The Arm function

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We give an expression of the Arm function
which gives rise to the definition of
the Arm binomial coefficient
and the corresponding
Arm triangle.

Introduction

First, the famous triangle constitued of binomial coefficient is a major breakthrough of mathematical sciences. Also, this triangle is useful in mathematics since it allows us to find each power $(a+b)^n$of the sum of two real numbers a, b with the mean of the binomial coefficient formula. By the way, real numbers can commutate one with an other which is very practical in the daily life.

However, since the pionnering works in noncommutative algebras, we know that there is operators which do not commutate. For example, we know that the two main operators of the quantum mechanics, the momentum operator $p = \frac{\partial}{\partial x}$ and the position operator $q = x$, form a noncommutative algebra with the defining relation

$$[p, q] = 1 \tag{0.1}$$

Moreover, we know that each function of $\mathbb{C}[x]$ can be develop by the serie expansion formula at a point a or x_0. Next, varying this point a and replacing it with y, we can rewrite the serie expansion formula as :

$$f(x) \ = \ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left((y-x)^k \ \frac{\partial^k}{\partial y^k} \right) f(y) \tag{0.2}$$

In that case, observing this formula (0.2), we want to see the begining of a serie expansion. But the factor $(y-x)^k \ \frac{\partial^k}{\partial y^k}$ is not the same as $((y-x) \ \frac{\partial}{\partial y})^k$ because those operators are noncommutative. Therefore, we decide to search a way to write the operator $(y-x)^k \ \frac{\partial^k}{\partial y^k}$ as a power of something. As a result, we find an equivalent of the binomial formula for this noncommutative algebra which we call Arm binomial formula and constitued the main result of this paper :

$$\left(x \ \frac{\partial}{\partial x} \right)^n \ = \ \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \ x^k \ \frac{\partial^k}{\partial x^k} \tag{0.3}$$

where appear a coefficient $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ which is an equivalent of the binomial coefficient for this algebra. We give an expression for this Arm coefficient $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ and this corresponding induction relation.

In fact the drawing of those Arm coefficient $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ reveal an Arm triangle which we can give here the first lines

1									
1	1								
1	3	1							
1	7	6	1						
1	15	25	10	1					
1	31	90	65	15	1				
1	63	301	350	140	21	1			
1	127	966	1701	1050	266	28	1		
1	255	3025	7770	6951	2646	462	36	1	
1	511	9330	34105	42525	22827	5880	750	45	1

with its induction relation

$$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = (k+1) \begin{Bmatrix} n \\ k+1 \end{Bmatrix} + \begin{Bmatrix} n \\ k \end{Bmatrix} \tag{0.4}$$

Therefore, we use the Arm binomial formula to express the factor $(y-x)^k \frac{\partial^k}{\partial y^k}$ of (0.2) wich gives us the definition of the expansion serie of the Arm function.

In a first time, we expose the problem and we rewrite the serie expansion formula and give the goal of finding the Arm formula. In a second time, we give and show the Arm binomial theorem which is a strict equivalent of the binomial one and give somes examples. In a third time, we solve the problem in inversing the Arm binomial triangle and gives the defining equation of the Arm function. As examples, we give first series expansions powers of this function.

1 The Problem

First, we write the Taylor serie formula for the function $f(x)$ at point a :

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \lim_{x \to a} \frac{\partial^k}{\partial x^k} f(x) \tag{1.1}$$

Instead of using the letter a, we choose to use the letter y, which give :

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{(x-y)^k}{k!} \lim_{x \to y} \frac{\partial^k}{\partial x^k} f(x) \tag{1.2}$$

Here, ecause the limit of the derivative when x tend to y is the same as derivate by y, we can replace (1.2) by

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{(x-y)^k}{k!} \frac{\partial^k}{\partial y^k} f(y) \tag{1.3}$$

what we can also write as

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left((y-x)^k \frac{\partial^k}{\partial y^k} \right) f(y) \tag{1.4}$$

Here we see that the begining of the sum seems to a serie expansion, but to have a serie expansion of the operator $(y-x)\frac{\partial}{\partial y}$, we need to find how to convert $(y-x)^k \frac{\partial^k}{\partial y^k}$ in $((y-x)\frac{\partial}{\partial y})^k$. We can easily guess that this is the same problem as converting $y^k \frac{\partial^k}{\partial y^k}$ in $(y\frac{\partial}{\partial y})^k$.

