Fock - Hua - Wolf - Schwinger - Wyler Quantum Theory

Frank Dodd (Tony) Smith, Jr . - 2015 - vixra 1507.xxxx

Fock (1931 - see Appendix I) showed that Fundamental Quantum Theory requires Linear Operators "... represented by a definite integral [of a]... kernel ... function ...".

Hua (1958 - see Appendix II) showed Kernel Functions for Complex Classical Domains.

Schwinger (1951 - see Schweber, PNAS 102, 7783-7788) "... introduced a description in terms of Green's functions, what Feynman had called propagators ... The Green's functions are vacuum expectation values of time-ordered Heisenberg operators, and the field theory can be defined non-perturbatively in terms of these functions ...[which]... gave deep structural insights into QFTs; in particular ... the structure of the Green's functions when their variables are analytically continued to complex values ...".

Wolf (J. Math. Mech 14 (1965) 1033-1047) showed that the Classical Domains
(complete simply connected Riemannian symmetric spaces)
representing 4-dim Spacetime with Quaternionic Structure are:
S1 x S1 x S1 x S1 = 4 copies of U(1)
S2 x S2 = 2 copies of SU(2)
CP2 = SU(3) / SU(2)xU(1)
S4 = Spin(5) / Spin(4) = Euclidean version of Spin(2,3) / Spin(1,3)

Armand Wyler (1971 - C. R. Acad. Sc. Paris, t. 271, 186-188) showed how to use
Green's Functions = Kernel Functions of Classical Domain structures characterizing
Sources = Leptons, Quarks, and Gauge Bosons,
to calculate Particle Masses and Force Strengths (see also viXra 1405.0030).

Schwinger (1969 - see physics/0610054) said: "... operator field theory ... replace[s] the particle with ... properties ... distributed througout ... small volumes of three-dimensional space ... particles ... must be created ... even though we vary a number of experimental parameters ... The properties of the particle ... remain the same ... We introduce a quantitative description of the particle source in terms of a source function ...
we do not have to claim that we can make the source arbitrarily small ...
the experimeter... must detect the particles ...[by]... collision that annihilates the particle ... the source ... can be ... an abstraction of an annilhilation collision,
with the source acting negatively, as a sink ... The basic things are ... the source functions ... describing the intermediate propagation of the particle ...".

Creation and Annihilation operators indicate a Clifford Algebra, and 8-Periodicity shows that the basic Clifford Algebra is formed by tensor products of 256-dim Cl(8) such as Cl(8) x Cl(8) = Cl(16) containing 248-dim E8 = 120-dim D8 + 128-dim D8 half-spinor whose maximal contraction is a realistic generalized Heisenberg Algebra h92 x A7 =
= 5-graded 28 + 64 + ((SL(8,R)+1) + 64 + 28 see viXra 1507.0069 and 1405.0030).

Appendix I

Excerpts from

V. A. Fock,
Fundamentals of Quantum Mechanics
Mir, 1931, 1978, 1982

Part I

BASIC CONCEPTS OF QUANTUM MECHANICS

Chapter I

THE PHYSICAL AND EPISTEMOLOGICAL BASES OF QUANTUM MECHANICS

1. The need for new methods and concepts in describing atomic phenomena

Quantum mechanics appeared during the first few decades of the century on the basis of studies of atomic phenomena. The structure of the atom, the properties of electrons and atomic nuclei, the very stability of a system consisting of a positively charged nucleus and negatively charged electrons, the radiation of light by atoms and molecules, and, last, the diffraction of electrons — all these properties and phenomena require for their explanation ideas and physical concepts that differ substantially from the ideas and concepts of classical physics.

A precise formulation of the new concepts demands new mathematical tools, and we will familiarize ourselves with these in subsequent chapters. But we will try to explain the principal difference between quantum mechanics and classical mechanics in this introductory chapter.

2. The classical description of phenomena

When we describe various phenomena in terms of classical physics, we assume that physical processes are independent of the conditions of observation. Thus we take it for granted that we can always "spy" on the process and yet not interfere with it or influence it. True enough, if we "spy" on a physical process from different view points (and correspondingly use different frames of reference for its description), it will appear to us in different ways. For instance, the free fall of a body may proceed in a straight line in one frame of reference and in a parabola in another. But the dependence of the form of a phenomenon on the frame of reference has always been taken into consideration, namely, by transforming from the coordinates of one frame of reference to the coordinates of another. Such a change in form introduces nothing new into the phenomenon. For this reason in

classical physics we can speak of the independence of a phenomenon from the manner of observation.

Quantum mechanics has shown that in microprocesses this is not the case. Here the very possibility of observation presupposes definite physical conditions that may be related to the essence of the process. Specifying these conditions does not mean simply indicating a particular frame of reference but requires more detailed elaboration.

Neglect of these considerations leads to an abstraction that we may call the *absolutization of physical processes*. If we accept this abstraction, it becomes possible to consider physical processes as occurring by themselves regardless of whether there is a real possibility of their observation (that is, whether the appropriate physical conditions exist for such processes).

The use of this abstraction is justified in studying macroscopic phenomena, for in these the influence produced by a measurement is to all practical purposes negligible. The absolutization of such phenomena seemed so natural that before the appearance of quantum mechanics it was never specifically stated. It went without saying that physical processes occur by themselves, which considerably simplified their description since there was no need to specify the conditions of observation. All of classical physics is based on the absolutization of physical processes. This abstraction is one of its characteristic features.

Another abstraction permitted in classical physics is the possibility of unlimited amendment of observation. By this we mean not only an increasingly precise measurement of a specific quantity but simultaneously the measurement of any other quantity related to the observed object or phenomenon. This can be called the *particularization of measurements*. Even when measuring different quantities requires different conditions of observation, classical physics considers it possible to combine the results in an overall picture describing the physical process under investigation. There is a logical connection between allowing for the independence of the physical process from the conditions of observation, that is, absolutization of the process, and allowing for the possibility of encompassing different aspects and characteristics of the behaviour of an object in the physical process.

The concepts of classical physics prompt the idea that not only an absolute but an exhaustive description of the state of motion of a physical system (with certain degrees of freedom) is possible. And an exhaustive description is assumed to be achieved if there is full particularization of observations and further observations can add nothing new.

3. Range of application of the classical way of describing phenomena. Heisenberg's and Bohr's uncertainty relations

Such fundamental facts as the wave-corpuscular duality of light and of particles of matter prove that the classical way of describing phenomena is unsuitable for micro-objects. At the same time we cannot dismiss it completely, since to describe phenomena objectively we must rely, directly or indirectly, on something that does not require reservations concerning the manner of observation. And this is the case with the "absolute" manner of description used in classical physics.

To apply the classical (absolute) manner of description intelligently we must first establish its limits. If we assume that the mathematical apparatus of quantum mechanics is known, the relations of classical physics derive from it in the form of a certain approximation, and the limits of application of the classical manner of description prove to be the conditions of applicability of this approximation. But in our discourse we proceed from classical mechanics and can only use the simplest quantum relationships.

Let us consider a simple case, the motion of a mass point with a mass m. In classical mechanics the state of motion of a mass point at any given moment of time is determined by its position (x, y, z) and momentum (p_x, p_y, p_z). It would be incorrect, however, to consider the two sets simultaneously without referring to the possibility of their measurement, which is limited by quantum effects.

As Werner Heisenberg proved, the localization of a particle in space demands conditions that are not favourable for measuring its momentum, that is, for the localization of the particle in momentum space. Conversely, conditions that are needed to measure the momentum of a particle preclude the possibility of localizing the particle in ordinary space.

Quantum effects, which limit the possibility of measurement, manifest themselves, for instance, when light quanta irradiate a particle. What is important here is that a photon, which is characterized by wave parameters, is at the same time a bearer of definite energy and momentum, which makes it a "particle of light". The wave parameters are: the frequency ν (or the angular frequency $\omega = 2\pi\nu$), the wavelength $\lambda = c/\nu$ (c is the velocity of light), and the wave vector $\mathbf{k}$, which determines the direction of the wave's propagation (the absolute value of $\mathbf{k}$ is $k = 2\pi/\lambda = 2\pi\nu/c = \omega/c$). If we define $\hbar$ as *Planck's constant* h divided by 2π, that is, $h = 2\pi\hbar$, the energy of the photon, E, and its

momentum, p, will be related to the wave parameters as

$$E = \hbar\omega, \qquad p = \hbar k \quad (p = 2\pi\hbar/\lambda) \tag{3.1}$$

where

$$\hbar = 1.054 \times 10^{-27} \text{ erg s} \tag{3.2}$$

It follows from Eqs. (3.1) that using light of short wavelengths favourable for localizing a particle in ordinary space means using high-energy photons that are capable of transferring a great impact (momentum) to the particle and thereby upsetting its localization in momentum space. Using low-energy photons means using light of long wavelengths, and this in turn broadens the diffraction bands and reduces the precision of localizing a particle in ordinary (coordinate) space.

