Inflation sans Singularity in “Standard” Transformed FLRW

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Abstract

The calculations of Oppenheimer and Snyder showed that quasi-Newtonian cycloidal metric and energy density singularities in the behavior of an initially stationary uniform dust ball in “comoving” coordinates fail to carry over to “standard” coordinates, where that contracting dust ball at no finite time attains a radius (quite) as small as its Schwarzschild radius. This physical behavior disparity reflects the singular nature of the “comoving” to “standard” transformation, whose cause is that “comoving time” requires the clocks of an infinite number of different observers, making that “time” inherently physically unobservable. Notwithstanding the warning implicit in the Oppenheimer-Snyder example, checking other “comoving” dust ball results by transforming them to physically reliable coordinates is seldom emulated. We here consider the analytically simplest case of a dust ball whose energy density always decreases; its “comoving” result has a well-known singularity at a sufficiently early time. But after transformation to “standard” coordinates, that singularity no longer occurs at any finite time, nor is this expanding dust ball at any finite time (quite) as small as its Schwarzschild radius. But this dust ball’s expansion rate peaks at a substantial fraction of the speed of light when its radius equals a few times the Schwarzchild value, and the “standard” time when this inflationary expansion peak occurs is roughly equal to the “comoving” time of the “occurrence” of the unphysical “comoving” singularity.

Introduction

In “comoving coordinates” the definition of “time” requires the clocks *of an infinite number of different observers* [1], which obviously renders “comoving time” *physically unobservable*.

The salient *consequence* of physically unobservable “comoving time” for “comoving-coordinate” metric tensors, such as, for example, the spherically-symmetric general form [2],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2), \tag{1}$$

is that they *all* conform to the condition [3],

$$g_{00} = 1,$$

which is *inconsistent* with two general physical properties of g_{00}, namely that in the static weak-field limit $(g_{00} - 1)/2$ is the Newtonian gravitational potential ϕ [4], and that in the static limit $(g_{00})^{-\frac{1}{2}}$ is the gravitational time-dilation factor [5].

It is clear from the foregoing discussion that “results” presented *exclusively* in terms of “comoving coordinates” *cannot* be assumed to be physically interpretable. For example, an Oppenheimer-Snyder finite-radius ball of dust with a *stationary* initial uniform finite energy density that is treated in “comoving coordinates” will cycloidally transit through a state whose metric tensor is *singular* and whose energy density is *infinite* [6], but that dust-ball system at *no* finite time manifests metric or energy-density singularities *after* Oppenheimer and Snyder’s transformation of its “comoving coordinate” metric solution to “standard” coordinates [7]. The mechanism which causes this *disparity* is that the O-S transformation maps an infinite-length interval of *physically unobservable* “comoving time” to *infinite* “standard” time.

Similarly, a finite-radius ball of dust with a *decreasing* initial uniform finite energy density that is treated in “comoving coordinates” will *invariably* transit *at a sufficiently early time* through a state whose metric tensor is singular and whose energy density is infinite [8]. But despite the *caveat* of the preceding paragraph that “results” presented exclusively in terms of “comoving coordinates” cannot be assumed to be physically interpretable, *precautionary* transformation of the “comoving-coordinate” results for dust balls with decreasing initial uniform finite energy density to another coordinate system which *doesn’t* have g_{00} fixed to unity *is seldom carried out*. In conjunction with this incautious omission the *early singularity* in the “comoving” result for dust with *decreasing* initial uniform finite energy density is typically *assumed to be physical* [8]; for example, Steven Weinberg declares that [9], “. . . the time elapsed since this singularity . . . may justly be called the age of the universe.”

Such a bold pronouncement of course inattentively ignores the *physically unobservable* character of “comoving time”. Furthermore, it *wouldn’t* be expected that *mathematical singularities* could play a role in *classical* theoretical physics; indeed a *physical* metric tensor in the context of GR that is *singular* would be inconsistent with the combination of the Principle of Equivalence and the tensor contraction theorem [10].

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To show this we note that the proof of the tensor contraction theorem for a given space-time transformation $\bar{x}^{\alpha}(x^{\mu})$ (and its inverse transformation $x^{\nu}(\bar{x}^{\alpha})$) hinges on the validity of the relationship [11],

$$(\partial\bar{x}^{\alpha}/\partial x^{\mu})(\partial x^{\nu}/\partial\bar{x}^{\alpha}) = \delta^{\nu}_{\mu}, \tag{2}$$

which, if each component of the Jacobian matrix $\partial\bar{x}^{\alpha}/\partial x^{\mu}$ is well-defined in terms of the finite real numbers at a given space-time point x^{μ}, and *also* each component of its inverse matrix is thus well-defined in terms of the finite real numbers, follows at that space-time point from the chain rule of the calculus. However, because the right-hand side δ^{ν}_{μ} of Eq. (2) is *always* well-defined in terms of the finite real numbers, Eq. (2) *becomes self-inconsistent* at any space-time point x^{μ} where any component of the Jacobian matrix $\partial\bar{x}^{\alpha}/\partial x^{\mu}$ or any component of its inverse matrix *fails to be a well-defined finite real number*. Because the tensor contraction theorem *is indispensable to GR* (e.g., the Einstein tensor is constructed from *contractions* of the Riemann curvature tensor), a space-time transformation can be considered *physical* in the context of GR *only* at space-time points where *each* component of its Jacobian matrix *is a well-defined finite real number* and the *same* is true of *each* component of the *inverse* of that Jacobian matrix.