Then this is the purpose of this paper to find the serie expansion of the Arm function $A(X)$ such that

$$f(x) \;=\; A\left((y-x) \frac{\partial}{\partial y} \right) f(y) \tag{1.5}$$

with its series expansion

$$A(X) \;=\; \sum_{k=0}^{\infty} \alpha_k \, X^k \tag{1.6}$$

and finally identify

$$A\left((y-x) \frac{\partial}{\partial y} \right) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left((y-x)^k \frac{\partial^k}{\partial y^k} \right) \tag{1.7}$$

2 The Arm Binomial Theorem

First, we introduce the Arm binomial coefficient and the corresponding Arm triangle.

Proposition 1. *The element of the Arm triangle is such that*

$$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = (k+1) \begin{Bmatrix} n \\ k+1 \end{Bmatrix} + \begin{Bmatrix} n \\ k \end{Bmatrix} \tag{2.8}$$

and we have the definition of the Arm binomial coefficient :

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!} \left(\sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k} \, p^n \right) \tag{2.9}$$

where $\binom{n}{k}$ *is the binomial coefficient.*

Proof :
We show by induction that the only element which respect (2.8) is the sequence (2.9 :
Basic :

$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{1}{1!} \sum_{p=1}^{1} \binom{1}{p} (-1)^{p-1} \, p^1 = 1 \tag{2.10}$$

Inductive step : We suppose the relation

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!} \left(\sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k} \, p^n \right) \tag{2.11}$$

for all $k \in [1, n]$ and we show that the relation

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = \frac{1}{k!} \left(\sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k} \, p^{n+1} \right) \tag{2.12}$$

is true for all $k \in [1, n+1]$.
So the relation we have to respect is

$$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = (k+1) \begin{Bmatrix} n \\ k+1 \end{Bmatrix} + \begin{Bmatrix} n \\ k \end{Bmatrix} \tag{2.13}$$

Using the induction hypothesis, we have :

$$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = (k+1) \frac{1}{(k+1)!} \left(\sum_{p=1}^{k+1} \binom{k+1}{p} (-1)^{p-k-1} \, p^n \right) + \frac{1}{k!} \left(\sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k} \, p^n \right) \tag{2.14}$$

and we obtain

$$\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \frac{1}{k!} \left(\sum_{p=1}^{k+1} \left[\binom{k}{p} + \binom{k}{p-1} \right] (-1)^{p-k-1} \, p^n + \sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k} \, p^n \right) \tag{2.15}$$

since the binomial coefficient relation is true $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
The relation (2.15) becomes

$$\left\{ {n+1 \atop k+1} \right\} \;=\; \frac{1}{k!} \left(\sum_{p=1}^{k+1} \binom{k}{p-1} (-1)^{p-k-1} \, p^n \right) \tag{2.16}$$

Using the fact that $\binom{k+1}{p} = \frac{k+1}{p} \binom{k}{p-1}$, we obtain

$$\left\{ {n+1 \atop k+1} \right\} \;=\; \frac{1}{k!} \left(\sum_{p=1}^{k+1} \frac{p}{k+1} \binom{k+1}{p} (-1)^{p-k-1} \, p^n \right) \tag{2.17}$$

which is what we want to show :

$$\left\{ {n+1 \atop k+1} \right\} \;=\; \frac{1}{(k+1)!} \left(\sum_{p=1}^{k+1} \binom{k+1}{p} (-1)^{p-k-1} \, p^{n+1} \right) \tag{2.18}$$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all natural n. Q.E.D.