Equations (3.1) relate the wave properties of a photon to its corpuscular properties. Their right-hand members contain ω and k, which are determined by the diffraction pattern, and their left-hand members, E and p, describe the photon as a particle. Hence Eqs. (3.1) reflect the wave-corpuscular duality of a photon.

The *wave-corpuscular duality* proves to be a general property not only of photons but of all particles. This makes it possible to correlate the concepts of the electron as a particle and as a wave. The first to suggest the idea of the wave property of matter was Louis de Broglie, and proof came later when the diffraction of electrons was discovered. A more precise statement of this idea is contained in a proper interpretation of the mathematical apparatus of quantum mechanics.

We can express the results of Heisenberg's reasoning, just elaborated, concerning the limits of precision of measurement in the form of the following inequalities:

$$\Delta x \, \Delta p_x \geqslant \hbar, \qquad \Delta y \, \Delta p_y \geqslant \hbar, \qquad \Delta z \, \Delta p_z \geqslant \hbar \tag{3.3}$$

in which Δx, Δy, Δz characterize the size of the region in coordinate space (x, y, z) containing the particle, and Δp_x, Δp_y, Δp_z the size of the region in momentum space (p_x, p_y, p_z) containing the particle. The inequalities show that the very nature of a particle makes it impossible to localize it simultaneously in coordinate space and in momentum space. They are called the *Heisenberg uncertainty relations*, or simply the *uncertainty relations*. The word "uncertainty" is understood to mean the regions of localization, $(\Delta x, \Delta y, \Delta z)$ and $(\Delta p_x, \Delta p_y, \Delta p_z)$, in the corresponding coordinate and momentum spaces.

We can couple the uncertainty relations (3.3) with

$$\Delta t \, \Delta (E' - E) \geqslant \hbar \tag{3.4}$$

which links the uncertainty in the change of energy of a particle, $E' - E$, with the uncertainty in the time during which this change

occurred. According to (3.4), the transfer of energy cannot be localized precisely in time. Relation (3.4) can be called the *Heisenberg-Bohr uncertainty relation*.

The uncertainty relations (3.3) and (3.4) characterize the range of application of the classical (absolute) manner of describing phenomena. Since Planck's constant is small, this manner of description can unquestionably be used in referring to macroscopic bodies and their interactions. But this does not exhaust its significance. It is important in describing quantum processes because it is applied to the instruments used to study atomic objects. Experiments (with atomic objects as well) are always described in the classical (absolute) way.

The instruments and other means of observation, including the human senses (which, so to say, play the role of instruments built into the human body), are the necessary intermediaries between the human brain and the atomic object under consideration. We can now define more accurately what is meant by means of observation by indicating the manner of their description:

The means of observation must be described in classical terms but with due regard to the uncertainty relations (3.3) and (3.4).

4. Relativity with respect to the means of observation as the basis for the quantum way of describing phenomena

The new, quantum manner of describing phenomena must allow for the possibility of actual measurement of the properties of a micro-object. We must not ascribe to any object properties and states of motion that cannot be justified. For this reason particular attention should be given to the way in which we specify properties and states of motion. We must bear in mind the design and operation of the instruments that create the conditions to which the object is subjected. As has been said, the instruments and the external conditions must be described in the classical manner by indicating their parameters. It stands to reason that these parameters can be defined only to an accuracy permitted by the uncertainty relations. Otherwise we will be exceeding the actual potential of the measuring instruments.

A micro-object is revealed in its interaction with an instrument. For instance, the path of a charged particle becomes visible in the irreversible snowballing process that takes place in a cloud chamber or in the emulsion of a photographic plate (the particle loses its energy in ionizing the vapour or the chemicals of the emulsion; hence, its momentum becomes uncertain). The results of the interaction of an atomic object with a measuring instrument

(which is described classically) are the main experimental elements the systematization of which, based on the assumptions about the properties of the object, makes up the aim of the theory: from a study of such interactions we can deduce the properties of the atomic object, and the predictions of the theory are formulated as the expected results of these interactions.

Such a statement of the problem allows the introduction of quantities that characterize the object irrespective of the measuring instrument (electric charge, mass, and properties described by quantum mechanical operators) and at the same time makes possible a comprehensive approach to the object: the object can be viewed from the aspect (wave or corpuscular, for instance) necessitated by the instrument and by the external conditions the instrument creates.

The new statement of the problem makes it possible to consider the case when the various aspects and properties of an object do not manifest themselves simultaneously, that is, when *particularization of the object's behaviour* is impossible. This will be so if incompatible external conditions are needed for the manifestation of the object's properties (for instance, wave and corpuscular).

We can act on the proposal of Niels Bohr and call *complementary* the properties that reveal themselves in their pure form only in different experiments held in mutually exclusive conditions, whereas in conditions of one and the same experiment they manifest themselves only in an incomplete, modified form (for instance, the incomplete localization in the coordinate and the momentum space permitted by the uncertainty relations). There is no sense in considering complementary properties simultaneously (in the pure form), which explains the absence of a contradiction in the concept of wave-corpuscular duality.

By making the results of the interaction of a micro-object and a measuring instrument the basis of the new manner of description we introduced an important concept, the concept of *relativity with respect to the means of observation*, which generalizes the well-known concept of relativity with respect to the frame of reference. Such a manner of description does not at all mean that we are ascribing a lesser degree of reality to the micro-object than to the measuring instrument or that we are reducing the properties of the micro-object to the properties of the instrument. On the contrary, a description on the basis of the concept of relativity with respect to the means of observation gives a much deeper, more refined, and more objective picture of the micro-object than was possible on the basis of the idealizations of classical physics. Such a picture also requires a more sophisticated mathematical apparatus, namely, the theory of linear operators, including eigenfunctions and eigenvalues, the theory of groups, and other math-

ematical concepts. The use of this apparatus in quantum physics made it possible to give a theoretical explanation of some fundamental properties of matter that could not be explained in the classical way and also to calculate the values of many quantities observed in experiments (for instance, the frequencies in atomic spectra). But more than that — and this is no less important to us — the physical interpretation of the mathematical concepts used in quantum mechanics leads to a number of profound and principled conclusions; for one, generalization of the concept of the state of a system on the basis of the concepts of probability and potential possibility.

5. Potential possibility in quantum mechanics

If we take the act of interaction between an atomic object and a measuring instrument as the source of our judgements about the object's properties and if in studying phenomena we allow for the concept of relativity with respect to the means of observation, we are introducing a substantially new element into the description of the atomic object and its state and behaviour, that is, the idea of probability and thereby the idea of potential possibility. The need to consider the concept of probability as a substantial element of description rather than a sign of incompleteness of our knowledge follows from the fact that for given external conditions the result of the object's interaction with the instrument is not, generally speaking, predetermined uniquely but only has a certain probability of occurring. With a fixed initial state of the object and with given external conditions a series of such interactions results in a statistics that corresponds to a certain probability distribution. This probability distribution reflects the potential possibilities that exist in the given conditions.

Let us consider an experiment with a physical system that would enable us to make predictions about the results of future interactions between the system and measuring instruments of various kinds. Such an *initial experiment* must include a certain *preparation of the system* (for instance, preparation of a monochromatic beam of electrons) and the creation of certain external conditions in which the system will be placed after the preparation (for instance, the passage of the electron beam through a crystal). At times it is advisable to consider the preparation of the system and the creation of external conditions as two different stages of the experiment, but the two stages can also be considered one initial experiment, the purpose of which is to obtain predictions:

The initial experiment is always addressed to the future.

