Neutrino magnetic moment and mass: an update

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ABSTRACT

In 1977 an expression for the magnetic moment of a massive Dirac neutrino was deduced in the context of electroweak interactions at the one-loop level. A linear dependence on the neutrino mass was found. Alternatively, a magnetic moment for a massive neutrino arising from gravitational origin is predicted by the so-called Wilson-Blackett law. The latter relation may also be deduced from a gravitomagnetic interpretation of the Einstein equations. Both formulas for the magnetic moment can be combined, yielding a value for the smallest neutrino mass m_1.

The gravitomagnetic moment, i.e., the magnetic moment from gravitational origin, may contain different g-factors for the massive neutrino eigenstates m_1, m_2 and m_3, respectively. Starting from the Dirac equation, a g-factor g = 2 has been deduced for a neutrino in first order, related to the derivation of the g-factor of charged leptons. When a value g = 2 is inserted, a value of 1.530 meV is obtained for the lightest neutrino mass m_1, the main result of this work. The remaining neutrino masses can then be calculated from observed neutrino oscillations. The so-called geometric mean mass relation between the three neutrino masses appears to be in fair agreement with our results. A possible dependence of the neutrino mass on the electroweak coupling constant is discussed.

The neutrino with the smallest mass m_1 may also possess the smallest magnetic moment of all known elementary particles. Its gravitomagnetic formula is a combination of three Planck units.

1. INTRODUCTION

It is generally accepted that an electrically neutral neutrino may possess electromagnetic properties through electroweak interactions with photons. The neutrino magnetic moment arises at the one-loop level from a minimal extension of the standard model with right-handed neutrinos. For a left-handed Dirac neutrino with a positive mass m_i (i = 1, 2, 3) the following electromagnetic moment $\boldsymbol{\mu}_i$(em) has been deduced [1, 2]

$$\boldsymbol{\mu}_i(\mathrm{em}) = \frac{3|e|G_{\mathrm{F}} m_i c^4 \hbar}{8\pi^2\sqrt{2}}\boldsymbol{\sigma} = \frac{3G_{\mathrm{F}} m_i m_e c^4 \mu_B}{4\pi^2\sqrt{2}}\boldsymbol{\sigma} = 3.2026\times10^{-22}\left(\frac{m_i}{\mathrm{meV}}\right)\mu_B\boldsymbol{\sigma}, \qquad (1.1)$$

where $G_{\mathrm{F}} = 1.16638\times10^{-5}$ GeV^{-2} is the Fermi coupling constant, c is the velocity of light, $\hbar$ is the reduced Planck constant, $\boldsymbol{\sigma}$ is the Pauli matrix and $\mu_B = |e|\hbar/2m_e$ is the Bohr magneton. Equation (1.1) predicts that the neutrino magnetic moment is proportional to the neutrino mass m_i, but the value of m_i does not follow from the calculation. The formula for $\boldsymbol{\mu}_i$(em) has been deduced from the one-loop contributions to the neutrino electromagnetic vertex function. To leading order in m_l^2/m_W^2, the result is independent of the charged lepton masses m_l ($l = e, \mu, \tau$) and of the neutrino mixing matrix U [1, 2].

Observed neutrino oscillations from different sources (Sun, Earth's atmosphere and samples in the laboratory) provide strong indications for the existence of massive neutrinos. A description of neutrino oscillations [3, 4] is possible by connecting three neutrino flavour states neutrinos ν_α ($\alpha = e, \mu, \tau$) to three massive eigenstates ν_i with masses m_i (i = 1, 2, 3). In that case three different magnetic moments $\boldsymbol{\mu}_i$(em) may exist, corresponding to the three neutrino masses m_i.

In this work the electromagnetic moment $\boldsymbol{\mu}_i$(em) of (1.1) for the neutrino will be compared with the so-called gravitomagnetic moment $\boldsymbol{\mu}_i$(gm). As will be discussed in section 2, it is assumed that the magnetic fields from $\boldsymbol{\mu}_i$(em) and $\boldsymbol{\mu}_i$(gm) are equivalent. For an elementary particle like a neutrino with mass m_i (i = 1, 2, 3) and angular momentum $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ the gravitomagnetic moment $\boldsymbol{\mu}_i$(gm) may be written as

$$\boldsymbol{\mu}_i(\mathrm{gm}) = -\frac{g_i\beta}{2}\left(\frac{G}{k}\right)^{1/2}\mathbf{S} = -\frac{g_i\beta}{4}\left(\frac{G}{k}\right)^{1/2}\hbar\,\boldsymbol{\sigma}, \tag{1.2}$$

where G is the gravitational constant and $k = (4\pi\varepsilon_0)^{-1}$ is Coulomb's constant. The parameter g_i (i = 1, 2, 3) is a dimensionless quantity of order unity, related to the g_l -factor for charged leptons ($l = e, \mu, \tau$). In addition, another unknown dimensionless constant β has been added to $\boldsymbol{\mu}_i$(gm). Note that $\boldsymbol{\mu}_i$(gm) does not explicitly depend on neutrino mass.

The gravitomagnetic moment $\boldsymbol{\mu}_i$(gm) of (1.2) for a neutrino with mass m_i may be distinguished by different g_i-factors. Starting from the Dirac equation, however, in first order the same factor $g_i = +2$ is deduced in section 3 for all neutrinos m_i, analogously to the factor $g_l = +2$ for all charged leptons.