♦

Here we give the first twelfth lines of the Arm triangle constitued by the Arm coefficients $\left\{ {n \atop k} \right\}_{1 \le k \le n \le 12}$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$
$n=1$	1											
$n=2$	1	1										
$n=3$	1	3	1									
$n=4$	1	7	6	1								
$n=5$	1	15	25	10	1							
$n=6$	1	31	90	65	15	1						
$n=7$	1	63	301	350	140	21	1					
$n=8$	1	127	966	1701	1050	266	28	1				
$n=9$	1	255	3025	7770	6951	2646	462	36	1			
$n=10$	1	511	9330	34105	42525	22827	5880	750	45	1		
$n=11$	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1	
$n=12$	1	2047	86526	611501	1379400	1323652	327396	159027	22275	1705	66	1

Here you can easily check the validity of the induction relation (2.8) :

$$\left\{ {n+1 \atop k+1} \right\} \;=\; (k+1) \left\{ {n \atop k+1} \right\} \;+\; \left\{ {n \atop k} \right\} \tag{2.19}$$

As an example, we try it with $n = 5$ and $k = 3$

$$\begin{aligned} \left\{ {5 \atop 3} \right\} &= 3\left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 2} \right\} \\ 25 &= 3 \times 6 + 7 \end{aligned} \tag{2.20}$$

As an other example, we try $n = 8$ and $k = 4$

$$\begin{aligned} \left\{ {8 \atop 4} \right\} &= 4\left\{ {7 \atop 4} \right\} + \left\{ {7 \atop 3} \right\} \\ 1701 &= 4 \times 350 + 301 \end{aligned} \tag{2.21}$$

Now we check the validity of the relation (2.9) :

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \left(\sum_{p=1}^{k} \binom{k}{p} (-1)^{p-k}\, p^n \right) \tag{2.22}$$

As an example, we try it with $n = 5$ and $k = 3$:

$$\begin{aligned} \left\{ {5 \atop 3} \right\} &= \frac{1}{3!} \left(\sum_{p=1}^{3} \binom{3}{p} (-1)^{p-3}\, p^5 \right) \\ \left\{ {5 \atop 3} \right\} &= \frac{1}{6} \left(\binom{3}{1} 1^5 - \binom{3}{2} 2^5 + \binom{3}{3} 3^5 \right) \\ 25 &= \frac{1}{6} \left(3 \times 1^5 - 3 \times 32 + 1 \times 243 \right) \end{aligned} \tag{2.23}$$

As an other example, we try it with $n = 8$ and $k = 4$:

$$\begin{aligned} \left\{ {8 \atop 4} \right\} &= \frac{1}{4!} \left(\sum_{p=1}^{4} \binom{4}{p} (-1)^{p-4}\, p^8 \right) \\ \left\{ {8 \atop 4} \right\} &= \frac{1}{24} \left(- \binom{4}{1} 1^8 + \binom{4}{2} 2^8 - \binom{4}{3} 3^8 + \binom{4}{4} 4^8 \right) \\ 1701 &= \frac{1}{24} \left(- \; 4 \times 1^8 + 6 \times 256 - 4 \times 6561 \; + 1 \times 65536 \right) \end{aligned} \tag{2.24}$$

Now, we introduce the Arm binom theorem

Theorem 1. *The Arm binomial theorem is given by*

$$\left(x \, \frac{\partial}{\partial x} \right)^n = \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k \, \frac{\partial^k}{\partial x^k} \tag{2.25}$$

where $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ *is the Arm binomial coefficient (2.9).*

Proof :
We show the relation (2.25) by induction.
Basic :

$$\left(x \, \frac{\partial}{\partial x} \right)^1 = \sum_{k=1}^{1} \left\{ \begin{matrix} 1 \\ k \end{matrix} \right\} x^k \, \frac{\partial^k}{\partial x^k} \tag{2.26}$$

Inductive step :
We suppose that the relation

$$\left(x \, \frac{\partial}{\partial x} \right)^n = \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k \, \frac{\partial^k}{\partial x^k} \tag{2.27}$$

is true at the n-th step.
Furthermore, we show that the relation

$$\left(x \, \frac{\partial}{\partial x} \right)^{n+1} = \sum_{k=1}^{n+1} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} x^k \, \frac{\partial^k}{\partial x^k} \tag{2.28}$$

for the $n+1$-th step.

$$\left(x \, \frac{\partial}{\partial x} \right)^{n+1} = x \, \frac{\partial}{\partial x} \left(x \, \frac{\partial}{\partial x} \right)^n$$