The Principle of Equivalence implies that a metric tensor is at each space-time point the congruence transformation of the Minkowski metric tensor with the Jacobian matrix of some space-time transformation [12]. Therefore in light of the foregoing discussion of physical space-time transformations in the context of GR, a metric tensor can *only* be considered *physical* in the context of GR at those space-time points where both it and its inverse consist solely of components which are well-defined finite real numbers and its signature is equal to the $\{+,-,-,-\}$ signature of the Minkowski metric tensor [13].

Mathematical singularities thus have no place in space-time transformations or metric tensors which are *physical* in the context of GR; their occurrence in "comoving" metrics arises from the *physically unobservable* character of "comoving time", which of course renders "comoving" metrics *unphysical*. Relatedly, the singular non-bijective (and therefore *unphysical*) character of the space-time transformation of Oppenheimer and Snyder is the consequence of its being *a purely abstract mathematical bridge* between the "standard" and the manifestly *unphysical* "comoving" coordinates for the initially stationary finite-radius uniform dust ball system.

The next section of this article is concerned with carrying out in detail the incautiously neglected analogue of the transformation of Oppenheimer and Snyder to "standard" coordinates of the "comoving" coordinate metric result for the analytically simplest case of an initially *decreasing* energy density finite-radius uniform dust ball. Just as in the initially *stationary* energy density case which was transformed to "standard" coordinates by Oppenheimer and Snyder, it is found that any metric and energy-density *singularities* which are present in unphysical "comoving" coordinates are of course *absent* at any finite time in "standard" coordinates.

In "standard" time the "age" of the expanding dust-ball "universe" is in fact infinite, precisely as the "the time of contraction" of an Oppenheimer-Snyder dust-ball system is infinite [7], and the "size" of the expanding dust-ball "universe" is at no finite time (quite) as small as its Schwarzschild radius, just as an Oppenheimer-Snyder dust-ball system never contracts to being (quite) as small as its Schwarzschild radius [7]. Also, just as there exists a particular time interval during which an Oppenheimer-Snyder dust-ball *contracts especially rapidly* [14], the expanding dust-ball "universe" correspondingly experiences *a peak expansion rate*, which is a substantial fraction of the speed of light, when its radius reaches a few times the Schwarzschild value, i.e., "inflation" is a natural attribute of an expanding dust ball. The "standard" time when that peak expansion rate occurs is approximately equal to the "comoving" time of the "occurrence" of the unphysical "comoving singularity."

Finally, although dust models don't per se incorporate any quantum effects, nor will we try to deal with those in this article, it is worth bearing in mind that a system's intrinsic uncertainty energy decreases when its size increases. Therefore an *expanding* system can be expected to experience a small nonclassical kinetic-energy "push" from the release of part of its store of uncertainty energy.

Uniform energy-density dust balls in "comoving" and "standard" coordinates

From the "comoving coordinate" Einstein-equation work of Friedmann, Lemaître (also Tolman), Robertson and Walker (FLRW), we have that the $0 \le r \le a$ interior region of a uniform energy-density dust ball of finite radius a is described by the spherically-symmetric metric of Eq. (1) with [15, 16],

$$V(r,t) = R^2(t)r^2, \tag{3a}$$

and,

$$U(r,t) = R^2(t)/(1 + \Omega(r^2/c^2)), \tag{3b}$$

where the dimensionless function $R(t)$ satisfies the initial condition,

$$R(t_0) = 1, \tag{3c}$$

and the first-order in time equation of motion,

$$(\dot{R}(t))^2 = (\omega^2/R(t)) + \Omega. \tag{3d}$$

Here the constant ω^2 reflects the universal gravitational constant G and the dust ball interior's initial uniform energy density $\rho(t_0)$, or, alternatively, its effective mass M and radius a,

$$\omega^2 \stackrel{\text{def}}{=} (8/3)\pi G\rho(t_0)/c^2 = 2GM/a^3, \tag{3e}$$

while the constant Ω, which occurs in *both* of Eqs. (3d) and (3b), is readily evaluated from Eq. (3d) in terms of ω^2 and the initial value of the time derivative of $R(t)$, namely,

$$\Omega = (\dot{R}(t_0))^2 - \omega^2. \tag{3f}$$

In *addition* to the relations given by Eqs. (3a) through (3f), which determine the "comoving" metric of Eq. (1) in the $0 \le r \le a$ *interior* region of the dust ball, we *also* have that in "comoving coordinates" the energy density $\rho(r,t)$ of the dust ball is given *everywhere* in space-time by,

$$\rho(r,t) = \begin{cases} \rho(t_0)/(R(t))^3 & \text{if } 0 \le r \le a, \\ 0 & \text{if } r > a. \end{cases} \tag{3g}$$

The fact that the region $r > a$ *is empty space* permits us to make use of the Birkhoff theorem *on that region's* $r = a$ *boundary* to *aid* us in working out the space-time transformation of the "comoving-coordinate" metric of Eqs. (1) and (3) to "standard" coordinates [17, 16].

