Running Coupling Constant Made Simple

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Abstract – Running coupling constants both of QED and QCD are studied in a heuristic way. We estimate two branches of the running coupling of the QCD, the first going from low to moderate energies, and the second running at high energies. The intercept of the high energy branch of QCD with the high energy curve of the QED-coupling is used, as a means to estimate the Grand Unification Theory (GUT) scale.

1 – Introduction

It is a consequence of the Maxwell's equations that the electromagnetic wave propagates in vacuum with a speed given by the inverse of the product of the electric permittivity and the magnetic permeability of vacuum. In classical electromagnetism, all these quantities are constants. However the quantum field theory (QFT) considers that the quantum vacuum behaves as an active medium, whose physical properties depends on the energy (wavelength) of the instrument used to probe it.

As asked by Kane [K]: "Why is the electromagnetic interaction due to a massless spin-one particle, the photon, being exchanged between charged objects?" Yet, as was pointed out by Kane [K], answer to these questions was done, at least partially, within the framework of the gauge theories. These are theories where the interaction is determined, taken in account some internal symmetry. Because of these symmetries the theory is invariant under some local transformations.

In this paper we will work out two of these theories, namely the Quantum Electrodynamics (QED) and the Quantum Chromodynamics (QCD), in order to describe the dependence of their coupling constants on the energy scale used to probe them. Besides this we will intend to unify the description of both theories with respect to the dependence of their coupling constants on the energy scale. We also will look at the energy scale where the strength of these two interactions becomes of equal magnitude (the electroweak and strong interactions unification scale).

2- Running Coupling constant of the QED

The lagrangian of the quantum electrodynamics can be written as [K, M, H]

$$L = i\underline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - m\underline{\Psi}\Psi - \frac{1}{4} F^{\mu\upsilon} F_{\mu\upsilon} + ie\underline{\Psi}\gamma^{\mu}A_{\mu}\Psi, \qquad (1)$$

With

$$F^{\mu\upsilon} = \partial^{\mu}A^{\upsilon} - \partial^{\upsilon}A^{\mu} \qquad \text{and} \qquad \underline{\Psi} = \Psi^{+}\gamma^{0}. \qquad (2)$$

In (1) Ψ is a fermionic field, e and m are respectively electron charge and rest mass, A_{μ} is the four-vector electromagnetic potential and γ^{μ} are the Dirac's γ-matrices. We observe that the gauge invariance of (1) is warranted by the coupling between the fermionic field and the vector potential.

In order to pursue further let us start from the energy density stored in a classical electric field E_{cl}. It is given by

$$u_{cl} = \frac{1}{2}\, \varepsilon_0\, E_{cl}^{2}. \qquad (3)$$

If we think in terms of a point charge, (3) takes the form

$$u_{cl} = \tfrac{1}{2}[1/(4\pi)](\alpha\hbar c)/r^4, \quad (4)$$

where α is the fine structure constant.

Meanwhile the demand for gauge invariance implies in a coupling between the fermionic field Ψ and the classical field E_{cl}. We can interpret this coupling as the the fermionic field imposing a quantum fluctuation over the classical field and write (being ξ a constant)

$$E = E_{cl} + \xi\,\Psi. \quad (5)$$

When the fermionic field is "turned off", the classical value of the electric field is recovered.

Now if we take the average (symbol <>) of the squared fields, we have

$$\langle E^2\rangle = \langle E_{cl}{}^2\rangle + \xi^2\,\langle\Psi^2\rangle. \quad (6)$$

The cross product of the two kinds of field averages out to zero due to the fluctuation character of the Ψ-field.