Using the induction hypothesis, we have

$$
\begin{aligned}
\left(x \, \frac{\partial}{\partial x} \right)^{n+1} &= x \, \frac{\partial}{\partial x} \Big(\sum_{k=1}^{n} \left\{ {n \atop k} \right\} \, x^k \, \frac{\partial^k}{\partial x^k} \Big) \\
&= \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \, x \, \frac{\partial}{\partial x} \Big(\, x^k \, \frac{\partial^k}{\partial x^k} \Big) \\
&= \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \, x \, \Big(\, k x^{k-1} \, \frac{\partial^k}{\partial x^k} + x^k \frac{\partial^{k+1}}{\partial x^{k+1}} \Big) \\
&= \sum_{k=1}^{n} k \, \left\{ {n \atop k} \right\} \, x^k \, \frac{\partial^k}{\partial x^k} + \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \, x^{k+1} \, \frac{\partial^{k+1}}{\partial x^{k+1}} \\
&= \sum_{k=1}^{n} k \, \left\{ {n \atop k} \right\} \, x^k \, \frac{\partial^k}{\partial x^k} + \sum_{k=2}^{n+1} \left\{ {n \atop k-1} \right\} \, x^k \, \frac{\partial^k}{\partial x^k} \\
&= \sum_{k=1}^{n+1} \left[k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \, \right] \, x^k \, \frac{\partial^k}{\partial x^k} \\
\left(x \, \frac{\partial}{\partial x} \right)^{n+1} &= \sum_{k=1}^{n+1} \left\{ {n+1 \atop k} \right\} \, x^k \, \frac{\partial^k}{\partial x^k}
\end{aligned}
\tag{2.29}
$$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all natural n. Q.E.D.

♦

We can check the validity of this theorem for the first line :

$$
\left(x \, \frac{\partial}{\partial x} \right)^{1} = 1 \times x \, \frac{\partial}{\partial x} \tag{2.30}
$$

for the second line :

$$
\begin{aligned}
\left(x \, \frac{\partial}{\partial x} \right)^{2} &= x \, \frac{\partial}{\partial x} \left(x \, \frac{\partial}{\partial x} \right)^{1} \\
\left(x \, \frac{\partial}{\partial x} \right)^{2} &= 1 \times x \, \frac{\partial}{\partial x} + 1 \times x^2 \, \frac{\partial^2}{\partial x^2}
\end{aligned}
\tag{2.31}
$$

for the third line :

$$
\begin{aligned}
\left(x \, \frac{\partial}{\partial x} \right)^{3} &= x \, \frac{\partial}{\partial x} \left(x \, \frac{\partial}{\partial x} \right)^{2} \\
\left(x \, \frac{\partial}{\partial x} \right)^{3} &= x \, \frac{\partial}{\partial x} \left(x \, \frac{\partial}{\partial x} + x^2 \, \frac{\partial^2}{\partial x^2} \right) \\
\left(x \, \frac{\partial}{\partial x} \right)^{3} &= 1 \times x \, \frac{\partial}{\partial x} + 3 \times x^2 \, \frac{\partial^2}{\partial x^2} + 1 \times x^3 \, \frac{\partial^3}{\partial x^3}
\end{aligned}
\tag{2.32}
$$

and for the fourth line :

$$\begin{aligned}
\left(x \frac{\partial}{\partial x}\right)^4 &= x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x} \right)^3 \\
\left(x \frac{\partial}{\partial x} \right)^3 &= x \frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x} + 3x^2 \frac{\partial^2}{\partial x^2} + x^3 \frac{\partial^3}{\partial x^3} \right) \\
\left(x \frac{\partial}{\partial x} \right)^3 &= 1 \times x \frac{\partial}{\partial x} + 7 \times x^2 \frac{\partial^2}{\partial x^2} + 6 \times x^3 \frac{\partial^3}{\partial x^3} + 1 \times x^4 \frac{\partial^4}{\partial x^4} \qquad (2.33)
\end{aligned}$$

etc

3 Resolution Of The Problem

To solve our problem we need to know how converting $x^k \frac{\partial^k}{\partial x^k}$ in $(x\frac{\partial}{\partial x})^k$.

