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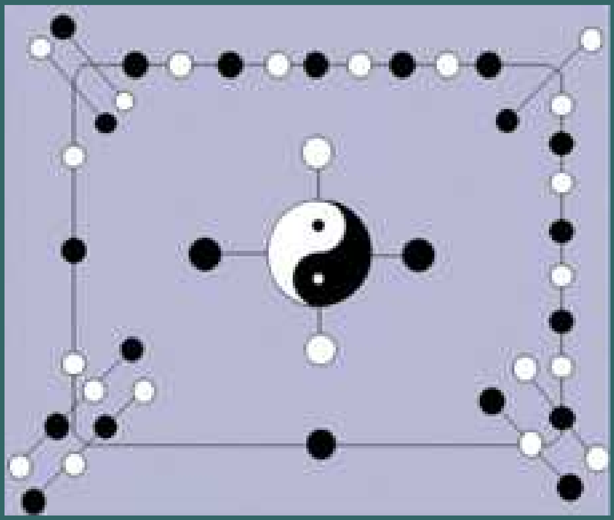
MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



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THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

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Famous Words:

A man is not old as long as he is seeking something. A man is not old until regrets take the place of dreams.

By J.Barrymore, an American actor.

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Mathematics After CC Conjecture

— Combinatorial Notions and Achievements

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Abstract: As a powerful technique for holding relations in things, combinatorics has experienced rapidly development in the past century, particularly, enumeration of configurations, combinatorial design and graph theory. However, the main objective for mathematics is to bring about a quantitative analysis for other sciences, which implies a natural question on combinatorics. Thus, *how combinatorics can contributes to other mathematical sciences, not just in discrete mathematics, but metric mathematics and physics?* After a long time speculation, I brought the CC conjecture for advancing mathematics by combinatorics, i.e., *any mathematical science can be reconstructed from or made by combinatorialization* in my postdoctoral report for Chinese Academy of Sciences in 2005, and reported it at a few academic conferences in China. After then, my surveying paper *Combinatorial Speculation and Combinatorial Conjecture for Mathematics* published in the first issue of *International Journal of Mathematical Combinatorics*, 2007. Clearly, CC conjecture is in fact a combinatorial notion and holds by a philosophical law, i.e., all things are inherently related, not isolated but it can greatly promote the developing of mathematical sciences. The main purpose of this report is to survey the roles of CC conjecture in developing mathematical sciences with notions, such as those of its contribution to algebra, topology, Euclidean geometry and differential geometry, non-solvable differential equations or classical mathematical systems with contradictions to mathematics, quantum fields after it appeared 10 years ago. All of these show the importance of combinatorics to mathematical sciences in the past and future.

Key Words: CC conjecture, Smarandache system, G^L-system, non-solvable system of equations, combinatorial manifold, geometry, quantum field.

AMS(2010): 03C05,05C15,51D20,51H20,51P05,83C05,83E50.

§1. Introduction

There are many techniques in combinatorics, particularly, the enumeration and counting with graph, a visible, also an abstract model on relations of things in the world. Among them,

[1]Reported at the *International Conference on Combinatorics, Graph Theory, Topology and Geometry*, January 29-31, 2015, Shanghai, P.R.China.

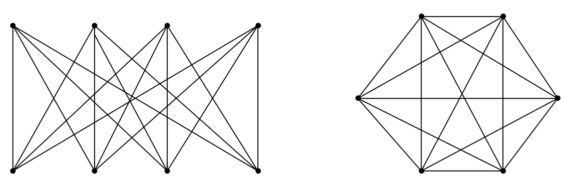
[2]Received October 20, 2014, Accepted May 8, 2015.

the most interested is the graph. A graph G is a 3-tuple (V, E, I) with finite sets V, E and a mapping $I: E \to V \times V$, and *simple* if it is without loops and multiple edges, denoted by $(V; E)$ for convenience. All elements v in V, e in E are said respectively *vertices* and *edges*.

A graph with given properties are particularly interested. For example, a *path* P_n in a graph G is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$, $e_i = (u_i, u_{i+1})$ with distinct vertices for an integer $n \geq 1$, and if $u_1 = u_{n+1}$, it is called a *circuit* or *cycle* C_n. For example, $v_1v_2v_3v_4$ and $v_1v_2v_3v_4v_1$ are respective path and circuit in Fig.1. A graph G is *connected* if for $u, v \in V(G)$, there are paths with end vertices u and v in G.

A *complete graph* $K_n = (V_c, E_c; I_c)$ is a simple graph with $V_c = \{v_1, v_2, \cdots, v_n\}$, $E_c = \{e_{ij}, 1 \leq i, j \leq n, i \neq j\}$ and $I_c(e_{ij}) = (v_i, v_j)$, or simply by a pair (V, E) with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_iv_j, 1 \leq i, j \leq n, i \neq j\}$.

A simple graph $G = (V, E)$ is *r-partite* for an integer $r \geq 1$ if it is possible to partition V into r subsets $V_1, V_2, \cdots, V_r$ such that for $\forall e(u, v) \in E$, there are integers $i \neq j, 1 \leq i, j \leq r$ such that $u \in V_i$ and $v \in V_j$. If there is an edge $e_{ij} \in E$ for $\forall v_i \in V_i$, $\forall v_j \in V_j$, where $1 \leq i, j \leq r, i \neq j$, then, G is called a *complete r-partite graph*, denoted by $G = K(|V_1|, |V_2|, \cdots, |V_r|)$. Thus a complete graph is nothing else but a complete 1-partite graph. For example, the bipartite graph $K(4, 4)$ and the complete graph K_6 are shown in Fig.1.



$K(4,4)$ K_6

Fig.1

Notice that a few edges in Fig.1 have intersections besides end vertices. Contrast to this case, a *planar graph* can be realized on a Euclidean plane $\mathbb{R}^2$ by letting points $p(v) \in \mathbb{R}^2$ for vertices $v \in V$ with $p(v_i) \neq p(v_j)$ if $v_i \neq v_j$, and letting curve $C(v_i, v_j) \subset \mathbb{R}^2$ connecting points $p(v_i)$ and $p(v_j)$ for edges $(v_i, v_j) \in E(G)$, such as those shown in Fig.2.

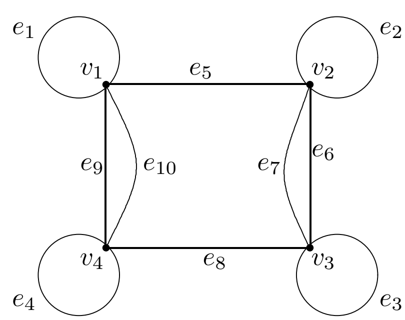


Fig.2

Generally, let $\mathscr{E}$ be a topological space. A graph G is said to be embeddable into $\mathscr{E}$ ([32])

if there is a $1-1$ continuous mapping $f: G \to \mathscr{E}$ with $f(p) \neq f(q)$ if $p \neq q$ for $\forall p, q \in G$, i.e., edges only intersect at vertices in $\mathscr{E}$. Such embedded graphs are called *topological graphs*.

There is a well-known result on embedding of graphs without loops and multiple edges in $\mathbb{R}^n$ for $n \geq 3$ ([32]), i.e., *there always exists such an embedding of G that all edges are straight segments in* $\mathbb{R}^n$, which enables us turn to characterize embeddings of graphs on $\mathbb{R}^2$ and its generalization, 2-*manifolds* or *surfaces* ([3]).

However, all these embeddings of G are established on an assumption that *each vertex of G is mapped exactly into one point of $\mathscr{E}$* in combinatorics for simplicity. If we put off this assumption, *what will happens*? *Are these resultants important for understanding the world*? The answer is certainly YES because this will enables us to pullback more characters of things, characterize more precisely and then hold the truly faces of things in the world.

All of us know an objective law in philosophy, namely, *the integral always consists of its parts and all of them are inherently related, not isolated*. This idea implies that every thing in the world is nothing else but a union of sub-things underlying a graph embedded in space of the world.

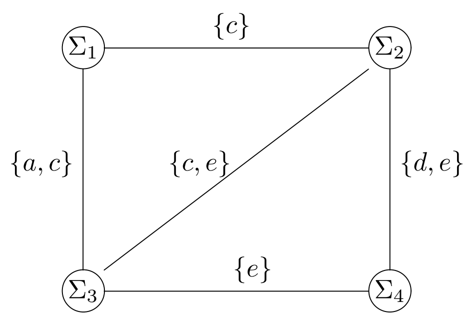


Fig.3

Formally, we introduce some conceptions following.

Definition 1.1([30]-[31], [12]) *Let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems, different two by two. A Smarandache multisystem $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, denoted by $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.*

Definition 1.2([11]-[13]) *For any integer $m \geq 1$, let $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ be a Smarandache multisystem consisting of m mathematical systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. An inherited topological structure $G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]$ of $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ is a topological vertex-edge labeled graph defined following:*

$$V\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{(\Sigma_i,\Sigma_j) | \Sigma_i \bigcap \Sigma_j \neq \emptyset,\ 1 \leq i \neq j \leq m\} \text{ with labeling}$$

$$L:\ \Sigma_i \to L(\Sigma_i) = \Sigma_i \quad \text{and} \quad L:\ (\Sigma_i,\Sigma_j) \to L(\Sigma_i,\Sigma_j) = \Sigma_i \bigcap \Sigma_j$$

for integers $1 \leq i \neq j \leq m$, also denoted by $G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]$ for $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

For example, let $\Sigma_1 = \{a, b, c\}$, $\Sigma_2 = \{c, d, e\}$, $\Sigma_3 = \{a, c, e\}$, $\Sigma_4 = \{d, e, f\}$ and $\mathcal{R}_i = \emptyset$ for integers $1 \leq i \leq 4$, i.e., all these system are sets. Then the multispace $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ with $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{4} \Sigma_i = \{a, b, c, d, e, f\}$ and $\widetilde{\mathscr{R}} = \emptyset$ underlying a topological graph $G^L\left[\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right]$ shown in Fig.3. Combinatorially, the Smarandache multisystems can be classified by their inherited topological structures, i.e., isomorphic labeled graphs following.

Definition 1.3 ([13]) *Let*

$$G_1{}^{L_1} = \left(\bigcup_{i=1}^{m} \Sigma_i^{(1)};\ \bigcup_{i=1}^{m} \mathcal{R}_i^{(1)}\right) \quad and \quad G_2{}^{L_2} = \left(\bigcup_{i=1}^{n} \Sigma_i^{(2)};\ \bigcup_{i=1}^{n} \mathcal{R}_i^{(2)}\right).$$

be two Smarandache multisystems underlying topological graphs G_1 and G_2, respectively. They are isomorphic if there is a bijection $\varpi : G_1{}^{L_1} \rightarrow G_2{}^{L_2}$ with $\varpi : \bigcup\limits_{i=1}^{m} \Sigma_i^{(1)} \rightarrow \bigcup\limits_{i=1}^{n} \Sigma_i^{(2)}$ and $\varpi : \bigcup\limits_{i=1}^{m} \mathcal{R}_i^{(1)} \rightarrow \bigcup\limits_{i=1}^{n} \mathcal{R}_i^{(2)}$ such that

$$\varpi|_{\Sigma_i}\left(a\mathcal{R}_i^{(1)}b\right) = \varpi|_{\Sigma_i}(a)\varpi|_{\Sigma_i}\left(\mathcal{R}_i^{(1)}\right)\varpi|_{\Sigma_i}(b)$$

for $\forall a, b \in \Sigma_i^{(1)}$, $1 \leq i \leq m$, where $\varpi|_{\Sigma_i}$ denotes the constraint of ϖ on $(\Sigma_i, \mathcal{R}_i)$.

Consequently, the previous discussion implies that

Every thing in the world is nothing else but a topological graph G^L in space of the world, and two things are similar if they are isomorphic.

After speculation over a long time, I presented the CC conjecture on mathematical sciences in the final chapter of my post-doctoral report for *Chinese Academy of Sciences* in 2005 ([9],[10]), and then reported at *The 2nd Conference on Combinatorics and Graph Theory of China* in 2006, which is in fact an inverse of the understand of things in the world.

CC Conjecture([9-10],[14]) *Any mathematical science can be reconstructed from or made by combinatorialization.*

Certainly, this conjecture is true in philosophy. It is in fact a *combinatorial notion* for developing mathematical sciences following.

Notion 1.1 *Finds the combinatorial structure, particularly, selects finite combinatorial rulers to reconstruct or make a generalization for a classical mathematical science.*

This notion appeared even in classical mathematics. For examples, Hilbert axiom system for Euclidean geometry, complexes in algebraic topology, particularly, 2-cell embeddings of graphs on surface are essentially the combinatorialization for Euclidean geometry, topological spaces and surfaces, respectively.

Notion 1.2 *Combine different mathematical sciences and establish new enveloping theory on topological graphs, with classical theory being a special one, and this combinatorial process will never end until it has been done for all mathematical sciences.*

A few fields can be also found in classical mathematics on this notion, for instance the topological groups, which is in fact a multi-space of topological space with groups, and similarly, the Lie groups, a multi-space of manifold with that of diffeomorphisms.

Even in the developing process of physics, the trace of Notions 1.1 and 1.2 can be also found. For examples, the many-world interpretation [2] on quantum mechanics by Everett in 1957 is essentially a multispace formulation of quantum state (See [35] for details), and the unifying the four known forces, i.e., gravity, electro-magnetism, the strong and weak nuclear force into one super force by many researchers, i.e., establish the *unified field theory* is nothing else but also a following of the combinatorial notions by letting Lagrangian $\mathscr{L}$ being that a combination of its subfields, for instance the *standard model* on electroweak interactions, etc..

Even so, the CC conjecture includes more deeply thoughts for developing mathematics by combinatorics i.e., *mathematical combinatorics* which extends the field of all existent mathematical sciences. After it was presented, more methods were suggested for developing mathematics in last decade. The main purpose of this report is to survey its contribution to algebra, topology and geometry, mathematical analysis, particularly, non-solvable algebraic and differential equations, theoretical physics with its producing notions in developing mathematical sciences.

All terminologies and notations used in this paper are standard. For those not mentioned here, we follow reference [5] and [32] for topology, [3] for topological graphs, [1] for algebraic systems, [4], [34] for differential equations and [12], [30]-[31] for Smarandache systems.

§2. Algebraic Combinatorics

Algebraic systems, such as those of groups, rings, fields and modules are combinatorial themselves. However, the CC conjecture also produces notions for their development following.

Notion 2.1 *For an algebraic system $(\mathscr{A};\mathcal{O})$, determine its underlying topological structure $G^L[\mathscr{A},\mathcal{O}]$ on subsystems, and then classify by graph isomorphism.*

Notion 2.2 *For an integer $m\geq 1$, let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ all be algebraic systems in Definition* 1.2 *and $\left(\widetilde{\mathscr{G}};\mathcal{O}\right)$ underlying $G^L\left[\widetilde{\mathscr{G}};\mathcal{O}\right]$ with $\widetilde{\mathscr{G}}=\bigcup\limits_{i=1}^{m}\Sigma_i$ and $\mathcal{O}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$, i.e., an algebraic multisystem. Characterize $\left(\widetilde{\mathscr{G}};\mathcal{O}\right)$ and establish algebraic theory, i.e., combinatorial algebra on $\left(\widetilde{\mathscr{G}};\mathcal{O}\right)$.*

For example, let

$$\langle\mathscr{G}_1;\circ_1\rangle=\left\langle a,b|a\circ_1 b=b\circ_1 a, a^2=b^n=1\right\rangle$$
$$\langle\mathscr{G}_2;\circ_2\rangle=\left\langle b,c|b\circ_2 c=c\circ_2 b, c^5=b^n=1\right\rangle$$
$$\langle\mathscr{G}_3;\circ_3\rangle=\left\langle c,d|c\circ_3 d=d\circ_3 c, d^2=c^5=1\right\rangle$$

be groups with respective operations $\circ_1$, $\circ_2$ and $\circ_3$. Then the set $(\widetilde{\mathscr{G}};\{\circ_1,\circ_2,\circ_3\})$ is an algebraic multisyatem with $\widetilde{\mathscr{G}}=\bigcup\limits_{i=1}^{3}\mathscr{G}_i$.

2.1 K_2^L-Systems

A K_2^L-system is such a multi-system consisting of exactly 2 algebraic systems underlying a topological graph K_2^L, including bigroups, birings, bifields and bimodules, etc.. For example, an algebraic field $(R;+,\cdot)$ is a K_2^L-system. Clearly, $(R;+,\cdot)$ consists of groups $(R;+)$ and $(R\backslash\{0\};\cdot)$ underlying K_2^L such as those shown in Fig.4, where $L:\ V\left(K_2^L\right)\rightarrow\{(R;+),(R\setminus\{0\};\cdot)\}$ and $L:\ E\left(K_2^L\right)\rightarrow\{R\setminus\{0\}\}$.