In a paper dealing with the critical behavior of the Ising model, C. J. Thompson [Th, N1] wrote an action as a means to study its critical behavior, supposed to be within the same universality class of the ϕ^4-theory. Inspired in Thompson's work [Th] it is possible to construct an action based on lagragian (1), by integrating it in four-volume (V_4) endowed with a spherical symmetry. One of the Thompson's prescriptions states that: by considering the integral of the lagrangian (1) in a coherence volume V_4, the modulus of each integrated term of it is separately equal to the unity. Next we use Thompson's recipe in order to study the running coupling constant in the QED. Applying this recipe to the first term of (1) we have

$$\left| \int i\, \underline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi \, dV_4 \right| = 1. \qquad (7)$$

Neglecting the spinorial character of the field Ψ, we can extract from (7) the average $<\underline{\Psi}\Psi>$, taking in account a 4-sphere of radius r. We get

$$<\underline{\Psi}\Psi>_r = [1/(2\pi^2)]\,(1/r^3). \qquad (8)$$

This result for the averaged squared quantum field is consistent with the calculated self energy of the electron, as discussed by Weisskopf [We].

From (60 a relation involving energy densities follows, namely

$$u = u_{cl} + u_q, \qquad (9)$$

where u_q stands for quantum contribution to the energy density, and using (8) and (4) it is possible to write

$$u_q = -\tfrac{1}{2}\,[1/(4\pi)]\,(\alpha\hbar c/r^4)\,(r/\lambda_C), \qquad (10)$$

where $\lambda_C \equiv \hbar/(mc)$is the reduced Compton wavelength of the electron. We observe that at $r = \lambda_C$, the two contributions for the energy density, namely classical and quantum contributions, have equal magnitude. The minus signal which appears in (10) is guided by the necessity of the current treatment to be consistent with the dielectric screening provoked by the presence of virtual electron-positron pairs in the quantum vacuum. These pairs behave as fluctuating electric dipoles.

Next we write an energy balance equation

$$c^2\,dm = u_q\,dV_3 = u_q\,4\pi r^2 dr. \qquad (11)$$

Inserting (10) into (11) leads to

$$dm/m = -\tfrac{1}{2}\,\alpha\,(dr/r) = \tfrac{1}{2}\,\alpha\,(d\mu/\mu). \qquad (12)$$

In the last equality of (12) we took in account that the energy scale μ goes as $1/r$.

At this point it is convenient to write the relation for the mass energy of the electron, namely

$$m\,c^2 = \alpha\hbar c/R, \qquad (13)$$

where it is supposed that the α-coupling also depends on the variable R.

Next we write

$$dm = (\partial m/\partial\alpha)\,d\alpha + (\partial m/\partial R)\,dR. \qquad (14)$$

Therefore using (13) we have

$$c^2\,dm = (\alpha\hbar c/R)\,(d\alpha/\alpha) - (\alpha\hbar c/R)\,(dR/R). \qquad (15)$$

It seems to be a plausible hypothesis to assume that

$$d\alpha / \alpha = - dR / R, \tag{16}$$

taking in account the screening properties of the vacuum filled with virtual electric dipoles. Using (13) and (16) into (15) yields

$$d\alpha / \alpha = ½ (dm / m). \tag{17}$$

Putting (12) into (17) we get

$$d\alpha / \alpha = ¼ \alpha (d\mu / \mu). \tag{18}$$

Finally we can rewrite (18) in the usual form

$$\mu (d\alpha / d\mu) = (¼) \alpha^2. \tag{19}$$

When the β-function of the QED_4 is calculated by computing all the Feynman

Diagrams present at one-loop level, it is found that the numerical coefficient multiplying the α^2-term in (19) is $(2 / 3\pi)$ instead of the ¼ obtained in the present work (please see references [N1,R]).

3 – Running coupling constant of the QCD

Quantum Chromodynamics (QCD) is the most fundamental theory of the strong interactions, where quarks endowed with color-charges interact through the exchange of gluons. This non-abelian theory exhibits an SU(3)internal symmetry (please see reference [Wil]).

