left(\frac{g(q)}{q}\right), \left(\frac{u}{v}\right).\left(\frac{h(r)}{r}\right)\right\}$. Corollary 8.3.2, requires that the numerical magnitudes of these products lie within the open interval $(0, 1)$, so that a dynamic hyperbola exist. While $\{u, v, p, q, r\}$ are fixed constants, $\{f(p), g(q), h(r)\}$ are variables with fixed maximums $\{p, q, r\}$. Recall, that the variables $\{f(p), g(q), h(r)\}$ depend on the shape and orientation of the stimulating wavefront. Therefore, the maximum value that each of the products: $\left\{\left(\frac{u}{v}\right).\left(\frac{f(p)}{p}\right), \left(\frac{u}{v}\right).\left(\frac{g(q)}{q}\right), \left(\frac{u}{v}\right).\left(\frac{h(r)}{r}\right)\right\}$ can take for any shape and orientation of a wavefront, (or equivalently, different sequences and intervals of source stimulations) is $\frac{u}{v}$. It may thus, be concluded that the absolute condition to be satisfied, for the locus of all the instantaneous points of concurrence to form a simple and closed curve (a.k.a. Jordan curve), is $0 < \frac{u}{v} < 1$.

8.3.5 Proposition

When the propagation speed v of the stimulating wavefront, is varied with respect to the fixed expansion rate u of the dynamic circles, keeping the condition $0 < \frac{u}{v} < 1$ *satisfied, a family of concentric PWC curves of the Jordan kind are obtained for different sequences and intervals of source stimulations. These curves are centered about the circumcenter of the triangle, at whose vertices the sources of the dynamic circles lie.*

8.3.6 Proposition

As $\frac{u}{v} \to 0$*, the area of the region enclosed by the innermost member of the family of concentric PWC curves of the Jordan kind, progressively diminishes, till collapse of the innermost member to a single geometrical point occurs at* $\frac{u}{v} = 0$*. This point is the circumcenter of the triangle, at whose vertices the sources of the dynamic circles lie.*

8.3.7 Proposition

As $\frac{u}{v} \rightarrow 1$, *the area of the region enclosed by the outermost member of the family of concentric PWC curves of the Jordan kind, progressively enlarges, till a break in the geometrical continuity of the outermost member occurs at* $\frac{u}{v} = 1$.

8.3.8 Proposition

The PWC curve for which $\frac{u}{v} \geq 1$, *is piece-wise continuous and therefore, non-Jordan.*

8.3.9 Proposition

The number of lines of reflection symmetry of a given PWC curve of the Jordan kind, is equal to the number of lines of reflection symmetry of the triangle, at whose vertices the sources of the dynamic circles lie. More elaborately stated, a tri-symmetrical PWC curve of the Jordan kind, is obtained when the triangular source configuration is equilateral; a mono-symmetrical PWC curve of the Jordan kind is obtained when the triangular source configuration is isosceles; an asymmetrical PWC curve of the Jordan kind is obtained when the triangular configuration is scalene.

8.3.10 Proposition

A PWC curve of the Jordan kind, has in total three vertices – one corresponding to each of the three vertices of the triangle from which that curve is obtained.

8.3.11 Proposition

The vertex of a PWC curve of the Jordan kind, may be defined as that point V' on one of its arc segments, for which the distance VV' is maximum; V is any one of the three vertices of the triangle from which that curve is obtained. V' and V may therefore, be referred to as corresponding vertices.

8.3.12 Proposition

If $V_1', V_2'\ and V_3'$ *be the vertices of a PWC curve of the Jordan kind and* $V_1, V_2\ and\ V_3$ *be the vertices of the triangle from which that curve is obtained, then the three lines* $V_1'V_1, V_2'V_2\ and\ V_3'V_3$ *joining corresponding vertices share a point of concurrence. This common point may be called the Point of Geometric Inversion, denoted by* $\boldsymbol{I}$.

8.3.13 Proposition

The vertex V' of a PWC curve of the Jordan kind and the corresponding vertex V of the triangle from which that curve is obtained, bear a geometrically inverse relationship with respect to the point of concurrence, $\boldsymbol{I}$.

8.3.14 Proposition

A convex PWC curve of the Jordan kind is obtained when a triangular configuration of sources is stimulated by a linear wavefront.

8.3.15 Proposition

A concave PWC curve of the Jordan kind is obtained when a triangular configuration of sources is stimulated by a circular wavefront. The concavity of the curve becomes increasingly evident as $\frac{u}{v} \rightarrow 1$.

8.3.16 Conjecture

A PWC curve of the Jordan kind that is obtained for a linear stimulating wavefront and a triangular configuration of sources, is a Trifocal Ellipse. The coordinate positions of the three foci of the Trifocal Ellipse may be found by inverting the vertices of the triangular configuration about the Point of Geometric Inversion $\boldsymbol{I}$.

8.3.17 Conjecture

The area of the region enclosed by a PWC curve of the Jordan kind, may be expressed as some function of the ratio $\frac{u}{v}$*, such that (i) as* $\frac{u}{v} \rightarrow 0$*, the area function progressively decreases till it reaches zero for* $\frac{u}{v} = 0$*, and (ii) as* $\frac{u}{v} \rightarrow 1$*, the area function progressively increases till it reaches infinity for* $\frac{u}{v} = 1$.

One candidate function fitting this description is $S = k_1 . \frac{\left(\frac{u}{v}\right)^p .(s)^{k_2 . \left(\frac{u}{v}\right)^q}}{\left(1-\frac{u}{v}\right)^r}$*, where S is the area of the region enclosed by the PWC curve of the Jordan kind, s is the area of the triangle from which that curve is obtained and* $\{p, q, r, k_1, k_2\} \in \mathbb{R}^+$

8.4 Prior Results

In preceding work by the same author, two additional types of PWC curves were constructed, besides those in § 7. These curves were obtained for an equilateral triangle source configuration and a rigid, semicircular (both convex and concave) stimulating wavefront. For the convex, semicircular stimulating wavefront, the locus of the instantaneous points of concurrence forms a Jordan curve. However, for the concave, semicircular stimulating wavefront, the locus of the instantaneous points of concurrence forms a piece-wise continuous curve.

According to the nomenclature and classification system introduced in § 8.1, these curves should fall under a new category Type-3c, which encompasses subdivisions Type-3c-i for convex, semicircular stimulating wavefront and Type-3c-ii for concave, semicircular stimulating wavefront. Further, details on Type-3c PWC curves can be found in the paper titled "A Mathematical Treatise on Polychronous Wavefront Computation and its Application into Modeling Neurosensory Systems (2014)".

8.5 Future Directions

I. The conjectures stated in § 8.3 are to be furnished with rigorous proof/disproof.
II. The algebraic equations of each type of PWC curve of the Jordan kind are to be determined. This will thereby, help establish their observed geometric properties.
III. In principle, it should be possible to construct PWC curves, when there are more than three sources involved.
IV. The geometrical framework of PWC developed here, may find later utility in modeling biological systems [11], [12].