When the magnetic moments $\boldsymbol{\mu}_i$(em) from (1.1) and $\boldsymbol{\mu}_i$(gm) from (1.2) are taken equal, the following expression for mass m_i results

$$m_i = -\frac{2\pi^2\sqrt{2}}{3|e|G_F c^4}\, g_i\beta\left(\frac{G}{k}\right)^{1/2}. \tag{1.3}$$

Note that $\boldsymbol{\mu}_i$(em) from (1.1) and $\boldsymbol{\mu}_i$(gm) of (1.2) have the same direction for a negative value of the product $g_i\beta$. Since a positive value $g_i = +2$ is deduced from the Dirac equation in section 3, β must be negative. This result is important, for the sign of the β-factor was unknown, so far. Insertion of the value $g_1 = +2$ and a value $\beta = -1$ into (1.3) yields a value of 1.530 meV for neutrino mass m_1, the main result of this work.

At present, no magnetic moment of any neutrino has been measured. The tightest constraint on μ_i comes from studies of a possible delay of helium ignition in the core of red giants in globular clusters. From the lack of observational evidence of this effect a limit of $\mu_i < 3\times10^{-12}\mu_B$ has been extracted [5]. This limit still exceeds the value $\mu_i = 3.2\times10^{-22}(m_i/\mathrm{meV})\mu_B$ from (1.1) by many orders of magnitude. Conformation of the proposed value of mass m_1, however, may provide a first indication of the existence of non-zero neutrino magnetic moments (1.1) and (1.2).

According to the neutrino oscillation theory [3, 4], the masses of the three neutrino flavour states ν_α ($\alpha = e, \mu, \tau$) can be expressed as a superposition of three massive eigenstates ν_i with masses m_i (i = 1, 2, 3). In addition, mass-squared splittings $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$ and $\Delta m_{31}^2 \equiv m_3^2 - m_1^2$ follow from observations. So, two relations between the masses m_1, m_2, and m_3 are available, whereas three masses m_i are initially unknown. Thus, when the neutrino mass m_1 is known, the remaining masses m_2 and m_3 can be calculated. In section 4 such a calculation has been performed. In section 5 our results are compared with results following from the so-called "geometric mean neutrino mass relation". In addition, a possible dependence of the masses m_i on the weak coupling constant α_W is discussed. Conclusions are drawn in section 6. In section 2, however, we first consider the deduction of relation (1.2) and the corresponding electromagnetic magnetic moments.

2. GRAVITOMAGNETIC AND ELECTROMAGNETIC MOMENTS

Since 1891 many authors have discussed a gravitational origin of the magnetic field of rotating celestial bodies. Particularly, the so-called Wilson-Blackett formula has often been considered [6–15]

$$\boldsymbol{\mu}(\text{gm}) = -\frac{\beta}{2}\left(\frac{G}{k}\right)^{1/2}\mathbf{S}, \tag{2.1}$$

where $\boldsymbol{\mu}$(gm) is the gravitomagnetic moment of the massive body with angular momentum **S**. For a sphere with a homogeneous mass density the angular momentum **S** is given by **S** = 2/5 $mr^2\boldsymbol{\omega}$, where m is the mass of the sphere of radius r and $\boldsymbol{\omega}$ its angular velocity. Note that $\boldsymbol{\mu}$(gm) is proportional to the mass m. The parameter β is assumed to be a dimensionless constant of order unity. So far, the sign and value of β are unknown, however (see ref. [14] for an ample discussion of this point).

Analogously to the electromagnetic counterpart $\boldsymbol{\mu}$(em) of (1.1), the gravitomagnetic moment $\boldsymbol{\mu}$(gm) leads to a dipolar gravitomagnetic field at distance **R** of magnitude

$$\mathbf{B}(\text{gm}) = \frac{\mu_0}{4\pi}\left(\frac{3\boldsymbol{\mu}(\text{gm})\cdot\mathbf{R}}{R^5}\mathbf{R} - \frac{\boldsymbol{\mu}(\text{gm})}{R^3}\right). \tag{2.2}$$

According to the Wilson-Blackett relation, the field **B**(gm) may be identified as an electromagnetic induction field. In that case the magnetic moments $\boldsymbol{\mu}$(em) of (1.1) and $\boldsymbol{\mu}$(gm) of (1.2) are equivalent.

Attempts to derive (2.1) from a more general theory have been given by several authors (see, e.g., [10–14, 16] and references therein). For example, Bennet *et al.* [10] already gave a five-dimensional theory resulting into (2.1). Luchak [11] found a relation related to (2.1) from another five-dimensional theory. He gave a relativistic generalization of the Maxwell equations by combining them with gravitational fields. Other authors like Biemond [12–14] and Widom and Ahluwalia [16], tried to explain equation (2.1) as a consequence of general relativity. The former author [12–14] started from the Einstein equations in the slow motion and weak field approximation and deduced a set of four gravitomagnetic equations, analogous to the four Maxwell equations. The so-called "magnetic-type" gravitational field in these equations is identified as a magnetic induction field, resulting into the gravitomagnetic moment $\boldsymbol{\mu}$(gm) of (2.1).