The QCD lagrangian can be written in the form [H, M, N1, Wil]

$$L_{QCD} = \sum_j \underline{\Psi}_j(i\gamma_\mu D^\mu - m_j\Psi_j) - ¼\, G_{\mu\upsilon}{}^a G^{\mu\upsilon}{}_a, \qquad (20A)$$

Where

$$D^\mu = \partial^\mu + ½\, ig\, \lambda_a A^\mu{}_a, \qquad G^{\mu\upsilon}{}_a = \partial^\mu A^\upsilon{}_a - \partial^\upsilon A^\mu{}_a - g\, f_{abc} A^\mu{}_b A^\upsilon{}_c. \qquad (20B)$$

In (20), m_j and Ψ_j are the mass and the fermionic field of the quark of flavor j, A is the gluonic field, μ and υ are space-time indexes and a,b,c are color indexes.

As we are dealing with a non-abelian field theory, besides the fermionic corrections to the averaged color-electric field we will also have the bosonic corrections to it and we write

$$<E^2> = <E_{cl}{}^2> + <(\Delta E_F)^2> + <(\Delta E_B)^2>. \qquad (21)$$

The total energy density u, reads

$$u = u_{cl} + u_F + u_B. \qquad (22)$$

In (21), ΔE_F and ΔE_B stand for the fermionic and bosonic contributions to the color-electric field.

As in the QED case, we can apply Thompson's prescription [Th] to the kinetic term of the action constructed by using lagrangian (20). We have

$$\left| \int i\, \underline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi\, dV_4 \right| \sim 1 \qquad \Rightarrow \qquad <\underline{\Psi}_j\Psi_j> \propto (1/r^3). \qquad (23)$$

The coupling between the quark fields Ψj and the color potential $A^{\mu}{}_{a}$ warrants the gauge invariance of the theory. Pursuing further we write

$$<(\Delta E_F)^2> \propto <\underline{\Psi}_j\Psi_j> \propto (1/r^3). \qquad (24)$$

However the non-abelian character of the QCD implies in the interaction among the components of the gauge field. This leads to a quantum fluctuation contribution to the energy density of a bosonic kind (u_B). As will see, this contribution has its signal changed relative to that found in the fermionic contribution, the only present in the QED case.

Next we are going to examine the aspect of the bosonic contribution through an "auxiliary" scalar field (ϕ). To do this we write an action which lagrangian exhibits a non-linear term in the field-ϕ, mimicking in this way the non-abelian feature of the problem.

3A – Color Paramagnetism

Inspired in the work of Nielsen [Ni], please see also Moryasu [M], we write the action ($\mathcal{A}$) for the field-ϕ

$$\mathcal{A} = \int d^4r\, \{[(\mathbf{\nabla} - (e_S/c)\mathbf{A})\phi]^2 + (1/c^2)(\partial\phi/\partial t)^2 + ¼\, k\, \phi^4\}. \qquad (25)$$

In (25) e_S is the strong charge, and taking $\delta\mathcal{A} = 0$ we get

$$\nabla^2\phi - (1/c^2)\partial^2\phi/\partial t^2 - (e_S/c)(\mathbf{\nabla.A} + \mathbf{A.\nabla})\phi = (e_S{}^2/c^2)A^2\phi + k\, \phi^3. \qquad (26)$$

Next we define

$$\mathbf{A} = \mathbf{H} \times \mathbf{r}, \tag{27}$$

being **H** a fluctuating color-magnetic field, and performing averages we get

$$\langle A \rangle = 0 \qquad \text{and} \qquad \langle A^2 \rangle = H^2 \lambda^2. \tag{28}$$

Before pursuing further we must renormalize the k-coupling. We adopt the Thompson's recipe and write

$$\int_1^2 (k\phi^4) d^4r = 1 \qquad \Rightarrow \qquad \langle k \rangle \int_1^2 (1/r^4)(r^3 dr) = \langle k \rangle \ln(R_2/R_1) = 1. \tag{29}$$