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Supplementary Material

MATLAB Code

1. Linear Stimulating Wavefront and Scalene Triangle Configuration of Sources (Fig. 7.1)

```
% Sequence A to C to B

x_11 = [13.5548 13.8602 14.1591 14.4494 14.7286 14.9937 15.2418 15.4700 15.6750
15.8533 16.0022 16.1181 16.1987 16.2411 16.2430 16.2029 16.1194 15.9915 15.8191
15.6022] ;
y_11 = [8.8151 9.3502 9.8911 10.4372 10.9876 11.5418 12.0984 12.6564 13.2145
13.7707 14.3234 14.8703 15.4092 15.9375 16.4526 16.9517 17.4323 17.8913 18.3262
18.7348] ;

x_12 = [10.7500 10.9033 11.0467 11.1797 11.3012 11.4102 11.5058 11.5872 11.6634
11.7034 11.7363 11.7516 11.7482 11.7257 11.6837 11.6214 11.5390 11.4361 11.3127
11.1691];
y_12 = [9.7500 10.0850 10.4232 10.7638 11.1060 11.4487 11.7912 12.1325 12.5316
12.8078 13.1398 13.4668 13.7875 14.1012 14.4068 14.7032 14.9894 15.2649 15.5284
15.7794];

x_13 = [9.3451 9.4455 9.5377 9.6208 9.6945 9.7585 9.8121 9.8548 9.8864 9.9062
9.9140 9.9094 9.8922 9.8619 9.8186 9.7619 9.6919 9.6085 9.5118 9.4020] ;
y_13 = [10.2183 10.4610 10.7060 10.9528 11.2006 11.4489 11.6970 11.9441 12.1897
12.4331 12.6734 12.9103 13.1429 13.3706 13.5928 13.8089 14.0184 14.2206 14.4151
14.6014] ;



% Sequence A to B to C

x_21 = [5.000 5.6631 6.3294 6.9972 7.6649 8.3301 8.9906 9.6438 10.2864 10.9151
11.5266 12.1164 12.6812 13.2165 13.7184 14.1830 14.6068 14.9863 15.3187 15.6022]
;
y_21 = [20.6973 20.8023 20.8988 20.9845 21.0571 21.1135 21.1509 21.1660 21.1556
21.1168 21.0465 20.9423 20.8019 20.6237 20.4062 20.1492 19.8526 19.5171 19.1440
18.7348] ;

x_22 = [5.0000 5.4009 5.8015 6.2005 6.5962 6.9877 7.3730 7.7509 8.1197 8.4780
8.8244 9.1571 9.4748 9.7760 10.0595 10.3238 10.5677 10.7880 10.9912 11.1691];
y_22 = [17.7452 17.7780 17.7989 17.8072 17.8023 17.7837 17.7505 17.7021 17.6378
17.5569 17.4591 17.3438 17.2107 17.0596 16.8904 16.7030 16.4978 16.2780 16.0355
15.7794];

x_23 = [5.0000 5.2861 5.5715 5.8555 6.1367 6.4146 6.6876 6.9552 7.2162 7.4697
7.7147 7.9504 8.1756 8.3898 8.5922 8.7818 8.9581 9.1206 9.2687 9.4020] ;
y_23 = [16.2596 16.2744 16.2789 16.2728 16.2561 16.2283 16.1891 16.1384 16.0761
16.0018 15.9154 15.8170 15.7063 15.5835 15.4487 15.3020 15.1435 14.9737 14.7928
14.6014] ;
```

```
% Sequence B to A to C

x_31 = [5 4.1639 3.3378 2.5239 1.7245 0.9426 0.1807 -0.5576 -1.2685 -1.9475 -
2.5899 -3.1903 -3.7433 -4.2434 -4.6852 -5.0642 -5.3763 -5.6185 -5.7893  -5.8881]
;
y_31 = [20.6973 20.5543 20.4031 20.2451 20.0807 19.9090 19.7282 19.5359 19.3288
19.1033 18.8560 18.5833 18.2823 17.9507 17.5868 17.1902 16.7612 16.3010 15.8118
15.2960] ;

x_32 = [5.0000 4.4921 3.9885 3.4918 3.0044 2.5290 2.0679 1.6236 1.1985 0.7950
0.4153 0.0616 -0.2639 -0.5594 -0.8231 -1.0536 -1.2496 -1.4104 -1.5357 -1.6250];
y_32 = [17.7452 17.6872 17.6117 17.5197 17.4115 17.2875 17.1479 16.9926 16.8216
16.6345 16.4313 16.2117 15.9754 15.7227 15.4534 15.1678 14.8665 14.5501 14.2193
13.8750];

x_33 = [5.0000 4.6371 4.2767 3.9209 3.5716 3.2304 2.8995 2.5803 2.2748 1.9843
1.7107 1.4550 1.2188 1.0033 0.8093 0.6377 0.4895 0.3649 0.2646 0.1882] ;
y_33 = [16.2596 16.2265 16.1777 16.1136 16.0347 15.9413 15.8339 15.7125 15.5774
15.4288 15.2671 15.0924 14.9049 14.7051 14.4931 14.2697 14.0350 13.7899 13.5348
13.2706] ;


% Sequence B to C to A

x_41 = [-2.0124 -2.3380 -2.6604 -2.9785 -3.2910 -3.5963 -3.8925 -4.1778 -4.4498
-4.7064 -4.9448 -5.1630 -5.3581 -5.5277 -5.6695 -5.7810 -5.8605 -5.9056 -5.9154
-5.8881] ;
y_41 = [6.9917 7.4395 7.8889 8.3395 8.7907 9.2423 9.6938 10.1447 10.5949 11.0438
11.4908 11.9354 12.3769 12.8144 13.2470 13.6736 14.0931 14.5042 14.9056 15.296]
;

x_42 = (-1).*[-0.1772 0.0179 0.2061 0.3866 0.5589 0.7223 0.8759 1.0189 1.1506
1.2700 1.3764 1.4689 1.5468 1.6092 1.6557 1.6852 1.6974 1.6917 1.6677 1.6250];
y_42 = [8.4515 8.7347 9.0216 9.3116 9.6039 9.8980 10.1932 10.4891 10.7848
11.0800 11.3740 11.6662 11.9563 12.2433 12.5270 12.8067 13.0818 13.3519 13.6165
13.8750];

x_43 = [1.3159 1.1767 1.0439 0.9179 0.7993 0.6884 0.5856 0.4914 0.4065 0.3318
0.2657 0.2106 0.1664 0.1335 0.1121 0.1025 0.1052 0.1202 0.1479 0.1882] ;
y_43 = [9.2106 9.4174 9.6277 9.8410 10.0568 10.2746 10.4939 10.7142 10.9350
11.1558 11.3761 11.5958 11.8141 12.0306 12.2449 12.4568 12.6656 12.8711 13.0728
13.2706] ;


% Sequence C to A to B

x_51 = [13.5548 13.1286 12.6979 12.2657 11.8339 11.4030 10.9733 10.5438 10.1135
9.6804 9.2432 8.8002 8.3503 7.8928 7.4264 6.9538 6.4716 5.9815 5.4885  5] ;
y_51 = [8.8151 8.0909 7.3800 6.6841 6.0050 5.3450 4.7068 4.0939 3.5103 2.9604
2.4494 1.9824 1.5652 1.2030 0.9008 0.6632 0.4942 0.3950 0.3685 0.4138] ;

x_52 = [10.7500 10.5253 10.2859 10.0334 9.7695 9.4953 9.2120 8.9206 8.6218
8.3161 8.0043 7.6867 7.3638 7.0361 6.7041 6.3684 6.0295 5.6879 5.3451 5.0000];
y_52 = [9.7500 9.2968 8.8534 8.4216 8.0038 7.6020 7.2181 6.8544 6.5131 6.1961
5.9057 5.6437 5.4121 5.2126 5.0466 4.9156 4.8204 4.7617 4.7398 4.7548];

x_53 = [9.3451 9.1943 9.0297 8.8526 8.6638 8.4645 8.2556 8.0380 7.8124 7.5797
7.3403 7.0954 6.8452 6.5904 6.3316 6.0696 5.8048 5.5379 5.2696 5] ;
y_53 = [10.2183 9.8903 9.5697 9.2580 8.9570 8.6681 8.3929 8.1330 7.8896 7.6642
7.4580 7.2725 7.1085 6.9671 6.8491 6.7554 6.6863 6.6421 6.6231 6.6293] ;
```

```
% Sequence C to B to A

x_61 = [-2.0124 -1.6582 -1.3023 -0.9453 -0.5874 -0.2288 0.1306 0.4911 0.8531
1.2168 1.5826 1.9511 2.3225 2.6971 3.0748 3.4558 3.8381 4.2270 4.6165 5] ;
y_61 = [6.9917 6.5107 6.0335 5.5611 5.0951 4.6370 4.1888 3.7525 3.3304 2.9252
2.5394 2.1761 1.8380 1.5281 1.2495 1.0049 0.7966 0.6278 0.4995 0.4138] ;

x_62 = [0.1771 0.3949 0.6193 0.8497 1.0855 1.3261 1.5713 1.8204 2.0731 2.3291
2.5880 2.8496 3.1136 3.3794 3.6471 3.9160 4.1861 4.4569 4.7281 5.0000];
y_62 = [8.4514 8.1504 7.8555 7.5673 7.2872 7.0161 6.7550 6.5049 6.2671 6.0425
5.8324 5.6377 5.4596 5.2992 5.1572 5.0348 4.9326 4.8513 4.7921 4.7548];

x_63 = [1.3159 1.4730 1.6366 1.8061 1.9811 2.1612 2.3459 2.5348 2.7276 2.9238
3.1231 3.3251 3.5294 3.7359 3.9440 4.1537 4.3643 4.5757 4.7875 5] ;
y_63 = [9.2106 8.9918 8.7784 8.5711 8.3706 8.1776 7.9927 7.8169 7.6506 7.4945
7.3494 7.2157 7.0944 6.9858 6.8905 6.8090 6.7417 6.6892 6.6516 6.6293] ;


hold on
plot(x_11,y_11,'r',x_31,y_31,'g',x_51,y_51,'b')
plot(x_12,y_12,'r',x_32,y_32,'g',x_52,y_52,'b')
plot(x_13,y_13,'r',x_33,y_33,'g',x_53,y_53,'b')
plot(x_21,y_21,'r',x_41,y_41,'g',x_61,y_61,'b')
plot(x_22,y_22,'r',x_42,y_42,'g',x_62,y_62,'b')
plot(x_23,y_23,'r',x_43,y_43,'g',x_63,y_63,'b')
plot(linspace(-10,20,1000),0,'k',linspace(-10,0,1000),3.*linspace(-
10,0,1000)+30,'k',linspace(0,20,1000),-(1.5).*linspace(0,20,1000)+30,'k')
axis equal
axis([-30 40 -5 40])
text(-20,-2,'A(-10,0)','color','r')
text(-10,29,'C(0,30)','color','b')
text(23,-2,'B(20,0)','color','g')
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')
plot(linspace(9,22,20),2,'k')
text(23,2,'u/v = 1/2')
plot(linspace(8,22,20),6,'k')
text(23,6,'u/v = 1/3')
plot(linspace(9.3,22,20),10,'k')
text(23,10,'u/v = 1/4')
title('Linear Stimulating Wavefront to a Scalene Triangle Source Configuration')
hold off
```