The manner of preparation and the external conditions in an initial experiment are described in the language of classical physics, but its result, which must give a full catalogue of the potential possibilities for the given conditions, requires new, quantum mechanical means for formulation. To have an idea of why we must use these means, let us consider how the potential possibilities existing in the given conditions materialize.

First of all we must bear in mind that a *final experiment*, in which the potential possibilities materialize, may be conducted in different ways: the registering instrument may be of different construction (as a rule, one excludes another). As in the initial experiment, the construction and operation of the instrument are described in the classical way. The different versions of the final experiment and the corresponding instruments can be characterized by the type of the quantity they measure (position, momentum, etc.).

Thus, with the initial experiment given, there is first of all a possibility of choosing different types of instruments for the final experiment. In any case,

The final experiment is always addressed to the past

(and not to the future in contrast to the initial experiment). It can be called the *verifying experiment* because it enables us to verify the predictions of the initial experiment.

Let us assume that the type of verifying experiment has been chosen. How do we formulate its result? We must always remember that we are talking about potential possibilities, which are created in the initial experiment and realized in the verifying experiment. For a given type of verifying experiment these potential possibilities are expressed as probability distributions for the given quantity (more precisely, for the values of the quantity that can be obtained in the verifying experiment). Hence it is the probability distribution we seek to verify. Clearly, this cannot be done by a simple measurement but requires many repetitions of the entire experiment (with the same preparation of the object and the same external conditions). The statistics obtained in this process of repetition makes it possible to draw a conclusion about the probability distribution that is to be studied.

A *total experiment* (an experiment that is carried out to the end and permits a comparison with theory) consists of the initial and verifying experiments combined and performed many times over. Here it is in place to note once more that for a given initial experiment (for given initial conditions) the final experiment may be set up in different ways (the measured quantities may differ)

and every type of final experiment has its own probability distribution.

Thus a theory must describe the initial state of a system in such a way as to make it possible to obtain probability distributions for any type of final experiment from this state. In this way we secure a full description of the potential possibilities that follow from the initial experiment.

Since a final experiment may take place later than the initial experiment, a theory must also give the time dependence of probabilities and potential possibilities. The establishment of this dependence will play the same role as the discovery of the laws of motion did in classical physics.

THE MATHEMATICAL APPARATUS
OF QUANTUM MECHANICS

1. Quantum mechanics
and the linear-operator problems

An important step towards the creation of present-day quantum mechanics was Bohr's postulation of two principles characterizing the properties of atomic systems.

The first principle asserts that atomic systems have stationary states, in which they do not radiate or absorb energy. In these states an atomic system possesses energy values that form a discrete sequence E_1, E_2, ..., E_n, ... (the *energy levels* of the system).

According to the second principle, radiation emitted or absorbed by an atomic system in the transition from one energy level to another has a definite frequency ν determined by the condition

$$E_m - E_n = h\nu$$

where h is Planck's constant, and E_m and E_n are the energy levels.

These principles conflict with classical mechanics and electrodynamics but are fully confirmed in experiments. It is a natural idea, therefore, to propose replacing the classical theory by a theory that would harmonize with Bohr's principles and be logically consistent.

The problem of determining the stationary states of an atomic system, states that are described by definite energy values (and certain other constants of integration), is analogous to the problem of mathematical physics where definite states of a system are chosen from the whole set of states, namely, the problem of eigenfrequencies of oscillations, or, more generally, the *linear-operator problem* and the associated *eigenvalue problem*. In the problem of this kind a sequence of values of a given quantity would emerge automatically from the whole set of values. Quantum mechanics has substantiated this idea of *quantization* ever since the historic paper of Erwin Schrödinger (1926) concerning quantization as an eigenvalue problem. A certain linear operator is related to each physical quantity, and the theory of linear operators is the mathematical apparatus of quantum mechanics.

2. The operator concept and examples

As in the case of a function, which is an instruction as to how, knowing number x, we can find number $y = f(x)$, an operator will map a given function $\varphi(x)$ into a new function

$$\psi(x) = L[\varphi(x)] \tag{2.1}$$

A *linear operator* has the properties that, for any functions φ_1, φ_2, φ,

$$L(\varphi_1 + \varphi_2) = L(\varphi_1) + L(\varphi_2)$$

$$L(a\varphi) = aL(\varphi) \tag{2.2}$$

where a is an arbitrary complex number. Since we will deal only with linear operators, the word "linear" will be often omitted.

Operators act on functions of one or several variables. The variables can be either continuous, which is the case for the coordinates (position) of an object, or discontinuous, that is, assuming only discrete values, which is the case for energy levels or the number that labels these levels. Continuous variables can either take on any value or change within certain domains. Discontinuous variables can take on both finite and infinite sequences of values. We will always assume that the values of the independent variables (or arguments) of a function are real numbers, whereas the functions themselves, which the operators act on, can be complex-valued. When specifying an operator, we must always indicate on the functions of what variables it acts.

Typical operators that act on functions of a continuous variable x are the multiplication of a function into x and the differentiation with respect to x:

$$L[f(x)] = xf(x), \qquad L[f(x)] = \frac{d}{dx} f(x)$$

In the first case x plays a double role: it is the argument of $f(x)$ and it is the operator itself.

Another example is the Laplacian operator ∇^2.

$$\nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

There is also a class of linear operators that can be represented by a definite integral:

$$L[f(x)] = \int_a^b K(x, \xi) f(\xi) d\xi \tag{2.3}$$

where the function $K(x, \xi)$ is called the *kernel* of the operator. As one example of a kernel let us consider Poisson's differential equation

$$\nabla^2 F = f$$

Appendix II

Excerpts from

L. K. Hua,
Harmonic Analysis of Functions of Several Complex Variables
in the Classical Domains
AMS, 1958, 1959, 1963

INTRODUCTION

I. Classical domains. By a classical domain we shall understand an irreducible bounded symmetric domain (in the space of several complex variables) of one of the following four types:

(1) The domain $\Re_{\mathrm{I}}$ of $m \times n$ matrices with complex entries satisfying the condition

$$I^{(m)} - Z\overline{Z'} > 0.$$

Here $I^{(m)}$ is the identity matrix of order m, $\overline{Z'}$ is the complex conjugate of the transposed matrix Z'. ($H > 0$ for a hermitian matrix H denotes, as usual, that H is positive definite.)

(2) The domain $\Re_{\mathrm{II}}$ of symmetric matrices of order n (with complex entries) satisfying the condition

$$I^{(n)} - Z\overline{Z} > 0.$$

(3) The domain $\Re_{\mathrm{III}}$ of skew-symmetric matrices of order n (with complex entries) satisfying the condition

$$I^{(n)} + Z\overline{Z} > 0.$$

(4) The domain $\Re_{\mathrm{IV}}$ of n-dimensional ($n > 2$) vectors

$$z = (z_1, z_2, \cdots, z_n)$$

(z_k are complex numbers) satisfying the conditions[2]

$$|zz'|^2 + 1 - 2zz' > 0, \qquad |zz'| < 1.$$

The complex dimension of these four domains is mn, $n(n+1)/2$, $n(n-1)/2$, n, respectively.

The author has shown (cf. L. K. Hua [3]) that $\Re_{\mathrm{IV}}$ can also be regarded as a homogeneous space of $2 \times n$ real matrices. Therefore, the study of all these domains can be reduced to a study of the geometry of matrices.

In 1935, E. Cartan [1] proved that there exist only six types of irreducible homogeneous bounded symmetric domains. Beside the above four types, there exist only two; their dimensions are 16 and 27. Of course

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[2] Translator's note (n.b., unless otherwise noted, these words refer to the Russian translator). Here and throughout, the author considers a vector as a matrix of one row and n columns. So z' is a matrix of one column and n rows (the transpose of the matrix z).

these two types are rather special. The problem of the explicit description of these two types is still open.

The purpose of the present book is to study harmonic analysis on the classical domains. (The exact content of this harmonic analysis will be outlined later.)

II. Characteristic manifolds.

Let $\mathfrak{R}$ be a bounded homogeneous domain in the $2n$-dimensional Euclidean space of n complex variables $z = (z_1, z_2, \cdots, z_n)$, and $f(z)$ an analytic function of z, regular in $\mathfrak{R}$. It is known that the maximum of the modulus of the function $f(z)$ is assumed on the boundary of $\mathfrak{R}$. Let $\mathfrak{C}$ be a manifold on the boundary of $\mathfrak{R}$ having the following properties:

(a) The modulus of every analytic function regular in $\mathfrak{R}$ assumes its maximum on $\mathfrak{C}$.