Since charges in rotating bodies may affect the value of the parameter β in many different ways, one can hardly expect that the observed value of β is a constant. Different values for the empirical value of β have indeed been found for about fourteen rotating bodies: metallic cylinders in the laboratory, moons, planets, stars and the Galaxy [13, ch. 1]. For pulsars a separate analysis has been given in ref. [17]. From a linear regression analysis of the series of the fourteen rotating bodies an almost linear relationship between the observed magnetic moment $|\boldsymbol{\mu}(\text{obs})|$ and the angular momentum $|\mathbf{S}|$ was found. Such a linear relationship between $\boldsymbol{\mu}$(gm) and **S** is predicted by (2.1). From this analysis an average value of $|\beta|$ = 0.076 was calculated. Although this result is distinctly different from a gravitomagnetic prediction for a theoretical value like $|\beta|$ = 1 in (2.1), the correct order of magnitude of β for so many, *strongly different*, rotating bodies is amazing (values of $|\boldsymbol{\mu}(\text{obs})|$ and $|\mathbf{S}|$ vary over an interval of *sixty decades!*). So, the gravitomagnetic hypothesis, embodied in the Wilson-Blackett law (2.1), may be basically valid.

For a macroscopic rotating sphere of mass m with a homogeneous charge density and a total charge Q, the magnetic moment is given by (see, e.g., ref. [18, 19])

$$\boldsymbol{\mu}(\text{em}) = \frac{Q}{2m}\mathbf{S}. \tag{2.3}$$

It is noted that the derivation of (2.3) from the Maxwell equations and the deduction of (2.1) from the gravitomagnetic equations are analogous. Recently, Barrow and Gibbons

[20] suggested that for charged rotating black holes un upper bound $(G/k)^{½}$ may hold for the quantity $|Q|/(2m)$.

For elementary particles like charged leptons with masses m_l ($l = e, \mu, \tau$) the z-component of the angular momentum **S** is given by $S_z = ½\hbar$, as has been discussed by Pauli [21]. As an example, for an electron with mass m_e and charge e the z-component of the electromagnetic moment $\boldsymbol{\mu}$(em) of (2.3) transforms into

$$\mu_z(\mathrm{em}) = \frac{g_e e}{2m_e}\frac{\hbar}{2}. \tag{2.4}$$

Since more contributions to the dimensionless factor g_e can be distinguished, the g_e-factor is usually written as a series expansion

$$g_e = 2 + \frac{\alpha}{\pi} + ... = 2\left(1 + \frac{\alpha}{2\pi} + ...\right) = 2\left(1 + 0.00116141 + ...\right), \tag{2.5}$$

where $\alpha = ke^2/\hbar c = 1/137.036$ is the fine-structure constant. The leading term in the series expansion of g_e, $g_e = +2$, has been deduced by Dirac [22]. Later on, Schwinger [23] deduced the first and largest one-loop correction α/π to g_e from the theory of quantum electrodynamics (QED).

Analogous to μ_z(em) of (2.4), the following gravitomagnetic moment μ_z(gm) can be obtained, both for charged leptons and neutrinos (compare to (1.2))

$$\mu_z(\mathrm{gm}) = -\frac{g\beta}{2}\left(\frac{G}{k}\right)^{½}\frac{\hbar}{2}. \tag{2.6}$$

Whereas $\boldsymbol{\mu}_i$(em) of (1.1) is proportional to neutrino mass m_i, μ_z(gm) of (2.6) does not explicitly depend on mass. For this reason, this mass dependence will be combined with the factor g-factor in this work.

As an example, for a neutrino of mass m_1 we substitute $g_1 = +2$ and $\beta = -1$ into (2.6), and for an electron $g_e = +2$ into (2.4), respectively. From (2.6) and (2.4) one then obtains the following expression for the ratio $|\mu_z(\mathrm{gm})|/|\mu_z(\mathrm{em})|$

$$\frac{|\mu_z(\mathrm{gm})|}{|\mu_z(\mathrm{em})|} = \left(\frac{G}{k}\right)^{½}\frac{m_e}{|e|} = +4.899\times 10^{-22}. \tag{2.7}$$

It appears that the gravitomagnetic moment $|\mu_z(\mathrm{gm})|$ for the neutrino is extremely small compared to the electromagnetic moment $|\mu_z(\mathrm{em})|$ of the electron. This results shows that a magnetic induction field from gravitomagnetic origin may easily be masked by the electromagnetic field generated by a small amount of charge. Note that the products $G^{½}m_e$ and $k^{½}e$ in (2.7) have the same dimension.

The leading QED correction to μ_z(em) in (2.4) (see (2.5)) equals to $\delta\mu_z(\mathrm{em}) \approx (\alpha/2\pi)\,\mu_z(\mathrm{em})$. When $g_1 = +2$ and $\beta_1 = -1$ are again substituted into (2.6), and $g_e = +2$ is again inserted into (2.4), the ratio $|\mu_z(\mathrm{gm})|/|\delta\mu_z(\mathrm{em})|$ appears to be

$$\frac{|\mu_z(\mathrm{gm})|}{\delta|\mu_z(\mathrm{em})|} \approx \left(\frac{G}{k}\right)^{½}\frac{m_e}{|e|}\frac{2\pi}{\alpha} = +4.22\times 10^{-19}. \tag{2.8}$$

So, for the electron the gravitomagnetic moment $|\mu_z(\mathrm{gm})|$ is still much smaller than the contribution $\delta\mu_z$(em).