2. Linear Stimulating Wavefront and Right Isosceles Triangle Source Configuration (Fig. 7.2)

```
% Sequence A to C to B

x_11 = [3.8188 4.0397 4.2328 4.4023 4.5514 4.6826 4.7985 4.8915 4.9732 5.0316
5.0658 5.0723 5.0474 4.9875 4.8901 4.7533 4.5767 4.3609 4.1075 3.8188] ;
y_11 = [-3.8188 -3.2157 -2.6164 -2.0282 -1.4572 -0.9074 -0.3810 0.1175 0.5919
1.0376 1.4552 1.8440 2.2030 2.5312 2.8277 3.0915 3.3220 3.5194 3.6845 3.8188] ;

x_12 = [2.4296 2.6037 2.7595 2.8968 3.0161 3.1168 3.2000 3.2615 3.3035 3.3332
3.3402 3.3248 3.2888 3.2308 3.1511 3.0493 2.9257 2.7807 2.6150 2.4296];
y_12 = [-2.4295 -2.1138 -1.7897 -1.4616 -1.1332 -0.8074 -0.4870 -0.1755 0.1240
0.4157 0.6918 0.9529 1.1982 1.4269 1.6383 1.8320 2.0077 2.1657 2.3062 2.4295];

x_13 = [1.7972 1.9325 2.0546 2.1680 2.2558 2.3348 2.3983 2.4450 2.4790 2.4960
2.4995 2.4862 2.4555 2.4088 2.3453 2.2661 2.1712 2.0608 1.9360 1.7972];
y_13 = [-1.7970 -1.5852 -1.3654 -1.1271 -0.9108 -0.6811 -0.4526 -0.2280 -0.0075
0.2080 0.4145 0.6128 0.8010 0.9784 1.1449 1.2996 1.4423 1.5728 1.6909 1.7970];


% Sequence A to B to C

x_21 = [0 0.2241 0.4474 0.6701 0.8919 1.1121 1.3307 1.5471 1.7608 1.9715 2.1790
2.3826 2.5818 2.7764 2.9659 3.1497 3.3273 3.4984 3.6624 3.8181] ;
y_21 = [4.1667 4.1665 4.1657 4.1644 4.1624 4.1595 4.1554 4.1498 4.1425 4.1329
4.1209 4.1058 4.0871 4.0645 4.0376 4.0055 3.9681 3.9249 3.8751 3.8181] ;

x_22 = [0 0.1455 0.2874 0.4297 0.5715 0.7123 0.8517 0.9897 1.1256 1.2597 1.3914
1.5203 1.6464 1.7695 1.8892 2.0053 2.1176 2.2259 2.3300 2.4296];
y_22 = [2.9167 2.9155 2.9120 2.9061 2.8981 2.8875 2.8744 2.8588 2.8405 2.8194
2.7954 2.7684 2.7383 2.7050 2.6684 2.6281 2.5842 2.5366 2.4851 2.4296];

x_23 = [0.0000 0.1055 0.2119 0.3172 0.4220 0.5258 0.6288 0.7307 0.8311 0.9298
1.0272 1.1225 1.2157 1.3066 1.3954 1.4815 1.5648 1.6451 1.7227 1.7972];
y_23 = [2.2500 2.2488 2.2453 2.2393 2.2311 2.2203 2.2072 2.1916 2.1734 2.1529
2.1296 2.1038 2.0753 2.0441 2.0100 1.9733 1.9336 1.8910 1.8455 1.7972];


% Sequence B to A to C

x_31 = (-1).*[0 0.2241 0.4474 0.6701 0.8919 1.1121 1.3307 1.5471 1.7608 1.9715
2.1790 2.3826 2.5818 2.7764 2.9659 3.1497 3.3273 3.4984 3.6624 3.8181] ;
y_31 = [4.1667 4.1665 4.1657 4.1644 4.1624 4.1595 4.1554 4.1498 4.1425 4.1329
4.1209 4.1058 4.0871 4.0645 4.0376 4.0055 3.9681 3.9249 3.8751 3.8181] ;

x_32 = (-1).*[0 0.1455 0.2874 0.4297 0.5715 0.7123 0.8517 0.9897 1.1256 1.2597
1.3914 1.5203 1.6464 1.7695 1.8892 2.0053 2.1176 2.2259 2.3300 2.4296];
y_32 = [2.9167 2.9155 2.9120 2.9061 2.8981 2.8875 2.8744 2.8588 2.8405 2.8194
2.7954 2.7684 2.7383 2.7050 2.6684 2.6281 2.5842 2.5366 2.4851 2.4296];

x_33 = (-1).*[0.0000 0.1055 0.2119 0.3172 0.4220 0.5258 0.6288 0.7307 0.8311
0.9298 1.0272 1.1225 1.2157 1.3066 1.3954 1.4815 1.5648 1.6451 1.7227 1.7972];
y_33 = [2.2500 2.2488 2.2453 2.2393 2.2311 2.2203 2.2072 2.1916 2.1734 2.1529
2.1296 2.1038 2.0753 2.0441 2.0100 1.9733 1.9336 1.8910 1.8455 1.7972];
```

```
% Sequence B to C to A

x_41 = (-1).*[3.8188 4.0397 4.2328 4.4023 4.5514 4.6826 4.7985 4.8915 4.9732
5.0316 5.0658 5.0723 5.0474 4.9875 4.8901 4.7533 4.5767 4.3609 4.1075 3.8188] ;
y_41 = [-3.8188 -3.2157 -2.6164 -2.0282 -1.4572 -0.9074 -0.3810 0.1175 0.5919
1.0376 1.4552 1.8440 2.2030 2.5312 2.8277 3.0915 3.3220 3.5194 3.6845 3.8188] ;

x_42 = (-1).*[2.4296 2.6037 2.7595 2.8968 3.0161 3.1168 3.2000 3.2615 3.3035
3.3332 3.3402 3.3248 3.2888 3.2308 3.1511 3.0493 2.9257 2.7807 2.6150 2.4296];
y_42 = [-2.4295 -2.1138 -1.7897 -1.4616 -1.1332 -0.8074 -0.4870 -0.1755 0.1240
0.4157 0.6918 0.9529 1.1982 1.4269 1.6383 1.8320 2.0077 2.1657 2.3062 2.4295];

x_43 = (-1).*[1.7972 1.9325 2.0546 2.1680 2.2558 2.3348 2.3983 2.4450 2.4790
2.4960 2.4995 2.4862 2.4555 2.4088 2.3453 2.2661 2.1712 2.0608 1.9360 1.7972];
y_43 = [-1.7970 -1.5852 -1.3654 -1.1271 -0.9108 -0.6811 -0.4526 -0.2280 -0.0075
0.2080 0.4145 0.6128 0.8010 0.9784 1.1449 1.2996 1.4423 1.5728 1.6909 1.7970];


% Sequence C to A to B

x_51 = [3.8188 3.6967 3.5661 3.4265 3.2777 3.1192 2.9508 2.7727 2.5845 2.3866
2.1793 1.9627 1.7378 1.5052 1.2655 1.0199 0.7693 0.5150 0.2181 0] ;
y_51 = (-1).*[3.8188 4.1188 4.4163 4.7092 4.9964 5.2763 5.5470 5.8072 6.0545
6.2875 6.5045 6.7037 6.8845 7.0428 7.1797 7.2938 7.3834 7.4479 7.4906 7.5] ;

x_52 = [2.4296 2.3356 2.2372 2.1341 2.0266 1.9148 1.7985 1.6782 1.5539 1.4259
1.2942 1.1594 1.0213 0.8808 0.7377 0.5926 0.4461 0.2983 0.1494 0];
y_52 = (-1).*[2.4295 2.5830 2.7327 2.8781 3.0188 3.1538 3.2826 3.4048 3.5197
3.6269 3.7255 3.8153 3.8957 3.9664 4.0269 4.0769 4.1159 4.1441 4.1610 4.1668];

x_53 = [1.7972 1.7244 1.6486 1.5697 1.4878 1.4030 1.3155 1.2252 1.1326 1.0375
0.9403 0.8409 0.7401 0.6375 0.5335 0.4284 0.3222 0.2150 0.1074 0];
y_53 = (-1).*[1.7970 1.8991 1.9978 2.0934 2.1852 2.2729 2.3563 2.4348 2.5083
2.5766 2.6393 2.6961 2.7469 2.7914 2.8294 2.8606 2.8850 2.9026 2.9132 2.9166];


% Sequence C to B to A

x_61 = (-1).*[3.8188 3.6967 3.5661 3.4265 3.2777 3.1192 2.9508 2.7727 2.5845
2.3866 2.1793 1.9627 1.7378 1.5052 1.2655 1.0199 0.7693 0.5150 0.2181 0] ;
y_61 = (-1).*[3.8188 4.1188 4.4163 4.7092 4.9964 5.2763 5.5470 5.8072 6.0545
6.2875 6.5045 6.7037 6.8845 7.0428 7.1797 7.2938 7.3834 7.4479 7.4906 7.5] ;

x_62 = (-1).*[2.4296 2.3356 2.2372 2.1341 2.0266 1.9148 1.7985 1.6782 1.5539
1.4259 1.2942 1.1594 1.0213 0.8808 0.7377 0.5926 0.4461 0.2983 0.1494 0];
y_62 = (-1).*[2.4295 2.5830 2.7327 2.8781 3.0188 3.1538 3.2826 3.4048 3.5197
3.6269 3.7255 3.8153 3.8957 3.9664 4.0269 4.0769 4.1159 4.1441 4.1610 4.1668];

x_63 = (-1).*[1.7972 1.7244 1.6486 1.5697 1.4878 1.4030 1.3155 1.2252 1.1326
1.0375 0.9403 0.8409 0.7401 0.6375 0.5335 0.4284 0.3222 0.2150 0.1074 0];
y_63 = (-1).*[1.7970 1.8991 1.9978 2.0934 2.1852 2.2729 2.3563 2.4348 2.5083
2.5766 2.6393 2.6961 2.7469 2.7914 2.8294 2.8606 2.8850 2.9026 2.9132 2.9166];
```

```
hold on
plot(x_11,y_11,'r',x_31,y_31,'g',x_51,y_51,'b')
plot(x_12,y_12,'r',x_32,y_32,'g',x_52,y_52,'b')
plot(x_13,y_13,'r',x_33,y_33,'g',x_53,y_53,'b')
plot(x_21,y_21,'r',x_41,y_41,'g',x_61,y_61,'b')
plot(x_22,y_22,'r',x_42,y_42,'g',x_62,y_62,'b')
plot(x_23,y_23,'r',x_43,y_43,'g',x_63,y_63,'b')
plot(linspace(-10,10,1000),0,'k',linspace(0,10,1000),10-
linspace(0,10,1000),'k',linspace(-10,0,1000),10+linspace(-10,0,1000),'k')
axis equal
axis([-20 20 -10 20])
text(-15,1,'A(-10,0)','color','r')
text(-5,10,'C(0,10)','color','b')
text(11,1,'B(10,0)','color','g')
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')
plot(linspace(3,5,5),-5.5,'k')
text(5.5,-5.5,'u/v = 1/2')
plot(linspace(0.8,5,9),-4,'k')
text(5.5,-4,'u/v = 1/3')
plot(linspace(1.7,5,7),-2,'k')
text(5.5,-2,'u/v = 1/4')
title('Linear Stimulating Wavefront to a Right Isosceles Triangle Source
Configuration')
hold off
```