Proposition 2. *The inverse Arm Binomial theorem is given by*

$$x^n \ \frac{\partial^n}{\partial x^n} \ = \ \sum_{k=1}^{n} \ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{-1} \ \left(\ x \ \frac{\partial}{\partial x} \ \right)^k \tag{3.34}$$

where the inverse if the Arm triangle is define with the induction relation

$$\left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}^{-1} \ = \ -n\left\{ \begin{matrix} n \\ k+1 \end{matrix} \right\}^{-1} + \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{-1} \tag{3.35}$$

with the definition of the comatrice of cofactors and each $p \in \mathbb{N}$.

Proof :
We show (3.34) by induction
Basic :

$$x^1 \ \frac{\partial^1}{\partial x^1} \ = \ \sum_{k=1}^{1} \left\{ \begin{matrix} 1 \\ k \end{matrix} \right\}^{-1} \ \left(\ x \ \frac{\partial}{\partial x} \ \right)^k \tag{3.36}$$

Inductive step : We suppose the relation

$$x^n \ \frac{\partial^n}{\partial x^n} \ = \ \sum_{k=1}^{n} \ \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^{-1} \ \left(\ x \ \frac{\partial}{\partial x} \ \right)^k \tag{3.37}$$

for all $k \in [1, n]$ and we show that the relation

$$x^{n+1} \ \frac{\partial^{n+1}}{\partial x^{n+1}} \ = \ \sum_{k=1}^{n+1} \ \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}^{-1} \ \left(\ x \ \frac{\partial}{\partial x} \ \right)^k \tag{3.38}$$

is true for all $k \in [1, n+1]$.
We have the relation

$$\frac{\partial}{\partial x}\left(x^n \frac{\partial^n}{\partial x^n} \right) = nx^{n-1}\frac{\partial^n}{\partial x^n} + x^n \frac{\partial^{n+1}}{\partial x^{n+1}} \tag{3.39}$$

and so

$$x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} = x\frac{\partial}{\partial x}\left(x^n \frac{\partial^n}{\partial x^n} \right) - nx^n \frac{\partial^n}{\partial x^n} \tag{3.40}$$

Using the induction hypothesis, we have :

$$
\begin{aligned}
x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} &= x\frac{\partial}{\partial x}\left(\sum_{k=1}^{n}\left\{{n \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}\right) - n\sum_{k=1}^{n}\left\{{n \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}\\
x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} &= \left(\sum_{k=1}^{n}\left\{{n \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k+1}\right) - n\sum_{k=1}^{n}\left\{{n \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}\\
x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} &= \left(\sum_{k=2}^{n+1}\left\{{n \atop k-1}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}\right) - n\sum_{k=1}^{n}\left\{{n \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}\\
x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} &= \sum_{k=1}^{n+1}\left(\left\{{n \atop k-1}\right\}^{-1} - n\left\{{n \atop k}\right\}^{-1}\right)\left(x\,\frac{\partial}{\partial x}\right)^{k}\\
x^{n+1}\frac{\partial^{n+1}}{\partial x^{n+1}} &= \sum_{k=1}^{n+1}\left\{{n+1 \atop k}\right\}^{-1}\left(x\,\frac{\partial}{\partial x}\right)^{k}
\end{aligned}
$$

Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all natural n. Since both the basis and the inductive step have been performed, by mathematical induction, the statement holds for all natural n.

♦

Here we give the first tenth lines of the inverse Arm triangle $\left\{{n \atop k}\right\}^{-1}_{1\le k\le n\le 10}$:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$n=1$	1	0	0	0	0	0	0	0		
$n=2$	-1	1	0	0	0	0	0	0	0	0
$n=3$	2	-3	1	0	0	0	0	0	0	0
$n=4$	-6	11	-6	1	0	0	0	0	0	0
$n=5$	24	-50	35	-10	1	0	0	0	0	0
$n=6$	-120	274	-225	85	-15	1	0	0	0	0
$n=7$	720	-1764	1624	-735	175	-21	1	0	0	0
$n=8$	-5040	13068	-13132	6769	-1960	322	-28	1	0	0
$n=9$	40320	-109584	118124	-67284	22449	-4536	546	-36	1	0
$n=10$	-362880	1026576	-1172700	723680	-269325	63273	-9450	870	-45	1

Now to solve our main problem, we take back the equation (1.4)