Solving for $\langle k \rangle$, we find

$$\langle k \rangle = 1/\ln(R_2/R_1) = 1/\ln(E_1/E_2). \tag{30}$$

Taking the average of the differential equation (26), linear terms in $\langle A \rangle$ vanish, and making the requirement of a free wave propagation for the ϕ-field we obtain

$$\nabla^2\phi - (1/c^2)\partial^2\phi/\partial t^2 = (e_S{}^2/c^2)H^2\lambda^2\,\phi + \langle k \rangle\,\phi^3 = 0. \tag{31}$$

Relations (30) and (31) imply

$$<\phi^2> = - (e_S{}^2/c^2)H^2\lambda^2 \ln(E_1/E_2). \quad (32)$$

A energy density may be associated to $<\phi^2>$ as follows

$$E_\phi = \int_V(\xi<\phi^2>)dV_3 = \xi<\phi^2>V = - (e_S{}^2/c^2)H^2V \ln(E_1/E_2). \quad (33)$$

On the other hand, the paramagnetism of a system in a color-magnetic field permit us to write

$$E_{paramag} = - HMV = - \chi H^2V, \quad (34)$$

where χ is the color-magnetic susceptibility, and M is the color-magnetization.

Comparing (33) and (34) we find

$$\chi = (e_S{}^2/c^2) \ln(E_1/E_2). \quad (35)$$

Besides this the color-magnetic permeability is given by

$$\mu = 1 + 4\pi\chi = 1 + 4\pi(e_S{}^2/c^2)\ln(E_1/E_2). \quad (36)$$

Taking in account that

$$\mu\varepsilon = c^2 = 1, \quad (37)$$

we obtain the relation for the color-electric permittivity, namely

$$\varepsilon = 1 / [1 + 4\pi(e_S{}^2 / c^2)\ln(E_1 / E_2)], \quad E_1 > E_2. \tag{38}$$

In (38) we have a single "mode" contribution from the bosonic field. By considering

$$\alpha_S = \alpha_{S0} / \varepsilon, \tag{39}$$

we have

$$\alpha_S(E_1) / \alpha_S(E_2) = 1 / [1 + 4\pi(e_S{}^2 / c^2)\ln(E_1 / E_2)]. \tag{40}$$

In the limit $E_1 \rightarrow \infty$, $\alpha_S(E_1) \rightarrow 0$,we verify the occurrence of the asymptotic freedom. However, here we have not yet considered the effect of the fermionic fields which leads the coupling to grow with increasing energies. The competition between these two fields will be treated in the following.

3B – QCD running coupling

Now we turn to the estimations of the quantum contribution for the energy density. We write

$$u_q = u_F + u_B = ½[1 / (4\pi)]\alpha_S \hbar c(1 / r^4)(r / \lambda_C). \tag{41}$$

We observe that at $r = \lambda_C$, we have $u_q = u_{classic}$. Pursuing further we have

$$u_q dV_3 = ½(\alpha_S \hbar c / \lambda_C)(dr / r) = - ½(\alpha_S \hbar c / \lambda_C)(d\mu / \mu). \tag{42}$$

Performing the integration of (42) we get

$$\Delta m\, c^2 = -\tfrac{1}{2} \int_{\mu 0}^{\mu} (\alpha_S\, m\, c^2)(d\mu / \mu), \qquad \text{where} \ \ \lambda_C = \hbar / (mc). \qquad (43)$$

We extract from the integrand the averages of α_S and m and we obtain

$$\Delta m = -\tfrac{1}{2} \ \langle\alpha_S\rangle \langle m\rangle \int_{\mu 0}^{\mu} (d\mu / \mu). \qquad (44)$$

Next we make a very plausible proposition, namely

$$\Delta\alpha_S / \langle\alpha_S\rangle = \Delta m / \langle m\rangle. \qquad (45)$$

Relations (44) and (45) yield

$$\Delta\alpha_S / \langle\alpha_S\rangle = -\tfrac{1}{2} \langle\alpha_S\rangle \ln(\mu / \mu_0). \qquad (46)$$