3. Linear Stimulating Wavefront and Equilateral Triangle Configuration of Sources (Fig. 7.3)

```
% Sequence A to C to B

x_11 = [4.5833 4.7874 4.9881 5.1844 5.3754 5.5594 5.7347 5.8994 6.0512 6.1880
6.3072 6.4065 6.4835 6.5362 6.5622 6.5602 6.5287 6.4667 6.3739 6.2499] ;
y_11 = [3.1273 3.4839 3.8414 4.1991 4.5567 4.9134 5.2689 5.6226 5.9740 6.3224
6.6668 7.0063 7.3397 7.6657 7.9829 8.2896 8.5845 8.8658 9.1320 9.3819] ;

x_12 = [3.1251 3.2479 3.3643 3.4736 3.5752 3.6687 3.7533 3.8282 3.8929 3.9466
3.9886 4.0182 4.0347 4.0377 4.0268 4.0013 3.9609 3.9057 3.8354 3.7501] ;
y_12 = [3.9693 4.1890 4.4115 4.6359 4.8616 5.0879 5.3139 5.5391 5.7628 5.9844
6.2032 6.4184 6.6295 6.8357 7.0366 7.2314 7.4194 7.6003 7.7735 7.9386] ;

x_13 = [2.3751 2.4627 2.5445 2.6202 2.6892 2.7513 2.8060 2.8532 2.8922 2.9229
2.9447 2.9574 2.9609 2.9547 2.9388 2.9129 2.8770 2.8308 2.7746 2.7084] ;
y_13 = [4.4023 4.5605 4.7213 4.8842 5.0485 5.2138 5.3792 5.5447 5.7093 5.8728
6.0345 6.1941 6.3509 6.5046 6.6546 6.8007 6.9423 7.0792 7.2109 7.3372] ;



% Sequence A to B to C

x_21 = [0 0.4109 0.8208 1.2287 1.6339 2.0348 2.4303 2.8190 3.1992 3.5693 3.9272
4.2709 4.5981 4.9068 5.1945 5.4592 5.6988 5.9113 6.0955 6.2499] ;
y_21 = [11.0658 11.0643 11.0593 11.0505 11.0371 11.0181 10.9922 10.9580 10.9138
10.8581 10.7891 10.7053 10.6053 10.4879 10.3518 10.1967 10.0220 9.8277 9.6141
9.3819] ;

x_22 = [0 0.2518 0.5026 0.7516 0.9979 1.2406 1.4786 1.7111 1.9372 2.1559 2.3664
2.5676 2.7586 2.9388 3.1072 3.2632 3.4059 3.5349 3.6497 3.7501] ;
y_22 = [9.3820 9.3785 9.3680 9.3505 9.3257 9.2935 9.2537 9.2060 9.1502 9.0859
9.0128 8.9308 8.8396 8.7392 8.6293 8.5098 8.3808 8.2425 8.0950 7.9386] ;

x_23 = [0.0000 0.1809 0.3611 0.5399 0.7167 0.8909 1.0616 1.2284 1.3905 1.5475
1.6984 1.8429 1.9805 2.1105 2.2324 2.3460 2.4507 2.5462 2.6321 2.7084] ;
y_23 = [8.5160 8.5127 8.5032 8.4873 8.4649 8.4360 8.4007 8.3588 8.3103 8.2551
8.1932 8.1244 8.0490 7.9668 7.8780 7.7825 7.6806 7.5722 7.4576 7.3372] ;



% Sequence B to A to C

x_31 = [0 -0.4109 -0.8208 -1.2287 -1.6339 -2.0348 -2.4303 -2.8190 -3.1992 -
3.5693 -3.9272 -4.2709 -4.5981 -4.9068 -5.1945 -5.4592 -5.6988 -5.9113 -6.0955 -
6.2499] ;
y_31 = [11.0658 11.0643 11.0593 11.0505 11.0371 11.0181 10.9922 10.9580 10.9138
10.8581 10.7891 10.7053 10.6053 10.4879 10.3518 10.1967 10.0220 9.8277 9.6141
9.3819] ;

x_32 = (-1).*[0 0.2517 0.5027 0.7516 0.9979 1.2406 1.4786 1.7112 1.9373 2.156
2.3665 2.5676 2.7587 2.9389 3.1072 3.2632 3.4060 3.5350 3.6498 3.7501] ;
y_32 = [9.3820 9.3785 9.3680 9.3505 9.3257 9.2935 9.2537 9.2060 9.1502 9.0859
9.0128 8.9308 8.8396 8.7392 8.6293 8.5098 8.3808 8.2425 8.0950 7.9386] ;

x_33 = (-1).*[0.0000 0.1809 0.3611 0.5399 0.7167 0.8909 1.0616 1.2284 1.3905
1.5475 1.6984 1.8429 1.9805 2.1105 2.2324 2.3460 2.4507 2.5462 2.6321 2.7084] ;
y_33 =       [8.5160 8.5127 8.5032 8.4873 8.4649 8.4360 8.4007 8.3588 8.3103
8.2551 8.1932 8.1244 8.0490 7.9668 7.8780 7.7825 7.6806 7.5722 7.4576 7.3372] ;
```

```
% Sequence B to C to A

x_41 = [-4.5833 -4.7874 -4.9881 -5.1844 -5.3754 -5.5594 -5.7347 -5.8994 -6.0512
-6.1880 -6.3072 -6.4065 -6.4835 -6.5362 -6.5622 -6.5602 -6.5287 -6.4667 -6.3739
-6.2499] ;
y_41 = [3.1273 3.4839 3.8414 4.1991 4.5567 4.9134 5.2689 5.6226 5.9740 6.3224
6.6668 7.0063 7.3397 7.6657 7.9829 8.2896 8.5845 8.8658 9.1320 9.3819] ;

x_42 = (-1).*[3.1251 3.2479 3.3643 3.4736 3.5752 3.6687 3.7533 3.8282 3.8929