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left((y-x)^k\,\frac{\partial^k}{\partial y^k}\right)f(y) \tag{3.41}$$

and using (3.34), we obtain

$$f(x) = f(y) + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\sum_{i=1}^{k}\left\{{k \atop i}\right\}^{-1}\left((y-x)\,\frac{\partial}{\partial y}\right)^{i}\right)f(y) \tag{3.42}$$

which we can write as

$$f(x) \;=\; \left(1 \;+\; \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{(-1)^k}{k!} \quad \begin{Bmatrix} k \\ i \end{Bmatrix}^{-1} \; \left((y-x) \; \frac{\partial}{\partial y} \right)^i \right) f(y) \tag{3.43}$$

Finally, we can see a definition of the Arm function

$$A(X) \;=\; 1 \;+\; \sum_{k=1}^{\infty} \sum_{i=1}^{k} \frac{(-1)^k}{k!} \quad \begin{Bmatrix} k \\ i \end{Bmatrix}^{-1} \; X^i \tag{3.44}$$

which as serie expansion

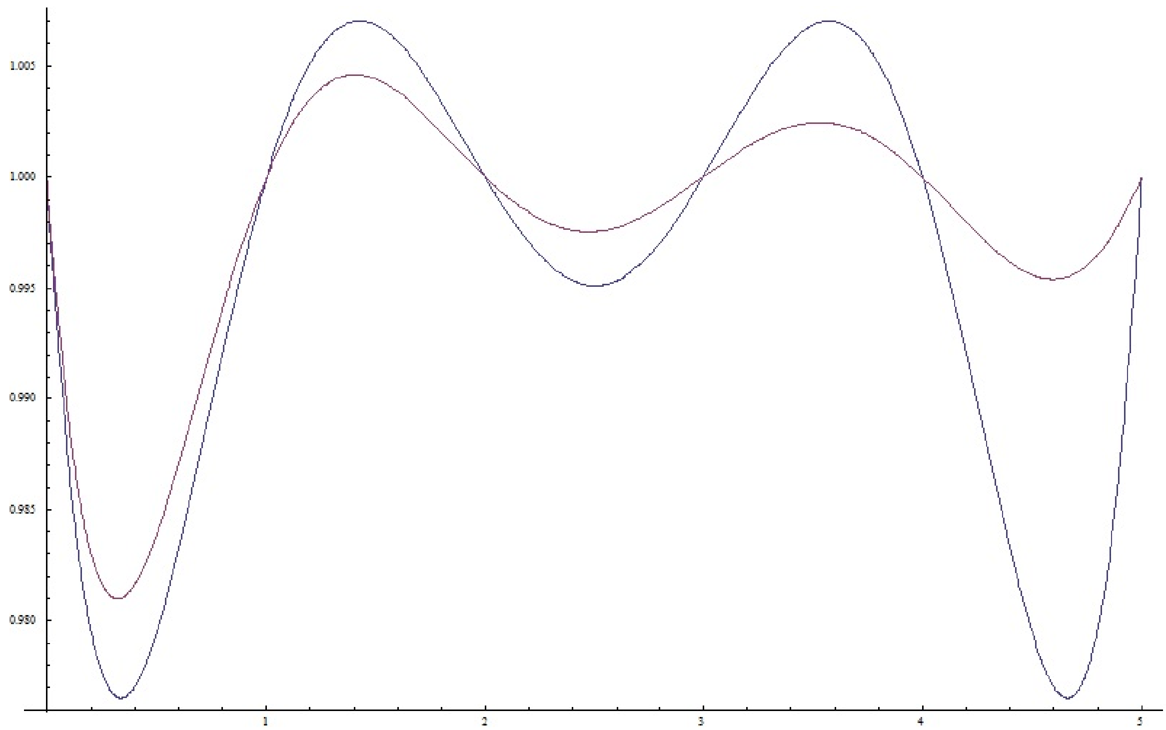

FIGURE 1 – The Arm function for $k = 6$ and $k = 7$

$$\begin{array}{ll}
k=0 & A(X) = 1 \\
k=1 & A(x) = 1 - X \\
k=2 & A(x) = 1 - \frac{1}{2}X + \frac{1}{2}X^2 \\
k=3 & A(x) = 1 - \frac{1}{3}X + \frac{1}{2}X^2 - \frac{1}{6}X^3 \\
k=4 & A(x) = 1 - \frac{1}{4}X + \frac{11}{24}X^2 - \frac{1}{4}X^3 + \frac{1}{24}X^4 \\
k=5 & A(x) = 1 - \frac{1}{5}X + \frac{5}{12}X^2 - \frac{7}{24}X^3 + \frac{1}{12}X^4 - \frac{1}{120}X^5 \\
k=6 & A(x) = 1 - \frac{1}{6}X + \frac{137}{360}X^2 - \frac{5}{16}X^3 + \frac{17}{144}X^4 - \frac{}{48}X^5 + \frac{1}{720}X^6
\end{array}$$

4 Binomial Groups And Algebras

We first define the binomial algebra

Proposition 3. *We define the binomial algebra which gives the definition of the translation group*

$$((x+t)^k)_{0\le k\le n} = \exp(t.\mathfrak{b}_n)(x^k)_{0\le k\le n} \tag{4.45}$$

where $\mathfrak{b}_n$ *is the binomial algebra defined by :*

$$(\mathfrak{b}_n)_{i+1,i} \;=\; i \tag{4.46}$$

and zero elsewhere or

$$\mathfrak{b}_n \;=\; \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix} \tag{4.47}$$

Proof :

The matrice $\binom{n-1}{k}_{0\le k\le n-1}$ is given by :