It is notable that multiplying Eq. (3d) by the factor $ma^2/2$, where m is a test mass, *formally* makes it the *Newtonian* gravitational equation of that test mass' *purely radial motion* in its radius variable $r(t) = aR(t)$ due to *the point mass* M, namely, $m(\dot{r}(t))^2/2 - GmM/r(t) = m(\dot{r}(t_0))^2/2 - GmM/a$ with $r(t_0) = a$. This *formal equivalence* of "comoving" coordinate FLRW "GR physics" to *the entirely Newtonian* gravitational physics of a test mass *which is unconstrained from coming arbitrarily close to a point mass* again spotlights *the profoundly unphysical character of "comoving" coordinates*.

In the Oppenheimer-Snyder case, we have that the energy density $\rho(r,t)$ *is initially stationary*, i.e.,

$$\dot{\rho}(r,t_0) = 0, \tag{4a}$$

From Eq. (3g) we see that Eq. (4a) is assured if,

$$\dot{R}(t_0) = 0. \tag{4b}$$

Eqs. (4b) and (3f) imply that,

$$\Omega = -\omega^2, \tag{4c}$$

which *specializes* Eq. (3d) to the following *time-cycloidal* form [18],

$$(\dot{R}(t))^2 = \omega^2[(1/R(t)) - 1]. \tag{4d}$$

The nonnegative *continuous* (although not continuously differentiable) time-cycloidal $R(t)$ which satisfies Eq. (4d) and Eq. (3c) (namely $R(t_0) = 1$) is periodic with period π/ω, and *vanishes* at $t = t_0 + (n+1/2)(\pi/\omega)$, $n = 0, \pm1, \pm2, \ldots$ [16], with the consequence that the "comoving" metric described by Eqs. (1), (3a), (3b), (3c), (3e), (4c) and (4d) is periodically *singular* at those times, as is the energy density $\rho(r,t)$ of Eq. (3g).

In light of the inherent character of General Relativity (or *even* in light of the inherent character of classical theoretical physics in general) these periodic *singularities* are *artifacts* of the unphysical "comoving coordinate system" (specifically of *physically unobservable* "comoving time") in which the metric described by Eqs. (1), (3a), (3b), (3c), (3e), (4c) and (4d) is expressed; *indeed these singularities are completely absent* *after Oppenheimer and Snyder's (singular) space-time transformation of that metric from "comoving" to "standard" coordinates* [16].

Instead of the initially *stationary* energy density of Oppenheimer and Snyder, we here are interested in initially *decreasing* energy density $\rho(r,t)$. We see from Eq. (3g) that such initial decrease in $\rho(r,t)$ is assured if $R(t)$ *increases* at the initial time t_0. The analytically *simplest* and most *convenient* way to ensure such increase in $R(t)$ at $t = t_0$ is to set $\dot{R}(t_0)$ equal to ω, which causes Eq. (3f) to become simply,

$$\Omega = 0, \tag{5a}$$

and which furthermore, in conjunction with the initial condition $R(t_0) = 1$ of Eq. (3c), *specializes* Eq. (3d) to,

$$\dot{R}(t) = \omega/(R(t))^{\frac{1}{2}}. \tag{5b}$$

The solution of Eq. (5b) which satisfies the $R(t_0) = 1$ initial condition is,

$$R(t) = (1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{5c}$$

which strictly increases from the initial time t_0 onward.

The $R(t)$ of Eq. (5c), however, *vanishes* at one particular time t_s, and t_s is *earlier* than the *initial* time t_0 for which $\dot{R}(t_0) = \omega$, namely,

$$t_s = t_0 - \tfrac{2}{3}\omega^{-1}. \tag{5d}$$

Therefore the "comoving" metric of Eqs. (1), (3a), (3b), (3e), (5a) and (5c) is *singular* at that *earlier* time t_s, as is the energy density $\rho(r,t)$ of Eq. (3g).

In light of the inherent character of General Relativity (or *even* in light of the inherent character of classical theoretical physics in general) this metric *singularity* at the earlier time t_s must be an *artifact* of the unphysical "comoving" coordinates in which the metric is expressed. To *check* that expectation, we now launch into the intricate and lengthy procedure needed to work out the space-time transformation of the metric of Eqs. (1), (3a), (3b), (3e), (5a) and (5c) from "comoving" to "standard" coordinates.

We need to transform the "comoving coordinates" (t, r, θ, ϕ), in terms of which the invariant line element ds^2 is given by Eq. (1), into "standard coordinates" [19] $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$, in terms of which the *same* invariant line element ds^2 is *also* given by,

$$\begin{aligned} ds^2 &= B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2((d\bar{\theta})^2 + (\sin\bar{\theta}d\bar{\phi})^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \tag{6a}$$

Inspection in Eq. (6a) *of the rightmost two terms* of the line element ds^2 in *both* its "standard" and its "comoving" form immediately reveals three very convenient transformation choices,

$$\bar{\theta} = \theta,\ \bar{\phi} = \phi \text{ and } \bar{r} = (V(r,t))^{\frac{1}{2}} = r(1 + \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{6b}$$

where we have evaluated $(V(r,t))^{\frac{1}{2}}$ by using Eqs. (3a) and (5c).