3.9466 3.9886 4.0182 4.0347 4.0377 4.0268 4.0013 3.9609 3.9057 3.8354 3.7501] ;
y_42 =       [3.9693 4.1890 4.4115 4.6359 4.8616 5.0879 5.3139 5.5391 5.7628
5.9844 6.2032 6.4184 6.6295 6.8357 7.0366 7.2314 7.4194 7.6003 7.7735 7.9386] ;

x_43 = (-1).*[2.3751 2.4626 2.5445 2.6202 2.6892 2.7513 2.8060 2.8531 2.8922
2.9229 2.9447 2.9575 2.9609 2.9547 2.9388 2.9129 2.8770 2.8308 2.7746 2.7084] ;
y_43 =       [4.4023 4.5605 4.7213 4.8842 5.0485 5.2138 5.3792 5.5447 5.7093
5.8728 6.0345 6.1941 6.3509 6.5046 6.6546 6.8007 6.9423 7.0792 7.2109 7.3372] ;



% Sequence C to A to B

x_51 = [4.5833 4.3765 4.1673 3.9557 3.7415 3.5248 3.3048 3.0813 2.8515 2.6150
2.3775 2.1345 1.8864 1.6297 1.3688 1.1013 0.8307 0.5561 0.2777 0.0000] ;
y_51 = [3.1273 2.7723 2.4197 2.0709 1.7267 1.3890 1.0594 0.7400 0.4328 0.1400 -
0.1350 -0.3920 -0.6245 -0.8331 -1.0144 -1.1659 -1.2858 -1.3730 -1.4258 -1.4433]
;

x_52 = [3.1251 2.9962 2.8617 2.7220 2.5773 2.4282 2.2748 2.1171 1.9557 1.7908
1.6223 1.4507 1.2764 1.0991 0.9201 0.7388 0.5558 0.3711 0.1842 0] ;
y_52 = [3.9693 3.7530 3.5409 3.3341 3.1332 2.9391 2.7529 2.5752 2.4075 2.2502
2.1045 1.9712 1.8514 1.7456 1.6547 1.5792 1.5202 1.4777 1.4520 1.4433] ;

x_53 = [2.3751 2.2818 2.1835 2.0803 1.9725 1.8604 1.7445 1.6248 1.5017 1.3755
1.2464 1.1146 0.9806 0.8443 0.7065 0.5669 0.4265 0.2853 0.1445 0.0000] ;
y_53 = [4.4023 4.2473 4.0960 3.9490 3.8071 3.6707 3.5405 3.4170 3.3009 3.1926
3.0927 3.0020 2.9206 2.8491 2.7878 2.7373 2.6975 2.6691 2.6520 2.6461] ;



% Sequence C to B to A

x_61 = [-4.5833 -4.3765 -4.1673 -3.9557 -3.7415 -3.5248 -3.3048 -3.0813 -2.8515
-2.6150 -2.3775 -2.1345 -1.8864 -1.6297 -1.3688 -1.1013 -0.8307 -0.5561 -0.2777
0] ;
y_61 = [3.1273 2.7723 2.4197 2.0709 1.7267 1.3890 1.0594 0.7400 0.4328 0.1400 -
0.1350 -0.3920 -0.6245 -0.8331 -1.0144 -1.1659 -1.2858 -1.3730 -1.4258 -1.4433]
;

x_62 = (-1).*[3.1251 2.9962 2.8617 2.7220 2.5773 2.4282 2.2748 2.1171 1.9557
1.7908 1.6223 1.4507 1.2764 1.0991 0.9201 0.7388 0.5558 0.3711 0.1842 0]  ;
y_62 = [3.9693 3.7530 3.5409 3.3341 3.1332 2.9391 2.7529 2.5752 2.4075 2.2502
2.1045 1.9712 1.8514 1.7456 1.6547 1.5792 1.5202 1.4777 1.4520 1.4433] ;

x_63 = (-1).*[2.3751 2.2818 2.1835 2.0803 1.9725 1.8604 1.7445 1.6248 1.5017
1.3755 1.2464 1.1146 0.9806 0.8443 0.7065 0.5669 0.4265 0.2853 0.1445 0.0000] ;
y_63 =       [4.4023 4.2473 4.0960 3.9490 3.8071 3.6707 3.5405 3.4170 3.3009
3.1926 3.0927 3.0020 2.9206 2.8491 2.7878 2.7373 2.6975 2.6691 2.6520 2.6461] ;
```

```
hold on
%curve-1
plot(x_11,y_11,'r',x_31,y_31,'g',x_51,y_51,'b')
plot(x_21,y_21,'r',x_41,y_41,'g',x_61,y_61,'b')
%curve-2
plot(x_12,y_12,'r',x_32,y_32,'g',x_52,y_52,'b')
plot(x_22,y_22,'r',x_42,y_42,'g',x_62,y_62,'b')
%curve-3
plot(x_13,y_13,'r',x_33,y_33,'g',x_53,y_53,'b')
plot(x_23,y_23,'r',x_43,y_43,'g',x_63,y_63,'b')
plot(linspace(-10,10,1000),0,'k',linspace(-10,0,1000),sqrt(3).*(linspace(-
10,0,1000)+10),'k',linspace(0,10,1000),-sqrt(3).*(linspace(0,10,1000)-10),'k')
axis equal
axis([-20 20 -5 25])
text(-16,-1,'A(-10,0)','color','r')
text(-7,17,'C(0,10{\surd3})','color','b')
text(12,-1,'B(10,0)','color','g')
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')
plot(linspace(4,10.5,10),2,'k')
text(11.3,2,'u/v = 1/2')
plot(linspace(3.3,10.5,10),4,'k')
text(11.3,4,'u/v = 1/3')
plot(linspace(3,10.5,10),6,'k')
text(11.3,6,'u/v = 1/4')
title('Linear Stimulating Wavefront to an Equilateral Triangle Source
Configuration')
hold off
```