$$\begin{pmatrix} 1 & 0 & 0 & \dots & \\ 1 & 1 & 0 & \dots & \\ 1 & 2 & 1 & 0 & \dots \\ \vdots & & \ddots & \ddots & \\ \binom{n-1}{0} & \binom{n-1}{1} & \dots & n-1 & 1 \end{pmatrix} \tag{4.48}$$

We take the t-th power of $\binom{n-1}{k}_{0\le k\le n-1}$:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & \\ 1t & 1 & 0 & \dots & \\ 1t^2 & 2t & 1 & 0 & \dots \\ \vdots & & \ddots & \ddots & \\ \binom{n-1}{0}t^{n-1} & \binom{n-1}{1}t^{n-2} & \dots & (n-1)t & 1 \end{pmatrix} \tag{4.49}$$

which we call $\binom{n-1}{k}^t_{0\le k\le n}$.

Here we see that

$$((x+t)^k)_{0\le k\le n-1} = \binom{n-1}{k}^t_{0\le k\le n-1}(x^k)_{0\le k\le n-1} \tag{4.50}$$

Then we see that the corresponding algebra of $\binom{n-1}{k}_{0\le k\le n-1}$ is

$$\mathfrak{b}_n \;=\; \lim_{t\to 0}\frac{\partial}{\partial t}\binom{n-1}{k}^t_{0\le k\le n-1} \;=\; \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix} \tag{4.51}$$

So we can deduce that $\exp(t\mathfrak{b}_n) = \binom{n-1}{k}^t_{0\leq k\leq n-1}$ and $\exp(-t\mathfrak{b}_n) = \binom{n-1}{k}^{-t}_{0\leq k\leq n}$

♦

<u>Example</u> : for $n = 4$, we have

$$\exp(t.\mathfrak{b}_4) = \exp\left(t\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{pmatrix}^0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{pmatrix}^1$$

$$+\frac{1}{2!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{pmatrix}^3$$

$$\exp(t.\mathfrak{b}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2t^2 & 0 & 0 & 0 \\ 0 & 6t^2 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6t^3 & 0 & 0 & 0 \end{pmatrix}$$

So we find that

$$\exp(t.\mathfrak{b}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1t & 1 & 0 & 0 \\ 1t^2 & 2t & 1 & 0 \\ 1t^3 & 3t^2 & 3t & 1 \end{pmatrix} \tag{4.52}$$

and thus

$$\begin{pmatrix} (x+t)^0 \\ (x+t)^1 \\ (x+t)^2 \\ (x+t)^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2 & 2t & 1 & 0 \\ t^3 & 3t^2 & 3t & 1 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{4.53}$$