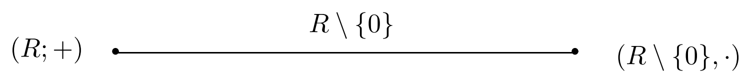


Fig.4

A generalization of field is replace $R\setminus\{0\}$ by any subset $H\leq R$ in Fig.4. Then a bigroup comes into being, which was introduced by Maggu [8] for industrial systems in 1994, and then Vasantha Kandasmy [33] further generalizes it to bialgebraic structures.

Definition 2.3 *A bigroup (biring, bifield, bimodule, $\cdots$) is a 2-system $(\mathscr{G};\circ,\cdot)$ such that*

(1) $\mathscr{G}=\mathscr{G}_1\bigcup\mathscr{G}_2$;

(2) $(\mathscr{G}_1;\circ)$ *and* $(\mathscr{G}_2;\cdot)$ *both are groups (rings, fields, modules,$\cdots$).*

For example, let $\widetilde{\mathscr{P}}$ be a permutation multigroup action on $\widetilde{\Omega}$ with

$$\widetilde{\mathscr{P}}=\mathscr{P}_1\bigcup\mathscr{P}_2\ \text{ and }\ \widetilde{\Omega}=\{1,2,3,4,5,6,7,8\}\bigcup\{1,2,5,6,9,10,11,12\},$$

where $\mathscr{P}_1=\langle(1,2,3,4),(5,6,7,8)\rangle$ and $\mathscr{P}_2=\langle(1,5,9,10),(2,6,11,12)\rangle$. Clearly, $\widetilde{\mathscr{P}}$ is a permutation bigroup.

Let $(\mathscr{G}_1;\circ_1,\cdot_1)$ and $((\mathscr{G}_2;\circ_2,\cdot_2))$ be bigroups. A mapping pair (ϕ,ι) with $\phi:\ \mathscr{G}_1\rightarrow\mathscr{G}_2$ and $\iota:\{\circ_1,\cdot_1\}\rightarrow\{\circ_2,\cdot_2\}$ is a *homomorphism* if

$$\phi(a\bullet b)=\phi(a)\iota(\bullet)\phi(b)$$

for $\forall a,b\in\mathscr{G}_1$ and $\bullet\in\{\circ_1,\cdot_1\}$ provided $a\bullet b$ existing in $(\mathscr{G}_1;\circ_1,\cdot_1)$. Define the *image* $\mathrm{Im}(\phi,\iota)$ and *kernel* $\mathrm{Ker}(\phi,\iota)$ respectively by

$$\begin{aligned}\mathrm{Im}(\phi,\iota)&=\{\ \phi(g)\mid g\in\mathscr{G}_1\ \},\\ \mathrm{Ker}(\phi,\iota)&=\{\ g\in\mathscr{G}_1|\ \phi(g)=1_\bullet,\forall\bullet\in\{\circ_2,\cdot_2\}\},\end{aligned}$$

where $1_\bullet$ denotes the unit of $(\mathscr{G}_\bullet;\bullet)$ with $\mathscr{G}_\bullet$ a maximal closed subset of $\mathscr{G}$ on operation $\bullet$.

For subsets $\widetilde{H}\subset\widetilde{G}$, $O\subset\mathcal{O}$, define $(\widetilde{H};O)$ to be a submultisystem of $\left(\widetilde{G};\mathcal{O}\right)$ if $(\widetilde{H};O)$ is multisystem itself, denoted by $\left(\widetilde{H};O\right)\leq\left(\widetilde{G};\mathcal{O}\right)$, and a subbigroup $(\mathscr{H};\circ,\cdot)$ of $(\mathscr{G};\circ,\cdot)$ is *normal*, denoted by $\mathscr{H} \lhd \mathscr{G}$ if for $\forall g \in \mathscr{G}$,

$$g \bullet \mathscr{H} = \mathscr{H} \bullet g,$$

where $g \bullet \mathscr{H} = \{g \bullet h | h \in \mathscr{H}$ provided $g \bullet h$ existing$\}$ and $\mathscr{H} \bullet g = \{h \bullet g | h \in \mathscr{H}$ provided $h \bullet g$ existing$\}$ for $\forall \bullet \in \{\circ, \cdot\}$. The next result is a generalization of isomorphism theorem of group in [33].

Theorem 2.4([11]) *Let $(\phi, \iota) : (\mathscr{G}_1; \{\circ_1, \cdot_1\}) \to (\mathscr{G}_2; \{\circ_2, \cdot_2\})$ be a homomorphism. Then*

$$G_1/\mathrm{Ker}(\phi, \iota) \simeq \mathrm{Im}(\phi, \iota).$$

Particularly, if $(\mathscr{G}_2; \{\circ_2, \cdot_2\})$ is a group $(\mathscr{A}; \circ)$, we know the corollary following.

Corollary 2.5 *Let $(\phi, \iota) : (\mathscr{G}; \{\circ, \cdot\}) \to (\mathscr{A}; \circ)$ be an epimorphism. Then*

$$\mathscr{G}_1/\mathrm{Ker}(\phi, \iota) \simeq (\mathscr{A}; \circ).$$

Similarly, a bigroup $(\mathscr{G}; \circ, \cdot)$ is *distributive* if

$$a \cdot (b \circ c) = a \cdot b \circ a \cdot c$$

hold for all $a, b, c \in \mathscr{G}$. Then, we know the following result.

Theorem 2.6([11]) *Let $(\mathscr{G}; \circ, \cdot)$ be a distributive bigroup of order≥ 2 with $\mathscr{G} = \mathscr{A}_1 \cup \mathscr{A}_2$ such that $(\mathscr{A}_1; \circ)$ and $(\mathscr{A}_2; \cdot)$ are groups. Then there must be $\mathscr{A}_1 \neq \mathscr{A}_2$. consequently, if $(\mathscr{G}; \circ)$ it a non-trivial group, there are no operations $\cdot \neq \circ$ on $\mathscr{G}$ such that $(\mathscr{G}; \circ, \cdot)$ is a distributive bigroup.*

2.2 G^L-Systems

Definition 2.2 is easily generalized also to multigroups, i.e., consisting of m groups underlying a topological graph G^L, and similarly, define conceptions of homomorphism, submultigroup and normal submultigroup, $\cdots$ of a multigroup without any difficult.

For example, a normal submultigroup of $(\widetilde{\mathscr{G}}; \widetilde{O})$ is such submutigroup $(\widetilde{\mathscr{H}}; O)$ that holds

$$g \circ \widetilde{\mathscr{H}} = \widetilde{\mathscr{H}} \circ g$$

for $\forall g \in \widetilde{\mathscr{G}}$, $\forall \circ \in O$, and generalize Theorem 2.3 to the following.

Theorem 2.7([16]) *Let $(\phi, \iota) : (\widetilde{\mathscr{G}}_1; \widetilde{O}_1) \to (\widetilde{\mathscr{G}}_2; \widetilde{O}_2)$ be a homomorphism. Then*

$$\widetilde{\mathscr{G}}_1/\mathrm{Ker}(\phi, \iota) \simeq \mathrm{Im}(\phi, \iota).$$

Particularly, for the transitive of multigroup action on a set $\widetilde{\Omega}$, let $\widetilde{\mathscr{P}}$ be a permutation multigroup action on $\widetilde{\Omega}$ with $\widetilde{\mathscr{P}} = \bigcup\limits_{i=1}^{m} \mathscr{P}_i, \widetilde{\Omega} = \bigcup\limits_{i=1}^{m} \Omega_i$ and for each integer $i, 1 \leq i \leq m$, the permutation group $\mathscr{P}_i$ acts on Ω_i, which is *globally k-transitive* for an integer $k \geq 1$ if for any two k-tuples $x_1, x_2, \cdots, x_k \in \Omega_i$ and $y_1, y_2, \cdots, y_k \in \Omega_j$, where $1 \leq i, j \leq m$, there are permutations $\pi_1, \pi_2, \cdots, \pi_n$ such that

$$x_1^{\pi_1\pi_2\cdots\pi_n} = y_1,\ x_2^{\pi_1\pi_2\cdots\pi_n} = y_2,\ \cdots,\ x_k^{\pi_1\pi_2\cdots\pi_n} = y_k$$

and abbreviate the globally 1-transitive to that *globally transitive* of a permutation multigroup. The following result characterizes transitive multigroup.

Theorem 2.8([17]) *Let $\widetilde{\mathscr{P}}$ be a permutation multigroup action on $\widetilde{\Omega}$ with*

$$\widetilde{\mathscr{P}} = \bigcup_{i=1}^{m} \mathscr{P}_i \ \ \textit{and} \ \ \widetilde{\Omega} = \bigcup_{i=1}^{m} \Omega_i,$$

where, each permutation group $\mathscr{P}_i$ transitively acts on Ω_i for each integers $1 \leq i \leq m$. Then $\widetilde{\mathscr{P}}$ is globally transitive on $\widetilde{\Omega}$ if and only if the graph $G^L\left[\widetilde{\Omega}\right]$ is connected.

Similarly, let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a completed multisystem with a double operation set $\mathcal{O}\left(\widetilde{R}\right) = \mathcal{O}_1 \bigcup \mathcal{O}_2$, where $\mathcal{O}_1 = \{\ \cdot_i, 1 \leq i \leq m\}$, $\mathcal{O}_2 = \{+_i, 1 \leq i \leq m\}$. If for any integers $i, 1 \leq i \leq m$, $(R_i; +_i, \cdot_i)$ is a ring, then $\widetilde{R}$ is called a multiring, denoted by $\left(\widetilde{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2\right)$ and $(+_i, \cdot_i)$ a double operation for any integer i, which is *integral* if for $\forall a, b \in \widetilde{R}$ and an integer $i,\ 1 \leq i \leq m$, $a \cdot_i b = b \cdot_i a$, $1_{\cdot_i} \neq 0_{+i}$ and $a \cdot_i b = 0_{+_i}$ implies that $a = 0_{+_i}$ or $b = 0_{+_i}$. Such a multiring $\left(\widetilde{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2\right)$ is called a *skew multifield* or a *multifield* if each $(R; +_i, \cdot_i)$ is a skew field or a field for integers $1 \leq i \leq m$. The next result is a generalization of finitely integral ring.

Theorem 2.9([16]) *A finitely integral multiring is a multifield.*

For multimodule, let $\mathcal{O} = \{\ +_i \mid 1 \leq i \leq m\}$, $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq m\}$ and $\mathcal{O}_2 = \{\dot{+}_i | 1 \leq i \leq m\}$ be operation sets, $(\mathscr{M}; \mathcal{O})$ a commutative multigroup with units 0_{+_i} and $(\mathscr{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ a multiring with a unit $1.$ for $\forall \cdot \in \mathcal{O}_1$. A pair $(\mathscr{M}; \mathcal{O})$ is said to be a *multimodule over* $(\mathscr{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ if for any integer $i,\ 1 \leq i \leq m$, a binary operation $\times_i : \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ is defined by $a \times_i x$ for $a \in \mathscr{R}$, $x \in \mathscr{M}$ such that the conditions following

(1) $\ a \times_i (x +_i y) = a \times_i x +_i a \times_i y$;
(2) $(a \dot{+}_i b) \times_i x = a \times_i x +_i b \times_i x$;
(3) $(a \cdot_i b) \times_i x = a \times_i (b \times_i x)$;
(4) $1_{\cdot_i} \times_i x = x$.

hold for $\forall a, b \in \mathscr{R}$, $\forall x, y \in \mathscr{M}$, denoted by $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$. Then we know the following result for finitely multimodules.

Theorem 2.10([16]) *Let $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) = \left\langle \widehat{S} | \mathscr{R} \right\rangle$ be a finitely generated multimodule with $\widehat{S} = \{u_1, u_2, \cdots, u_n\}$. Then*

$$\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) \ \cong \ \mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)),$$

where $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ *is a multimodule on* $\mathscr{R}^{(n)} = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathscr{R}, 1 \leq i \leq n\}$ *with*

$$(x_1, x_2, \cdots, x_n) +_i (y_1, y_2, \cdots, y_n) = (x_1 \dot{+}_i y_1, x_2 \dot{+}_i y_2, \cdots, x_n \dot{+}_i y_n),$$
$$a \times_i (x_1, x_2, \cdots, x_n) = (a \cdot_i x_1, a \cdot_i x_2, \cdots, a \cdot_i x_n)$$

for $\forall a \in \mathscr{R}$, *integers* $1 \leq i \leq m$. *Particularly, a finitely module over a commutative ring* $(\mathscr{R}; +, \cdot)$ *generated by* n *elements is isomorphic to the module* $\mathscr{R}^n$ *over* $(\mathscr{R}; +, \cdot)$.

§3. Geometrical Combinatorics

Classical geometry, such as those of Euclidean or non-Euclidean geometry, or projective geometry are not combinatorial. Whence, the CC conjecture produces combinatorial notions for their development further, for instance the topological space shown in Fig.5 following.

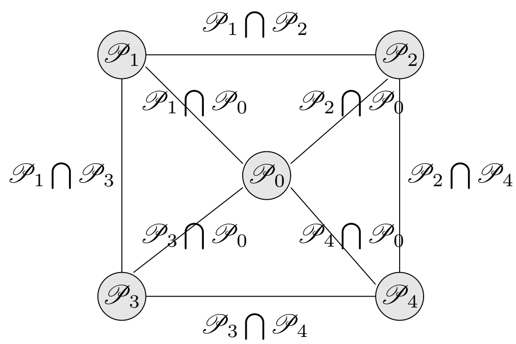


Fig.5

Notion 3.1 *For a geometrical space* $\mathscr{P}$, *determine its underlying topological structure* $G^L[\mathscr{A}, \mathcal{O}]$ *on subspaces, for instance,* n*-manifolds and classify them by graph isomorphisms.*

Notion 3.2 *For an integer* $m \geq 1$, *let* $\mathscr{P}_1$, $\mathscr{P}_2$, $\cdots$, $\mathscr{P}_m$ *all be geometrical spaces in Definition 1.2 and* $\widetilde{\mathscr{P}}$ *underlying* $G^L\left[\widetilde{\mathscr{P}}\right]$ *with* $\widetilde{\mathscr{P}} = \bigcup\limits_{i=1}^{m} \mathscr{P}_i$, *i.e., a geometrical multispace. Characterize* $\widetilde{\mathscr{P}}$ *and establish geometrical theory, i.e., combinatorial geometry on* $\widetilde{\mathscr{P}}$.

3.1 Euclidean Spaces

Let $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$ be the normal basis of a Euclidean space $\mathbb{R}^n$ in a general position, i.e., for two Euclidean spaces $\mathbb{R}^{n_\mu}, \mathbb{R}^{n_\nu}$, $\mathbb{R}^{n_\mu} \cap \mathbb{R}^{n_\nu} \neq \mathbb{R}^{\min\{n_\mu, n_\nu\}}$. In this case, let $\mathcal{X}_{v_\mu}$ be the set of orthogonal orientations in $\mathbb{R}^{n_{v_\mu}}$, $\mu \in \Lambda$. Then $\mathbb{R}^{n_\mu} \cap \mathbb{R}^{n_\nu} = \mathcal{X}_{v_\mu} \cap \mathcal{X}_{v_\nu}$, which enables us to construct topological spaces by the combination.

For an index set Λ, a *combinatorial Euclidean space* $\mathscr{E}_{G^L}(n_\nu; \nu \in \Lambda)$ underlying a connected graph G^L is a topological spaces consisting of Euclidean spaces $\mathbb{R}^{n_\nu}$, $\nu \in \Lambda$ such that

$$V\left(G^L\right) = \{\ \mathbb{R}^{n_\nu} \mid \nu \in \Lambda\ \};$$

$E\left(G^L\right)=\{\ (\mathbb{R}^{n_\mu},\mathbb{R}^{n_\nu})\ \mid \mathbb{R}^{n_\mu}\cap\mathbb{R}^{n_\nu}\neq\emptyset,\mu,\nu\in\Lambda\ \}$ and labeling

$L:\mathbb{R}^{n_\nu}\to\mathbb{R}^{n_\nu}$ and $L:(\mathbb{R}^{n_\mu},\mathbb{R}^{n_\nu})\to\mathbb{R}^{n_\mu}\bigcap\mathbb{R}^{n_\nu}$

for $(\mathbb{R}^{n_\mu},\mathbb{R}^{n_\nu})\in E\left(G^L\right)$, $\nu,\mu\in\Lambda$.

Clearly, for any graph G, we are easily construct a combinatorial Euclidean space underlying G, which induces a problem following.