4. Circular Stimulating Wavefront and Scalene Triangle Configuration of Sources (Fig. 7.4)

```
% Sequence A to C to B

x_11 = [4.6171 4.1383 3.6531 3.1571 2.6458 2.1156 1.5633 0.9867 0.3841];
y_11 = [0.2534 1.0740 1.9654 2.8881 3.8193 4.7460 5.6612 6.5612 7.4444];

x_12 = [4.7325 4.3950 4.0501 3.6980 3.3376 2.9689 2.5918 2.2066 1.8140] ;
y_12 = [4.6629 5.0890 5.5682 6.0808 6.6136 7.1584 7.7095 8.2628 8.8157] ;

x_13 = [4.7090 4.5275 4.2595 3.9870 3.7100 3.4289 3.1439 2.8557 2.5647] ;
y_13 = [6.5617 6.8546 7.1870 7.5461 7.9229 8.3113 8.7073 9.1078 9.5106] ;



% Sequence A to B to C

x_21 = [5.0916 5.5683 6.0412 6.5078 6.9674 7.4236 7.8821 8.3505 8.8348 9.3403
9.8709 10.4299 11.0193 11.6398];
y_21 = [-0.4316 -0.8894 -1.0270 -0.8018 -0.2570 0.5132 1.4187 2.3966 3.4095
4.4365 5.4674 6.4970 7.5244 8.5500];

x_22 = [5.7169 5.3920 5.0648 9.5196 9.1507 8.7846 8.4230 8.0665 7.7157 7.3708
7.0316 6.6975 6.3679 6.0415] ;
y_22 = [4.0374 4.0979 4.3194 9.6009 8.9575 8.3179 7.6853 7.0632 6.4570 5.8750
5.3298 4.8406 4.4363 4.1559] ;

x_23 = [5.5611 5.3066 5.0504 8.4302 8.1688 7.9048 7.6398 7.3747 7.1104 6.8474
6.5862 6.3272 6.0702 5.8151] ;
y_23 = [6.1421 6.1799 6.3283 10.1194 9.6520 9.1883 8.7309 8.2830 7.8490 7.4349
7.0493 6.7055 6.4229 6.2269] ;



% Sequence B to A to C

x_31 = [18.3664 18.3612 18.1371 17.7322 17.1903 16.5536 15.8586 15.1342 14.4023
13.6780 12.9722 12.2914];
y_31 = [20.3398 19.6135 18.7649 17.8274 16.8318 15.8035 14.7607 13.7148 12.6721
11.6349 10.6031 9.5755];

x_32 = [12.3988 12.4604 12.4298 12.3179 12.1374 11.9009 11.6210 11.3085 10.9724
10.6199 10.2575 9.8894] ;
y_32 = [16.4586 16.0680 15.6187 15.1195 14.5794 14.0071 13.4105 12.7958 12.1684
11.5321 10.8905 10.2460] ;

x_33 = [10.2119 10.2727 10.2744 10.2229 10.1251 9.9886 9.8202 9.6258 9.4113
9.1810 8.9387 8.6876] ;
y_33 = [15.0371 14.7571 14.4385 14.0857 13.7036 13.2977 12.8726 12.4324 11.9810
11.5214 11.0564 10.5883] ;
```

```
% Sequence B to C to A

x_41 = [5.1875 6.4129 7.6492 8.8955 10.1489 11.4022 12.6425 13.8494 14.9941
16.0378 16.9371 17.6461 18.1291];
y_41 = [19.4127 19.6758 19.9623 20.2659 20.5774 20.8837 21.1668 21.4040 21.5682
21.6279 21.5536 21.3202 20.9147];

x_42 = [5.1150 5.8624 6.6057 7.3403 8.0612 8.7620 9.4344 10.0685 10.6529 11.1742
11.6198 11.9775 12.2384] ;
y_42 = [16.8773 16.9932 17.1045 17.2081 17.2998 17.3743 17.4251 17.4448 17.4249
17.3563 17.2309 17.0416 16.7843] ;

x_43 = [5.0827 5.6189 6.1500 6.6721 7.1813 7.6729 8.1416 8.5808 8.9837 9.3426
9.6503 9.9004 10.0882] ;
y_43 = [15.6004 15.6712 15.7344 15.7880 15.8295 15.8560 15.8639 15.8491 15.8069
15.7325 15.6214 15.4698 15.2751] ;


% Sequence C to A to B

x_51 = (-1).*[0.2455 0.9021 1.5850 2.2920 3.0191 3.7591 4.5009 5.2258 5.9049
6.4943 6.9337];
y_51 = [8.3105 9.1599 9.9932 10.8118 11.6169 12.4096 13.1902 13.9568 14.7046
15.4230 16.0938];

x_52 = [1.4145 1.0098 0.6014 0.1921 -0.2144 -0.6131 -0.9971 -1.3563 -1.6765 -
1.9384] ;
y_52 = [9.3658 9.9117 10.4518 10.9850 11.5097 12.0246 12.5274 13.0153 13.4843
13.9287] ;

x_53 = [2.2723 1.9794 1.6877 1.3989 1.1161 0.8426 0.5832 0.3447 0.1363 -0.0296]
;
y_53 = [9.9137 10.3157 10.7151 11.1105 11.5007 11.8841 12.2588 12.6224 12.9722
13.3039] ;


% Sequence C to B to A

x_61 = [-7.1520 -7.0877 -6.7175 -6.0706 -5.2113 -4.2087 -3.1184 -1.9779 -0.8098
0.3732 1.5655 2.7652 3.9720];
y_61 = [16.6917 17.1909 17.5769 17.8557 18.0502 18.1898 18.3010 18.4045 18.5149
18.6420 18.7924 18.9702 19.1769];

x_62 = [-2.1171 -2.1839 -2.1148 -1.8994 -1.5469 -1.0804 -0.5272 0.0889 0.7494
1.4411 2.1545 2.8831 3.6220 4.3671] ;
y_62 = [14.3409 14.7122 15.0354 15.3081 15.5343 15.7228 15.8837 16.0261 16.1568
16.2810 16.4017 16.5211 16.6402 16.7591] ;

x_63 = [-0.1370 -0.1677 -0.1066 0.0533 0.3063 0.6381 1.0309 1.4693 1.9411 2.4367
2.9497 3.4746 4.0075 4.5445] ;
y_63 = [13.6126 13.8930 14.1407 14.3544 14.5365 14.6927 14.8291 14.9507 15.0617
15.1647 15.2617 15.3536 15.4408 15.5232] ;
```

```
hold on
%curve-1
plot(x_11,y_11,'r.',x_31,y_31,'g.',x_51,y_51,'b.')
plot(x_21,y_21,'r.',x_41,y_41,'g.',x_61,y_61,'b.')
%curve-2
plot(x_12,y_12,'r.',x_32,y_32,'g.',x_52,y_52,'b.')
plot(x_22,y_22,'r.',x_42,y_42,'g.',x_62,y_62,'b.')
%curve-3
plot(x_13,y_13,'r.',x_33,y_33,'g.',x_53,y_53,'b.')
plot(x_23,y_23,'r.',x_43,y_43,'g.',x_63,y_63,'b.')

plot(linspace(-10,20,100),0,'k',linspace(-10,0,100),3.*linspace(-
10,0,100)+30,'k',linspace(0,20,100),-(1.5).*linspace(0,20,100)+30,'k')
axis equal
axis([-15 35 -5 40])
text(-3,31,'A(0,30)','color','r')
text(20,-1,'C(20,0)','color','b')
text(-11,-1,'B(-10,0)','color','g')
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')
plot(linspace(8.2,22,50),2,'k')
text(23,2,'u/v = 1/2')
plot(linspace(7.5,22,50),6,'k')
text(23,6,'u/v = 1/3')
plot(linspace(8.4,22,50),10,'k')
text(23,10,'u/v = 1/4')
title('Cicruclar Stimulating Wavefront to a Scalene Triangle Source
Configuration')
hold off
```