Then we define the Arm binomial algebra

Proposition 4. *We define the Arm binomial algebra as*

$$\left(\left(x\frac{\partial}{\partial x}\right)^k\right)_{0\le k\le n} = \exp\left(\mathfrak{a}_n\right)\left(x^k\frac{\partial^k}{\partial x^k}\right)_{0\le k\le n} \tag{4.54}$$

where $\mathfrak{a}_n$ is the binomial algebra defined by :

$$\mathfrak{a}_n \;=\; \lim_{t\to 0}\frac{\partial}{\partial t}\begin{Bmatrix} n \\ k \end{Bmatrix}^t_{0\le k\le n} \tag{4.55}$$

and zero elsewhere or

$$\mathfrak{a}_n \;=\; \begin{pmatrix} 1 & & & \\ -\frac{1}{2} & 3 & & \\ \frac{1}{2} & -2 & 6 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.56}$$

Proof :
Admitted

♦

Example : for $n = 4$, we have

$$\begin{aligned}
\exp(t.\mathfrak{a}_4) = &\quad \exp\left(t\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 3 & 0 & 0 \\ \frac{1}{2} & -2 & 6 & 0 \end{pmatrix}\right) \\
= &\quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ -\frac{1}{2}t & 3t & 0 & 0 \\ \frac{1}{2}t & -2t & 6t & 0 \end{pmatrix}^0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ -\frac{1}{2}t & 3t & 0 & 0 \\ \frac{1}{2}t & -2t & 6t & 0 \end{pmatrix}^1 \\
&+\frac{1}{2!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ -\frac{1}{2}t & 3t & 0 & 0 \\ \frac{1}{2}t & -2t & 6t & 0 \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ -\frac{1}{2}t & 3t & 0 & 0 \\ \frac{1}{2}t & -2t & 6t & 0 \end{pmatrix}^3 \\
\exp(t.\mathfrak{a}_4) = &\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^0 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1t & 0 & 0 & 0 \\ -\frac{1}{2}t & 3t & 0 & 0 \\ \frac{1}{2}t & -2t & 6t & 0 \end{pmatrix} \\
&+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3t^2 & 0 & 0 & 0 \\ -5t^2 & 18t^2 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 18t^3 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

So we find that

$$\exp(t.\mathfrak{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1t & 1 & 0 & 0 \\ \frac{1}{2}t(-1+3t) & 3t & 1 & 0 \\ \frac{1}{2}t(1-5t+6t^2) & t(-2+9t) & 6t & 1 \end{pmatrix} \tag{4.57}$$

and thus

$$\exp(\mathfrak{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \tag{4.58}$$

and

$$\exp(-\mathfrak{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} \tag{4.59}$$

$$\begin{pmatrix} \left(x\frac{\partial}{\partial x}\right)^0 \\ \left(x\frac{\partial}{\partial x}\right)^1 \\ \left(x\frac{\partial}{\partial x}\right)^2 \\ \left(x\frac{\partial}{\partial x}\right)^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} x^0\frac{\partial^0}{\partial x^0} \\ x^1\frac{\partial^1}{\partial x^1} \\ x^2\frac{\partial^2}{\partial x^2} \\ x^3\frac{\partial^3}{\partial x^3} \end{pmatrix} \tag{4.60}$$

$$\begin{pmatrix} x^0\frac{\partial^0}{\partial x^0} \\ x^1\frac{\partial^1}{\partial x^1} \\ x^2\frac{\partial^2}{\partial x^2} \\ x^3\frac{\partial^3}{\partial x^3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} \begin{pmatrix} \left(x\frac{\partial}{\partial x}\right)^0 \\ \left(x\frac{\partial}{\partial x}\right)^1 \\ \left(x\frac{\partial}{\partial x}\right)^2 \\ \left(x\frac{\partial}{\partial x}\right)^3 \end{pmatrix} \tag{4.61}$$

Conclusion

In the third section, we do not give a formal expression for the inverse $\left\{ n \atop k \right\}^{-1}$ of the Arm triangle. We just define it with the definition of the comatrix of cofactors. Maybe in a future version, we will find a mathematical expression for this coefficient.

In fact, the inverse of the usual binomial triangle also exists but it is not interesting to study it since it is the same as the binomial triangle with an additional minus.