Next we would like to obtain $\bar{t}$ as a function of r and t, as has been done in Eq. (6b) for $\bar{r}$. Inspection of Eq. (6a), however, shows that that task is completely enmeshed with the determination of B and A as functions of r and t; moreover $\bar{t}(r,t)$ *itself* cannot be extracted from Eq. (6a), *only certain combinations of its partial derivatives* $(c(\partial\bar{t}/\partial r))$ *and* $(\partial\bar{t}/\partial t)$ *can*. We thus must solve *both* simultaneous algebraic *and* first-order partial differential equations to obtain $\bar{t}(r,t)$.

We now present in greater detail *that part of* Eq. (6a) *which isn't rendered redundant by the three transformation choices of* Eq. (6b), namely,

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{6c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (6c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{7a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \tag{7b}$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \tag{7c}$$

We now eliminate A and B from Eqs. (7) in order to obtain the partial differential equation for $\bar{t}$. Solving Eq. (7b) for A yields,

$$A = \frac{B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \tag{8a}$$

We now insert this value of A into each one of Eqs. (7a) and (7c), and follow that by solving each one for $(1/B)$,

$$(1/B) = \frac{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t)[(\partial\bar{r}/\partial r)(\partial\bar{t}/\partial t)-((1/c)(\partial\bar{r}/\partial t))(c(\partial\bar{t}/\partial r))]}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}, \tag{8b}$$

$$(1/B) = \frac{(1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))[(\partial\bar{r}/\partial r)(\partial\bar{t}/\partial t)-((1/c)(\partial\bar{r}/\partial t))(c(\partial\bar{t}/\partial r))]}{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}. \tag{8c}$$

If the factor in square brackets which occurs in the numerators of the right-hand sides of *both* Eq. (8b) and Eq. (8c) were equal to *zero*, then $(1/B) = 0$. But since we do *not* want the metric function B to be *singular*, we must assume that this common factor is *nonzero*. With that assumption, equating the right-hand side of Eq. (8b) to that of Eq. (8c) produces the relation,

$$((1/c)(\partial\bar{r}/\partial t))(\partial\bar{t}/\partial t) = (1/U)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \tag{9a}$$

From Eqs. (3b), (5a) and (5c), we obtain that,

$$U(r,t) = (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{4}{3}}, \tag{9b}$$

and from Eq. (6b) we obtain both that,

$$(\partial\bar{r}/\partial r) = (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}, \tag{9c}$$

and that,

$$((1/c)(\partial\bar{r}/\partial t)) = (\omega r/c)/(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{1}{3}}. \tag{9d}$$

Putting Eqs. (9b), (9c) and (9d) into Eq. (9a) yields the following first-order linear partial differential equation for $\bar{t}(r,t)$,

$$(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{1}{3}}(\partial\bar{t}/\partial t) = (c^2/(\omega r))(\partial\bar{t}/\partial r), \tag{9e}$$

which is separable in r and t. Putting,

$$\bar{t}(r,t) = \omega^{-1}\alpha(r)\beta(t), \tag{10a}$$

into Eq. (9e) yields,

$$(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{1}{3}}(\dot{\beta}(t)/\beta(t)) = (c^2/(\omega r))(\alpha'(r)/\alpha(r)) = -\omega p, \tag{10b}$$

where p is an arbitrary dimensionless constant. Eq. (10b) is satisfied by,

$$\beta(t) = b_0(p)\exp(-p[(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}]), \tag{10c}$$

and,

$$\alpha(r) = a_0(p)\exp(-p[\tfrac{1}{2}(\omega r/c)^2]), \tag{10d}$$

where $b_0(p)$ and $a_0(p)$ are arbitrary dimensionless constants that can vary with p.

Because the Eq. (9e) partial differential equation is *linear*, any linear combination of its solutions will be a solution as well. Combining this linear superposition property of the solutions of Eq. (9e) with Eqs. (10a), (10c) and (10d), we see that the general solution of Eq. (9e) will be of the form,

$$\bar{t}(r,t) = \omega^{-1}\int dp\, a_0(p)b_0(p)\exp(-p[\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}]). \tag{10e}$$

Eq. (10e) tells us that $\bar{t}(r,t)$ is equal to ω^{-1} times the Laplace representation of a general dimensionless function of the dimensionless variable $(\frac{1}{2}(\omega r/c)^2 + (1 + \frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})$.

Therefore, given *any* differentiable dimensionless function $\phi(u)$ of the dimensionless variable u, the function $\bar{t}(r,t)$ which is given by,

$$\bar{t}(r,t) = \omega^{-1}\phi(\tfrac{1}{2}(\omega r/c)^2 + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}), \tag{10f}$$

satisfies the Eq. (9e) partial differential equation, and also has the dimensions of time. Note that it is straightforward to verify by direct substitution that the general form of $\bar{t}(r,t)$ which is specifically described by Eq. (10f) satisfies the Eq. (9e) partial differential equation.