(b) For every point a on $\mathfrak{C}$ there exists a function $f(z)$, regular on $\mathfrak{R}$, such that the modulus of $f(z)$ assumes its maximum at $z = a$.

Such a manifold $\mathfrak{C}$ is called a characteristic manifold of the domain $\mathfrak{R}$. We should mention that $\mathfrak{C}$ is in general a proper subset of the boundary, and that the dimension of $\mathfrak{C}$ may be much less than $2n - 1$. It is clear that $\mathfrak{C}$ is uniquely determined by $\mathfrak{R}$. It is easy to show that $\mathfrak{C}$ is closed, and that an analytic function which is regular in a neighborhood of each point of $\mathfrak{C}$ is uniquely determined by its values on $\mathfrak{C}$. Hence it follows that the real dimension of $\mathfrak{C}$ is not less than n. We shall denote by ξ the variable on $\mathfrak{C}$, and by $d\xi \overline{d\xi'}$ and $\dot{\xi}$ the metric and the element of volume of $\mathfrak{C}$.

Clearly, in the definition of $\mathfrak{C}$ it is enough to consider only linear functions instead of all analytic functions.

We describe the characteristic manifolds of the classical domains.

(1) $\mathfrak{C}_{\mathrm{I}}$ consists of the $m \times n$ matrices U satisfying the condition

$$U\overline{U}' = I^{(m)}.$$

(2) $\mathfrak{C}_{\mathrm{II}}$ consists of all symmetric unitary matrices of order n.

(3) $\mathfrak{C}_{\mathrm{III}}$ is defined differently for even and odd n. If n is even, then $\mathfrak{C}_{\mathrm{III}}$ consists of all skew-symmetric unitary matrices of order n. If n is odd, then $\mathfrak{C}_{\mathrm{III}}$ consists of all matrices of the form

$$UDU',$$

where U is an arbitrary unitary matrix and

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus \cdots \dotplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dotplus 0.$$

(4) $\mathfrak{C}_{IV}$ consists of the vectors $e^{i\theta}x$, where x is a real vector such that $xx' = 1$.

The manifolds $\mathfrak{C}_I$, $\mathfrak{C}_{II}$, $\mathfrak{C}_{III}$ and $\mathfrak{C}_{IV}$ have real dimension $m(2n-m)$, $n(n+1)/2$, $n(n-1)/2+(1+(-1)^n)(n-1)/2$ and n, respectively.

These characteristic manifolds are homogeneous spaces. Furthermore, any point of $\mathfrak{C}$ can be carried into any other point of $\mathfrak{C}$ by a transformation leaving a given point of $\mathfrak{R}$ invariant. The general theory of harmonic analysis on homogeneous spaces has been developed earlier (cf. E. Cartan [1], H. Weyl [1]); however, the method presented in this book gives more precise and more useful results.

III. **Heuristic considerations.** Suppose that we have a sequence of analytic functions in $\mathfrak{R}$

$$\{\varphi_\nu(z)\}, \qquad \nu = 0, 1, 2, \cdots,$$

such that any analytic function $f(z)$ in $\mathfrak{R}$ can be developed in a series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(z),$$

convergent in $\mathfrak{R}$. We define the two infinite hermitian matrices

$$H_1 = \left(\int_{\mathfrak{C}} \varphi_\nu(\xi) \overline{\varphi_\mu(\xi)} \, \dot{\xi} \right)_{\nu, \mu=0, 1, 2, \ldots}$$

and

$$H_2 = \left(\int_{\mathfrak{R}} \varphi_\nu(z) \overline{\varphi_\mu(z)} \, \dot{z} \right)_{\nu, \mu=0, 1, 2, \ldots}$$

The basis $\{\varphi_\nu(z)\}$ can be chosen to be orthonormal, such that

$$\int_{\mathfrak{C}} \varphi_\nu(\xi) \overline{\varphi_\mu(\xi)} \, \dot{\xi} = \delta_{\nu\mu}$$

and

$$\int_{\mathfrak{R}} \varphi_\nu(z) \overline{\varphi_\mu(z)} \, \dot{z} = \lambda_\nu \delta_{\nu\mu}.$$

The eigenvalues $\lambda_0, \lambda_1, \lambda_2, \cdots$ are pseudoconformal invariants, i.e., they do not depend on the choice of the basis $\{\varphi_\nu(z)\}$ and are preserved under analytic mappings transforming $\mathfrak{R}$ and $\mathfrak{C}$ into $\mathfrak{R}_1$ and $\mathfrak{C}_1$, respectively.

The existence of a system $\{\varphi_\nu(z)\}$ is known from a theorem of H. Cartan

[1] on complete circular domains.[3]

Now setting

$$K(z, \bar{w}) = \sum_{\nu=0}^{\infty} \frac{\varphi_\nu(z)\overline{\varphi_\nu(w)}}{\lambda_\nu},$$

we obtain the Bergman kernel which has the following reproducing property. For any function $f(z)$ analytic in $\mathfrak{R}$ we have

$$f(z) = \int_{\mathfrak{R}} K(z, \bar{w}) f(w) \dot{w}.$$

Setting

$$H(z, \bar{\xi}) = \sum_{\nu=0}^{\infty} \varphi_\nu(z) \overline{\varphi_\nu(\xi)},$$

we obtain the Cauchy kernel of the domain $\mathfrak{R}$. This kernel has the reproducing property that for any analytic function $f(z)$ with a series development

$$f(\xi) = \sum_{\nu=0}^{\infty} a_\nu \varphi_\nu(\xi),$$

on $\mathfrak{C}$ we have

$$f(z) = \int_{\mathfrak{C}} H(z, \bar{\xi}) f(\xi) \dot{\xi}.$$

Setting

$$f(z) = u(z) H(z, \bar{w}),$$

we have

$$u(z) = \int_{\mathfrak{C}} \frac{H(z, \bar{\xi}) H(\xi, \bar{w})}{H(z, \bar{w})} u(\xi) \dot{\xi}.$$

The function

$$P(z, \xi) = \frac{H(z, \bar{\xi}) H(\xi, \bar{z})}{H(z, \bar{z})}$$

is called the Poisson kernel for the domain $\mathfrak{R}$. It is positive.

It is clear that the system of functions $\{\varphi_\nu(\xi)\}$, $\nu = 0, 1, 2, \cdots$ is not complete in the space of continuous functions on $\mathfrak{C}$. We complete it to a com-

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[3] Translator's note. The domain $\mathfrak{R}$ in the space of several complex variables is said to be a circular domain (with center at the origin) if together with any point z in $\mathfrak{R}$ the point $ze^{i\varphi}$ is in $\mathfrak{R}$ for any real φ. If together with any point z in $\mathfrak{R}$ also the point $rze^{i\varphi}$ is in $\mathfrak{R}$ for any real φ and $0 \leq r \leq 1$, then $\mathfrak{R}$ is said to be a complete circular domain.

plete orthonormal system

$$\{\varphi_\nu(\xi)\}, \qquad \nu = 0, \ \pm 1, \ \pm 2, \cdots,$$

and develop the function $P(z, \xi)$ into a Fourier series with respect to this new system

$$P(z, \xi) = \sum_{\nu=-\infty}^{\infty} \Phi_\nu(z) \overline{\varphi_\nu(\xi)}, \quad \Phi_\nu(z) = \int_{\mathfrak{C}} P(z, \xi) \varphi_\nu(\xi) \, \dot{\xi}.$$

If

$$\lim_{z \to \xi} \Phi_\nu(z) = \varphi_\nu(\xi),$$

then the functions on $\mathfrak{C}$ having a Fourier series development

$$\varphi(\xi) = \sum_{\nu=-\infty}^{\infty} c_\nu \varphi_\nu(\xi), \quad c_\nu = \int_{\mathfrak{C}} \varphi(\xi) \overline{\varphi_\nu(\xi)} \, \dot{\xi}$$

can be put in correspondence with the class of functions

$$\Phi(z) = \int_{\mathfrak{C}} P(z, \xi) \varphi(\xi) \, \dot{\xi} = \sum_{\nu=-\infty}^{\infty} c_\nu \Phi_\nu(z),$$

which we shall call harmonic functions in the domain $\mathfrak{R}$.