Problem 3.3 *Determine the dimension of a combinatorial Euclidean space consisting of m Euclidean spaces $\mathbb{R}^{n_1},\mathbb{R}^{n_2},\cdots,\mathbb{R}^{n_m}$.*

Generally, the combinatorial Euclidean spaces $\mathscr{E}_{G^L}(n_1,n_2,\cdots,n_m)$ are not unique and to determine $\dim\mathscr{E}_{G^L}(n_1,n_2,\cdots,n_m)$ converts to calculate the cardinality of $|X_{n_1}\cup X_{n_2}\cup\cdots\cup X_{n_m}|$, where X_{n_i} is the set of orthogonal orientations in $\mathbb{R}^{n_i}$ for integers $1\leq i\leq m$, which can be determined by the inclusion-exclusion principle, particularly, the maximum dimension following.

Theorem 3.4([21]) $\dim\mathscr{E}_{G^L}(n_1,\cdots,n_m)\leq 1-m+\sum\limits_{i=1}^{m}n_i$ *and with the equality holds if and only if* $\dim\left(\mathbb{R}^{n_i}\cap\mathbb{R}^{n_j}\right)=1$ *for* $\forall\left(\mathbb{R}^{n_i},\mathbb{R}^{n_j}\right)\in E\left(G^L\right),1\leq i,j\leq m$.

To determine the minimum $\dim\mathscr{E}_{G^L}(n_1,\cdots,n_m)$ is still open. However, we know this number for $G=K_m$ and $n_i=r$ for integers $1\leq i\leq m$, i.e., $\mathscr{E}_{K_m}(r)$ by following results.

Theorem 3.5([21]) *For any integer $r\geq 2$, let $\mathscr{E}_{K_m}(r)$ be a combinatorial Euclidean space of $\underbrace{\mathbb{R}^r,\cdots,\mathbb{R}^r}_{m}$, and there exists an integer s, $0\leq s\leq r-1$ such that*

$$\begin{pmatrix} r+s-1\\ r\end{pmatrix}<m\leq\begin{pmatrix} r+s\\ r\end{pmatrix}.$$

Then

$$\dim_{min}\mathscr{E}_{K_m}(r)=r+s.$$

Particularly,

$$\dim_{min}\mathscr{E}_{K_m}(3)=\begin{cases}3, & \text{if}\quad m=1,\\ 4, & \text{if}\quad 2\leq m\leq 4,\\ 5, & \text{if}\quad 5\leq m\leq 10,\\ 2+\lceil\sqrt{m}\rceil, & \text{if}\quad m\geq 11.\end{cases}$$

3.2 Manifolds

An n-manifold is a second countable Hausdorff space of locally Euclidean n-space without boundary, which is in fact a combinatorial Euclidean space $\mathscr{E}_{G^L}(n)$. Thus, we can further replace these Euclidean spaces by manifolds and to get topological spaces underlying a graph, such as those shown in Fig.6.

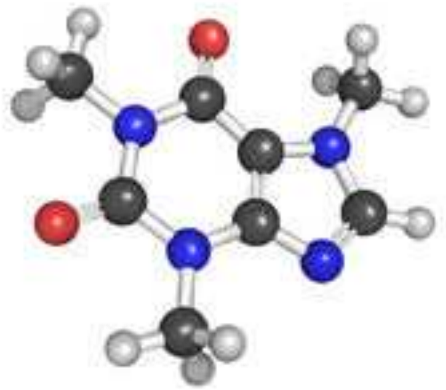


Fig.6

Definition 3.6([22]) *Let $\widetilde{M}$ be a topological space consisting of finite manifolds M_μ, $\mu \in \Lambda$. An inherent graph $G^{in}\left[\widetilde{M}\right]$ of $\widetilde{M}$ is such a graph with*

$$V\left(G^{in}\left[\widetilde{M}\right]\right) = \{M_\mu,\ \mu \in \Lambda\};$$

$$E\left(G^{in}\left[\widetilde{M}\right]\right) = \{(M_\mu, M_\nu)_i\,, 1 \le i \le \kappa_{\mu\nu}+1 | M_\mu \cap M_\nu \neq \emptyset,\ \mu,\nu \in \Lambda\},$$

where $\kappa_{\mu\nu}+1$ is the number of arcwise connected components in $M_\mu \cap M_\nu$ for $\mu,\nu \in \Lambda$.

Notice that $G^{in}\left[\widetilde{M}\right]$ is a multiple graph. If replace all multiple edges $(M_\mu, M_\nu)_i$, $1 \le i \le \kappa_{\mu\nu}+1$ by (M_μ, M_ν), such a graph is denoted by $G[\widetilde{M}]$, also an underlying graph of $\widetilde{M}$.

Clearly, if $m = 1$, then $\widetilde{M}(n_i,\ i \in \Lambda)$ is nothing else but exactly an n_1-manifold by definition. Even so, Notion 3.1 enables us characterizing manifolds by graphs. The following result shows that every manifold is in fact homeomorphic to combinatorial Euclidean space.

Theorem 3.7([22]) *Any locally compact n-manifold M with an alta $\mathscr{A} = \{\ (U_\lambda; \varphi_\lambda)|\ \lambda \in \Lambda\}$ is a combinatorial manifold $\widetilde{M}$ homeomorphic to a combinatorial Euclidean space $\mathscr{E}_{G^L}(n, \lambda \in \Lambda)$ with countable graphs $G^{in}[M] \cong G$.*

Topologically, a Euclidean space $\mathbb{R}^n$ is homeomorphic to an opened ball $\mathbb{B}^n(R) = \{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 < R\}$. Thus, we can view a combinatorial Euclidean space $\mathscr{E}_G(n, \lambda \in \Lambda)$ as a graph with vertices and edges replaced by ball $\mathbb{B}^n(R)$ in space, such as those shown in Fig.6, a 3-dimensional graph.

Definition 3.8 *An n-dimensional graph $\widetilde{M}^n[G]$ is a combinatorial ball space $\widetilde{B}$ of B^n, $\mu \in \Lambda$ underlying a combinatorial structure G such that*

(1) *$V(G)$ is discrete consisting of B^n, i.e., $\forall v \in V(G)$ is an open ball B_v^n;*

(2) *$\widetilde{M}^n[G] \setminus V(\widetilde{M}^n[G])$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^n;*

(3) *the boundary* $\overline{e}_i - e_i$ *of* e_i *consists of one or two* B^n *and each pair* $(\overline{e}_i, e_i)$ *is homeomorphic to the pair* $(\overline{B}^n, B^n)$*;*

(4) *a subset* $A \subset \widetilde{M}^n[G]$ *is open if and only if* $A \cap \overline{e}_i$ *is open for* $1 \leq i \leq m$.

Particularly, a topological graph $\mathscr{T}[G]$ *of a graph* G *embedded in a topological space* $\mathscr{P}$ *is* 1-*dimensional graph.*

According to Theorem 3.7, an n-manifold is homeomorphic to a combinatorial Euclidean space, i.e., n-dimensional graph. This enables us knowing a result following on manifolds.

Theorem 3.9([22]) *Let* $\mathscr{A}[M] = \{\ (U_\lambda; \varphi_\lambda) \mid \lambda \in \Lambda\}$ *be a atlas of a locally compact* n*-manifold* M*. Then the labeled graph* $G^L_{|\Lambda|}$ *of* M *is a topological invariant on* $|\Lambda|$*, i.e., if* $H^{L_1}_{|\Lambda|}$ *and* $G^{L_2}_{|\Lambda|}$ *are two labeled* n*-dimensional graphs of* M*, then there exists a self-homeomorphism* $h : M \to M$ *such that* $h : H^{L_1}_{|\Lambda|} \to G^{L_2}_{|\Lambda|}$ *naturally induces an isomorphism of graph.*

Theorem 3.9 enables us listing manifolds by two parameters, the dimensions and inherited graph. For example, let $|\Lambda| = 2$ and then $\mathscr{A}_{min}[M] = \{(U_1; \varphi_1),\ (U_2; \varphi_2)\}$, i.e., M is double covered underlying a graphs $D^L_{0,\kappa_{12}+1,0}$ shown in Fig.7,

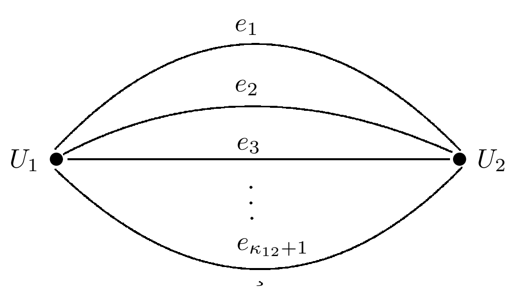


Fig.7

For example, let $U_1 = \mathbb{R}^2$, $\varphi_1 = z$, $U_2 = (\mathbb{R}^2 \setminus \{(0,0)\} \cup \{\infty\}$, $\varphi_2 = 1/z$ and $\kappa_{12} = 0$. Then the 2-manifold is nothing else but the Riemannian sphere.

The G^L-structure on combinatorial manifold $\widetilde{M}$ can be also applied for characterizing a few topological invariants, such as those fundamental groups, for instance the conclusion following.

Theorem 3.10([23]) *For* $\forall (M_1, M_2) \in E\left(G^L\left[\widetilde{M}\right]\right)$, *if* $M_1 \cap M_2$ *is simply connected, then*

$$\pi_1\left(\widetilde{M}\right) \cong \left(\bigotimes_{M \in V\left(G\left[\widetilde{M}\right]\right)} \pi_1(M)\right) \bigotimes \pi_1\left(G^{in}\left[\widetilde{M}\right]\right).$$

Particularly, for a compact n*-manifold* M *with charts* $\{(U_\lambda, \varphi_\lambda) |\ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda\}$, *if* $U_\mu \cap U_\nu$ *is simply connected for* $\forall \mu, \nu \in \Lambda$*, then*

$$\pi_1(M) \cong \pi_1\left(G^{in}[M]\right).$$

3.3 Algebraic Geometry

The topological group, particularly, Lie group is a typical example of K_2^L-systems that of algebra with geometry. Generally, let

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (LEq)$$

be a linear equation system with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers $m,\ n \geq 1$, and all equations in (LEq) are non-trivial, i.e., there are no numbers λ such that $(a_{i1}, a_{i2}, \cdots, a_{in}, b_i) = \lambda(a_{j1}, a_{j2}, \cdots, a_{jn}, b_j)$ for any integers $1 \leq i, j \leq m$.

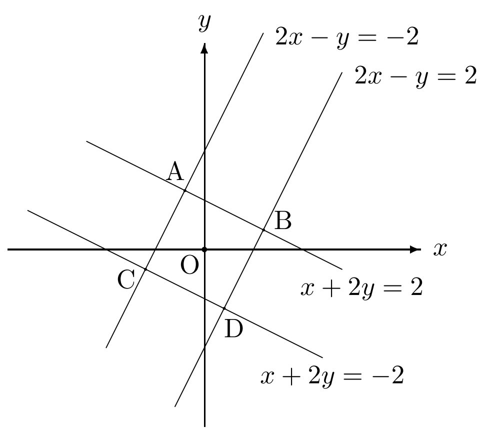


Fig.8

It should be noted that the geometry of a linear equation in n variables is a plane in $\mathbb{R}^n$. Whence, a linear system (LEq) is non-solvable or not dependent on their intersection is empty or not. For example, the linear system shown in Fig.8 is non-solvable because their intersection is empty.

Definition 3.11 *For any integers $1 \leq i, j \leq m,\ i \neq j$, the linear equations*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

are called parallel if there no solution $x_1, x_2, \cdots, x_n$ hold both with the 2 equations.

Define a graph $G^L[LEq]$ on linear system (LEq) following:

$V\left(G^L[LEq]\right) = \{$ the solution space S_i of ith equation $|1 \leq i \leq m\}$,

$E\left(G^L[Eq]\right) = \{\ (S_i, S_j) \mid S_i \bigcap S_j \neq \emptyset,\ 1 \leq i, j \leq m\}$ and with labels

$L : S_i \to S_i$ and $L ; (S_i, S_j) \to S_i \bigcap S_j$

for $\forall S_i \in V\left(G^L[LEq]\right)$, $(S_i, S_j) \in E\left(G^L[LEq]\right)$. For example, the system of equations shown in Fig.8 is

$$\begin{cases} x + 2y &= 2 \\ x + 2y &= -2 \\ 2x - y &= -2 \\ 2x - y &= 2 \end{cases}$$

and C_4^L is its underlying graph $G^L[LEq]$ shown in Fig.9.

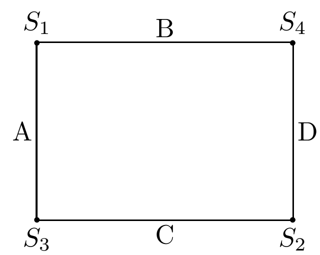


Fig.9

Let L_i be the ith linear equation. By definition we divide these equations $L_i,\ 1 \leq i \leq m$ into *parallel families*

$$\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$$

by the property that all equations in a family $\mathscr{C}_i$ are parallel and there are no other equations parallel to lines in $\mathscr{C}_i$ for integers $1 \leq i \leq s$. Denoted by $|\mathscr{C}_i| = n_i,\ 1 \leq i \leq s$. Then, we can characterize $G^L[LEq]$ following.

Theorem 3.12([24]) *Let (LEq) be a linear equation system for integers $m, n \geq 1$. Then*

$$G^L[LEq] \simeq K^L_{n_1, n_2, \cdots, n_s}$$

with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i$ is the parallel family with $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) and (LEq) is non-solvable if $s \geq 2$.

Notice that this result is not sufficient, i.e., even if $G^L[LEq] \simeq K_{n_1, n_2, \cdots, n_s}$, we can not claim that (LEq) is solvable or not. How ever, if $n = 2$, we can get a necessary and sufficient condition on non-solvable linear equations.

Let H be a planar graph with each edge a straight segment on $\mathbb{R}^2$. Its *c-line graph* $L_C(H)$ is defined by

$V(L_C(H)) = \{$straight lines $L = e_1 e_2 \cdots e_l, s \geq 1$ in $H\}$;

$E(L_C(H)) = \{(L_1, L_2) |$ if e_i^1 and e_j^2 are adjacent in H *for* $L_1 = e_1^1 e_2^1 \cdots e_l^1,\ L_2 = e_1^2 e_2^2 \cdots e_s^2,\ l, s \geq 1\}$.

Theorem 3.13([24]) *A linear equation system* $(LEq2)$ *is non-solvable if and only if* $G^L[LEq2] \simeq L_C(H))$, *where* H *is a planar graph of order* $|H| \geq 2$ *on* $\mathbb{R}^2$ *with each edge a straight segment*

Similarly, let

$$P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x}) \qquad (ES_m^{n+1})$$

be m homogeneous polynomials in $n+1$ variables with coefficients in $\mathbb{C}$ and each equation $P_i(\overline{x}) = 0$ determine a hypersurface $M_i,\ 1 \leq i \leq m$ in $\mathbb{R}^{n+1}$, particularly, a curve C_i if $n = 2$. We introduce the parallel property following.

Definition 3.14 *Let* $P(\overline{x}), Q(\overline{x})$ *be two complex homogeneous polynomials of degree* d *in* $n+1$ *variables and* $I(P,Q)$ *the set of intersection points of* $P(\overline{x})$ *with* $Q(\overline{x})$. *They are said to be parallel, denoted by* $P \parallel Q$ *if* $d > 1$ *and there are constants* $a, b, \cdots, c$ *(not all zero) such that for* $\forall \overline{x} \in I(P,Q)$, $ax_1 + bx_2 + \cdots + cx_{n+1} = 0$, *i.e., all intersections of* $P(\overline{x})$ *with* $Q(\overline{x})$ *appear at a hyperplane on* $\mathbb{P}^n\mathbf{C}$, *or* $d = 1$ *with all intersections at the infinite* $x_{n+1} = 0$. *Otherwise,* $P(\overline{x})$ *are not parallel to* $Q(\overline{x})$, *denoted by* $P \nparallel Q$.

Define a topological graph $G^L\left[ES_m^{n+1}\right]$ in $\mathbb{C}^{n+1}$ by

$$V\left(G^L\left[ES_m^{n+1}\right]\right) = \{P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})\};$$
$$E\left(G^L\left[ES_m^{n+1}\right]\right) = \{(P_i(\overline{x}), P_j(\overline{x})) | P_i \nparallel P_j,\ 1 \leq i, j \leq m\}$$

with a labeling

$$L:\ P_i(\overline{x}) \rightarrow P_i(\overline{x}), \qquad (P_i(\overline{x}), P_j(\overline{x})) \rightarrow I(P_i, P_j),$$

where $1 \leq i \neq j \leq m$, and the topological graph of $G^L\left[ES_m^{n+1}\right]$ without labels is denoted by $G\left[ES_m^{n+1}\right]$. The following result generalizes Theorem 3.12 to homogeneous polynomials.