5. Circular Stimulating Wavefront and Right Isosceles Triangle Source Configuration (Fig.7.5)

```
% Sequence A to C to B

x_11 = (-1).*[0 0.4825 0.9534 1.4026 1.8217 2.2058 2.5522 2.8618 3.1373 3.3827
3.6034 0];
y_11 = (-1).*[9.4319 9.3575 9.1365 8.7803 8.3057 7.7330 7.0845 6.3821 5.6458
4.8937 4.1404 9.4319];

x_12 = (-1).*[0 0.2617 0.5207 0.7746 1.0212 1.2586 1.4856 1.7012 1.9049 2.0966
2.2763] ;
y_12 = (-1).*[4.9472 4.9186 4.8329 4.6924 4.5012 4.2639 3.9859 3.6735 3.3326
2.9698 2.5910] ;

x_13 = (-1).*[0 0.1860 0.3706 0.5529 0.7315 0.9055 1.0741 1.2364 1.3919 1.5401
1.6804 0] ;
y_13 = (-1).*[3.4074 3.3899 3.3376 3.2515 3.1335 2.9854 2.8105 2.6114 2.3914
2.1540 1.9028 3.4074] ;



% Sequence A to B to C

x_21 = [0 3.6034 3.3827 3.1373 2.8618 2.5522 2.2057 1.8217 1.4026 0.9534 0.4825
0];
y_21 = (-1).*[9.4319 4.1404 4.8937 5.6458 6.3821 7.0845 7.7330 8.3057 8.7803
9.1364 9.3575 9.4319];

x_22 = [0 2.2763 2.0966 1.9049 1.7012 1.4856 1.2586 1.0212 0.7746 0.5207 0.2617
0] ;
y_22 = (-1).*[4.9472 2.5910 2.9698 3.3327 3.6735 3.9859 4.2639 4.5012 4.6924
4.8329 4.9186 4.9472] ;

x_23 = [0 1.6804 1.5401 1.3919 1.2365 1.0741 0.9055 0.7315 0.5529 0.3706 0.1860]
;
y_23 = (-1).*[3.4074 1.9028 2.1540 2.3915 2.6114 2.8105 2.9854 3.1335 3.2515
3.3376 3.3899] ;



% Sequence B to A to C

x_31 = [3.8683 4.2284 4.5376 4.7891 4.9778 4.9778 5.1022 5.1645 5.1700 5.1271
5.0453 4.9341 4.8020 4.6554 4.4999 4.3372 4.1683 3.9917 3.8048];
y_31 = [3.6436 3.6138 3.5421 3.4211 3.2449 3.2449 3.0107 2.7191 2.3721 1.9736
1.5276 1.0381 0.5083 -0.0599 -0.6639 -1.3023 -1.9730 -2.6726 -3.3972];

x_32 = [2.4690 2.6935 2.8881 3.0500 3.1776 3.2704 3.3293 3.3558 3.3527 3.3229
3.2698 3.1955 3.1050 2.9984 2.8781 2.7453 2.6006 2.4443] ;
y_32 = [2.3050 2.2446 2.1607 2.0498 1.9092 1.7372 1.5336 1.2988 1.0345 0.7426
0.4262 0.0882 -0.2679 -0.6393 -1.0223 -1.4131 -1.8076 -2.2020] ;

x_33 = [1.8283 1.9949 2.1400 2.2617 2.3589 2.4314 2.4793 2.5035 2.5056 2.4873
2.4504 2.3955 2.3288 2.2475 2.1541 2.0499 1.9359 1.8125] ;
y_33 = [1.6984 1.6388 1.5622 1.4666 1.3504 1.2128 1.0534 0.8730 0.6726 0.4542
0.2198 -0.0282 -0.2864 -0.5528 -0.8245 -1.0985 -1.3716 -1.6409] ;
```

```
% Sequence B to C to A

x_41 = [0 0.5275 1.0513 1.5676 2.0722 2.5605 3.0269 3.4652];
y_41 = [3.4351 3.4424 3.4635 3.4961 3.5363 3.5789 3.6165 3.6413];

x_42 = [0 0.3419 0.6807 1.0135 1.3369 1.6477 1.9424 2.2174] ;
y_42 = [2.3906 2.3912 2.3923 2.3929 2.3914 2.3852 2.3711 2.3457] ;

x_43 = [0 0.2536 0.5048 0.7513 0.9908 1.2208 1.4387 1.6420] ;
y_43 = [1.8371 1.8359 1.8326 1.8262 1.8160 1.7999 1.7765 1.7434] ;



% Sequence C to A to B

x_51 = (-1).*[3.8048 3.9918 4.1683 4.3373 4.4999 4.6555 4.8020 4.9341 5.0453
5.1271 5.1700 5.1645 5.1022 4.9778 4.7891 4.5376 4.2284 3.8683];
y_51 = [-3.3972 -2.6725 -1.9730 -1.3023 -0.6639 -0.0598 0.5083 1.0381 1.5276
1.9735 2.3721 2.7191 3.0107 3.2449 3.4211 3.5421 3.6138 3.6436];

x_52 = (-1).*[2.4443 2.6006 2.7453 2.8781 2.9984 3.1050 3.1956 3.2698 3.3228
3.3527 3.3558 3.3293 3.2705 3.1776 3.0500 2.8881 2.6935 2.4690] ;
y_52 = [-2.2020 -1.8076 -1.4131 -1.0223 -0.6393 -0.2679 0.0882 0.4262 0.7427
1.0344 1.2988 1.5336 1.7373 1.9092 2.0498 2.1607 2.2446 2.3049] ;

x_53 = (-1).*[1.8125 1.9359 2.0500 2.1541 2.2476 2.3288 2.3965 2.4504 2.4873
2.5056 2.5035 2.4792 2.4314 2.3589 2.2616 2.1400 1.9949 1.8283] ;
y_53 = [-1.6408 -1.3716 -1.0985 -0.8245 -0.5528 -0.2864 -0.0285 0.2198 0.4542
0.6727 0.8730 1.0534 1.2128 1.3504 1.4666 1.5622 1.6388 1.6984] ;



% Sequence C to B to A

x_61 = (-1).*[3.4652 3.0269 2.5605 2.0722 1.5676 1.0513 0.5275 0];
y_61 = [3.6413 3.6165 3.5789 3.5363 3.4961 3.4635 3.4424 3.4351];

x_62 = (-1).*[2.2174 1.9424 1.6477 1.3369 1.0134 0.6807 0.3419 0] ;
y_62 = [2.3457 2.3711 2.3852 2.3914 2.3930 2.3923 2.3912 2.3906] ;

x_63 = (-1).*[1.6420 1.4387 1.2208 0.9908 0.7513 0.5047 0.2536 0] ;
y_63 = [1.7434 1.7765 1.7999 1.8160 1.8262 1.8326 1.8360 1.8371] ;


hold on
%curve-1
plot(x_11,y_11,'r.',x_31,y_31,'g.',x_51,y_51,'b.')
plot(x_21,y_21,'r.',x_41,y_41,'g.',x_61,y_61,'b.')
%curve-2
plot(x_12,y_12,'r.',x_32,y_32,'g.',x_52,y_52,'b.')
plot(x_22,y_22,'r.',x_42,y_42,'g.',x_62,y_62,'b.')
%curve-2
plot(x_13,y_13,'r.',x_33,y_33,'g.',x_53,y_53,'b.')
plot(x_23,y_23,'r.',x_43,y_43,'g.',x_63,y_63,'b.')

plot(linspace(-10,10,1000),0,'k',linspace(0,10,1000),10-
linspace(0,10,1000),'k',linspace(-10,0,1000),10+linspace(-10,0,1000),'k')
axis equal
axis([-20 20 -13 18])
text(-15,1,'B(-10,0)','color','g')
text(-5,10,'A(0,10)','color','r')
```

```
text(11,1,'C(10,0)','color','b')
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')

plot(linspace(3.4,5,15),-5.5,'k')
text(5.5,-5.5,'u/v = 1/2')
plot(linspace(1.3,5,15),-4,'k')
text(5.5,-4,'u/v = 1/3')
plot(linspace(1.8,5,15),-2,'k')
text(5.5,-2,'u/v = 1/4')
title('Cicruclar Stimulating Wavefront to a Right Isosceles Triangle Source
Configuration')
hold off
```