The harmonic functions can also be defined as solutions of a second order partial differential equation. This equation can be obtained from the following considerations.

The Bergman kernel yields a Riemannian metric on the space $\mathfrak{R}$:

$$d\bar{d} \ln K(z, \bar{z}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial z_i \, \partial \bar{z}_j} \ln K(z, \bar{z}) \, dz_i \, d\bar{z}_j = \sum_{i, j=1}^{n} h_{ij} \, dz_i \, d\bar{z}_j.$$

Corresponding to the tensor h^{ij} we have a differential operator

$$\sum_{i=1}^{n} \sum_{j=1}^{n} h^{ij} \frac{\partial^2}{\partial z_i \, \partial \bar{z}_j},$$

which we call the Laplace operator of this space.

We can talk about a Dirichlet problem with respect to this operator.

All the facts mentioned in this section will be treated in detail later in the book.

IV. Remarks on the methods to be used.

(a) *The machinery of group representation theory.* It is known that the classical domains of all four types are complete circular domains (cf. the

But if Q is a unitary skew-symmetric matrix, then a unitary matrix V_0 can be found such that $V_0 Q V_0' = F$. Hence it is also possible to find a unitary matrix U_0 such that

$$\bar{U}_0' K_1 \bar{U}_0 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}.$$

Setting $h = k\bar{U}_0$, we obtain

$$h \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}' = k\bar{U}_0 U_0' \bar{K}_1' U_0 = 0.$$

This means that

$$h = [e^{i\theta}, \ 0, \ \ldots, \ 0].$$

Thus the inner integral in formula (4.7.10) is equal to

$$\int_0^{2\pi} f\left[\begin{pmatrix} 0 & hU_0' \\ -U_0 h' & K_1 \end{pmatrix}\right] d\theta = 2\pi f\left[\begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix}\right]$$

and we obtain (4.7.8) from (4.7.10).

(We used the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi\left(z_1 e^{i\theta}, \ \ldots, \ z_n e^{i\theta}\right) d\theta = \varphi(0, \ 0, \ \ldots, \ 0),$$

which holds for $\varphi(z)$ analytic in a closed circular domain with its center at the origin and z lying inside the domain.)

4°. The characteristic manifold of the domain $\mathfrak{R}_{IV}$ consists of vectors of the form $e^{i\theta} x$, where $0 \leq \theta \leq \pi$, and $x = (x_1, \cdots, x_n)$ is a real vector which satisfies the condition $xx' = 1$.

$$H(z, \ \theta, \ x) = \frac{1}{V(\mathfrak{C}_{IV}) \left[(x - e^{-i\theta} z)(x - e^{-i\theta} z)'\right]^{n/2}}, \qquad (4.7.11)$$

It is easy to calculate the magnitude of the volume $V(\mathfrak{C}_{IV})$:

$$V(\mathfrak{C}_{IV}) = \frac{2\pi^{\frac{n}{2}+1}}{\Gamma\left(\frac{n}{2}\right)}.$$

4.8. **The Poisson kernel for circular domains.** Suppose that $\mathfrak{R}$, just as in §4.5, is a star-shaped circular domain, and $\mathfrak{C}$ its characteristic manifold, transitive with respect to the group Γ_0 of motions of $\mathfrak{R}$ which leave the origin unchanged. Then, by Theorem 4.6.1, there exists a Cauchy kernel

for the domain $\Re$, and Cauchy's formula holds for any function $f(z)$ which is analytic in $\Re$ and on its boundary.

Setting, in particular,

$$f(z) = H(z, \overline{w}) g(z),$$

where $g(z)$ is an arbitrary function which is analytic in $\Re$ and on its boundary, we have

$$H(z, \overline{w}) g(z) = \int_{\mathfrak{C}} H(z, \overline{\xi}) H(\xi, \overline{w}) g(\xi) \dot{\xi}.$$

For $w = z$, we obtain Poisson's formula

$$g(z) = \int_{\mathfrak{C}} P(z, \xi) g(\xi) \dot{\xi}, \tag{4.8.1}$$

where the kernel

$$P(z, \xi) = \frac{H(z, \overline{\xi}) \, H(\xi, \overline{z})}{H(z, \overline{z})} \tag{4.8.2}$$

has occured above under the name of the Poisson kernel of the domain $\Re$.

Up to now we have established that formula (4.8.1) is valid for analytic $g(z)$; yet it can be extended to other classes of functions too (see §5.8). For any continuous function $u(\xi)$ the integral

$$u(z) = \int_{\mathfrak{C}} P(z, \xi) u(\xi) \dot{\xi} \tag{4.8.3}$$

defines a certain function. It can be proved (see §5.8) that $u(z) \to u(\xi)$ for $z \to \xi$. Functions of the form (4.8.3) we shall call harmonic functions in $\Re$. It is reasonable to expect that if there exists a complete orthonormal system $\{\psi_\nu(\xi)\}$ on $\mathfrak{C}$, then the set of functions which are harmonic in $\Re$ is the closure of the linear span of the system $\{\psi_\nu(z)\}$ (see §5.10).

If $\Re$ satisfies the conditions of Theorem 4.6.3, then the Poisson kernel can be written in the following simple form:

$$P(z, \xi) = \frac{1}{V(\mathfrak{C})} \cdot |B(\xi, z, U)|. \tag{4.8.4}$$

In conclusion let us list the Poisson kernels for the classical domains.
(1) For $\Re_{\mathrm{I}}$

$$P(Z, U) = \frac{1}{V(\mathfrak{C}_{\mathrm{I}})} \cdot \frac{[\det(I - Z\overline{Z}')]^n}{|\det(I - Z\overline{U}')|^{2n}}, \tag{4.8.5}$$

where $U \in \mathfrak{C}_{\mathrm{I}}$. In particular, for $m = n$, one can also write

$$P(Z, U) = \frac{1}{V(\mathfrak{C}_{\mathrm{I}})} \cdot \frac{[\det (I - Z\overline{Z}')]^n}{|\det (Z - U)|^{2n}}.$$

(2) For $\mathfrak{R}_{\mathrm{II}}$

$$P(Z, U) = \frac{1}{V(\mathfrak{C}_{\mathrm{II}})} \cdot \frac{[\det (I - Z\overline{Z})]^{\frac{n+1}{2}}}{|\det (I - Z\overline{U})|^{n+1}}, \qquad (4.8.6)$$

where $U \in \mathfrak{C}_{\mathrm{II}}$.

(3) For $\mathfrak{R}_{\mathrm{III}}$ with even n

$$P(Z, K) = \frac{1}{V(\mathfrak{C}_{\mathrm{III}})} \cdot \frac{[\det (I + Z\overline{Z})]^{\frac{n-1}{2}}}{|\det (I + Z\overline{K})|^{n-1}}, \qquad (4.8.7)$$

and with odd n

$$P(Z, K) = \frac{1}{V(\mathfrak{C}_{\mathrm{III}})} \cdot \frac{[\det (I + Z\overline{Z})]^{\frac{n}{2}}}{|\det (I + Z\overline{K})|^{n}}. \qquad (4.8.8)$$

In both cases $K \in \mathfrak{C}_{\mathrm{III}}$.

(4) For $\mathfrak{R}_{\mathrm{IV}}$

$$P(z, \xi) = \frac{1}{V(\mathfrak{C}_{\mathrm{IV}})} \cdot \frac{(1 + |zz'|^2 - 2\bar{z}z')^{\frac{n}{2}}}{|(z - \xi)(z - \xi)'|^{n}}, \qquad (4.8.9)$$

where $\xi \in \mathfrak{C}_{\mathrm{IV}}$.

THEOREM 4.1.1. *For a complete circular domain $\mathfrak{R}$ the system of functions*

$$\left(\beta_\nu^f\right)^{-\frac{1}{2}} \varphi_\nu^f(z), \quad \begin{matrix} f = 0, 1, 2, \ldots, \\ \nu = 1, 2, \ldots, N_f, \end{matrix} \tag{4.1.6}$$

is a complete orthonormal system in the domain $\mathfrak{R}$. On the other hand, the system $\{\varphi_\nu^f(\xi)\}$ is orthonormal, but in general not complete in the space of functions which are continuous on $\mathfrak{C}$.