With the Eq. (10f) form of $\bar{t}(r,t)$ in hand, we are now almost in a position to obtain the two "standard" metric functions B and A in terms of r and t from Eqs. (8), but first we need to calculate the two partial derivatives $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ from Eq. (10f),

$$(\partial\bar{t}/\partial t) = \phi'(\tfrac{1}{2}(\omega r/c)^2 + (1+\tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}})/(1+\tfrac{3}{2}\omega(t-t_0))^{\frac{1}{3}}, \tag{11a}$$

$$(c(\partial\bar{t}/\partial r)) = (\omega r/c)\phi'(\tfrac{1}{2}(\omega r/c)^2 + (1+\tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}). \tag{11b}$$

We can now substitute Eqs. (11a) and (11b), along with Eqs. (9c) and (9d), into Eq. (8a), with the result,

$$A(r,t) = \frac{B(r,t)[\phi'(\frac{1}{2}(\omega r/c)^2+(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})]^2}{(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}. \tag{12a}$$

We can make the same four substitutions into Eq. (8b) (or, with exactly the same effect, make those same four substitutions together with the substitution of Eq. (9b), into Eq. (8c)), with the result,

$$B(r,t) = \frac{(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}{[\phi'(\frac{1}{2}(\omega r/c)^2+(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})]^2[1-(\omega r/c)^2/(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}]}. \tag{12b}$$

The substitution of Eq. (12b) into Eq. (12a) then yields the "standard" metric function $A(r,t)$,

$$A(r,t) = \frac{1}{1-(\omega r/c)^2/(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}}. \tag{12c}$$

We note that while in Eq. (12b) $B(r,t)$ is expressed in terms of a function $\phi'(\frac{1}{2}(\omega r/c)^2+(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})$ which has not yet been determined, the expression in Eq. (12c) for $A(r,t)$ is the finished product. This expression is valid *only* for $0 \le r \le a$, where dust of uniform energy density $\rho(t) = \rho(t_0)/(R(t))^3$ *is present* (and where, of course, $R(t) = (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$). However, we *also* know that the region $r > a$ *is empty space*, so we *in addition* expect the Birkhoff theorem to apply on the $r = a$ *boundary* of that empty-space region. In order to *check* whether that is in fact the case for the $A(r,t)$ of Eq. (12c), we first eliminate the "comoving time coordinate" t from $A(r,t)$ in favor of the "standard radial coordinate" $\bar{r} = r(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$ (see Eq. (6b)), then we *fix* r to its *boundary value* a, and then we *compare* A as a function of $\bar{r}$ with the static Schwarzschild "standard form" of this metric component for a system of effective mass $M = (4/3)\pi a^3\rho(t_0)/c^2$. For the $A(r,t)$ of Eq. (12c), this prescription yields,

$$A(r=a,\bar{r}) = \frac{1}{1-((a^3\omega^2)/(c^2\bar{r}))}, \tag{12d}$$

and from Eq. (3e),

$$\omega^2 = (8/3)\pi G\rho(t_0)/c^2 = 2GM/a^3, \tag{12e}$$

so that Eq. (12d) reads,

$$A(r=a,\bar{r}) = \frac{1}{1-((2GM)/(c^2\bar{r}))}, \tag{12f}$$

which is indeed the Schwarzschild "standard form" of this metric component.

The *same* approach permits one *to pin down* the not yet determined function $\phi'(u)$ which appears in the Eq. (12b) expression for the metric component $B(r,t)$. On one hand we of course require $B(r=a,\bar{r})$ to have its static Schwarzschild "standard form", namely,

$$B(r=a,\bar{r}) = 1-((2GM)/(c^2\bar{r})) = 1-((a^3\omega^2)/(c^2\bar{r})) = 1-((\omega a/c)^2/(\bar{r}/a)), \tag{13a}$$

where we have used Eq. (12e). On the other hand, applying the relation,

$$(1+\tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}} = \bar{r}/r, \tag{13b}$$

which follows from Eq. (6b), to the Eq. (12b) expression for $B(r,t)$ permits one to obtain the result that,

$$B(r=a,\bar{r}) = \frac{(\bar{r}/a)}{[\phi'(\frac{1}{2}(\omega a/c)^2+(\bar{r}/a))]^2[1-((\omega a/c)^2/(\bar{r}/a))]}. \tag{13c}$$

Equating the right-hand side of Eq. (13c) to the expression which appears after the last equal sign on the right-hand side of Eq. (13a) then permits one to solve for $\phi'(\frac{1}{2}(\omega a/c)^2 + (\bar{r}/a))$,

$$\phi'(\tfrac{1}{2}(\omega a/c)^2 + (\bar{r}/a)) = \tfrac{(\bar{r}/a)^{\frac{3}{2}}}{(\bar{r}/a)-(\omega a/c)^2}. \tag{13d}$$

We now obtain $\phi'(u)$ by setting $(\bar{r}/a)$ to $u - \frac{1}{2}(\omega a/c)^2$ in both sides of Eq. (13d),

$$\phi'(u) = \tfrac{(u-\frac{1}{2}(\omega a/c)^2)^{\frac{3}{2}}}{u-\frac{3}{2}(\omega a/c)^2} \tag{13e}$$

In order to obtain $\bar{t}(r,t) = \omega^{-1}\phi(\frac{1}{2}(\omega r/c)^2 + (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}})$ as per Eq. (10f), we must *integrate* the specific $\phi'(u)$ which we have obtained in Eq. (13e),

$$\phi(u) = \phi(u_0) + \textstyle\int_{u_0}^{u} du'\, \tfrac{(u'-\frac{1}{2}(\omega a/c)^2)^{\frac{3}{2}}}{u'-\frac{3}{2}(\omega a/c)^2} = \phi(u_0) + \textstyle\int_{u_0-\frac{1}{2}(\omega a/c)^2}^{u-\frac{1}{2}(\omega a/c)^2} ds\, \tfrac{s^{\frac{3}{2}}}{s-(\omega a/c)^2}. \tag{14}$$