Putting $\Delta\alpha_S = \alpha_S - \alpha_{S0}$ in (46), we obtain

$$\alpha_S = \alpha_{S0} - \tfrac{1}{2} (\langle\alpha_S\rangle)^2 \ln(\mu / \mu_0). \qquad (47)$$

Now we define

$$(\langle\alpha_S\rangle)^2 = \alpha_S\,\alpha_{ref}\,, \qquad (48)$$

where α_{ref} stands for reference α_S-coupling. Finally using (48) in (47) we get

$$\alpha_S = \alpha_{S0} / [1 + ½\alpha_{ref} \ln(\mu / \mu_0)]. \qquad (49)$$

Relation (49) satisfies the differential equation

$$\mu d\alpha_S / d\mu = -\,½(\alpha_{ref} / \alpha_{S0})\, \alpha_S^2. \qquad (50)$$

It is interesting to do an analysis of the signal of the term $½(\alpha_{ref} / \alpha_{S0})$ of the equation (50). As discussed before, in QCD there is a competition between the bosonic and the fermionic fields and we can write

$$\alpha_{ref} / (2\alpha_{S0}) = (3N - 2n_f) / Q. \qquad (51)$$

In (51), N is the number of the bosonic contributions and 3 is the degeneracy number of a spin-1field (2s+1=3), and n_f is the number of fermionic fields (2s+1=2).

Q is a number to be determined. In the QED, we have N=0 and n_f=1, and by considering the β-function at the one-loop level, we find Q = 3π. Therefore we have

$$\alpha_{ref} / (2\alpha_{S0}) = (3N - 2n_f) / (3\pi). \qquad (52)$$

At lower and moderate energies it is convenient to do an alternative analysis.

Looking at relation (48) we observe that if $\alpha_S<1$, the maximum value of α_{ref} which is consistent with this equation is $\alpha_{ref} = 1$. We take as reference energy that of the quark constituent mass, equal to one-third of the nucleon mass. Doing this eq. (49) assumes the form

$$\alpha_S = 0.41922/[1 + 0.5 \ln(\mu/\mu_0)]. \quad (53)$$

Table 1 displays some values for the running coupling of the QCD as function of the energy, according eq. (53).

Energy μ(GeV)	QCD running coupling α_S
0.313	1
1	0.41922
2	0.311
3	0.271
4	0.248
5	0.232
10	0.195
15	0.178
20	0.168
30	0.155
40	0.147
80	0.131375
0.939	0.43284

Table 1-Running coupling constant of the QCD

3- Grand Unification (GUT) Scale

It seems that relation (53) is more appropriate to represent the running coupling constant of the QCD at lower and intermediate energy scales. At very high energies, when approaching the GUT scale we proceed as follows.

With respect to the number N of bosons which appear in eq. (52), we observe that before reaching the Gut scale we have met the electro-weak scale and therefore we have 8 bosons coming from the SU(3) theory (QCD) plus 3bosons 0f the SU(2) (QFD-Quantum Flavor Dynamics), given N = 11. Besides this n_f = 6 (six quark flavors). In this way we propose a new branch of the strong coupling starting at the energy of 80 GeV.