It is also well known that the series

$$\sum_{f=0}^{\infty} \sum_{\nu=1}^{N_f} \frac{\varphi_\nu^f(z) \overline{\varphi_\nu^f(w)}}{\beta_\nu^f} = K(z, \overline{w})$$

converges uniformly for any z and w which lie in the interior of $\mathfrak{R}$, representing there a function called the Bergman kernel.[12]

The sum of the series (if it converges)

$$\sum_{f=0}^{\infty} \sum_{\nu=0}^{N_f} \varphi_\nu^f(z) \overline{\varphi_\nu^f(\xi)} = H(z, \overline{\xi})$$

we shall call the Cauchy kernel for the domain $\mathfrak{R}$.

Finally, we shall call the function

$$P(z, \xi) = \frac{|H(z, \overline{\xi})|^2}{H(z, \overline{z})}$$

the Poisson kernel for the domain $\mathfrak{R}$.

This chapter deals with the direct methods of determination of these kernels.

4.2. **The Bergman kernel.** Let $\mathfrak{R}$ be a bounded domain which contains the origin, Γ a group of analytic mappings of $\mathfrak{R}$ onto itself, and Γ_0 a subgroup of Γ which leaves the origin fixed. It is well known (H. Cartan [1]) that an element of Γ_0 is fully determined by its linear terms in the neighborhood of the origin, i.e., the mapping of $\mathfrak{R}$ onto itself which has the form

$$w_i = \sum_{j=1}^{n} u_{ij} z_j + \sum_{\substack{m_1 \cdots m_n \\ m_1 + \cdots + m_n \geqslant 2}} a_{m_1 \cdots m_n}^{(i)} z_1^{m_1} \cdots z_n^{m_n}, \tag{4.2.1}$$

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[12] Translator's note. In the Russian literature the Bergman kernel is usually called the "kernel function of the domain". In this book such a designation would not be very appropriate since we are dealing with three kernels to which this designation could apply.

is fully determined if the matrix $(u_{ij})_1^n$ is given. As it is well known that Γ_0 is compact, it can be assumed without loss of generality that the matrices $(u_{ij}) = U$ which form the representation of Γ_0 are unitary. The letter U we shall also use for denoting the nonlinear transformation (4.2.1) itself, determined by the linear element U.

Let us now consider the set of cosets of Γ/Γ_0. All group transformations belonging to one and the same coset carry into the origin one and the same point a. The totality of all such points a forms in $\mathfrak{R}$ some set $\mathfrak{M}$. It is called a transitive set with respect to the group Γ which contains the origin. Thus any element of Γ is uniquely determined by a point a of $\mathfrak{M}$, and by the unitary matrix U of Γ_0. We shall write the transformations determined by the elements of Γ in the form

$$w = f(z, a, U), \quad a \in \mathfrak{M}, \quad U \in \Gamma_0. \tag{4.2.2}$$

Suppose

$$z = f(x, b, V), \quad b \in \mathfrak{M}, \quad V \in \Gamma_0, \tag{4.2.3}$$

is another transformation, and

$$w = f(f(x, b, V), a, U) = f(x, c, W) \tag{4.2.4}$$

is the product of the transformations (4.2.2) and (4.2.3). Setting $w = 0$, we at once obtain

$$a = f(c, b, V). \tag{4.2.5}$$

Differentiation of (4.2.4) yields

$$\frac{\partial f_i(x, c, W)}{\partial x_j} = \sum_{k=1}^n \frac{\partial f_i(z, a, U)}{\partial z_k} \frac{\partial f_k(x, b, V)}{\partial x_j}. \tag{4.2.6}$$

We shall denote the Jacobian of the transformation (4.2.2) by

$$J(z, a, U) = (a_{ij}), \quad a_{ij} = \frac{\partial f_i(z, a, U)}{\partial z_j}.$$

By setting $x = c$ in (4.2.6) we obtain $z = a$. Hence

$$J(c, c, W) = J(a, a, U) \cdot J(c, b, V).$$

By a change of notation we obtain

$$J(x, x, W) = J(z, z, U) J(x, b, V). \tag{4.2.7}$$

This formula is valid for x and z of $\mathfrak{M}$ which satisfy the relation

$$z = f(x, b, V). \tag{4.2.8}$$

If we have another transformation

$$u = f(x, b, V_0),$$

then the mapping of u into z leaves the origin unchanged. Hence

$$U_0 = \left(\frac{\partial z}{\partial u}\right)_{z=0}$$

is a unitary matrix. Whence follows that

$$\{J(x, b, V)\}_{x=b} = \left(\frac{\partial z}{\partial u}\right)_{z=0} \cdot \{J(x, b, V_0)\}_{x=b},$$

so that we have

$$J(b, b, V) = U_0 J(b, b, V_0), \tag{4.2.9}$$

where U_0 is the unitary matrix of Γ_0. Thus

$$\overline{J(z, z, V)}' \cdot J(z, z, V) = \overline{J(z, z, V_0)}' \cdot J(z, z, V_0). \tag{4.2.10}$$

This shows that $\overline{J}'J$ depends on the coset of Γ/Γ_0 but does not depend on the choice of representative of this coset. Hence we can write

$$|\det J(z, z, V)|^2 = Q(z, \bar{z}).$$

From (4.2.7) we obtain for z and x, lying in $\mathfrak{M}$ and satisfying the relation (4.2.8), the formula

$$Q(x, \bar{x}) = Q(z, \bar{z})|\det J(x, b, V)|^2. \tag{4.2.11}$$

Bergman [1] has proved that under the transformation (4.2.8) the Bergman kernel of the domain $\mathfrak{R}$ changes according to the law

$$K(x, \bar{x}) = K(z, \bar{z})|\det J(x, b, V)|^2. \tag{4.2.12}$$

Thus for z and x of $\mathfrak{M}$

$$\frac{K(x, \bar{x})}{Q(x, \bar{x})} = \frac{K(z, \bar{z})}{Q(z, \bar{z})}. \tag{4.2.13}$$

THEOREM 4.2.1. *If $\mathfrak{R}$ is a bounded circular domain, then for z lying in $\mathfrak{M}$, we have*

$$K(z, \bar{z}) = \frac{1}{\Omega} Q(z, \bar{z}),$$

where Ω is the complete volume of $\mathfrak{R}$.

PROOF. In view of §4.1, we can propose the following process of constructing an orthonormal system of functions.

We orthonormalize the terms

$$z_1^{a_1} \cdot z_2^{a_2} \cdot \ldots \cdot z_n^{a_n}, \qquad a_1 + \ldots + a_n = m,$$

for a given m, and we take the totality of all such functions for $m = 0, 1, 2, \cdots$. This totality forms a complete orthonormal system.

Among the functions $\varphi_\nu(z)$ obtained by this process, we have the constant $\Omega^{-1/2}$, whereas the other functions are homogeneous forms of order $m \geq 1$. Hence

$$\varphi_0(z) = \Omega^{-\frac{1}{2}}, \qquad \varphi_\nu(0) = 0, \qquad \nu \geq 1.$$

Therefore from equation

$$K(z, \bar{z}) = \sum_{\nu=0}^{\infty} \varphi_\nu(z) \, \overline{\varphi_\nu(z)}$$

we obtain at once

$$K(0, 0) = \frac{1}{\Omega}.$$

On the other hand, by the definition of $Q(z, \bar{z})$ we have $Q(0,0) = 1$. Hence the theorem follows from (4.2.13).

Assuming now that $\Re$ is a transitive domain (i.e., $\Re \equiv \mathfrak{M}$), let us ascertain the geometrical properties of $Q(z, \bar{z})$. From (4.2.7) we have

$$J(x, x, W)dx' = J(z, z, U)dz',$$

hence

$$\overline{dx} \cdot \overline{J(x, x, W)}' \cdot J(x, x, W) \cdot dx' = \overline{dz} \cdot \overline{J(z, z, U)}' \cdot J(z, z, U) \cdot dz'.$$

This invariant form can be considered as the metric of our space. The volume element in this metric is

$$|\det J(z, z, U)|^2 \, \dot{z} = Q(z, \bar{z}) \cdot \dot{z},$$

so that $Q(z, \bar{z})$ can be called the *volume density*.

From Theorem 4.2.1 we obtain the following proposition:

The Bergman kernel for any transitive circular region is equal to the ratio of the volume density to the Euclidean volume of the domain.