3. CALCULATION OF THE *g*-FACTOR FOR LEPTONS

For comparison, we first review the derivation of the g_l-factor of a charged lepton before considering the g_i-factor of a neutrino. For a system of particles with charge e and mass m, moving with velocity $\mathbf{v}$ through an external uniform magnetic induction field $\mathbf{B}$(em), the following term has to be added to the Lagrangian (see, e.g., Landau and Lifshitz [24, ch. 45])

$$L' = \sum e\mathbf{A}(\mathrm{em})\cdot\mathbf{v} = \sum \frac{e}{2}\{\mathbf{B}(\mathrm{em})\times\mathbf{r}\}\cdot\mathbf{v} = \sum \frac{e}{2}(\mathbf{r}\times\mathbf{v})\cdot\mathbf{B}(\mathrm{em}). \tag{3.1}$$

In deriving (3.1), use has been made of the expression for the external uniform electromagnetic vector potential $\mathbf{A}(\mathrm{em}) = \frac{1}{2}\mathbf{B}(\mathrm{em})\times\mathbf{r}$. If all charges of the system have the same ratio of charge to mass and the velocities $|\mathbf{v}|$ of all charges are much smaller than c, then (3.1) can be rewritten as

$$L' = \sum \frac{e}{2m}(\mathbf{r}\times m\mathbf{v})\cdot\mathbf{B}(\mathrm{em}) = \frac{e}{2m}\mathbf{S}\cdot\mathbf{B}(\mathrm{em}) = \boldsymbol{\mu}(\mathrm{em})\cdot\mathbf{B}(\mathrm{em}), \tag{3.2}$$

where $\mathbf{S} = \Sigma\, \mathbf{r}\times\mathbf{p}$ is the angular momentum of the system, $\mathbf{p} = m\mathbf{v}$ is the momentum of a particle and $\boldsymbol{\mu}$(em) is the magnetic moment of the system.

For a lepton with charge e and mass m the contribution to the Hamiltonian, H', corresponding to (3.2), is given by

$$H' = -\frac{g_l e}{2m}\mathbf{S}\cdot\mathbf{B}(\mathrm{em}) = -\boldsymbol{\mu}_l(\mathrm{em})\cdot\mathbf{B}(\mathrm{em}), \tag{3.3}$$

where $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ is now the angular momentum of the charged lepton and $\boldsymbol{\mu}_l$(em) its magnetic moment. In order to calculate the g_l-value in (3.3) the Dirac equation will now be considered below.

The Dirac equation in the presence of an external electromagnetic covariant four-vector potential $A_\mu(\mathrm{em}) = \{A_0(\mathrm{em}), -\mathbf{A}(\mathrm{em})\}$ (or alternatively written in terms of the contravariant four-vector $A^\mu(\mathrm{em}) = \{A^0(\mathrm{em}), \mathbf{A}(\mathrm{em})\}$) for a charged lepton is given by [22]

$$\left[\gamma^\mu\{p_\mu - eA_\mu(\mathrm{em})\} - mc\right]\psi = 0. \tag{3.4}$$

Here the matrices γ^μ (μ = 0, 1, 2, 3) are 4×4 matrices and p_μ is a four-vector defined by $p_\mu \equiv i\hbar\partial/\partial x^\mu$ with $x^\mu = (ct, \mathbf{r})$, whereas the wave function ψ is a four-component column matrix. When the electromagnetic vector potential $\mathbf{A}(\mathrm{em}) = \{A^1(\mathrm{em}), A^2(\mathrm{em}), A^3(\mathrm{em})\}$ and the scalar potential $A_0(\mathrm{em}) = A^0(\mathrm{em}) = \phi/c$ in (3.4) are relatively small, the components of the wave functions ψ are approximately solutions of the Dirac equation for the free particle (this approximation is usually denoted as the principle of minimal coupling). Analogously to the Schrödinger equation, the Dirac equation (3.4) can then be written as a differential equation first order in time

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\{\mathbf{p} - e\mathbf{A}(\mathrm{em})\} + e\phi + \beta mc^2\right]\psi = H_{\mathrm{D}}\psi. \tag{3.5}$$

Here the components of $\boldsymbol{\alpha}$ are defined by the 4×4 matrices $\alpha^i \equiv \gamma^0\gamma^i$ (i = 1, 2, 3) and the components of momentum $\mathbf{p}$ by $p^i \equiv (p^1, p^2, p^3)$, respectively. Note that α^i is no spatial part of a four-vector (there is no α^0); so, its superscript index i is no contravariant index. The 4×4 matrix β is defined by $\beta \equiv \gamma^0$ and $\mathbf{A}$(em) is the vector potential. From (3.5) the expression for the Dirac Hamiltonian H_D follows.

Since the time dependence of the wave function is governed by the energy (energy eigenstates have the time dependence $e^{-iEt/\hbar}$), a factor of $e^{-imc^2t/\hbar}$ may be split off from the Dirac wave function ψ in first order. In addition, the four-component spinor ψ may be decomposed into two two-component spinors φ and χ. So, ψ will be rewritten as

$$\psi = e^{-imc^2t/\hbar}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{3.6}$$

In addition, the four 4×4 matrices γ^μ in the so-called Dirac representation and α^i are partitioned into 2×2 matrices

$$\gamma^0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix} \equiv \beta, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix}; \qquad \alpha^i \equiv \gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}, \tag{3.7}$$

where I is a 2×2 unit matrix and σ_i are the Pauli 2×2 matrices. These matrices are, respectively, given by

$$I = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}, \tag{3.8}$$

where the 4×4 matrix β is replaced by the 2×2 matrix I and the 4×4 matrices of $\boldsymbol{\alpha}$ are replaced by the 2×2 Pauli matrices σ_1, σ_2 and σ_3.