Theorem 3.15([26]) *Let* $n \geq 2$ *be an integer. For a system* (ES_m^{n+1}) *of homogeneous polynomials with a parallel maximal classification* $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$,

$$G[ES_m^{n+1}] \leq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$$

and with equality holds if and only if $P_i \parallel P_j$ *and* $P_s \nparallel P_i$ *implies that* $P_s \nparallel P_j$, *where* $K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$ *denotes a complete* l*-partite graphs*

Conversely, for any subgraph $G \leq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l)$, *there are systems* (ES_m^{n+1}) *of homogeneous polynomials with a parallel maximal classification* $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_l$ *such that*

$$G \simeq G[ES_m^{n+1}].$$

Particularly, if $n = 2$, i.e., an (ES_m^3) system, we get the condition following.

Theorem 3.16([26]) *Let* G^L *be a topological graph labeled with* $I(e)$ *for* $\forall e \in E\left(G^L\right)$. *Then there is a system* $\left(ES_m^3\right)$ *of homogeneous polynomials such that* $G^L\left[ES_m^3\right] \simeq G^L$ *if and only if* *there are homogeneous polynomials* $P_{v_i}(x,y,z), 1 \le i \le \rho(v)$ *for* $\forall v \in V\left(G^L\right)$ *such that*

$$I(e) = I\left(\prod_{i=1}^{\rho(u)} P_{u_i}, \prod_{i=1}^{\rho(v)} P_{v_i}\right)$$

for $e = (u,v) \in E\left(G^L\right)$, *where* $\rho(v)$ *denotes the valency of vertex* v *in* G^L.

These G^L-system of homogeneous polynomials enables us to get combinatorial manifolds, for instance, the following result appeared in [26].

Theorem 3.17 *Let* (ES_m^{n+1}) *be a* G^L*-system consisting of homogeneous polynomials* $P_1(\overline{x}), P_2(\overline{x}), \cdots, P_m(\overline{x})$ *in* $n+1$ *variables with respectively hypersurfaces* $S_1, S_2, \cdots, S_m$. *Then there is a combinatorial manifold* $\widetilde{M}$ *in* $\mathbb{C}^{n+1}$ *such that* $\pi: \widetilde{M} \to \widetilde{S}$ *is* $1-1$ *with* $G^L\left[\widetilde{M}\right] \simeq G^L\left[\widetilde{S}\right]$, *where,* $\widetilde{S} = \bigcup\limits_{i=1}^{m} S_i$.

Particularly, if $n = 2$, we can further determine the genus of surface $g\left(\widetilde{S}\right)$ by closed formula as follows.

Theorem 3.18([26]) *Let* $C_1, C_2, \cdots, C_m$ *be complex curves determined by homogeneous polynomials* $P_1(x,y,z), P_2(x,y,z), \cdots, P_m(x,y,z)$ *without common component, and let*

$$R_{P_i,P_j} = \prod_{k=1}^{\deg(P_i)\deg(P_j)} \left(c_k^{ij} z - b_k^{ij} y\right)^{e_k^{ij}}, \quad \omega_{i,j} = \sum_{k=1}^{\deg(P_i)\deg(P_j)} \sum_{e_k^{ij} \neq 0} 1$$

be the resultant of $P_i(x,y,z), P_j(x,y,z)$ *for* $1 \le i \ne j \le m$. *Then there is an orientable surface* $\widetilde{S}$ *in* $\mathbb{R}^3$ *of genus*

$$\begin{aligned} g\left(\widetilde{S}\right) &= \beta\left(G\left\langle \widetilde{C} \right\rangle\right) + \sum_{i=1}^{m}\left(\frac{(\deg(P_i)-1)(\deg(P_i)-2)}{2} - \sum_{p^i \in \mathrm{Sing}(C_i)} \delta(p^i)\right) \\ &+ \sum_{1 \le i \ne j \le m} (\omega_{i,j}-1) + \sum_{i \ge 3} (-1)^i \sum_{C_{k_1} \bigcap \cdots \bigcap C_{k_i} \ne \emptyset} \left[c\left(C_{k_1} \bigcap \cdots \bigcap C_{k_i}\right) - 1\right] \end{aligned}$$

with a homeomorphism $\varphi: \widetilde{S} \to \widetilde{C} = \bigcup\limits_{i=1}^{m} C_i$. *Furthermore, if* $C_1, C_2, \cdots, C_m$ *are non-singular, then*

$$\begin{aligned} g\left(\widetilde{S}\right) &= \beta\left(G\left\langle \widetilde{C} \right\rangle\right) + \sum_{i=1}^{m} \frac{(\deg(P_i)-1)(\deg(P_i)-2)}{2} \\ &+ \sum_{1 \le i \ne j \le m} (\omega_{i,j}-1) + \sum_{i \ge 3} (-1)^i \sum_{C_{k_1} \bigcap \cdots \bigcap C_{k_i} \ne \emptyset} \left[c\left(C_{k_1} \bigcap \cdots \bigcap C_{k_i}\right) - 1\right], \end{aligned}$$

where

$$\delta(p^i) = \frac{1}{2}\left(I_{p^i}\left(P_i, \frac{\partial P_i}{\partial y}\right) - \nu_\phi(p^i) + |\pi^{-1}(p^i)|\right)$$

is a positive integer with a ramification index $\nu_\phi(p^i)$ *for* $p^i \in \mathrm{Sing}(C_i), 1 \leq i \leq m$.

Theorem 3.17 enables us to find interesting results in projective geometry, for instance the following result.

Corollary 3.19 *Let* $C_1, C_2, \cdots, C_m$ *be complex non-singular curves determined by homogeneous polynomials* $P_1(x,y,z), P_2(x,y,z), \cdots, P_m(x,y,z)$ *without common component and* $C_i \bigcap C_j = \bigcap\limits_{i=1}^{m} C_i$ *with* $\left|\bigcap\limits_{i=1}^{m} C_i\right| = \kappa > 0$ *for integers* $1 \leq i \neq j \leq m$. *Then the genus of normalization* $\widetilde{S}$ *of curves* $C_1, C_2, \cdots, C_m$ *is*

$$g(\widetilde{S}) = g(\widetilde{S}) = (\kappa - 1)(m-1) + \sum_{i=1}^{m} \frac{(\deg(P_i) - 1)(\deg(P_i) - 2)}{2}.$$

Particularly, if $C_1, C_2, \cdots, C_m$ *are distinct lines in* $\mathbb{P}^2\mathbf{C}$ *with respective normalizations of spheres* $S_1, S_2, \cdots, S_m$. *Then there is a normalization of surface* $\widetilde{S}$ *of* $C_1, C_2, \cdots, C_m$ *with genus* $\beta\left(G\left\langle\widetilde{L}\right\rangle\right)$. *Furthermore, if* $G\left\langle\widetilde{L}\right\rangle)$ *is a tree, then* $\widetilde{S}$ *is homeomorphic to a sphere.*

3.4 Combinatorial Geometry

Furthermore, we can establish combinatorial geometry by Notion 3.2. For example, we have 3 classical geometries, i.e., *Euclidean, hyperbolic geometry* and *Riemannian geometries* for describing behaviors of objects in spaces with different axioms following:

Euclid Geometry:

(A1) *There is a straight line between any two points.*

(A2) *A finite straight line can produce a infinite straight line continuously.*

(A3) *Any point and a distance can describe a circle.*

(A4) *All right angles are equal to one another.*

(A5) *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Hyperbolic Geometry:

Axioms $(A1)-(A4)$ and the axiom $(L5)$ following:

(L5) *there are infinitely many lines parallel to a given line passing through an exterior point.*

Riemannian Geometry:

Axioms $(A1)-(A4)$ and the axiom $(R5)$ following:

there is no parallel to a given line passing through an exterior point.

Then whether there is a geometry established by combining the 3 geometries, i.e., partially Euclidean and partially hyperbolic or Riemannian. Today, we have know such theory really exists, called *Smarandache geometry* defined following.

Definition 3.20([12]) *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969).

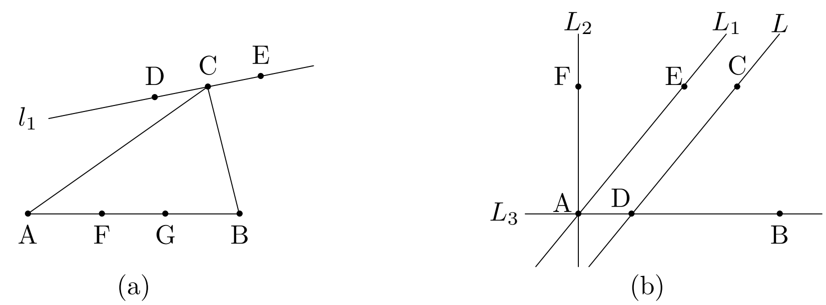


Fig.10

For example, let us consider a Euclidean plane $\mathbb{R}^2$ and three non-collinear points A, B and C shown in Fig.10. Define s-points as all usual Euclidean points on $\mathbb{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C. Then such a geometry is a Smarandache geometry by the following observations.

Observation 1. The axiom (E1) that through any two distinct points there exist one line passing through them is now replaced by: *one s-line* and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.10(a).

Observation 2. The axiom (E5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*. Let L be an s-line passes through C and D on L, and AE is parallel to CD in the Euclidean sense. Then there is one and only one line passing through E which is parallel to L, but passing a point not on AE, for instance, point F there are no lines parallel to L such as those shown in Fig.10(b).

Generally, we can construct a Smarandache geometry on smoothly combinatorial manifolds $\widetilde{M}$, i.e., combinatorial geometry because it is homeomorphic to combinatorial Euclidean space $\mathscr{E}_{G^L}(n_1, n_2, \cdots, n_m)$ by Definition 3.6 and Theorem 3.7. Such a theory is founded on the results for basis of tangent and cotangent vectors following.

Theorem 3.21([15]) *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix $\left[\frac{\partial}{\partial\overline{x}}\right]_{s(p)\times n_{s(p)}} =$

$$\left[\begin{array}{cccccccc} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{array}\right]$$

where $x^{il} = x^{jl}$ *for* $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, *namely there is a smoothly functional matrix* $[v_{ij}]_{s(p)\times n_{s(p)}}$ *such that for any tangent vector* $\overline{v}$ *at a point* p *of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$,

$$\overline{v} = \left\langle [v_{ij}]_{s(p)\times n_{s(p)}}, [\frac{\partial}{\partial\overline{x}}]_{s(p)\times n_{s(p)}} \right\rangle,$$

where $\langle [a_{ij}]_{k\times l}, [b_{ts}]_{k\times l}\rangle = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{l} a_{ij}b_{ij}$, *the inner product on matrixes.*

Theorem 3.22([15]) *For* $\forall p \in (\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ *with a local chart* $(U_p; [\varphi_p])$, *the dimension of* $T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$ *is*

$$\dim T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)}(n_i - \widehat{s}(p))$$

with a basis matrix $[d\overline{x}]_{s(p)\times n_{s(p)}} =$

$$\left[\begin{array}{cccccccc} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\widehat{s}(p)}}{s(p)} & dx^{1(\widehat{s}(p)+1)} & \cdots & dx^{1n_1} & \cdots & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\widehat{s}(p)}}{s(p)} & dx^{2(\widehat{s}(p)+1)} & \cdots & dx^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\widehat{s}(p)}}{s(p)} & dx^{s(p)(\widehat{s}(p)+1)} & \cdots & \cdots & dx^{s(p)n_{s(p)}-1} & dx^{s(p)n_{s(p)}} \end{array}\right]$$

where $x^{il} = x^{jl}$ *for* $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$, *namely for any co-tangent vector* d *at a point* p *of* $\widetilde{M}(n_1, n_2, \cdots, n_m)$, *there is a smoothly functional matrix* $[u_{ij}]_{s(p)\times s(p)}$ *such that,*

$$d = \left\langle [u_{ij}]_{s(p)\times n_{s(p)}}, [d\overline{x}]_{s(p)\times n_{s(p)}} \right\rangle.$$

Then we can establish tensor theory with connections on smoothly combinatorial manifolds ([15]). For example, we can establish the curvatures on smoothly combinatorial manifolds, and get the curvature $\widetilde{R}$ formula following.

Theorem 3.23([18]) *Let* $\widetilde{M}$ *be a finite combinatorial manifold,* $\widetilde{R} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to C^\infty(\widetilde{M})$ *a curvature on* $\widetilde{M}$. *Then for* $\forall p \in \widetilde{M}$ *with a local chart* $(U_p; [\varphi_p])$,

$$\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)}dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda},$$

where

$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

and $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

This enables us to characterize the combination of classical fields, such as the Einstein's gravitational fields and other fields on combinatorial spacetimes and hold their behaviors (See [19]-[20] for details).

§4. Differential Equation's Combinatorics

Let

$$(Eq_m) \quad \begin{cases} f_1(x_1, x_2, \cdots, x_{n+1}) = 0 \\ f_2(x_1, x_2, \cdots, x_{n+1}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, x_2, \cdots, x_{n+1}) = 0 \end{cases}$$

be a system of equations. It should be noted that the classical theory on equations is not combinatorics. However, the solutions of an equation usually form a manifold in the view of geometry. Thus, the CC conjecture bring us combinatorial notions for developing equation theory similar to that of geometry further.

Notion 4.1 *For a system (Eq_m) of equations, solvable or non-solvable, determine its underlying topological structure $G^L[Eq_m]$ on each solution manifold and classify them by graph isomorphisms and transformations.*

Notion 4.2 *For an integer $m \geq 1$, let $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_m$ be the solution manifolds of an equation system (Eq_m) in Definition 1.2 and $\widetilde{\mathscr{D}}$ underlying $G^L\left[\widetilde{\mathscr{D}}\right]$ with $\widetilde{\mathscr{D}} = \bigcup\limits_{i=1}^{m} \mathscr{D}_i$, i.e., a combinatorial solution manifold. Characterize the system (Eq_m) and establish an equation theory, i.e., equation's combinatorics on (Eq_m).*

Geometrically, let

$$S_{f_i} = \{(x_1, x_2, \cdots, x_{n+1}) | f_i(x_1, x_2, \cdots, x_{n+1}) = 0\} \subset \mathbb{R}^{n+1}$$

the solution-manifold in $\mathbb{R}^{n+1}$ for integers $1 \leq i \leq m$, where f_i is a function hold with conditions of the implicit function theorem for $1 \leq i \leq m$. Then we are easily finding criterions on the solubility of system (ES_m), i.e., it is solvable or not dependent on

$$\bigcap_{i=1}^{m} S_{f_i} \neq \emptyset \quad \text{or} \quad = \emptyset.$$

Whence, if the intersection is empty, i.e., (ES_m) is non-solvable, there are no meanings in classical theory on equations, but it is important for hold the global behaviors of a complex thing. For such an objective, Notions 4.1 and 4.2 are helpful.

Let us begin at a linear differential equations system such as those of

$$\dot{X} = A_1 X, \cdots, \dot{X} = A_k X, \cdots, \dot{X} = A_m X \qquad (LDES_m^1)$$

or

$$\begin{cases} x^{(n)} + a_{11}^{[0]} x^{(n-1)} + \cdots + a_{1n}^{[0]} x = 0 \\ x^{(n)} + a_{21}^{[0]} x^{(n-1)} + \cdots + a_{2n}^{[0]} x = 0 \\ \cdots\cdots\cdots\cdots \\ x^{(n)} + a_{m1}^{[0]} x^{(n-1)} + \cdots + a_{mn}^{[0]} x = 0 \end{cases} \qquad (LDE_m^n)$$

with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

For example, let (LDE_6^2) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} + 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} + 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} + 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} + 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} + 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} + 7\dot{x} + 6x = 0 & (6) \end{cases}$$

Certainly, it is non-solvable. However, we can easily solve equations (1)-(6) one by one and get their solution spaces as follows:

$$\begin{aligned} S_1 &= \langle e^{-t},\ e^{-2t} \rangle = \{C_1 e^{-t} +\ C_2 e^{-2t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 3\dot{x} + 2x = 0\} \\ S_2 &= \langle e^{-2t}, e^{-3t} \rangle = \{C_1 e^{-2t} + C_2 e^{-3t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 5\dot{x} + 6x = 0\} \\ S_3 &= \langle e^{-3t}, e^{-4t} \rangle = \{C_1 e^{-3t} + C_2 e^{-4t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 7\dot{x} + 12x = 0\} \\ S_4 &= \langle e^{-4t}, e^{-5t} \rangle = \{C_1 e^{-4t} + C_2 e^{-5t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 9\dot{x} + 20x = 0\} \\ S_5 &= \langle e^{-5t}, e^{-6t} \rangle = \{C_1 e^{-5t} + C_2 e^{-6t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 11\dot{x} + 30x = 0\} \\ S_6 &= \langle e^{-6t},\ e^{-t} \rangle = \{C_1 e^{-6t} +\ C_2 e^{-t} | C_1, C_2 \in \mathbb{R}\} = \{x | \ddot{x} + 7\dot{x} + 6x = 0\} \end{aligned}$$

Replacing each Σ_i by solution space S_i in Definition 1.2, we get a topological graph

$G^L[LDE_6^2]$ shown in Fig.11 on the linear homogeneous differential equation system (LDE_6^2). Thus we can solve a system of linear homogeneous differential equations on its underlying graph G^L, no matter it is solvable or not in the classical sense.