To conveniently have the lower limit of the integration over the variable s in Eq. (14) be equal to unity, we choose u_0 equal to $1 + \frac{1}{2}(\omega a/c)^2$. With that choice of u_0 we use the $\phi(u)$ of Eq. (14) to implement Eq. (10f) for $\bar{t}(r,t)$,

$$\bar{t}(r,t) = \omega^{-1}\phi(\tfrac{1}{2}(\omega r/c)^2 + (1+\tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}) = \bar{t}(r=a,t=t_0) + \omega^{-1}\textstyle\int_1^{S(r,t)} ds\, \tfrac{s^{\frac{3}{2}}}{s-(\omega a/c)^2}, \tag{15}$$

where $S(r,t) \stackrel{\text{def}}{=} \frac{1}{2}(\omega/c)^2(r^2-a^2) + (1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$. (Note that $\omega^{-1}\phi(1+\frac{1}{2}(\omega a/c)^2) = \bar{t}(r=a,t=t_0)$ because $S(r=a,t=t_0)=1$.)

The Eq. (15) expression for $\bar{t}(r,t)$ is, of course, only valid for $0 \le r \le a$, i.e., within the dust ball. The *most crucial feature* of $\bar{t}(r,t)$ is that it *diverges* when $0 \le r \le a$ and $S(r,t) \le (\omega a/c)^2 = (2GM)/(c^2 a)$, where the last equality follows from Eq. (12e).

In particular, when $t = t_s = t_0 - \frac{2}{3}\omega^{-1}$, we see that for $0 \le r \le a$, $S(r,t_s) = \frac{1}{2}(\omega/c)^2(r^2-a^2) \le 0$, and therefore $\bar{t}(r,t_s)$ *diverges* for $0 \le r \le a$. The "comoving coordinate singularity" at $t = t_s = t_0 - \frac{2}{3}\omega^{-1}$ is therefore *simply never reached* anywhere within the dust ball, as it would require *the infinite time* $\bar{t}(r,t_s)$ in "standard" coordinates to do so. In terms of the colorful language invoked by Steven Weinberg [9], the "age of the universe" *is infinite*; the dust-ball metric singularity at the "comoving time" $t_s = t_0 - \frac{2}{3}\omega^{-1}$ *is only an artifact* of the use of the *unphysical* "comoving coordinates", *in exactly the same way* that the Oppenheimer-Snyder periodic dust-ball metric singularities are all *likewise* only *artifacts* of the use of the *unphysical* "comoving coordinates" [16]. Of course *the physical nonexistence of metric singularities* is a readily demonstrated *general property* of General Relativity, as we have showed in detail in the Introduction. It would in any case overstretch physical credibility for *mathematical singularities* to actually play a *role* in classical theoretical physics.

It is as well of interest to look into what the requirement $S(r,t) > (\omega a/c)^2 = (2GM)/(c^2 a)$ implies for the $r = a$ outer surface of the dust ball in the "standard" coordinate system (in the unphysical "comoving" coordinate system every dust particle has zero three-velocity [20, 16], so the radius a of the dust ball's outer surface *never changes* in unphysical "comoving" coordinates; *only* the magnitude of the dust ball's uniform *energy density* can change with "comoving time" in unphysical "comoving coordinates"). If we express $S(r,t)$ in terms of the "standard" radial coordinate $\bar{r} = r(1+\frac{3}{2}\omega(t-t_0))^{\frac{2}{3}}$ in place of the "comoving" time t, we obtain $S(r,\bar{r}) = \frac{1}{2}(\omega/c)^2(r^2-a^2) + \bar{r}/r$. Specializing to the dust ball's outer surface $r = a$, we obtain $S(r=a,\bar{r}) = \bar{r}/a > (2GM)/(c^2 a)$, which implies that $\bar{r} > (2GM)/c^2$, namely the radius in "standard" coordinates of the dust ball's outer surface *must always exceed* that dust ball's Schwarzschild radius $(2GM)/c^2$—the dust ball's outer surface *actually attaining* its Schwarzschild radius is *impossible* in "standard" coordinates because that would require an *infinite* "standard" time, namely $\bar{t}(r=a,\bar{r})$ *diverges* when $\bar{r}/a = S(r=a,\bar{r}) \le (\omega a/c)^2 = (2GM)/(c^2 a)$. The result here that the radius in "standard" coordinates of the dust ball's outer surface *must always exceed* its Schwarzschild radius *is identical* to what is found in the Oppenheimer-Snyder case [16, 7], and it is *also*, of course, merely *another* instance of *the physical nonexistence of metric singularities* in General Relativity, which is proved in detail in the Introduction (not that it could even be physically credible for mathematical singularities to actually play a role in *any* branch of classical theoretical physics). In terms of the colorful language invoked by Steven Weinberg [9], *not only* is the "age of the universe" *infinite*, also the "size of the universe" could *never* in the past *have been as small as its Schwarzschild radius*.

Finally, just as in the Oppenheimer-Snyder case, analytic evaluation of the integral expression for $\bar{t}(r,t)$ (given here by Eq. (15)) can be carried out *in the region where it doesn't diverge*, namely for $S(r,t) > (\omega a/c)^2 = (2GM)/(c^2 a)$, and it is furthermore only valid when $0 \le r \le a$, i.e., within the dust ball. One must take care *not* to inadvertently *mentally analytically continue* that analytic result into the region $S(r,t) \le (\omega a/c)^2 = (2GM)/(c^2 a)$ *where* $\bar{t}(r,t)$ *diverges*.