In the subsequent sections we shall determine the Bergman kernel for our four types of classical domains on the basis of the above considerations

only, without the use of complete orthonormal systems.[13]

4.3. Bergman kernels for the domains $\Re_I$, $\Re_{II}$ and $\Re_{III}$.

1°. The group Γ for the domain $\Re_I$ consists of the following transformations (see L. K. Hua [1]):

$$Z_1 = (AZ + B)(CZ + D)^{-1}, \tag{4.3.1}$$

where A, B, C, D are matrices of dimensions $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively, satisfying the relations

$$\bar{A}A' - \bar{B}B' = I^{(m)}, \quad \bar{A}C' = \bar{B}D', \quad \bar{C}C' - \bar{D}D' = -I^{(n)}.$$

For $m = n$, we assume, moreover, that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = +1. \tag{4.3.2}$$

Let us find the transformations which carry an arbitrary point $Z = P$ into the origin. By the definition of the domain $\Re_I$ we have

$$I^{(m)} - \bar{P}P' > 0.$$

Thus it follows from Theorem 2.1.2 that also

$$I^{(n)} - P'\bar{P} > 0.$$

It is known that there exists an $m \times m$ matrix Q and $n \times n$ matrix R such that

$$\bar{Q}(I^{(m)} - \bar{P}P')Q' = I^{(m)}, \quad \bar{R}(I^{(n)} - P'\bar{P})R' = I^{(n)}. \tag{4.3.3}$$

The transformation

$$Z_1 = Q(Z - P)(I^{(n)} - \bar{P}'Z)^{-1}R^{-1} \tag{4.3.4}$$

carries P into the origin. It is easy to see that this transformation is of the form (4.3.1).

Differentiating (4.3.4), we obtain

$$dZ_1 = Q \left\{ dZ \cdot (I - \bar{P}'Z)^{-1} + (Z - P) d(I - \bar{P}'Z)^{-1} \right\} R^{-1}.$$

We shall set $Z = P$. Then

$$dZ_1 = Q \cdot dZ \cdot (I - \bar{P}'P)^{-1}R^{-1} = Q \cdot dZ \cdot \bar{R}',$$

i.e., at the point $Z = P$

$$\dot{Z}_1 = \left| (\det Q)^m \cdot (\det \bar{R}')^n \right|^2 \cdot \dot{Z} = \left\{ \det(I - P\bar{P}') \right\}^{-m-n} \cdot \dot{Z}.$$

Hence

$$Q(Z, \bar{Z}) = \{\det(I - Z\bar{Z}')\}^{-(m+n)}.$$

Using the results of §4.2, we obtain the following theorem.

THEOREM 4.3.1. *The Bergman kernel of the domain* $\Re_I$ *is*

$$\frac{1}{V(\Re_I)} \cdot \{\det(I - Z\bar{Z}')\}^{-(m+n)}, \qquad (4.3.5)$$

where, by (2.2.2)

$$V(\Re_I) = \frac{1!\,2! \ldots (m-1)!\,1!\,2! \ldots (n-1)!}{1!\,2! \ldots (m+n-1)!} \pi^{mn}.$$

2°. The group Γ of the domain $\Re_{II}$ consists of transformations of the form

$$Z_1 = (AZ + B)(\bar{B}Z + \bar{A})^{-1}, \qquad (4.3.6)$$

where

$$A'B = B'A, \quad \bar{A}A' - \bar{B}B' = I.$$

Suppose P is a point of $\Re_{II}$. A matrix R can be found such that

$$\bar{R}(I - \bar{P}P')R' = I. \qquad (4.3.7)$$

The transformation

$$Z_1 = R(Z - P)(I - \bar{P}Z)^{-1}\bar{R}^{-1}, \qquad (4.3.8)$$

belonging to Γ, carries the point P into the origin.

Differentiation of (4.3.8) yields

$$dZ_1 = R\{dZ \cdot (I - \bar{P}Z)^{-1} + (Z - P)\,d(I - \bar{P}Z)^{-1}\}\bar{R}^{-1}.$$

Setting $Z = P$, we obtain

$$dZ_1 = R \cdot dZ \cdot (I - \bar{P}P)^{-1}\bar{R}^{-1} = R \cdot dZ \cdot R'.$$

Hence at the point $Z = P$

$$\dot{Z}_1 = |(\det R)^{n+1}|^2 \cdot \dot{Z} = \{\det(I - P\bar{P})\}^{-(n+1)} \cdot \dot{Z}.$$

Thus

$$Q(Z, \bar{Z}) = \{\det(I - Z\bar{Z})\}^{-(n+1)}.$$

THEOREM 4.3.2. *The Bergman kernel of the domain* $\Re_{II}$ *is*

$$\frac{1}{V(\Re_{II})} \cdot \{\det(I - Z\bar{Z})\}^{-(n+1)}, \qquad (4.3.9)$$

where, by (2.3.2),

$$V(\Re_{\mathrm{II}}) = \pi^{\frac{n(n+1)}{2}} \cdot \frac{2!\,4!\,\ldots\,(2n-2)!}{n!\,(n+1)!\,\ldots\,(2n-1)!}.$$

3°. The group Γ of the domain $\Re_{\mathrm{III}}$ consists of transformations of the form

$$Z_1 = (AZ + B)(-\bar{B}Z + \bar{A})^{-1}, \qquad (4.3.10)$$

where

$$A'B = -B'A, \quad \bar{A}'A - \bar{B}'B = I.$$

Suppose P is a point of $\Re_{\mathrm{III}}$, i.e., $I + P\bar{P} > 0$. Then a matrix Q can be found such that

$$\bar{Q}(I + P\bar{P})Q' = I.$$

Then in Γ we have the transformation

$$Z_1 = Q(Z - P)(I + \bar{P}Z)^{-1}\bar{Q}^{-1}, \qquad (4.3.11)$$

which carries the point P into the origin.

By differentiation of (4.3.11) we have

$$dZ_1 = Q\{dZ \cdot (I + \bar{P}Z)^{-1} + (Z - P)\,d\,(I + \bar{P}Z)^{-1}\}\,\bar{Q}^{-1}.$$

For $Z = P$, we obtain

$$dZ_1 = Q \cdot dZ \cdot (I + \bar{P}P)^{-1} \cdot \bar{Q}^{-1} = Q \cdot dZ \cdot Q'.$$

Hence at the point $Z = P$

$$\dot{Z}_1 = |(\det Q)^{n-1}|^2\, \dot{Z} = \{\det(I + \bar{P}P)\}^{-n+1} \cdot \dot{Z}.$$

Therefore

$$Q(Z, \bar{Z}) = \{\det(I + Z\bar{Z})\}^{-n+1}.$$

THEOREM 4.3.3. *The Bergman kernel of the domain $\Re_{\mathrm{III}}$ is*

$$\frac{1}{V(\Re_{\mathrm{III}})}\{\det(I + Z\bar{Z})\}^{-n+1},$$

where, by (2.4.2),

$$V(\Re_{\mathrm{III}}) = \pi^{\frac{n(n-1)}{2}} \cdot \frac{2!\,4!\,\ldots\,(2n-4)!}{(n-1)!\,n!\,\ldots\,(2n-3)!}.$$

4.4. **The Bergman kernel for the domain $\Re_{\mathrm{IV}}$.** The group Γ of the domain $\Re_{\mathrm{IV}}$ consists of transformations of the form

$$w = \left\{ \left[\left(\tfrac{1}{2}(zz'+1), \ \tfrac{i}{2}(zz'-1) \right) A' + zB' \right] \binom{1}{i} \right\}^{-1}$$
$$\times \left\{ \left(\tfrac{1}{2}(zz'+1), \ \tfrac{i}{2}(zz'-1) \right) C' + zD' \right\}, \qquad (4.4.1)$$

where A, B, C and D are real matrices of dimensions 2×2, $2\times n$, $n\times 2$ and $n\times n$, respectively, satisfying the relations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}' = \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix} \qquad (4.4.2)$$

and

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = +1. \qquad (4.4.3)$$

We shall now find the transformations of Γ which carry the point z_0 into the origin. Proceeding from the vector z_0 we shall construct the $2\times n$ matrix X_0 as follows:

$$X_0 = 2 \begin{pmatrix} z_0 z_0' + 1 & i\left(z_0 z_0' - 1\right) \\ \overline{z_0 z_0'} + 1 & -i\left(\overline{z_0 z_0'} - 1\right) \end{pmatrix}^{-1} \begin{pmatrix} z_0 \\ \overline{z}_0 \end{pmatrix} = 2 A_0^{-1} \begin{pmatrix} z_0 \\ \overline{z}_0 \end{pmatrix}$$
$$= \frac{1}{1 - |z_0 z_0'|^2} \begin{pmatrix} z_0 + \overline{z}_0 - \left(\overline{z_0 z_0'} \cdot z_0 + z_0 z_0' \cdot \overline{z}_0\right) \\ i\left(z_0 - \overline{z}_0\right) + i\left(\overline{z_0 z_0'} \cdot z_0 - z_0 z_0' \cdot \overline{z}_0\right) \end{pmatrix}. \qquad (4.4.4)$$