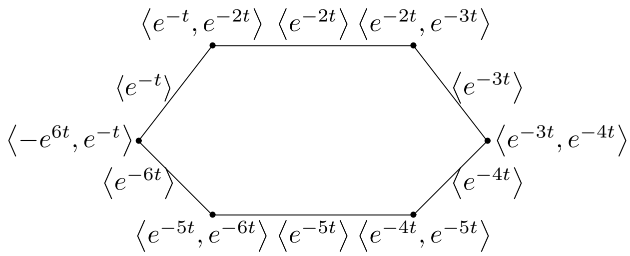


Fig.11

Generally, we know a result on G^L-solutions of homogeneous equations following.

Theorem 4.3([25]) *A linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) has a unique G^L-solution, and for every H^L labeled with linear spaces $\langle \overline{\beta}_i(t)e^{\alpha_i t}, 1 \leq i \leq n\rangle$ on vertices such that*

$$\langle \overline{\beta}_i(t)e^{\alpha_i t}, 1 \leq i \leq n\rangle \bigcap \langle \overline{\beta}_j(t)e^{\alpha_j t}, 1 \leq j \leq n\rangle \neq \emptyset$$

if and only if there is an edge whose end vertices labeled by $\langle \overline{\beta}_i(t)e^{\alpha_i t}, 1 \leq i \leq n\rangle$ and $\langle \overline{\beta}_j(t)e^{\alpha_j t}$, $1 \leq j \leq n\rangle$ respectively, then there is a unique linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) with G^L-solution H^L, where α_i is a k_i-fold zero of the characteristic equation, $k_1 + k_2 + \cdots + k_s = n$ and $\overline{\beta}_i(t)$ is a polynomial in t with degree$\leq k_i - 1$.

Applying G^L-solution, we classify such systems by graph isomorphisms.

Definition 4.4 *A vertex-edge labeling $\theta :\ G \to \mathbb{Z}^+$ is said to be integral if $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ}, and two integral labeled graphs $G_1^{I_\theta}$ and $G_2^{I_\tau}$ are called identical if $G_1 \overset{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1) \bigcup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$.*

For example, $G_1^{I_\theta} = G_2^{I_\tau}$ but $G_1^{I_\theta} \neq G_3^{I_\sigma}$ for integral graphs shown in Fig.12.

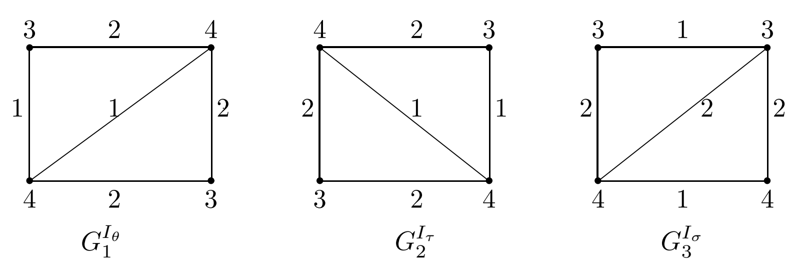


Fig.12

The following result classifies the systems $(LDES_m^1)$ and (LDE_m^n) by graphs.

Theorem 4.5([25]) *Let $(LDES_m^1)$, $(LDES_m^1)'$ (or (LDE_m^n), $(LDE_m^n)'$) be two linear homogeneous differential equation systems with integral labeled graphs H, H'. Then $(LDES_m^1) \overset{\mathcal{L}}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \overset{\mathcal{L}}{\simeq} (LDE_m^n)'$) if and only if $H = H'$.*

For partial differential equations, let

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \end{cases} \qquad (PDES_m)$$

be such a system of first order on a function $u(x_1, \cdots, x_n, t)$ with continuous $F_i : \mathbf{R}^n \to \mathbf{R}^n$ such that $F_i(\overline{0}) = \overline{0}$.

Definition 4.6 *The symbol of $(PDES_m)$ is determined by*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n) = 0, \end{cases}$$

i.e., substitutes $u_{x_1}, u_{x_2}, \cdots, u_{x_n}$ by $p_1, p_2, \cdots, p_n$ in $(PDES_m)$, and it is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.

For example, the system of partial differential equations following

$$\begin{cases} (z-y)u_x + (x-z)u_y + (y-x)u_z = 0 \\ zu_x + xu_y + yu_z = x^2 + y^2 + z^2 + 1 \\ yu_x + zu_y + xu_z = x^2 + y^2 + z^2 + 4 \end{cases}$$

is algebraically contradictory because its symbol

$$\begin{cases} (z-y)p_1 + (x-z)p_2 + (y-x)p_3 = 0 \\ zp_1 + xp_2 + yp_3 = x^2 + y^2 + z^2 + 1 \\ yp_1 + zp_2 + xp_3 = x^2 + y^2 + z^2 + 4 \end{cases}$$

is contradictory. Generally, we know a result for Cauchy problem on non-solvable systems of partial differential equations of first order following.

Theorem 4.7([28]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \leq k \leq m$$

is algebraically contradictory, in this case, there must be an integer k_0, $1 \leq k_0 \leq m$ such that

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer j_0, $1 \leq j_0 \leq n-1$ such that

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

According to Theorem 4.7, we know conditions for uniquely G^L-solution of Cauchy problem on system of partial differential equations of first order following.

Theorem 4.8([28]) *A Cauchy problem on system $(PDES_m)$ of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ for the kth equation in $(PDES_m)$, $1 \leq k \leq m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G^L-solvable, i.e., $G^L[PDES]$ is uniquely determined.

Applying the G^L-solution of a G^L-system (DES_m) of differential equations, the global stability, i.e, sum-stable or prod-stable of (DES_m) can be introduced. For example, the sum-stability of (DES_m) is defined following.

Definition 4.9 *Let (DES_m^C) be a Cauchy problem on a system of differential equations in $\mathbb{R}^n$, $H^L \leq G^L\left[DES_m^C\right]$ a spanning subgraph, and $u^{[v]}$ the solution of the vth equation with initial value $u_0^{[v]}$, $v \in V\left(H^L\right)$. It is sum-stable on the subgraph H^L if for any number $\varepsilon > 0$ there* *exists,* $\delta_v > 0,\ v \in V\left(H^L\right)$ *such that each* $G^L(t)$*-solution with*

$$\left|{u'}_0^{[v]} - u_0^{[v]}\right| < \delta_v, \qquad \forall v \in V\left(H^L\right)$$

exists for all $t \geq 0$ *and the inequality*

$$\left|\sum_{v\in V(H^L)} {u'}^{[v]} - \sum_{v\in V(H^L)} u^{[v]}\right| < \varepsilon$$

holds, denoted by $G^L[t] \overset{H}{\sim} G^L[0]$ *and* $G^L[t] \overset{\Sigma}{\sim} G^L[0]$ *if* $H^L = G^L\left[DES_m^C\right]$*. Furthermore, if there exists a number* $\beta_v > 0,\ v \in V\left(H^L\right)$ *such that every* $G^{L'}[t]$*-solution with*

$$\left|{u'}_0^{[v]} - u_0^{[v]}\right| < \beta_v, \qquad \forall v \in V\left(H^L\right)$$

satisfies

$$\lim_{t\to\infty}\left|\sum_{v\in V(H)} {u'}^{[v]} - \sum_{v\in V(H^L)} u^{[v]}\right| = 0,$$

then the $G^L[t]$*-solution is called asymptotically stable, denoted by* $G^L[t] \xrightarrow{H} G^L[0]$ *and* $G^L[t] \xrightarrow{\Sigma} G^L[0]$ *if* $H^L = G^L\left[DES_m^C\right]$*.*

For example, let the system $(SDES_m^C)$ be

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\}\ 1 \leq i \leq m \qquad \left(SDES_m^C\right)$$

and a point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for an integer $1 \leq i \leq m$ is *equilibrium* of the ith equation in $(SDES_m^C)$. A result on the sum-stability of $(SDES_m^C)$ is obtained in [30] following.

Theorem 4.10([28]) *Let* $X_0^{[i]}$ *be an equilibrium point of the* i*th equation in* $(SDES_m^C)$ *for each integer* $1 \leq i \leq m$*. If*

$$\sum_{i=1}^{m} H_i(X) > 0 \ \ \text{and} \ \ \sum_{i=1}^{m} \frac{\partial H_i}{\partial t} \leq 0$$

for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$*, then the system* $(SDES_m^C)$ *is sum-stable, i.e.,* $G^L[t] \overset{\Sigma}{\sim} G^L[0]$*.*

Furthermore, if

$$\sum_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0$$

for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$*, then* $G^L[t] \xrightarrow{\Sigma} G^L[0]$*.*

§5. Field's Combinatorics

The modern physics characterizes particles by fields, such as those of scalar field, Maxwell field, Weyl field, Dirac field, Yang-Mills field, Einstein gravitational field, $\cdots$, etc., which are in fact spacetime in geometry, isolated but non-combinatorics. Whence, the CC conjecture can bring us a combinatorial notion for developing field theory further, which enables us understanding the world and discussed extensively in the first edition of [13] in 2009, and references [18]-[20].

Notion 5.1 *Characterize the geometrical structure, particularly, the underlying topological structure $G^L[\mathscr{D}]$ of spacetime $\mathscr{D}$ on all fields appeared in theoretical physics.*

Notice that the essence of Notion 5.1 is to characterize the geometrical spaces of particles. Whence, it is in fact equivalent to Notion 3.1.

Notion 5.2 *For an integer $m \geq 1$, let $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_m$ be spacetimes in Definition* 1.2 *and $\widetilde{\mathscr{D}}$ underlying $G^L\left[\widetilde{\mathscr{D}}\right]$ with $\widetilde{\mathscr{D}} = \bigcup\limits_{i=1}^{m} \mathscr{D}_i$, i.e., a combinatorial spacetime. Select suitable Lagrangian or Hamiltonian density $\widetilde{\mathscr{L}}$ to determine field equations of $\widetilde{\mathscr{D}}$, hold with the principle of covariance and characterize its global behaviors.*

There are indeed such fields, for instance the gravitational waves in Fig.13.

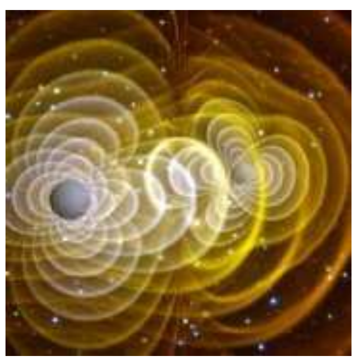


Fig.13

A combinatorial field $\widetilde{\mathscr{D}}$ is a combination of fields underlying a topological graph G^L with actions between fields. For this objective, a natural way is to characterize each field C_i, $1 \leq i \leq n$ of them by itself reference frame $\{\overline{x}\}$. Whence, the principles following are indispensable.

Action Principle of Fields. *There are always exist an action $\overrightarrow{A}$ between two fields C_1 and C_2 of a combinatorial field if $\dim(C_1 \cap C_2) \geq 1$, which can be found at any point on a spatial direction in their intersection.*

Thus, a combinatorial field depends on graph $G^L\left[\widetilde{\mathscr{D}}\right]$, such as those shown in Fig.14.

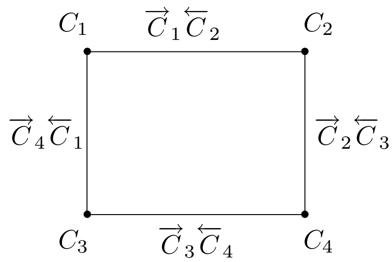


Fig.14

For understanding the world by combinatorial fields, the *anthropic principle*, i.e., *the born of human beings is not accidental but inevitable in the world* will applicable, which implies the generalized principle of covariance following.

Generalized Principle of Covariance([20]) *A physics law in a combinatorial field is invariant under all transformations on its coordinates, and all projections on its a subfield.*

Then, we can construct the Lagrangian density $\widetilde{\mathscr{L}}$ and find the field equations of combinatorial field $\widetilde{\mathscr{D}}$, which are divided into two cases ([13], first edition).

Case 1. *Linear*

In this case, the expression of the Lagrange density $\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}$ is

$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]} = \sum_{i=1}^{n} a_i \mathscr{L}_{\mathscr{D}_i} + \sum_{(\mathscr{D}_i,\mathscr{D}_j)\in E\left(G^L[\widetilde{\mathscr{D}}]\right)} b_{ij}\mathscr{T}_{ij} \ ,$$

where $a_i,\ b_{ij}$ are coupling constants determined only by experiments.

Case 2. *Non-Linear*

In this case, the Lagrange density $\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}$ is a non-linear function on $\mathscr{L}_{\mathscr{D}_i}$ and $\mathscr{T}_{ij}$ for $1 \le i, j \le n$. Let the minimum and maximum indexes j for $(M_i, M_j) \in E\left(G^L\left[\widetilde{\mathscr{D}}\right]\right)$ are i^l and i^u, respectively. Denote by

$$\overline{x} = (x_1, x_2, \cdots) = (\mathscr{L}_{\mathscr{D}_1}, \mathscr{L}_{\mathscr{D}_2}, \cdots, \mathscr{L}_{\mathscr{D}_n}, \mathscr{T}_{11^l}, \cdots, \mathscr{T}_{11^u}, \cdots, \mathscr{T}_{22^l}, \cdots,).$$

If $\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}$ is $k+1$ differentiable, $k \ge 0$, by Taylor's formula we know that

$$\begin{aligned}
\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]} \quad = \quad & \mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}(\overline{0}) + \sum_{i=1}^{n}\left[\frac{\partial \mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_i}\right]_{x_i=0} x_i + \frac{1}{2!}\sum_{i,j=1}^{n}\left[\frac{\partial^2 \mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_i \partial x_j}\right]_{x_i,x_j=0} x_i x_j \\
& + \cdots + \frac{1}{k!}\sum_{i_1,i_2,\cdots,i_k=1}^{n}\left[\frac{\partial^k \mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_{i_1}\partial x_{i_2}\cdots \partial x_{i_k}}\right]_{x_{i_j}=0, 1\le j\le k} x_{i_1}x_{i_2}\cdots x_{i_k} \\
& + R(x_1, x_2, \cdots),
\end{aligned}$$

where

$$\lim_{\|\overline{x}\|\to 0}\frac{R(x_1,x_2,\cdots)}{\|\overline{x}\|}=0,$$

and choose the first k terms

$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}(\overline{0})+\sum_{i=1}^{n}\left[\frac{\partial\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_i}\right]_{x_i=0}x_i+\frac{1}{2!}\sum_{i,j=1}^{n}\left[\frac{\partial^2\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_i\partial x_j}\right]_{x_i,x_j=0}x_ix_j$$
$$+\cdots+\frac{1}{k!}\sum_{i_1,i_2,\cdots,i_k=1}^{n}\left[\frac{\partial^k\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}\right]_{x_{i_j}=0,1\leq j\leq k}x_{i_1}x_{i_2}\cdots x_{i_k}$$

to be the asymptotic value of Lagrange density $\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}$, particularly, the linear parts

$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}(\overline{0})+\sum_{i=1}^{n}\left[\frac{\partial\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial\mathscr{L}_{\mathscr{D}_i}}\right]_{\mathscr{L}_{\mathscr{D}_i}=0}\mathscr{L}_{M_i}+\sum_{(M_i,M_j)\in E(G^L[\widetilde{M}])}\left[\frac{\partial\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial\mathscr{T}_{ij}}\right]_{\mathscr{T}_{ij}=0}\mathscr{T}_{ij}.$$

Notice that such a Lagrange density maybe intersects. We need to consider those of Lagrange densities without intersections. For example,

$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}=\sum_{i=1}^{4}\mathscr{L}^2_{C_i}-\sum_{i=1}^{4}\mathscr{L}_{\overrightarrow{C}_i\overleftarrow{C}_{i+1}}$$

for the combinatorial field shown in Fig.14.