Utilizing (3.6), (3.7) and (3.8), equation (3.5) transforms into

$$\left\{mc^2\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} + i\hbar\frac{\partial}{\partial t}\right\}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \left\{c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0\end{pmatrix} + e\phi + mc^2\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\right\}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \tag{3.9}$$

The generalized momentum $\boldsymbol{\pi}$ is defined by $\boldsymbol{\pi} \equiv \mathbf{p} - e\mathbf{A}$(em). Evaluation of (3.9) leads to two coupled equations

$$i\hbar\frac{\partial\varphi}{\partial t} = c(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\chi + e\phi\varphi, \tag{3.10}$$

$$i\hbar\frac{\partial\chi}{\partial t} = c(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\varphi + e\phi\chi - 2mc^2\chi. \tag{3.11}$$

When in the weak field limit $e\phi$ is small compared to rest energy mc^2 and the function χ slowly varies in time, i.e. $i\hbar\partial\chi/\partial t \approx 0$, relation (3.11) reduces to

$$\chi \approx \frac{1}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\varphi. \tag{3.12}$$

In the non-relativistic limit $\boldsymbol{\sigma}\cdot\boldsymbol{\pi} << mc$, so that $\chi << \varphi$. Substitution of (3.12) into (3.10) yields

$$i\hbar\frac{\partial\varphi}{\partial t} = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\varphi + e\phi\varphi. \tag{3.13}$$

This differential equation is known as the Pauli equation [21].

Equation (3.13) can be evaluated by the Pauli vector identity

$$\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)^2=\boldsymbol{\pi}^2+i\boldsymbol{\sigma}\cdot\left(\boldsymbol{\pi}\times\boldsymbol{\pi}\right). \tag{3.14}$$

In addition, the quantity $(\boldsymbol{\pi}\times\boldsymbol{\pi})\varphi$ can be shown to be

$$\left(\boldsymbol{\pi}\times\boldsymbol{\pi}\right)\varphi=ie\hbar\nabla\times\mathbf{A}(\mathrm{em})\,\varphi=ie\hbar\,\mathbf{B}(\mathrm{em})\,\varphi. \tag{3.15}$$

Combination of the equations (3.13), (3.14) and (3.15) yields

$$i\hbar\frac{\partial\varphi}{\partial t}=\left\{\frac{\boldsymbol{\pi}^2}{2m}+e\phi-\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}(\mathrm{em})\right\}\varphi. \tag{3.16}$$

Insertion of $\mathbf{S}=(\hbar/2)\boldsymbol{\sigma}$ into (3.16) leads to the Pauli Hamiltonian H_{P}

$$H_{\mathrm{P}}=\frac{\boldsymbol{\pi}^2}{2m}+e\phi-\frac{e}{m}\mathbf{S}\cdot\mathbf{B}(\mathrm{em})=\frac{\boldsymbol{\pi}^2}{2m}+e\phi-\boldsymbol{\mu}_l(\mathrm{em})\cdot\mathbf{B}(\mathrm{em}). \tag{3.17}$$

From comparison of the spin-dependent term in H_{P} of (3.17a) and that of H' in (3.3a) follows in first order that $g_l=+2$ for a charged lepton.

When a mass m is considered moving with a velocity $\mathbf{v}$ through an external uniform gravitomagnetic vector potential $\mathbf{A}$(gm), the following contribution to the Lagrangian has to be added (see refs. [13, 14]), corresponding to the electromagnetic Lagrangian L' of (3.1)

$$L'=\frac{4m}{\beta}\left(\frac{G}{k}\right)^{1/2}\mathbf{A}(\mathrm{gm})\cdot\mathbf{v},\quad A^{\alpha}(\mathrm{gm})\equiv-\frac{\beta}{4}\left(\frac{k}{G}\right)^{1/2}c\,g_{0\alpha}. \tag{3.18}$$

Here the three components of $\mathbf{A}$(gm) are given by A^{α}(gm) ($\alpha=x,y,z$). The components A^{α}(gm) can be connected to the metric components $g_{0\alpha}$ and have been deduced in the weak field and slow motion limit [13, 14].

Utilizing the expression for the vector potential $\mathbf{A}(\mathrm{gm})=\tfrac{1}{2}\mathbf{B}(\mathrm{gm})\times\mathbf{r}$ and (2.1), equation (3.18) can be rewritten as

$$L'=\frac{2}{\beta}\left(\frac{G}{k}\right)^{1/2}\mathbf{S}\cdot\mathbf{B}(\mathrm{gm})=-\frac{4}{\beta^2}\boldsymbol{\mu}(\mathrm{gm})\cdot\mathbf{B}(\mathrm{gm}), \tag{3.19}$$

where $\mathbf{S}$ is the angular momentum of mass m and $\boldsymbol{\mu}$(gm) its gravitomagnetic moment.

For an elementary particle like a neutrino, the contribution to the Hamiltonian, H', corresponding to (3.19) is then given by

$$H'=-\frac{2}{\beta}\left(\frac{G}{k}\right)^{1/2}\mathbf{S}\cdot\mathbf{B}(\mathrm{gm})=\frac{4}{\beta^2}\boldsymbol{\mu}_i(\mathrm{gm})\cdot\mathbf{B}(\mathrm{gm}), \tag{3.20}$$

where $\mathbf{S}=(\hbar/2)\boldsymbol{\sigma}$ is now the angular momentum and $\boldsymbol{\mu}_i$(gm) is the gravitomagnetic moment of the neutrino, respectively. When $\boldsymbol{\mu}_i$(gm) and $\mathbf{B}$(gm) are equivalent to their electromagnetic counterparts $\boldsymbol{\mu}_i$(em) and $\mathbf{B}$(em), a deduction, analogous to (3.3a) and (3.17a), leads to a value $g_i=+2$ for all neutrinos in first order.