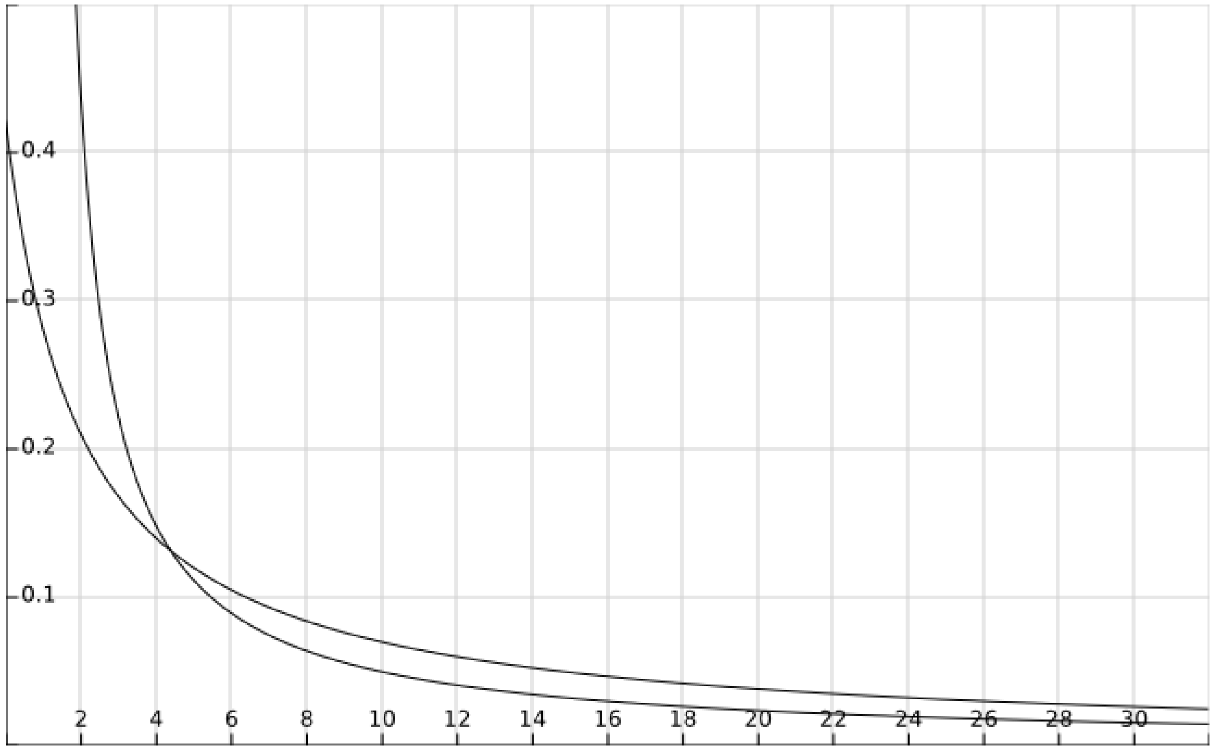
We have

$$\alpha_{ref} = \alpha_{S0}(80\ \text{GeV})(7/\pi) = 0.131375\text{x}7/\pi = .292726 \qquad (54)$$

Taking in account the above considerations, we can write

$$\alpha_S|_{\mu > 80\ \text{GeV}} = 0.131375/[1 + 0.292726\ \ln(\mu/80\ \text{GeV})]. \qquad (55)$$

Figure-1displays the two branches of the running coupling of QCD dicussed in this work.



$x = \ln(\mu/\mu_0)$, $\mu_0 = 1$ GeV

Figure 1 – Two regimes for the running coupling constant in QCD. The first describes α_S for $x \equiv \ln(\mu/\mu_0)$,in the range 0 to 4.3820266. The second describes α_S from x = 4.3820266 to 32. We must consider the lower branch of the curves in both regimes. The second branch is used to find the energy scale of the GUT.
First branch equation: .41922/(1+.5*x).
Second branch equation: .131375/(1+.292726*(x-4.3820266)).

In order to estimate the running coupling constant of the QED, we adopt a procedure advanced by Kane [K-page 235]. We assume that at high energies (μ> 80 GeV), the number of dipoles (n_d) contributing to the dielectric screening in QED is given by

$$n_d = n_l + 3(4/9)n_u + 3(1/9)n_d + n_W. \qquad (56)$$

In (56), n_l is the number of charged leptons, n_u the number of quarks up-like, n_d of quarks down-like and n_W =1stands for the W-particles. At sufficient higher energies we assume full contribution of all these particles, which gives n_d =9.