Then, applying the Euler-Lagrange equations, i.e.,

$$\partial_\mu\frac{\partial\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial\partial_\mu\phi_{\widetilde{\mathscr{D}}}}-\frac{\partial\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}}{\partial\phi_{\widetilde{\mathscr{D}}}}=0,$$

where $\phi_{\widetilde{\mathscr{D}}}$ is the wave function of combinatorial field $\widetilde{\mathscr{D}}(t)$, we are easily find the equations of combinatorial field $\widetilde{\mathscr{D}}$.

For example, for a combinatorial scalar field $\phi_{\widetilde{\mathscr{D}}}$, without loss of generality let

$$\phi_{\widetilde{\mathscr{D}}}=\sum_{i=1}^{n}c_i\phi_{\mathscr{D}_i}$$
$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}=\frac{1}{2}\sum_{i=1}^{n}(\partial_{\mu_i}\phi_{\mathscr{D}_i}\partial^{\mu_i}\phi_{\mathscr{D}_i}-m_i^2\phi^2_{\mathscr{D}_i})+\sum_{(\mathscr{D}_i,\mathscr{D}_j)\in E\left(G^L[\widetilde{\mathscr{D}}]\right)}b_{ij}\phi_{\mathscr{D}_i}\phi_{\mathscr{D}_j},$$

i.e., linear case

$$\mathscr{L}_{G^L[\widetilde{\mathscr{D}}]}=\sum_{i=1}^{n}\mathscr{L}_{\mathscr{D}_i}+\sum_{(\mathscr{D}_i,\mathscr{D}_j)\in E\left(G^L[\widetilde{\mathscr{D}}]\right)}b_{ij}\mathscr{T}_{ij}$$

with $\mathscr{L}_{\mathscr{D}_i}=\frac{1}{2}(\partial_{\mu_i}\phi_{\mathscr{D}_i}\partial^{\mu_i}\phi_{\mathscr{D}_i}-m_i^2\phi^2_{\mathscr{D}_i})$, $\mathscr{T}_{ij}=\phi_{\mathscr{D}_i}\phi_{\mathscr{D}_j}$, $\mu_i=\mu_{\mathscr{D}_i}$ and constants $b_{ij},\ m_i,\ c_i$ for integers $1 \leq i, j \leq n$. Then the equation of combinatorial scalar field is

$$\sum_{i=1}^{n} \frac{1}{c_i}(\partial_\mu \partial^{\mu_i} + m_i^2)\phi_{M_i} - \sum_{(M_i,M_j)\in E(G^L[\widetilde{M}])} b_{ij}\left(\frac{\phi_{M_j}}{c_i} + \frac{\phi_{M_i}}{c_j}\right) = 0.$$

Similarly, we can determine the equations on combinatorial Maxwell field, Weyl field, Dirac field, Yang-Mills field and Einstein gravitational field in theory. For more such conclusions, the reader is refers to references [13], [18]-[20] in details.

Notice that the *string theory* even if arguing endlessly by physicists, it is in fact a combinatorial field $\mathbb{R}^4 \times \mathbb{R}^7$ under supersymmetries, and the same also happens to the unified field theory such as those in the gauge field of Weinberg-Salam on Higgs mechanism. Even so, Notions 5.1 and 5.2 produce developing space for physics, merely with examining by experiment.

§6. Conclusions

The role of CC conjecture to mathematical sciences has been shown in previous sections by examples of results. Actually, it is a mathematical machinery of philosophical notion: *there always exist universal connection between things $\mathscr{T}$* with a disguise $G^L[\mathscr{T}]$ on connections, which enables us converting a mathematical system with contradictions to a compatible one ([27]), and opens thoroughly new ways for developing mathematical sciences. However, *is a topological graph an element of a mathematical system with measures, not only viewed as a geometrical figure*? The answer is YES!

Recently, the author introduces $\overrightarrow{G}$-flow in [29], i.e., an oriented graph $\overrightarrow{G}$ embedded in a topological space $\mathscr{S}$ associated with an injective mappings $L : (u,v) \to L(u,v) \in \mathscr{V}$ such that $L(u,v) = -L(v,u)$ for $\forall (u,v) \in X\left(\overrightarrow{G}\right)$ holding with conservation laws

$$\sum_{u\in N_G(v)} L(v,u) = \mathbf{0} \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right),$$

where V is a Banach space over a field $\mathscr{F}$ and showed all these $\overrightarrow{G}$-flows $\overrightarrow{G}^{\mathscr{V}}$ form a Banach space by defining

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(u,v)\in X\left(\overrightarrow{G}\right)} \|L(u,v)\|$$

for $\forall \overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{V}}$, and furthermore, Hilbert space by introducing inner product similarly, where $\|L(u,v)\|$ denotes the norm of $F(u^v)$ in $\mathscr{V}$, which enables us to get $\overrightarrow{G}$-flow solutions, i.e., combinatorial solutions on differential equations.

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Timelike-Spacelike Mannheim Pair Curves Spherical Indicators Geodesic Curvatures and Natural Lifts

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Abstract: In this paper, Mannheim curve is a timelike curve, by getting partner curve as a spacelike curve which has spacelike binormal, with respect to $\mathbb{IL}^3$ Lorentz Space, S_1^2 Lorentz sphere, or H_0^2 Hyperbolic sphere, we have calculated arc lengths of Mannheim partner curve's (T^*) , (N^*), (B^*) spherical indicator curves, arc length of (C^*) fixed pol curve, and we have calculated geodesic curves of them, and also we have figured some relations among them. In addition, if the natural lifts geodesic spray of spherical indicator curvatures of Mannheim partner curve is an integral curve, we have expressed how Mannheim Curve is supposed to be.

Key Words: Lorentz space, Mannheim curve, geodesic curvatures, geodesic spray, natural lift.

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§1. Introduction

There are a lot of researches to be done in 3-dimentional Euclidian Space on differential geometry of the curves. Especially, many theories were obtained by making connections two curves' mutual points and between Frenet Frames. Well known researches are Bertrand curves and Involute-Evolute curves, [6], [4], [7], [19]. Those curves were studied carefully in different spaces, therefore, so many results were gained. In Euclidian Space and Minkowski Space, Bertrand curves' Frenet frames and Involute-Evolute curves' Frenet frames create spherical indicator curves on unit sphere surface. Those spherical indicator curves' natural lifts and geodesic sprays are defined in [5], [16], [3], [8].

Mannheim curve was firstly defined by A. Mannheim in 1878. Any curve can be a Mannheim curve if and only if $\kappa = \lambda\left(\kappa^2 + \tau^2\right)$, λ is a nonzere constant, where curvature of curve is κ and curvature of torsion is τ. After a time, Manheim curve was redefined by Liu ve Wang. According to this new definition, when first curve's principal normal vector and second curve's binormal vector are linearly dependent, first curve was named as Mannheim curve, and second curve was named as Mannheim partner curve, [21], [10]. There are so many researches

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to be done by Mathematicians after Liu ve Wang's definition [12], [15], [15], [2].

§2. Preliminaries

Let$\alpha : I \to \mathbb{E}^3$, $\alpha(t) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$ be unit speed differentiable curve. If we symbolize $\alpha : I \to \mathbb{E}^3$ curve's Frenet as $\{T, N, B\}$, curvature as κ, and torsion as τ, there are some equations among them;

$$\begin{cases} T'(s) = \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) = -\tau(s)N(s). \end{cases} \tag{2.1}$$

By using (2.1), we can get W Darboux vector as;

$$W = \tau T + \kappa B \tag{2.2}$$

If φ is the angle which is between W and B, the unit Darboux vector is that;

$$C = \sin\varphi T + \cos\varphi B \tag{2.3}$$

Let X be differentiable vector space on M.(Note that M is any vector space)

$$\frac{d}{ds}\left(\alpha\left(s\right)\right) = X\left(\alpha\left(s\right)\right) \tag{2.4}$$

α curve is an integral curve of X if and only if $\frac{d}{ds}\left(\alpha\left(s\right)\right) = X\left(\alpha\left(s\right)\right)$.

Suppose that $TM = \bigcup_{P \in M} T_M\left(P\right)$, then,

$$\overline{\alpha} : I \to TM, \overline{\alpha}\left(s\right) = \left(\alpha\left(s\right), \alpha'\left(s\right)\right) \tag{2.5}$$

$\overline{\alpha} : I \to TM$ curve is natural lift of $\alpha : I \to M$ and for $v \in TM$

$$X\left(v\right) = -\left\langle v, S\left(v\right)\right\rangle \mathrm{N}\,|_P$$

X vector space is called geodesic spray [5], [17]. Where

$$\overline{D}_X Y = D_X Y + \left\langle S\left(X\right), Y\right\rangle \mathrm{N} \tag{2.6}$$

The equation of (2.6) is a Gauss equation on M. $(T), (N)$ and (B) spherical indicator curves' equations and (C) pol curve's equations are given respectively;

$$\begin{cases} \alpha_T(s) = T(s) \\ \alpha_N(s) = N(s) \\ \alpha_B(s) = B(s) \\ \alpha_C(s) = C(s) \end{cases}$$

With respect to $\mathbb{E}^3$, arc lengths and geodesic curvatures of those curves are given respectively;

$$s_T = \int_0^s \kappa ds,\ s_N = \int_0^s \|W\| ds,\ \ s_B = \int_0^s \tau ds,\ s_C = \int_0^s \varphi' ds \tag{2.7}$$

$$\begin{cases} k_T = \dfrac{1}{\cos\varphi}, \\ k_N = \sqrt{1 + \left(\frac{\varphi'}{\|W\|}\right)^2}, \\ k_B = \dfrac{1}{\sin\varphi}, \\ k_C = \sqrt{1 + \left(\frac{\|W\|}{\varphi'}\right)^2}. \end{cases} \tag{2.8}$$

With respect to S^2, geodesic curvatures are given;

$$\gamma_T = \tan\varphi,\ \gamma_N = \frac{\varphi'}{\|W\|},\ \gamma_B = \cot\varphi,\ \gamma_C = \frac{\|W\|}{\varphi'}\ \ (see[9]) \tag{2.9}$$

Let $\overline{\alpha} : I \to \chi(M)$ be natural lift of $\alpha : I \to M$. X geodesic spray is an integral curve if and only if there is a geometric curve on M, (see [5]).

$$g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\ ,\ g(X,Y) = x_1y_1 + x_2y_2 - x_3y_3$$

This inner product space is defined as Lorentz Space and symbolized as $\mathbb{IL}^3$.$X \in \mathbb{IL}^3$ vector's norm is $\|X\|_{\mathbb{IL}} = \sqrt{\ |g(X,X)|}$. For $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{IL}^3$

$$X \times Y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$$

$X \times Y$ is called vector product of X and Y, [1]. Let T be tangent vector of $\alpha : I \to \mathbb{IL}^3$. $\alpha : I \to \mathbb{IL}^3$ is respectively defined as:

(1) If $g(T,T) > 0$, αcurve is a spacelike curve;
(2) If $g(T,T) < 0$, αcurve is a timelike curve;
(3) If $g(T,T) = 0$, α curve is a lightlike or null curve, (see [11]).

Let $\alpha : I \to \mathbb{IL}^3$ be differentiable timelike curve. In this case, T is timelike, N and B are spacelike, and Frenet formulas are given;

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \tag{2.10}$$

(see [22]). Where

$$W = \tau T - \kappa B \tag{2.11}$$

(see [20]). In this situation, there are two cases for W Darboux vector; if W is spacelike, the Lorentzian timelike angle φ which is between $-B$ and W, then;

$$\kappa = \|W\| \cosh \varphi \ , \ \tau = \|W\| \sinh \varphi \tag{2.12}$$

$$C = \sinh \varphi T - \cosh \varphi B \tag{2.13}$$

and unit Darboux vector is;

If W timelike, κ and τ are formulized;

$$\kappa = \|W\| \sinh \varphi \ , \ \tau = \|W\| \cosh \varphi \tag{2.14}$$

and unit Darboux vector is;

$$C = \cosh \varphi T - \sinh \varphi B \tag{2.15}$$

Let $\alpha : I \rightarrow \mathbb{L}^3$ be spacelike curve which has spacelike binormal. In this case, α curve's Frenet vectors' vector product are respectively;
$T \times N = -B$, $N \times B = -T$, $B \times T = N$ and Frenet formulas are found as;

$$\begin{cases} T' = \kappa N \\ N' = \kappa T + \tau B \\ B' = \tau N \end{cases} \tag{2.16}$$

(see [22]).In this case, Darboux vector will be;

$$W = -\tau T + \kappa B, (see[20]). \tag{2.17}$$

Let φ be the angle which is between B and W. Then,

$$\kappa = \|W\| \cos \varphi \ , \ \tau = \|W\| \sin \varphi \tag{2.18}$$

and unit Darboux vector is given as;

$$C = -\sin \varphi T + \cos \varphi B \tag{2.19}$$

Let M be a Lorentz manifold, and $\overline{M}$ be a hypersurface of M. Suppose that S is a shape operator which is obtained from N normal of $\overline{M}$, D is the connection on M, $\bar{D}$ is the connection on $\overline{M}$,

For $X, Y \in \chi(\overline{M})$, Gauss Equation is;

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)N \tag{2.20}$$

Where $S(X) = -D_X N$ and $\varepsilon = g(N, N)$, [18].

$$S_1^2(r) = \left\{ X \in IR_1^3 \middle| \, g(X, X) = r^2, \ r \in IR, \ r = fixed \right\}$$

is defined as Lorentz sphere,

$$H_0^2(r) = \left\{ X \in IR_1^3 \middle| \, g(X, X) = -r^2, \ r \in IR, \ r = fixed \right\}$$

is defined as hyperbolic sphere.

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be two differentiable curves. Suppose that Frenet Frames on the points of $\alpha(s)$ and $\alpha^*(s)$ are respectively given as $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$. If α curve's principal normal vector and α^* curve's binormal vector are linearly dependent, α curve is named Mannheim curve, α^* curve is named Mannheim partner curve, and it is shown as (α, α^*), [21]. Mannheim curve's equation is given as;

$$\alpha^*(s^*) = \alpha(s) - \lambda N(s) \text{ or } \alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*) \ \ [12].$$

There are some following equations among those curves;

$$\begin{cases} T = \cos\theta T^* + \sin\theta N^* \\ N = B^* \\ B = -\sin\theta T^* + \cos\theta N^*. \end{cases} \tag{2.21}$$

$$\cos\theta = \frac{ds^*}{ds}, \ \ \sin\theta = \lambda\tau^* \frac{ds^*}{ds} \tag{2.22}$$

$$\begin{cases} T^* = \cos\theta T - \sin\theta B \\ N^* = \sin\theta T + \cos\theta B \\ B^* = N. \end{cases} \tag{2.23}$$

Where $\sphericalangle$ (T,T^*)=θ, [3] .Let κ be curvature of α, τ be torsion of α, and let κ^* be curvature of α^* , τ^* be torsion of α^*. Then, there are the following equations;

$$\begin{cases} \kappa = \tau^* \sin\theta \cdot \dfrac{ds^*}{ds} \\ \\ \tau = -\tau^* \cos\theta \cdot \dfrac{ds^*}{ds}. \end{cases} \tag{2.24}$$

$$\begin{cases} \kappa^* = \dfrac{d\theta}{ds^*} \\ \tau^* = (\kappa \sin\theta - \tau\cos\theta) \cdot \dfrac{ds^*}{ds}. \end{cases} \tag{2.25}$$

$$\tau^* = \frac{\kappa}{\lambda\tau}, \text{ [3]} \tag{2.26}$$

Let $\alpha : I \to \mathbb{E}^3$ be Mannheim curve, $\alpha^* : I \to \mathbb{E}^3$ be Mannheim partner curve. Suppose that Frenet frames are respectively given as $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$. Let θ be the angle which is between T and T^*, and let φ be the angle which is between B and W. In this case, the following equations hold.

$$C = T^* \tag{2.27}$$

$$\begin{cases} \sin\varphi = \cos\theta \\ \cos\varphi = -\sin\theta \end{cases} \tag{2.28}$$

(see [15]). If we consider (2.28), (2.8), (2.9), (2.22) and (2.23) will respectively turn the following equations;

$$\cos\theta = \frac{\tau}{\|W\|} \;\; , \; -\sin\theta = \frac{\kappa}{\|W\|} \tag{2.29}$$

$$\begin{cases} k_T = -\dfrac{1}{\sin\theta}, \\ k_N = \sqrt{1 + \left(\dfrac{\theta'}{\|W\|}\right)^2}, \\ k_B = \dfrac{1}{\cos\theta}, \;, \\ k_C = \sqrt{1 + \left(\dfrac{\|W\|}{\theta'}\right)^2} \qquad . \end{cases} \tag{2.30}$$

$$\gamma_T = -\cot\theta, \;\; \gamma_N = \frac{\theta'}{\|W\|}, \;\; \gamma_B = -\tan\theta, \;\; \gamma_C = \frac{\|W\|}{\theta'}$$

$$\sin\varphi = \frac{ds^*}{ds}, \;\; \cos\varphi = \lambda\tau^*\frac{ds^*}{ds} \tag{2.31}$$

(see [15]).