Many authors have also followed the gravitomagnetic approach, but without taking $\mathbf{B}$(gm) equivalent to $\mathbf{B}$(em). They implicitly choose different values for the *dimensionless* constant β like $\beta=+4$, $+2$, $+1$ and -1 without further comment (see ref. [14] for an ample discussion of the sign and value of β).

It appears that the sign of term $-\boldsymbol{\mu}_i(\text{em})\cdot\mathbf{B}(\text{em})$ in the electromagnetic Hamiltonian H_P of (3.17) differs from that of the term $4/\beta^2\boldsymbol{\mu}_i(\text{gm})\cdot\mathbf{B}(\text{gm})$ in the gravitomagnetic Hamiltonian H' of (3.20). A related difference in sign is found between the electromagnetic torque $\boldsymbol{\tau}(\text{em}) = \boldsymbol{\mu}(\text{em})\times\mathbf{B}(\text{em})$ and the corresponding gravitomagnetic torque $\boldsymbol{\tau}(\text{gm}) = -1/\beta^2\boldsymbol{\mu}(\text{gm})\times\mathbf{B}(\text{gm})$ (see [13]; compare to [24, p. 105]). Moreover, for $\beta = \pm1$ the term $4/\beta^2\boldsymbol{\mu}_i(\text{gm})\cdot\mathbf{B}(\text{gm})$ contains an extra factor of four compared to the corresponding electromagnetic term $-\boldsymbol{\mu}_i(\text{em})\cdot\mathbf{B}(\text{em})$. This factor stems from the factor of four in the gravitomagnetic Lagrangian L' of (3.19b), compared to the corresponding electromagnetic Lagrangian L' of (3.2).

It is noticed, that such a difference of a factor of four is also found between the gravitational and electromagnetic radiation formula. As an illustration, consider the system of two identical point masses moving at an angular frequency ω in a circular orbit of radius r around the common center of mass. The quadrupole formula for gravitational radiation for this system, $I_{\text{grav. rad.}}$, is given by (see, e.g., Landau and Lifshitz [24, p. 356])

$$I_{\text{grav. rad.}} = I_{\text{grav. magn. rad.}} = \frac{128}{5}\frac{Gm^2\omega^4r^6}{c^5}. \tag{3.21}$$

From the gravitomagnetic approach the same expression for the quadrupole radiation of this system, $I_{\text{grav. magn. rad.}}$, is obtained. It appears that this result does not depend on the constant β (see, ref. [13, ch. 3]). For comparison, the quadrupole radiation of two identical point charges e moving at an angular frequency ω in a circular orbit of radius r around the common center of mass, equals to

$$I_{\text{el. magn. rad.}} = \frac{32}{5}\frac{ke^2\omega^4r^6}{c^5}. \tag{3.22}$$

Note that the electromagnetic dipole radiation is zero is this case. It follows that in the ratio $I_{\text{grav. magn. rad.}}/I_{\text{el. magn. rad.}} = 4\ Gm^2/ke^2$ the same peculiar factor of four appears.

According to (1.1), $\boldsymbol{\mu}_i(\text{em})$ depends on mass m_i, whereas $\boldsymbol{\mu}_i(\text{gm})$ in (1.2) displays no explicit dependence on mass. Therefore, this mass dependence of $\boldsymbol{\mu}_i(\text{gm})$ of (1.2) will be combined with the g_i-factor.

4. CALCULATION OF THE NEUTRINO MASSES

In table 1 data deduced from the framework of three-neutrino oscillations are summarized. Best fit values are given for the two squared-mass differences Δm_{21}^2 and Δm_{31}^2 from de Salas *et al.* [25], Esteban *et al.* [26], and Capozzi *et al.* [27]. Three data series for the normal hierarchy are shown and one for the inverted hierarchy.

Table 1. Best fit Δm^2 values ($\pm1\sigma$) from three-neutrino oscillation analyses.

	normal hierarchy [25]	inverted hierarchy [25]	normal hierarchy [26]	normal hierarchy [27]
Δm^2	best fit $\pm1\sigma$	best fit $\pm1\sigma$	best fit $\pm1\sigma$	best fit $\pm1\sigma$
$\Delta m_{21}^2/10^{-5}$ eV^2	7.56	7.56	7.50	7.37
$\Delta m_{31}^2/10^{-3}$ eV^2	+ 2.550	– 2.492	+ 2.524	+ 2.562

Substitution of a value $g_1 = +2$ and $\beta = -1$ into (1.3) for the neutrino mass m_1, yields a value of $m_1 = +1.530$ meV. The accuracy of this result depends on the relative inaccuracy of the gravitational constant $G = 6.674\times10^{-11}$ kg^{-1}m^{3}s^{-2}. Subsequently, the masses m_2 and m_3 have been calculated from the values of $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$ and $\Delta m_{31}^2 \equiv m_3^2 - m_1^2$ for the three data series in the case of normal hierarchy given in table 1. Results are summarized in table 2. Note that all masses m_i have a positive sign. Since the mass m_1 is relativity small compared with masses m_2 and m_3, comparison of tables 1 and 2 shows, that the masses m_2 and m_3 are approximately given by the roots of Δm_{21}^2 and Δm_{31}^2, respectively.