6. Circular Stimulating Wavefront and Equilateral Triangle Source Configuration (Fig. 7.6)

```
% Sector-1: A to C to B

x_11 = [0 -0.1106 -0.2198 -0.3269 -0.4314 -0.5334 -0.6334 -0.7321 -0.8309 -
0.9306 -1.0323 -1.1364 -1.2433] ;
y_11 = [-0.6630 -0.6444 -0.5894 -0.5012 -0.3844 -0.2443 -0.0864 0.0848 0.2646
0.4497 0.6376 0.8262 1.0142] ;

x_12 = (-1).*[0 0.0724 0.1448 0.2169 0.2885 0.3598 0.4308 0.5013 0.5715 0.6413
0.7106 0.7791 0.8467] ;
y_12 = [0.3212 0.3299 0.3558 0.3981 0.4554 0.5261 0.6081 0.6998 0.7993 0.9048
1.0148 1.1280 1.2432] ;

x_13 = (-1).*[0 0.0556 0.1110 0.1665 0.2216 0.2765 0.3310 0.3850 0.4385 0.4913
0.5432 0.5938 0.6430] ;
y_13 = [0.7180 0.7238 0.7412 0.7695 0.8082 0.8561 0.9123 0.9756 1.0449 1.1191
1.1971 1.2780 1.3608] ;


% Sector-2: C to A to B

x_21 = (-1).*[1.2433 1.3527 1.4640 1.5759 1.6862 1.7926 1.8914 1.9782 2.0486
2.0975 2.1203 2.1134 2.0742] ;
y_21 = [1.0142 1.2008 1.3853 1.5673 1.7462 1.9216 2.0928 2.2584 2.4167 2.5656
2.7024 2.8246 2.9296] ;

x_22 = (-1).*[0.8467 0.9127 0.9764 1.0371 1.0935 1.1887 1.1446 1.2243 1.2498
1.2636 1.2642 1.2506 1.2218] ;
y_22 = [1.2432 1.3593 1.4753 1.5903 1.7035 1.9210 1.8141 2.0235 2.1205 2.2113
2.2948 2.3704 2.4375] ;

x_23 = (-1).*[0.6430 0.6901 0.7349 0.7765 0.8143 0.8476 0.8754 0.8969 0.9109
0.9168 0.9136 0.9010 0.8782] ;
y_23 = [1.3608 1.4448 1.5291 1.6130 1.6958 1.7768 1.8553 1.9306 2.0021 2.0692
2.1313 2.1881 2.2391] ;


% Sector-3: C to B to A

x_31 = (-1).*[2.0742 2.0029 1.9006 1.7707 1.6172 1.4449 1.2581 1.0605 0.8554
0.6453 0.4317 0.2163 0] ;
y_31 = [2.9296 3.0160 3.0831 3.1318 3.1639 3.1821 3.1897 3.1897 3.1853 3.1792
3.1733 3.1691 3.1677] ;

x_32 = (-1).*[1.2218 1.1782 1.1195 1.0468 0.9613 0.8645 0.7580 0.6433 0.5220
0.3958 0.2658 0.1336 0] ;
y_32 = [2.4375 2.4959 2.5455 2.5868 2.6202 2.6466 2.6671 2.6823 2.6934 2.7011
2.7060 2.7088 2.7097] ;

x_33 = (-1).*[0.8782 0.8454 0.8026 0.7504 0.6893 0.6204 0.5445 0.4625 0.3758
0.2851 0.1917 0.0963 0] ;
y_33 = [2.2391 2.2843 2.3236 2.3575 2.3859 2.4095 2.4286 2.4437 2.4553 2.4640
2.4699 2.4734 2.4745] ;
```

```
% Sector-4: B to C to A

x_41 = [0 0.2163 0.4317 0.6453 0.8554 1.0605 1.2581 1.4449 1.6172 1.7706 1.9006
2.0029 2.0742] ;
y_41 = [3.1677 3.1691 3.1733 3.1792 3.1853 3.1897 3.1897 3.1821 3.1639 3.1317
3.0831 3.0160 2.9296] ;

x_42 = [0 0.1336 0.2658 0.3958 0.5220 0.6433 0.7580 0.8645 0.9613 1.0468 1.1194
1.1782 1.2218] ;
y_42 = [2.7097 2.7088 2.7060 2.7011 2.6934 2.6823 2.6671 2.6466 2.6202 2.5868
2.5455 2.4959 2.4375] ;

x_43 = [0 0.0963 0.1917 0.2852 0.3758 0.4625 0.5445 0.6204 0.6893 0.7504 0.8026
0.8454 0.8782] ;
y_43 = [2.4745 2.4734 2.4699 2.4640 2.4553 2.4437 2.4286 2.4095 2.3859 2.3575
2.3236 2.2843 2.2391] ;


% Sector-5: B to A to C

x_51 = [2.0742 2.1133 2.1203 2.0975 2.0486 1.9782 1.8914 1.7926 1.6862 1.5759
1.4640 1.3527 1.2433] ;
y_51 = [2.9296 2.8246 2.7024 2.5656 2.4167 2.2584 2.0928 1.9216 1.7462 1.5673
1.3853 1.2008 1.0142] ;

x_52 = [1.2218 1.2506 1.2642 1.2636 1.2498 1.2243 1.1887 1.1446 1.0935 1.0371
0.9764 0.9127 0.8467] ;
y_52 = [2.4375 2.3704 2.2948 2.2113 2.1205 2.0235 1.9210 1.8140 1.7035 1.5903
1.4753 1.3593 1.2432] ;

x_53 = [0.8782 0.9010 0.9136 0.9168 0.9109 0.8969 0.8754 0.8476 0.8143 0.7764
0.7349 0.6901 0.6430] ;
y_53 = [2.2391 2.1881 2.1313 2.0692 2.0021 1.9306 1.8553 1.7768 1.6958 1.6130
1.5291 1.4448 1.3608] ;


% Sector-6: A to B to C

x_61 = [1.2433 1.1364 1.0323 0.9306 0.8309 0.7321 0.6334 0.5333 0.4314 0.3269
0.2198 0.1106 0.0000] ;
y_61 = [1.0142 0.8262 0.6376 0.4497 0.2646 0.0848 -0.0864 -0.2443 -0.3844 -
0.5013 -0.5894 -0.6444 -0.6630] ;

x_62 = [0.8467 0.7791 0.7106 0.6413 0.5715 0.5013 0.4308 0.3598 0.2885 0.2169
0.1448 0.0723 0.0000] ;
y_62 = [1.2432 1.1280 1.0148 0.9048 0.7993 0.6998 0.6081 0.5261 0.4554 0.3981
0.3558 0.3298 0.3212] ;

x_63 = [0.6430 0.5938 0.5432 0.4913 0.4385 0.3850 0.3310 0.2765 0.2216 0.1664
0.1110 0.0556 0] ;
y_63 = [1.3608 1.2780 1.1971 1.1191 1.0449 0.9756 0.9123 0.8561 0.8082 0.7695
0.7412 0.7238 0.7180] ;
```

```
hold on
% outer curve
plot(x_11,y_11,'r.',x_51,y_51,'g.',x_31,y_31,'b.')
plot(x_61,y_61,'r.',x_41,y_41,'g.',x_21,y_21,'b.')
% middle curve
plot(x_12,y_12,'r.',x_52,y_52,'g.',x_32,y_32,'b.')
plot(x_62,y_62,'r.',x_42,y_42,'g.',x_22,y_22,'b.')
% inner curve
plot(x_13,y_13,'r.',x_53,y_53,'g.',x_33,y_33,'b.')
plot(x_63,y_63,'r.',x_43,y_43,'g.',x_23,y_23,'b.')

plot(linspace(-3,3,100),0,'k',linspace(-3,0,100),sqrt(3).*(linspace(-
3,0,100)+3),'k',linspace(0,3,100),-sqrt(3).*(linspace(0,3,100)-3),'k')
text(-1.5,5.2,'A(0,3{\surd3})','color','r')
text(-4,0.3,'B(-3,0)','color','g')
text(3.1,0.3,'C(3,0)','color','b')
axis equal
axis([-5 5 -1 8])
legend('Seq. A-C-B & Seq. A-B-C','Seq. B-A-C & Seq. B-C-A','Seq. C-A-B & Seq. C-
B-A')
title('PWC Type-4 Curves')
text(2.5,2.55,'u/v = 1/2')
text(2.55,2,'u/v = 1/3')
text(2.55,1.5,'u/v = 1/4')
plot(linspace(0.7,2.4,10),1.5,'k')
plot(linspace(1.2,2.4,7),2,'k')
plot(linspace(2.1,2.4,3),2.5,'k')
title('Cicruclar Stimulating Wavefront to an Equilateral Triangle Source
Configuration')
hold off
```