§3. Timelike-Spacelike Mannheim Curve Pairs

Definition 3.1 *Let $\alpha : I \to \mathbb{IL}^3$ be timelike curve and let $\alpha^* : I \to \mathbb{IL}^3$ be spacelike curve which has spacelike binormal. Suppose that α curve's Frenet frames on the point of $\alpha(s)$ is*

$\{T(s), N(s), B(s)\}$ and α^* curve's Frenet frames on the point of $\alpha^*(s)$ is $\{T^*(s), N^*(s), B^*(s)\}$ If α curve's principal normal vector and α^* curve's binormal vektorare linearly dependent, α curve is called Mannheim curve and α^* curve is called Mannheim partner curve. This pair curve is briefly symbolized as (α, α^*) and it is named timelike-spacelike Mannheim curve pairs.

Theorem 3.1 *The distance which is between (α, α^*) timelike-spacelikeMannheim curve pairs is constant.*

Proof It can be written that;

$$\alpha(s) = \alpha^*(s^*) + \lambda(s^*)B^*(s^*).$$

If this equation is derived with respect to s^* parameter, we can write that;

$$T\frac{ds}{ds^*} = T^* + \lambda\tau^* N^* + \lambda' B^*$$

If we get inner product of the last equation and B^*, then;

$$\lambda' = 0.$$

From the definition of Euclidean distance, we can write that;

$$\begin{aligned} d(\alpha^*(s^*), \alpha(s)) &= \|\alpha(s) - \alpha^*(s^*)\| \\ &= |\lambda| = cons\tan t \end{aligned}$$

□

Theorem 3.2 *Let (α, α^*) be timelike-spacelike Mannheim curve pairs. Suppose that α curve's and α^* curve's Frenet frames are respectively $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$. In this case, there are the following equations;*

$$\begin{cases} T^* = -\sinh\theta T + \cosh\theta B \\ N^* = -\cosh\theta T + \sinh\theta B \\ B^* = N. \end{cases} \tag{3.1}$$

$$\sinh\theta = \frac{ds^*}{ds}, \quad \cosh\theta = -\lambda\tau^*\frac{ds^*}{ds} \tag{3.2}$$

$$\begin{cases} T = \sinh\theta T^* - \cosh\theta N^* \\ N = B^* \\ B = \cosh\theta T^* - \sinh\theta N^*. \end{cases} \tag{3.3}$$

Proof If we derive $\alpha^*(s^*) = \alpha(s) - \lambda N(s)$ with respect to s parameter, we can write that;

$$T^*\frac{ds^*}{ds} = (1 - \lambda\kappa)T(s) - \lambda\tau B \tag{3.4}$$

If we get inner product of (3.4) and T, then;

$$-\sinh\theta\frac{ds^*}{ds} = 1 - \lambda\kappa \tag{3.5}$$

If we get inner product of (3.4) and B, then;

$$\cosh\theta\frac{ds^*}{ds} = \lambda\tau \tag{3.6}$$

If (3.5) and (3.6) are plugged into (3.4), we can write that;

$$T^* = -\sinh\theta T + \cosh\theta B$$

From Frenet formulas, the following equations can be found.

$$N^* = -\cosh\theta T + \sinh\theta B,$$

$$B^* = N$$

Obviously, we have shown that the equation of (3.1). If we arrange this equation with respect to T and B, we can find the equation of (3.3). If $\alpha(s) = \alpha^*(s^*) + \lambda B^*(s^*)$ is derived with respect to $s$parameter, it can be found that;

$$T = T^*\frac{ds^*}{ds} + \lambda\tau^*\frac{ds^*}{ds}N^*$$

If we consider the corresponding value of T from (3.3), the equation of (3.2) is proven.

Theorem 3.3 *Let(α, α^*)be timelike-spacelike Mannheim curve pairs. Let κ be curvature of α, and let τ be torsion of α. In this case, there is the following equation*

$$\lambda\kappa - \mu\tau = 1$$

Proof If we divide (3.5) to (3.6), we can write that;

$$\tanh\theta = \frac{\lambda\kappa - 1}{\lambda\tau}$$

If we get $\mu = \lambda\tanh\theta$, the result is proven. □

Theorem 3.4 *Let (α, α^*) be timelike-spacelike Mannheim curve pairs. In this case, there are the following equations among curvatures.*

$$\kappa = -\tau^*\cosh\theta\frac{ds^*}{ds},\ \tau = -\tau^*\sinh\theta\frac{ds^*}{ds} \tag{3.7}$$

$$\kappa^* = \frac{d\theta}{ds^*} = \theta'\frac{ds}{ds^*},\ \tau^* = -\kappa\cosh\theta\frac{ds}{ds^*} + \tau\sinh\theta\frac{ds}{ds^*}. \tag{3.8}$$

Proof If $\langle T, B^*\rangle = 0$ is derived, then; $\kappa = -\tau^* \cosh\theta \dfrac{ds^*}{ds}$,

If $\langle B, B^*\rangle = 0$ is derived, then; $\tau = -\tau^* \sinh\theta \dfrac{ds^*}{ds}$,

If $\langle T, T^*\rangle = \sinh\theta$ is derived, then; $\kappa^* = \dfrac{d\theta}{ds^*} = \theta' \dfrac{ds}{ds^*}$,

If $\langle N, N^*\rangle = 0$ is derived, then; $\tau^* = -\kappa \cosh\theta \dfrac{ds}{ds^*} + \tau \sinh\theta \dfrac{ds}{ds^*}$. Therefore, the result is proven. □

Theorem 3.5 *Let(α, α^*) be timelike-spacelike Mannheim curve pairs. Let κ^* be curvature of α^*, let τ^* be torsion of α^*, and let τ be torsion of α. The following equation holds.*

$$\tau^* = -\frac{\kappa}{\lambda\tau}. \tag{3.9}$$

Proof If we get equations from (3.2) and if we multiply side by side (3.5) and (3.6), the result is proven. □

Theorem 3.6 *Let (α, α^*) be timelike-spacelike Mannheim curve pairs. There is the following equation between α curve's W Darboux vector and α^* curve's T^* tangent vector.*

$$W = \tau^* \frac{ds^*}{ds} T^* \tag{3.10}$$

Proof We know that $W = \tau T - \kappa B$. If we get the corresponding values of T and B from (3.3), and then if we plug into those values in (3.10),Then, if we get the corresponding values of κ and τ from (3.7), and then if we plug into those values in (3.10), the result is proven. □

Result 3.1 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. Let φ be the angle which is between α curve's Darboux vector and α curve's binormal vector. There are the following equations between θ and φ.

If W is a spacelike vector;

$$\begin{cases} \sinh\varphi = -\sinh\theta \\ -\cosh\varphi = \cosh\theta \end{cases} \tag{3.11}$$

If W is timelike vector;

$$\begin{cases} \cosh\varphi = -\sinh\theta \\ -\sinh\varphi = \cosh\theta \end{cases} \tag{3.12}$$

$$\varphi' = -\theta' \tag{3.13}$$

Proof Suppose that W is a spacelike vector, and the following equations hold because of the equations of (2.13) and (3.1),

$$C = \sinh\varphi T - \cos h\varphi B,$$

$$T^* = -\sinh\theta T + \cosh\theta B$$

If we consider that $C = T^*$, we can write that;

$$\begin{cases} \sinh\varphi = -\sinh\theta \\ -\cosh\varphi = \cosh\theta \end{cases},$$

Similarly, suppose that W is a timelike vector, and from the equation of (2.15), we can write that;

$$C = \cosh\varphi\, T - \sin h\varphi\ B$$

If we consider that $C = T^*$, we can write that;

$$\begin{cases} \cos h\varphi = -\sinh\theta \\ -\sin h\varphi = \cosh\theta \end{cases}$$

If we divide two equations to each other in the equation of (3.12), we can easily write that;

$$\cot h\varphi = \tanh\theta$$

If we derive last equation, the following equation holds.

$$\varphi' = -\theta'.$$

Theorem 3.7 *Let (α, α^*) be timelike-spacelike Mannheim curve pairs. There is the following equation which is between α curve's W Darboux vector and α^* curve's W^* Darboux vector;*

$$W^* = \frac{-1}{\sinh\theta} W - \frac{\theta'\kappa}{\lambda\kappa\,\|W\|} N \tag{3.14}$$

Proof It can be written $\tau^* T^* = -W^* + \kappa^* B^*$ because α^* Manheim partner curve's W^* vector is spacelike. If this equation is plugged into (3.10), and if we consider that B^*= N, the following equation holds.

$$W = \frac{ds^*}{ds}(-W^* + \kappa^* N)$$

If the corresponding value of $\dfrac{ds}{ds^*}$ from (3.2) is written in this equation, then;

$$W^* = \frac{-1}{\sinh\theta} W + \kappa^* N \tag{3.15}$$

On the other hand, from the equation of (3.7), we can write that $\dfrac{ds}{ds^*} = \dfrac{\tau^*}{\|W\|}$. If the corresponding value of τ^* in (3.9) is written in this equation, then;

$$\frac{ds}{ds^*} = \frac{-\kappa}{\lambda\tau\,\|W\|}.$$

If we plug this equation into $\kappa^* = \theta' \dfrac{ds}{ds^*}$, the following equation holds.

$$\kappa^* = -\theta' \frac{\kappa}{\lambda\tau \|W\|} \tag{3.16}$$

If we write this value of κ^* in (3.15), the result is proven.

Let s_T be $\alpha : I \to \mathbb{IL}^3$ timelike Mannheim curve's (T) tangent indicator's arc length, then;

$$s_T = \int_0^s \kappa ds \tag{3.17}$$

Similarly, (N) principal normal, (B) binormal, and (C) fixed pol curve's arc lengths are respectively;

$$s_N = \int_0^s \|W\| \, ds, \tag{3.18}$$

$$s_B = \int_0^s |\tau| \, ds, \tag{3.19}$$

$$s_C = \int_0^s |\phi'| \, ds \tag{3.20}$$

Let k_T be (T) tangent indicator's geodesic cuvature on $\mathbb{IL}^3$. Suppose that T_T is unit tangent vector of (T), then;

$$k_T = \|D_{T_T} T_T\|$$

If $\alpha_T(s) = T(s)$ tangent indicator is derived with respect to s_T parameter, we can write that;

$$T_T = N \tag{3.21}$$

If we derive one more time and simplify the equation, the following equation holds.

$$D_{T_T} T_T = T - \frac{\tau}{\kappa} B \tag{3.22}$$

From the definition of geodesic curvature;

$$k_T = \sqrt{\left| -1 + \frac{\tau^2}{\kappa^2} \right|} \tag{3.23}$$

If we consider the equations of (2.12) and (3.11), we can write that;

$$k_T = \frac{1}{\cosh\theta} \tag{3.24}$$

Similarly, if $\alpha_N(s) = N(s)$ principal normal indicatoris derived with respect to s_N parameter, and the equation of (2.12) is plugged into (3.25), and then the equation of (3.11) is considered, we can write that;

$$T_N = -\cosh\theta T + \sinh\theta B \tag{3.25}$$

If we derive one more time, the following equation holds.

$$D_{T_N}T_N = (\theta' \sinh\theta T + \|W\| N - \theta' \cosh\theta B)\frac{1}{\|W\|} \tag{3.26}$$

From the definition of geodesic curvature;

$$k_N = \sqrt{\left|1+\left(\frac{\theta'}{\|W\|}\right)^2\right|} \tag{3.27}$$

If $\alpha_B(s) = B(s)$ binormal indicator is derived with respect to s_B parameter, and if we chose the positive routing.

$$D_{T_B}T_B = \frac{\kappa}{\tau}T - B \tag{3.28}$$

From the equation of (2.12);

$$k_B = \frac{1}{\sinh\theta} \tag{3.29}$$

If $\alpha_C(s) = C(s)$ fixed pol curve is derived with respect to s_C parameter, we can write that;

$$T_C = \cosh\varphi T - \sinh\varphi B$$

If we derive one more time, and if we consider the equations of (3.11) and (3.13), we can write that;

$$D_{T_C}T_C = -\sinh\theta T + \cosh\theta B \pm \frac{\|W\|}{\theta'}N \tag{3.30}$$

From the definition of geodesic curvature;

$$k_C = \sqrt{\left|1+\left(\frac{\|W\|}{\theta'}\right)^2\right|} \tag{3.31}$$

Result 3.2 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. α Mannheim curve's spherical indicators are $(T), (N)$, (B) and also α Mannheim curve's fixed pol curve which is (C), with respect to $\mathbb{L}^3$, geodesic curvatures of $(T), (N)$,(B) and (C) are respectively;

$$\begin{cases} k_T = \dfrac{1}{\cosh\theta} \\ \\ k_N = \sqrt{\left|1+\left(\dfrac{\theta'}{\|W\|}\right)^2\right|} \end{cases},$$

$$\begin{cases} k_B = \dfrac{1}{\sinh\theta} \\ \\ k_C = \sqrt{\left|1+\left(\dfrac{\|W\|}{\theta'}\right)^2\right|}. \end{cases}$$

Let D be $\alpha : I \to \mathbb{L}^3$ Mannheim curve's connection on $\mathbb{L}^3$, let $\bar{D}$ be $\alpha : I \to \mathbb{L}^3$ Mannheim curve's connection on S_1^2, and let $\bar{\bar{D}}$ be $\alpha : I \to \mathbb{L}^3$ Mannheim curve's connection on H_0^2.

Suppose that ξ is unit normal vector space of S_1^2 and H_0^2, then;

$$D_X Y = \bar{D}_X Y + \varepsilon g(S(X), Y)\xi, \qquad \varepsilon = g(\xi, \xi)$$

$$D_X Y = \bar{\bar{D}}_X Y + \varepsilon g(S(X), Y)\xi, \qquad \varepsilon = g(\xi, \xi).$$

Where S is shape operator of S_1^2 and H_0^2, and corresponding matrix is;

$$S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(see[?]). Let γ_T be (T) tangent indicator's geodesic curvature on H_0^2, then;

$$\gamma_T = \left\| \bar{\bar{D}}_{T_T} T_T \right\|$$

From Gauss Equation, we can write that;

$$D_{T_T} T_T = \bar{\bar{D}}_{T_T} T_T + \varepsilon g(S(T_T), T_T) T$$

where

$$\varepsilon = g(T, T) = -1, \quad S(T_T) = -T_T, \ \text{ and } \mathrm{g}(S(T_T), T_T) = -1$$

If we write those values in Gauss equation, and if we consider (3.22), we can write that;

$$\bar{\bar{D}}_{T_T} T_T = -\frac{\tau}{\kappa} B. \tag{3.32}$$

And also from the equation of (2.12) and (3.11);

$$\gamma_T = \tanh\theta. \tag{3.33}$$

Similarly, Let γ_N be geodesic curvature of (N) principal normal indicator on S_1^2, then;

$$\gamma_N = \left\| \bar{D}_{T_N} T_N \right\|$$

From Gauss Equation, we can write that;

$$D_{T_N} T_N = \bar{D}_{T_N} T_N + \varepsilon g(S(T_N), T_N) N$$

where

$$\varepsilon = g(N, N) = +1, \quad S(T_N) = -T_N, \ \text{ and } \ \mathrm{g}(S(T_N), T_N) = +1.$$

If those values are written in Gauss equation, and if we consider (3.26), we can write that;

$$\bar{D}_{T_N} T_N = \frac{\theta'}{\|W\|}(\sinh\theta T - \cosh\theta B) \tag{3.34}$$

If we get norm of the equation;

$$\gamma_N = \frac{\theta'}{\|W\|} \tag{3.35}$$

Let γ_B be geodesic curvature of (B) binormal indicator on S_1^2, then;

$$\gamma_B = \left\| \bar{D}_{T_B} T_B \right\|.$$

From Gauss equation, we can write that;

$$D_{T_B} T_B = \bar{D}_{T_B} T_B + \varepsilon g(S(T_B), T_B) B$$

where

$$\varepsilon = g(B, B) = +1, \quad S(T_B) = -T_B \text{ and } \mathrm{g}(S(T_B), T_B) = -1.$$