When the calculated masses m_2 and m_3 are subsequently inserted into (1.3) and a value $\beta = -1$ is chosen, the empirical g_i-factors g_2 and g_3 can be calculated, respectively. For comparison, the g_i-factors have been expressed relative to $g_1 = +2$ by defining the relative factor $g'_i \equiv g_i/g_1$. So, $g'_1 = 1$ and from (1.3) follows that $g'_i = m_i/m_1$. The values of g'_i have been given in table 2.

When, instead of $\beta = -1$, a value $\beta = -2$ is introduced into (1.3) the value of m_1 doubles to $m_1 = 3.06$ meV. Combination of this value for m_1 and data from ref. [25] in table 1 then leads to the values $m_2 = 9.22$ meV and $m_3 = 50.6$ meV.

Table 2. Calculated neutrino masses m_2 and m_3 from data in table 1 for the normal hierarchy. All masses are given in units of meV. The g'_i-factor is defined by $g'_i \equiv m_i/m_1$ and the quantity $\Delta g'_i(\alpha_W^{-1})$ by $\Delta g'_i(\alpha_W^{-1}) \equiv (g'_i - g'_1)\times\alpha_W^{-1}$, respectively. For comment, see text.

ref.	[25]	[26]	[27]		[25]	[26]	[27]
m_1/meV	1.530	1.530	1.530	g'_1	1	1	1
m_2/meV	8.83	8.79	8.72	g'_2	5.77	5.75	5.70
m_3/meV	50.5	50.3	50.6	g'_3	33.0	32.9	33.1
$\Sigma_i\ m_i$/meV	60.9	60.6	60.9	$\Delta g'_1(\alpha_W^{-1})$	0	0	0
$R_\nu = (m_2)^2/(m_1 m_3)$	1.01	1.01	0.98	$\Delta g'_2(\alpha_W^{-1})$	0.146	0.145	0.144
				$\Delta g'_3(\alpha_W^{-1})$	0.979	0.973	0.981

[a] A value $\alpha_W^{-1} = 32.71$ has been used in the calculations. See section 5 for a discussion of this parameter.

Using the values Δm_{21}^2 and Δm_{31}^2 for the inverted hierarchy from ref. [25], the choice $m_3 = 1.530$ meV leads to the values $m_2 = 50.7$ meV and $m_1 = 49.9$ meV, resulting into $\Sigma_i\ m_i = 102$ meV. So, following our approach the possibility of an inverted hierarchy cannot be excluded.

From calculated values m_i in table 2 an average sum $\Sigma_i\ m_i = 60.8$ meV can be calculated. This value can be compared with cosmological constraints. So far, the tightest constraint of the sum $\Sigma_i\ m_i < 92.6$ meV at 90% C. L. has been given by Di Valentino *et al*. [28]. They extracted this bound by combining the full Planck measurements, Baryon Acoustic Oscillation and Planck clusters data. In addition, they imposed a low reionization redshift prior. The obtained value for $\Sigma_i\ m_i$ illustrates that a cosmological measurement of the neutrino mass hierarchy is at reach.

5. DISCUSSION OF THE RESULTS

In the past, many relations between the masses of leptons and quarks have been proposed in order to investigate the origin of mass of elementary particles. For example, He and Zee [29], and Sazdović [30] used the so-called "geometric mean neutrino mass relation" between the three active masses m_i (i = 1, 2, 3) of the neutrinos

$$R_\nu = \frac{m_2^2}{m_1 m_3} = 1. \tag{5.1}$$

The ratio (5.1) provides a third relation between the masses m_1, m_2 and m_3, so that all masses can be calculated. As an example, the values for Δm_{21}^2 and Δm_{31}^2 for the normal hierarchy from ref. [25] in our table 1 can be chosen. Combination of these data and relation (5.1) yields the following masses: $m_1 = 1.54$ meV, $m_2 = 8.83$ meV and $m_3 = 50.5$ meV. These values are close to the values $m_1 = 1.53$ meV, $m_2 = 8.83$ meV and $m_3 = 50.5$ meV deduced from data of ref. [25] (see table 2). Inserting the latter values into (5.1) yields a value $R_\nu = (m_2)^2/(m_1 m_3) = 1.01$. The other values for R_ν calculated from data of refs. [26, 27] are also given in table 2. The averaged value of these three results is $R_\nu = 0.999$, remarkably close to the value $R_\nu = 1$ of (5.1).

One can also try to express the masses m_2 and m_3 in units of the weak coupling constant $\alpha_W = kg^2/\hbar c$, a basic constant for electroweak interactions. This constant is analogous to the fine-structure constant $\alpha = ke^2/\hbar c = 1/(137.036)$ for electromagnetic interactions. The relation between the charges g and e is given by $g \sin\theta_W = e$, where θ_W is the weak mixing angle. From the modified minimal subtraction scheme $\overline{\mathrm{MS}}$ a value of $\sin^2\theta_W = 0.23868$ at low energies has been taken from ref. [31], resulting into a value of $\alpha_W = 1/(32.71)$. The averaged values of the masses m_2 and m_3 from table 2 can then be written as