The factor of three comes from the number of color charges. Therefore we have

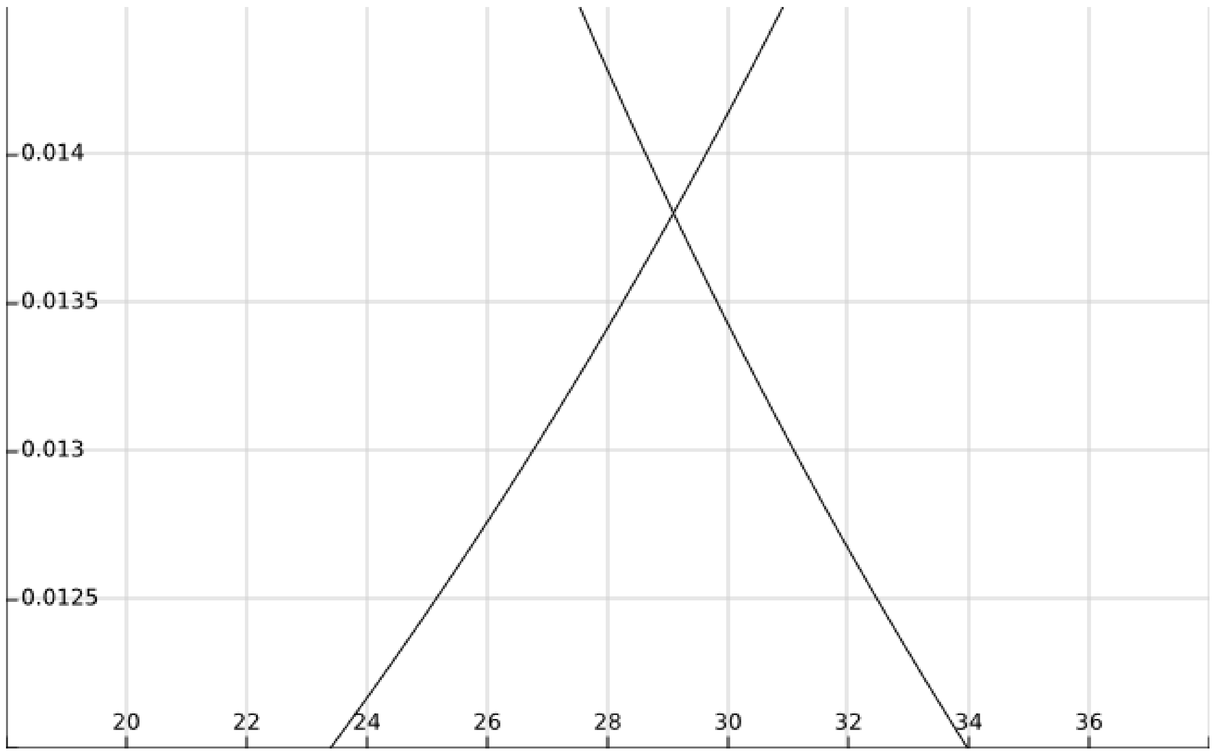
$$\alpha_{ref}/2|_{QED} = - \alpha_0 [2/(3\pi)] n_d = -(3/64)\pi, \quad \text{with } \alpha_0 = 1/128. \qquad (57)$$

For μ> 80 GeV, we write

$$\alpha|_{QED} = (1/128)/[1 - (3x)/(64\pi)], \quad \text{where } x = \ln(\mu/80 \text{ GeV}). \qquad (58)$$

Figure-2 displays the two running coupling constants, namely α_{QED} and α_S in the region close to the energy scale of GUT. The intercept of these two curves give

$\mu_{GUT} = 3.55 \times 10^{14}$ GeV.



x = ln(μ ⁄ (80 GeV))

Figure 2 – Plot of the running coupling constants of the QED, given by the equation: 1/128/(1-3*x/(64*pi)), and of the QCD, given by the equation:.131375/(1+.292726*x). The intercept of the two curves occurs at the point x = 29.12 and y = .0138. This corresponds to the energy of $\mu_{GUT} = 3.55 \times 10^{14}$ GeV, the energy scale of the GUT found in the present work.

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