To simplify notation during and after evaluation of the integral, we reexpress Eq. (15) in the streamlined form,

$$\omega(\bar{t}_\alpha(S) - \bar{t}_\alpha(S=1)) = \int_1^S ds\, s^{\frac{3}{2}}/(s-\alpha), \tag{16a}$$

where,

$$S \stackrel{\text{def}}{=} S(r,t) = \tfrac{1}{2}(\omega/c)^2(r^2 - a^2) + (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}, \tag{16b}$$

and,

$$\alpha \stackrel{\text{def}}{=} (\omega a/c)^2 = (2GM)/(c^2 a). \tag{16c}$$

Note that we consider *only* the case where $S > \alpha$; the integral on the right-hand side of Eq. (16a) *diverges* when $S \le \alpha$. Also, Eqs. (16a) through (16c) only apply when $0 \le r \le a$, namely within the dust ball. It may be useful to note that the ω which appears on the left-hand side of Eq. (16a), and in the expression for $S(r,t)$ in Eq. (16b), is related to α via $\omega = (c/a)\alpha^{\frac{1}{2}}$, and it may also be useful to note that $S(r=a, t=t_0) = 1$.

Evaluation of the integral on the right-hand side of Eq. (16a) requires only the simple change of variable $s = v^2$, followed by some mildly tedious algebra,

$$\int_1^S ds\, s^{\frac{3}{2}}/(s-\alpha) = 2\int_1^{S^{\frac{1}{2}}} dv\, v^4/(v^2-\alpha). \tag{16d}$$

Next we note that,

$$2v^4/(v^2-\alpha) = 2[(v^2-\alpha)+\alpha]^2/(v^2-\alpha) = 2v^2 + 2\alpha + \alpha^{\frac{3}{2}}[(1/(v-\alpha^{\frac{1}{2}})) - (1/(v+\alpha^{\frac{1}{2}}))].$$

We now need only carry out four elementary integrations to obtain the result,

$$\bar{t}_\alpha(S) = \bar{t}_\alpha(S=1) + \omega^{-1}\Big\{\tfrac{2}{3}(S^{\frac{3}{2}} - 1) + 2\alpha(S^{\frac{1}{2}} - 1) + \alpha^{\frac{3}{2}}\ln\Big[\tfrac{(S^{\frac{1}{2}}-\alpha^{\frac{1}{2}})(1+\alpha^{\frac{1}{2}})}{(1-\alpha^{\frac{1}{2}})(S^{\frac{1}{2}}+\alpha^{\frac{1}{2}})}\Big]\Big\}, \tag{16e}$$

and we reiterate that Eq. (16e) is valid *only* when $S > \alpha$, and also that $\bar{t}_\alpha(S)$ *diverges* when $S \le \alpha$.

Dynamical behavior of the dust ball's surface in "standard" coordinates

By using Eq. (16e) for $\bar{t}(r,t) = \bar{t}_\alpha(S(r,t))$ one can obtain in closed form the dependence of the local "standard" time at the surface of the dust ball, namely $\bar{t}_a(t) \stackrel{\text{def}}{=} \bar{t}(a,t) = \bar{t}_\alpha(S(a,t))$, on the dynamically increasing "standard" radial coordinate of that surface, which is,

$$\bar{r}_a(t) \stackrel{\text{def}}{=} \bar{r}(a,t) = a(1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}}, \tag{17a}$$

where the rightmost equality in Eq. (17a) follows from Eq. (6b). The *reason* that a closed-form result for $\bar{t}_a(\bar{r}_a)$ can be obtained from Eq. (16e) is that Eq. (16b) implies,

$$S(a,t) = (1 + \tfrac{3}{2}\omega(t-t_0))^{\frac{2}{3}} = \bar{r}_a(t)/a. \tag{17b}$$

and therefore,

$$\bar{t}_a(t) = \bar{t}_\alpha(\bar{r}_a(t)/a). \tag{17c}$$

From Eq. (16c) we note that the entity α is related to the dust ball's Schwarzschild radius r_s by,

$$\alpha = r_s/a, \tag{17d}$$

where,

$$r_s \stackrel{\text{def}}{=} 2GM/c^2 = \omega^2 a^3/c^2. \tag{17e}$$

We now use Eqs. (16e), (17c) and (17d) to obtain,

$$\bar{t}_a(\bar{r}_a) = \bar{t}_a(\bar{r}_a = a) + \omega^{-1}\Big\{\tfrac{2}{3}((\bar{r}_a/a)^{\frac{3}{2}} - 1) + 2(r_s/a)((\bar{r}_a/a)^{\frac{1}{2}} - 1) + (r_s/a)^{\frac{3}{2}}\ln\Big[\tfrac{((\bar{r}_a/a)^{\frac{1}{2}} - (r_s/a)^{\frac{1}{2}})(1+(r_s/a)^{\frac{1}{2}})}{(1-(r_s/a)^{\frac{1}{2}})((\bar{r}_a/a)^{\frac{1}{2}} + (r_s/a)^{\frac{1}{2}})}\Big]\Big\} \text{ for } \bar{r}_a > r_s. \tag{17f}$$

Note that because the $\bar{t}_\alpha(S)$ of Eq. (16e) *diverges* for $S \leq \alpha$, $\bar{t}_a(\bar{r}_a)$ *also diverges* for $\bar{r}_a \leq r_s$. That the $\bar{t}_\alpha(S = 1)$ in Eq. (16e) equals the $\bar{t}_a(\bar{r}_a = a)$ in Eq. (17f) follows from $S(a, t_0) = 1$ (see Eq. (17b)) and $\bar{r}_a(t_0) = a$ (see Eq. (17a)).