$$m_2 = (1 + 0.145\alpha_W^{-1})\, m_1, \tag{5.2}$$

$$m_3 = (1 + 0.978\alpha_W^{-1})\, m_1. \tag{5.3}$$

By defining a quantity $\Delta g'_i(\alpha_W^{-1}) \equiv (g'_i - g'_1)\times\alpha_W^{-1}$, the dependence of the neutrino masses on units of α_W^{-1} can be expressed. The values of $\Delta g'_i(\alpha_W^{-1})$ have been given separately in table 2. As an illustration, for $\Delta g'_3(\alpha_W^{-1}) = m_3/m_1 - 1$ follows from (5.3) that $\Delta g'_3(\alpha_W^{-1}) = 0.978\alpha_W^{-1}$, or otherwise stated $m_3 \approx (1 + \alpha_W^{-1})\, m_1$. For the on-shell electroweak mixing parameter $\sin^2\theta_W = 0.22333$ a value of $\alpha_W = 1/(30.60)$ is obtained. Results of $\Delta g'_i(\alpha_W^{-1})$ for this choice of α_W have earlier been given in ref. [15].

For comparison, the masses m_l ($l = e, \mu, \tau$) of the charged leptons can be expressed in terms of the fine-structure constant α

$$m_e = m_e, \quad m_\mu = \left(1 + \frac{3}{2}\alpha^{-1}\right) m_e \quad \text{and} \quad m_\tau = 17\left(-1 + \frac{3}{2}\alpha^{-1}\right) m_e. \tag{5.4}$$

Using the observed electron mass $m_e = 0.51099895$ MeV, the calculated muon mass $m_\mu = 105.54888$ MeV from (5.4) differs -0.104 % from the observed mass $m_\mu = 105.65837$ MeV, whereas the calculated tauon mass $m_\tau = 1776.96$ MeV differs $+0.0055$ % from the observed mass $m_\tau = 1776.86$ MeV (see for the observed data ref. [32]). Previously, Barut [33, 34] deduced a series expansion for the masses of the charged leptons. For m_μ he gave the same expression as in (5.4), whereas for m_τ he obtained the following formula

$$m_\tau = m_\mu + \left(2^4 \frac{3}{2}\alpha^{-1}\right) m_e = \left(1 + 17\frac{3}{2}\alpha^{-1}\right) m_e. \tag{5.5}$$

In this case the calculated tauon mass $m_\tau = 1786.15$ MeV differs $+0.523$ % from the observed mass $m_\tau = 1776.86$ MeV. Analogous to (5.1), the following ratio R_l between the masses m_l ($l = e, \mu, \tau$) of the charged leptons can be calculated. For the observed masses one obtains

$$R_l = \frac{m_\mu^2}{m_e m_\tau} = 12.2952. \tag{5.6}$$

Substitution of the expressions of (5.4) into (5.6) yields a value $R_l = 12.2690$ that is 0.213 % lower than the observed value. From the expressions of Barut from (5.4) and (5.5) a value of $R_l = 12.2059$ is obtained that is 0.726 % lower than the observed value. Finally, Sazdović [30] proposed a value $R_l = e^{5/2} = 12.1825$ that is 0.916 % lower than the observed value.

6. CONCLUSIONS

A value of 1.530 meV/c^2 = 2.727×10^{-39} kg for the mass of the lightest elementary particle, the neutrino m_1, is obtained in this work. This value is extracted from a combination of the magnetic moment of a massive Dirac neutrino [1, 2], deduced in the context of electroweak interactions at the one-loop level, and the magnetic moment from gravitational origin proposed by Wilson and Blackett [6–15]. The latter relation has also been obtained from a gravitomagnetic interpretation of the Einstein equations [12–14]. Combination with neutrino oscillation data yields the other masses m_2 and m_3 (see table 2). It is stressed that these results depend on the validity of the assumptions involved, especially the validity of the Wilson-Blackett relation (1.2).

Previously, the so-called geometric mean neutrino mass relation $(m_2)^2/(m_1m_3) = 1$ between the three active masses m_i (i = 1, 2, 3) of the neutrinos has been used by He and Zee [29], and Sazdović [30]. As an example, from data of ref. [25] given in our table 1 a value of m_1 = 1.54 meV/c^2 can be calculated from this relation, in fair agreement with our result. More research, both theoretically and observationally is necessary, however, to confirm the validity of the relation $(m_2)^2/(m_1m_3) = 1$.

It has been shown by Barut [33, 34] that the masses m_l ($l = e, \mu, \tau$) of the charged leptons can be expressed in terms of the fine-structure constant α. Analogously, the masses m_2 and m_3 of the neutrinos may be written in terms of the weak coupling constant $\alpha_W = kg^2/\hbar c$. It appears that the mass m_3 can approximately be written as $m_3 = (1+ \alpha_W^{-1})\, m_1$.

It is noticed that the neutrino m_1 with the smallest mass may also possess the smallest gravitomagnetic moment μ_1(gm) = 4.899×10^{-22} μ_B and may thus be the smallest magnet. So far, no magnetic moment of any neutrino has been measured, however (see, e.g., ref. [5]). According to (1.2), μ_{1z}(gm) for the lightest mass m_1 equals $\frac{1}{2}(G/k)^{1/2}\hbar$ for β = –1 and g_1 = +2, or $\mu_{1z}(\text{gm}) = \frac{1}{2}(G^{1/2}/c)\hbar$ in Gaussian units. The latter expression for μ_{1z}(gm), consists of a combination of the universal constants G, c and $\hbar$. Recently, renewed interest arose in the literature for such combinations of Planck units (see, e.g., Barrow and Gibbons [20]).

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