This matrix is evidently real. We have

$$I - X_0 X_0' = \overline{A}_0^{-1} \left(\overline{A}_0 A_0' - 4 \overline{\left(\frac{z_0}{\overline{z}_0}\right)} \left(\frac{z_0}{\overline{z}_0}\right)' \right) \cdot A_0'^{-1}$$
$$= 2 \overline{A}_0^{-1} \begin{pmatrix} 1 + |z_0 z_0'|^2 - 2\overline{z}_0 z_0' & 0 \\ 0 & 1 + |z_0 z_0'|^2 - 2\overline{z}_0 z_0' \end{pmatrix} \cdot A_0'^{-1}. \qquad (4.4.5)$$

Hence

$$(I - X_0 X_0')^{-1} = \frac{1}{2\left(1 + |z_0 z_0'|^2 - 2\overline{z}_0 z_0'\right)} A_0' \overline{A}_0 = \frac{1}{1 + |z_0 z_0'|^2 - 2\overline{z}_0 z_0'}$$
$$\times \begin{pmatrix} (z_0 z_0' + 1)(\overline{z_0 z_0'} + 1) & i\left(z_0 z_0' - \overline{z_0 z_0'}\right) \\ i\left(z_0 z_0' - \overline{z_0 z_0'}\right) & (z_0 z_0' - 1)(\overline{z_0 z_0'} - 1) \end{pmatrix} = A' A, \qquad (4.4.6)$$

where

$$A = \frac{1}{2}\left(1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'\right)^{-\frac{1}{2}} \begin{pmatrix} -i\left(z_0 z_0' - \overline{z_0 z_0'}\right) & z_0 z_0' + \overline{z_0 z_0'} - 2 \\ z_0 z_0' + \overline{z_0 z_0'} + 2 & i\left(z_0 z_0' - \overline{z_0 z_0'}\right) \end{pmatrix}. \tag{4.4.7}$$

We shall choose a D which satisfies the condition $D(I^{(n)} - X_0' X_0)D' = I^{(n)}$; then the transformation

$$w = \left\{\left[\left(\tfrac{1}{2}(zz'+1), \tfrac{i}{2}(zz'-1)\right)A' - zX_0'A'\right]\begin{pmatrix}1\\i\end{pmatrix}\right\}^{-1}$$
$$\times \left\{zD' - \left(\tfrac{1}{2}(zz'+1), \tfrac{i}{2}(zz'-1)\right)X_0 D'\right\} \tag{4.4.8}$$

has the form (4.4.1) and carries the point z_0 into the origin.

In addition,

$$\det A = \det D = \frac{1 - |z_0 z_0'|^2}{1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'}. \tag{4.4.9}$$

Differentiation of (4.4.8) yields $(z = (z^{(1)}, \cdots, z^{(n)}))$

$$dw = \left\{dz \cdot D' - \left(\sum_{p=1}^n z^{(p)} dz^{(p)}, i\sum_{p=1}^n z^{(p)} dz^{(p)}\right)X_0 D'\right\}$$
$$\times \left\{\left[\left(\tfrac{1}{2}(zz'+1), \tfrac{i}{2}(zz'-1)\right)A' - zX_0'A'\right]\begin{pmatrix}1\\i\end{pmatrix}\right\}^{-1}$$
$$+ \left\{zD' - \left(\tfrac{1}{2}(zz'+1), \tfrac{i}{2}(zz'-1)\right)X_0 D'\right\}$$
$$\times d\left\{\left[\left(\tfrac{1}{2}(zz'+1), \tfrac{i}{2}(zz'-1)\right)A' - zX_0'A'\right]\begin{pmatrix}1\\i\end{pmatrix}\right\}^{-1}.$$

Setting $z = z_0$, we obtain

$$dw = \left\{dz \cdot D' - \left(\sum_{p=1}^n z_0^{(p)} dz^{(p)}, i\sum_{p=1}^n z_0^{(p)} dz^{(p)}\right)X_0 D'\right\}$$
$$\times \left\{\left[\left(\tfrac{1}{2}(z_0 z_0'+1), \tfrac{i}{2}(z_0 z_0'-1)\right)A' - z_0 X_0'A'\right]\begin{pmatrix}1\\i\end{pmatrix}\right\}^{-1},$$

i.e.,

$$dw = \left\{dz \cdot D' - d(zz') \cdot (1, i) X_0 D'\right\}$$
$$\times \left\{\left[\left(\tfrac{1}{2}(z_0 z_0'+1), \tfrac{i}{2}(z_0 z_0'-1)\right)A' - z_0 X_0'A'\right]\begin{pmatrix}1\\i\end{pmatrix}\right\}^{-1}. \tag{4.4.10}$$

Using (4.4.4) and (4.4.7), we have

$$dw = -i\, dz \cdot \left\{I - 2\frac{z_0' \bar{z}_0 - \overline{z_0 z_0'} \cdot z_0' z_0}{1 - |z_0 z_0'|^2}\right\} \cdot D' \cdot \left(1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'\right)^{-\frac{1}{2}}. \tag{4.4.11}$$

From (4.4.9) we obtain

$$\det\left(\frac{\partial w}{\partial z}\right)_{z=z_0}$$

$$= \det\left\{I - 2\frac{z_0'\bar{z}_0 - \overline{z_0 z_0'} \cdot z_0' z_0}{1 - |z_0 z_0'|^2}\right\} \cdot \frac{1 - |z_0 z_0'|^2}{\left(1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'\right)}$$

$$\times \left(1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'\right)^{-\frac{n}{2}},$$

or

$$\left|\det\left(\frac{\partial w}{\partial z}\right)_{z=z_0}\right|^2 = \left(1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'\right)^{-n}. \qquad (4.4.12)$$

Here the identity

$$\det\left\{I - 2\frac{z_0'\bar{z}_0 - \overline{z_0 z_0'} \cdot z_0' z_0}{1 - |z_0 z_0'|^2}\right\} = \frac{1 + |z_0 z_0'|^2 - 2\bar{z}_0 z_0'}{1 - |z_0 z_0'|^2} \qquad (4.4.13)$$

is used, which follows from the relation $\det(I - u'\bar{v}) = 1 - \bar{v}u'$ (see Theorem 2.1.2).

Thus we arrived at the theorem:

THEOREM 4.4.1. *The Bergman kernel of the domain* $\mathfrak{R}_{IV}$ *is*

$$\frac{1}{V(\mathfrak{R}_{IV})}\left(1 + |zz'|^2 - 2\bar{z}z'\right)^{-n},$$

where, by (2.5.7),

$$V(\mathfrak{R}_{IV}) = \frac{\pi^n}{2^{n-1} \cdot n!}.$$

4.5. **The Cauchy kernel.** Let us now pass to the study of the Cauchy kernel

$$\sum_{f=0}^{\infty} \sum_{\nu=1}^{N_f} \varphi_\nu^f(z) \overline{\varphi_\nu^f(\xi)} = H(z, \bar{\xi}). \qquad (4.5.1)$$

(Here z belongs to $\mathfrak{R}$, and ξ to $\mathfrak{C}$.)

Suppose Γ_0 is a group of motions of $\mathfrak{R}$ which leave the origin unchanged. We shall assume that $\mathfrak{C}$ is transitive with respect to Γ_0, i.e., that any two points of $\mathfrak{C}$ can be carried into each other by a transformation which belongs to Γ_0.

THEOREM 4.5.1. *The series*