In the $c \to \infty$ *nonrelativistic limit*, $r_s \to 0$ (see Eq. (17e)), so from Eq. (17f), $(\bar{r}_a(\bar{t}_a)/a) = (1 + \frac{3}{2}\omega(\bar{t}_a - \bar{t}_a(\bar{r}_a = a)))^{\frac{2}{3}}$ when $c \to \infty$; i.e., *the unphysical "comoving"* Eq. (5c) *emerges from* Eq. (17f) *when* $c \to \infty$. We *can't* obtain $\bar{r}_a(\bar{t}_a)$ *in closed form* from the *full* "standard" Eq. (17f), but we *can* derive key facts about $d\bar{r}_a/d\bar{t}_a$ via (mildly tedious) *exact* calculation of $d\bar{t}_a/d\bar{r}_a$ from Eq. (17f),

$$d\bar{t}_a/d\bar{r}_a = (1/c)(\bar{r}_a/r_s)^{\frac{3}{2}}/((\bar{r}_a/r_s) - 1) \quad \text{for } \bar{r}_a > r_s, \tag{18a}$$

where we have used $\omega = c(r_s/a)^{\frac{1}{2}}/a$ (see Eq. (17e)). Note that $d\bar{t}_a/d\bar{r}_a$ has no physical meaning for $\bar{r}_a \leq r_s$ because $\bar{t}_a(\bar{r}_a)$ *diverges* at those values of $\bar{r}_a$. Eq. (18a) implies that the $\bar{t}_a(\bar{r}_a)$ of Eq. (17f) *is strictly increasing* for $\bar{r}_a > r_s$, and that $d\bar{r}_a/d\bar{t}_a$ as a function of $\bar{r}_a$ is given by,

$$d\bar{r}_a/d\bar{t}_a = c(r_s/\bar{r}_a)^{\frac{1}{2}}(1 - (r_s/\bar{r}_a)) \quad \text{for } \bar{r}_a > r_s. \tag{18b}$$

Of course $d\bar{r}_a/d\bar{t}_a$ has no physical meaning for $\bar{r}_a \leq r_s$, where $\bar{t}_a(\bar{r}_a)$ *diverges*. As $c \to \infty$, Eq. (18b) becomes $d(\bar{r}_a/a)/d\bar{t}_a = \omega/(\bar{r}_a/a)^{\frac{1}{2}}$, *the unphysical "comoving"* Eq. (5b). The *full* Eq. (18b) is easily shown to imply that $d\bar{r}_a/d\bar{t}_a$, the dust-ball surface's radial expansion rate, *peaks* at $\bar{r}_a = 3r_s$, where its value is a substantial fraction of c,

$$d\bar{r}_a/d\bar{t}_a\Big|_{\bar{r}_a = 3r_s} = 2c/\sqrt{27} = 0.3849c. \tag{18c}$$

Putting $\bar{r}_a = 3r_s$ into the right-hand side of Eq. (17f), we obtain that the local "standard" *time* when the dust-ball surface's radial expansion rate peaks is approximately,

$$\bar{t}_a(\bar{r}_a = 3r_s) \approx \bar{t}_a(\bar{r}_a = a) - \tfrac{2}{3}\omega^{-1}, \tag{18d}$$

with an error of the form $(1/\omega)O(r_s/a)$. Eq. (18d) *formally* approximately corresponds to the unphysical "comoving" time $t_s = t_0 - \frac{2}{3}\omega^{-1}$ of Eq. (5d) for the "occurrence" of the unphysical "comoving" *singularity*.

In précis, when the analytically simplest FLRW expanding dust-ball solution in *unphysical* "comoving" coordinates is transformed à la Oppenheimer and Snyder to "standard" coordinates, its *unphysical* "comoving" singularity *doesn't occur at any finite "standard" time*; instead, at approximately *the formally corresponding* local "standard" time, *an inflationary peak in the dust-ball surface's radial expansion rate* occurs, at which stage the dust ball's surface radius is a few times its Schwarzschild radius and is growing at a substantial fraction of c.

References

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[4] S. Weinberg, op. cit., Eq. (3.4.5), p. 78.

[5] S. Weinberg, op. cit., Eq. (3.5.2), p. 79.

[6] S. Weinberg, op. cit., Eqs. (11.9.25) and (11.9.26), p. 345.

[7] S. Weinberg, op. cit., Eq. (11.9.40), p. 347.

[8] S. Weinberg, op. cit., text leading up to Eq. (15.1.24), p. 473.

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[14] S. Weinberg, op. cit., paragraph below Eq. (11.9.46), p. 348.

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[19] S. Weinberg, op. cit., Eqs. (11.7.1), p. 336.

[20] S. Weinberg, op. cit., Eq. (11.8.9), p. 340 and Eq. (11.9.3), p. 342.