If we write those values in Gauss equation, and if we consider (3.28), we can write that;

$$\bar{D}_{T_B} T_B = \frac{\kappa}{\tau} T \tag{3.36}$$

From the equations of (2.12) and (3.11), it can be written that;

$$\gamma_B = \coth \theta \tag{3.37}$$

Let γ_C be geodesic curvature of (C) fixed pol curve on S_1^2, then;

$$\gamma_C = \left\| \bar{D}_{T_C} T_C \right\|.$$

From Gauss equation

$$D_{T_C} T_C = \bar{D}_{T_C} T_C + \varepsilon g(S(T_C), T_C) C$$

where

$$\varepsilon = g(C, C) = +1, \ S(T_C) = -T_C \text{ and } \mathrm{g}(S(T_C), T_C) = -1.$$

If we write those values in Gauss equation, and if we consider (2.13), (3.11) and (3.30), we can write that;

$$\gamma_C = \frac{\|W\|}{\theta'}. \tag{3.38}$$

Result 3.3 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. α Mannheim curve's spherical indicators are $(T), (N)$, (B) and also α Mannheim curve's fixed pol curve is (C), with respect to S_1^2 Lorentz sphere or H_0^2 hyperbolic sphere, geodesic curvatures of $(T), (N)$, (B) and (C) are respectively;

$$\gamma_T = \tanh \theta, \gamma_N = \frac{\theta'}{\|W\|}, \gamma_B = \coth \theta, \gamma_C = \frac{\|W\|}{\theta'}.$$

Let $\alpha^* : I \to \mathbb{IL}^3$ be spacelike Mannheim curve which has spacelike binormal, and let s_{T^*}

be $\alpha^* : I \to \mathbb{IL}^3$ curve's (T^*) tangent indicator's arc length, then,

$$s_{T^*} = \int_0^s \theta' ds \tag{3.39}$$

Similarly, arc lengths of (N^*), (B^*) and (C^*) are found;

$$s_{N^*} = \int_0^s \sqrt{\left|(\theta')^2 + \|W\|^2\right|} ds \tag{3.40}$$

$$s_{B^*} = \int_0^s \|W\| \, ds \tag{3.41}$$

$$s_{C^*} = \int_0^s (\varphi^*)' ds. \tag{3.42}$$

On the other hand, let φ^* be the angle which is between W^* and B^*. unit Darboux vector can be written as;

$$C^* = -\sin\varphi^* T^* + \cos\varphi^* B^*$$

where

$$\sin\varphi^* = \frac{\tau^*}{\|W^*\|} \text{ and } \cos\varphi^* = \frac{\kappa^*}{\|W^*\|} \Rightarrow \tan\varphi^* = \frac{\tau^*}{\kappa^*}.$$

C^* is derived and then if we simplify the equation, we can write that;

$$(\varphi^*)' = \frac{\left(\frac{\tau^*}{\kappa^*}\right)'}{1 + \left(\frac{\tau^*}{\kappa^*}\right)^2}$$

If the values of κ^* and τ^* are written in the equation, we can write that;

(Note that the values of κ^* and τ^* are corresponding values of (3.8) and (3.12)

$$(\varphi^*)' = \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1 + \left(\frac{\|W\|}{\theta'}\right)^2} \tag{3.43}$$

If (3.43) is written in the equation of (3.42), we can easily find that;

$$s_{C^*} = \int_0^s \frac{\left(\frac{\|W\|}{\theta'}\right)'}{1 + \left(\frac{\|W\|}{\theta'}\right)^2} ds \tag{3.44}$$

If (3.23) is considered, we can obtain that;

$$(\varphi^*)' = \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} \tag{3.45}$$

If (3.45) is written in the equation of (3.42), we can find that;

$$s_{C^*} = \int_0^s \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} ds. \tag{3.46}$$

Result 3.4 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. On the point of $\alpha^*(s)$, α^*curve's Frenet vectors' spherical indicator curves and (C^*) fixed pol curve which is drawn by the unit Darboux vector. In terms of $\mathbb{IL}^3$, (C^*) fixed pol curve's arc lengths are respectively;

$$\begin{cases} s_{T^*} = \displaystyle\int_0^s \theta' ds, \\ s_{N^*} = \displaystyle\int_0^s \sqrt{|(\theta')^2 + \|W\|^2} ds, \\ s_{B^*} = \displaystyle\int_0^s \|W\| ds, \\ s_{C^*} = \displaystyle\int_0^s \frac{(\frac{\|W\|}{\theta'})'}{1 + (\frac{\|W\|}{\theta'})^2} ds. \end{cases}$$

Result 3.5 Let (α, α^*) be timelike-spacelike Mannheim curve pairs, and let k_C be α timelike Mannheim curve's geodesic curvature. In this case, the arc length of (C^*) fixed pol curve is;

$$s_{C^*} = \int_0^s \frac{\left(\sqrt{k_C^2 - 1}\right)'}{k_C^2} ds.$$

In terms of $\mathbb{IL}^3$, Let k_{T^*} be $\alpha^*_{T^*}(s) = T^*(s)$ tangent indicator's geodesic curvature, let s_{T^*} be arc parameter , and let T_{T^*} be unit tangent vector . Then, we can say that;

$$T_{T^*} = -\cosh\theta T + \sinh\theta B \tag{3.47}$$

If we derive one more time the equation of (3.47), it can be written that;

$$D_{T_{T^*}} T_{T^*} = -\sinh\theta T + \cosh\theta B + \frac{\|W\|}{\theta'} N \tag{3.48}$$

or if we get norm of (3.48), we can write that;

$$k_{T^*} = \sqrt{\left|1 + \left(\frac{\|W\|}{\theta'}\right)^2\right|} \tag{3.49}$$

Similarly, in terms of $\mathbb{IL}^3$, Let k_{N^*} be $\alpha^*_{N^*}(s) = N^*(s)$ principal normal indicator's geodesic curvature, let s_{N^*} be arc parameter, and let T_{N^*} be unit tangent vector. Then, we can say that;

$$T_{N^*}=\frac{1}{\sqrt{\left|1+\left(\frac{\|W\|}{\theta'}\right)^2\right|}}(-\sinh\theta T+\cosh\theta B)+\frac{1}{\sqrt{\left|1+\left(\frac{\theta'}{\|W\|}\right)^2\right|}}N \tag{3.50}$$

If we consider the equations of (3.27) and (3.31), it can be written that;

$$T_{N^*}=\frac{1}{k_C}(-\sinh\theta T+\cosh\theta B)+\frac{1}{k_N}N \tag{3.51}$$

If we derive one more time the equation of (3.51), after simplifying, it can be written that;

$$\begin{aligned} D_{T_{N^*}}T_{N^*} \quad = \quad & \left(\left[\frac{\left(\frac{-\sinh\theta}{k_C}\right)'+\left(\frac{\kappa}{k_N}\right)}{\sqrt{|(\theta')^2+\|W\|^2|}}\right]T+\left[\frac{\left(\frac{1}{k_N}\right)'}{\sqrt{|(\theta')^2+\|W\|^2|}}\right]N\right. \\ & \left.+\left[\frac{\left(\frac{\cosh\theta}{k_C}\right)'-\frac{\tau}{k_N}}{\sqrt{|(\theta')^2+\|W\|^2|}}\right]B\right) \end{aligned} \tag{3.52}$$

If we get norm of (3.52), we can write that;

$$k_{N^*}=\sqrt{\frac{\left[\left(\frac{-\sinh\theta}{k_C}\right)'+\left(\frac{\kappa}{k_N}\right)\right]^2+\left[\left(\frac{1}{k_N}\right)'\right]^2+\left[\left(\frac{\cosh\theta}{k_C}\right)'-\frac{\tau}{k_N}\right]^2}{(\theta')^2+\|W\|^2}}. \tag{3.53}$$

In terms of $\mathbb{IL}^3$, Let k_{B^*} be $\alpha^*_{B^*}(s)=B^*(s)$ binormal indicator's geodesic curvature, let s_{B^*} be arc parameter, and let T_{B^*} be unit tangent vector. Then, we can say that;

$$T_{B^*}=\frac{\kappa}{\|W\|}T-\frac{\tau}{\|W\|}B \tag{3.54}$$

If we consider the equations of (2.12) and (3.11), it can be written that;

$$T_{B^*}=-\cosh\theta T+\sinh\theta B$$

If we derive, T_{B^*} after simplifying, it can be written that;

$$D_{T_{B^*}}T_{B^*}=\frac{\theta'}{\|W\|}(-\sinh\theta T+\cosh\theta B)+N \tag{3.55}$$

or if we get norm of T_{B^*}, we can write that;

$$k_{B^*}=\sqrt{1+\left(\frac{\theta'}{\|W\|}\right)^2} \tag{3.56}$$

In terms of $\mathbb{IL}^3$, Let k_{C^*} be $\alpha^*_{C^*}(s)=C^*(s)$ fixed pol curve's geodesic curvature, let s_{C^*} be arc parameter, and let T_{C^*} be unit tangent vector. Then, we can say that;

$$T_{C^*} = -\cos\varphi^* T^* + \sin\varphi^* B^* \tag{3.57}$$

If we derive one more time the equation of (3.57), it can be written that;

$$D_{T_{C^*}} T_{C^*} = -\sin\varphi^* T^* + \cos\varphi^* B^* - \frac{\|W^*\|}{(\varphi^*)'} N^* \tag{3.58}$$

If we get norm of (3.57), we can write that;

$$k_{C^*} = \sqrt{\left| 1 + \left(\frac{\|W^*\|}{(\varphi^*)'} \right)^2 \right|} \tag{3.59}$$

If the values of κ^* and τ^* are written in $\|W^*\| = \sqrt{|(\tau^*)^2 + (\kappa^*)^2|}$, we can find that (Note that the value of κ^* and τ^* are corresponding values of (3.8) and (3.9).)

$$\|W^*\| = \frac{\kappa}{\lambda\tau} \sqrt{\left| 1 + \left(\frac{\|W\|}{(\varphi)'} \right)^2 \right|}$$

From the equation of (3.31) and (3.45), it can be written that;

$$\frac{\|W^*\|}{(\varphi^*)'} = \frac{\kappa (k_C)^3}{\lambda\tau \left(\sqrt{k_C^2 - 1} \right)'} \tag{3.60}$$

If the value of (3.60) is written in, we can say that;

$$k_{C^*} = \sqrt{\left| 1 + \left(\frac{\kappa (k_C)^3}{\lambda\tau \left(\sqrt{k_C^2 - 1} \right)'} \right)^2 \right|}$$

Result 3.6 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. In terms of $\mathbb{IL}^3$, α^* curve's $(T^*), (N^*), (B^*)$ spherical indicator curves' and (C^*) fixed pol curve's geodesic curvatures are respectively;

$$\begin{cases} k_{T^*} = \sqrt{\left|1+(\dfrac{\|W\|}{\theta'})^2\right|}, \\ \\ k_{N^*} = \sqrt{\dfrac{[(\frac{-\sinh\theta}{k_C})'+(\frac{\kappa}{k_N})]^2+[(\frac{1}{k_N})']^2+[(\frac{\cosh\theta}{k_C})'-\frac{\tau}{k_N}]^2}{(\theta')^2+\|W\|^2}}, \\ \\ k_{B^*} = \sqrt{1+\Big(\dfrac{\theta'}{\|W\|}\Big)^2}, \\ \\ k_{C^*} = \sqrt{\left|1+\Big(\dfrac{\kappa(k_C)^3}{\lambda\tau(\sqrt{k_C^2-1})'}\Big)^2\right|}. \end{cases}$$

Let γ_{T^*} be $\alpha^* : I \to \mathbb{IL}^3$ spacelike binormal spacelike Mannheim partner curve's $\alpha^*_{T^*}(s) = T^*(s)$ tangent indicator's geodesic curvature in S_1^2. Then;

$$\gamma_{T^*} = \left\|\bar{D}_{T_{T^*}} T_{T^*}\right\|$$

From Gauss equation, it can be written that;

$$D_{T_{T^*}} T_{T^*} = \bar{D}_{T_{T^*}} T_{T^*} + \varepsilon g(S(T_{T^*}), T_{T*})T^*$$

where

$$\varepsilon = g(T^*, T^*) = +1,\ S(T_{T^*}) = -T_{T^*} \text{ and } \mathrm{g}(S(T_{T^*}), T_{T^*}) = +1.$$

If those values are written in Gauss equation, and if the equation of (3.1) and (3.48) are considered, we can say that;

$$\bar{D}_{T_{T^*}} T_{T^*} = \left(\frac{\|W\|}{\theta'}\right) N \tag{3.61}$$

If we get norm of (3.61), we can write that;

$$\gamma_{T^*} = \frac{\|W\|}{\theta'}$$

$\overline{D}_{T_{T^*}} T_{T^*} = 0$ if and only if $\left(\overline{T^*}\right)$ curve geodesic spray is an integral curve. From the equation of (3.61), we can find that; $\kappa = 0, \tau = 0$ which means α is a straight line.

Result 3.7 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. α Mannheim curve does not have any partner curve because α Mannheim curveis a straight line.

Let γ_{N^*} be geodesic indicator in H_0^2 for $\alpha^*_{N^*}(s) = N^*(s)$ principal normal indicator. We can write that;

$$\gamma_{N^*} = \left\|\bar{D}_{T_{N^*}} T_{N^*}\right\|$$

From Gauss equation, it can be written that;

$$D_{T_{N^*}}T_{N^*} = \bar{D}_{T_{N^*}}T_{N^*} + \varepsilon g(S(T_{N^*}), T_{N*})N^*$$

where

$$\varepsilon = g(N^*, N^*) = +1,\ S(T_{N^*}) = -T_{N^*} \text{ and } \mathrm{g}(S(T_{N^*}), T_{N^*}) = +1.$$

If those values are written in Gauss equation, and if the equations of (3.1) and (3.52) are considered, we can say that

$$\begin{aligned}\bar{D}_{T_{N^*}}T_{N^*} &= \left[\frac{(\frac{-\sinh\theta}{k_C})' + (\frac{\kappa}{k_N})}{\sqrt{|(\theta')^2 + \|W\|^2|}} + \cosh\theta\right]T + \qquad (3.62)\\ &\left[\frac{(\frac{1}{k_N})'}{\sqrt{|(\theta')^2 + \|W\|^2|}}\right]N + \left[\frac{(\frac{\cosh\theta}{k_C})' - \frac{\tau}{k_N}}{\sqrt{|(\theta')^2 + \|W\|^2|}} - \sinh\theta\right]B\end{aligned}$$

If we get norm of (0.1), we can write that;

$$\begin{aligned}\gamma_{N^*} &= \left(\left[\frac{(\frac{-\sinh\theta}{k_C})' + (\frac{\kappa}{k_N})}{\sqrt{|(\theta')^2 + \|W\|^2|}} + \cosh\theta\right]^2 + \left[\frac{(\frac{1}{k_N})'}{\sqrt{|(\theta')^2 + \|W\|^2|}}\right]^2\right.\\ &\left.+\left[\frac{(\frac{\cosh\theta}{k_C})' - \frac{\tau}{k_N}}{\sqrt{|(\theta')^2 + \|W\|^2|}} - \sinh\theta\right]^2\right)^{\frac{1}{2}}\end{aligned}$$

$\overline{D}_{T_{N^*}}T_{N^*} = 0$ if and only if $\left(\overline{N^*}\right)$ curve geodesic spray is an integral curve. In this case, we can write that;

$$\begin{cases}\dfrac{\left(\frac{-\sinh\theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{\left|(\theta')^2 + \|W\|^2\right|}} + \cosh\theta = 0,\\ \dfrac{\left(\frac{1}{k_N}\right)'}{\sqrt{\left|(\theta')^2 + \|W\|^2\right|}} = 0,\\ \dfrac{\left(\frac{\cosh\theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{\left|(\theta')^2 + \|W\|^2\right|}} - \sinh\theta = 0\end{cases}$$

This value cannot be 0.

Result 3.8 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. There is no Mannheim partner curve on hyperbolic sphere to be made α^* Mannheim partner curve's (N^*) principal normal indicator geodezic spray is an integral curve.

Let γ_{B^*} be geodesic curve in S_1^2 for $\alpha^*_{B^*}(s) = B^*(s)$ principal normal indicator. In this case, we can write that;

$$\gamma_{B^*} = \left\| \bar{\bar{D}}_{T_{B^*}} T_{B^*} \right\|$$

From Gauss Equation, it can be written that;

$$D_{T_{B^*}} T_{B^*} = \bar{\bar{D}}_{T_{B^*}} T_{B^*} + \varepsilon g(S(T_{B^*}), T_{B*}) B^*$$

where

$$\varepsilon = g(B^*, B^*) = -1, \; S(T_{B^*}) = -T_{B^*} \text{ and } \; \mathrm{g}(S(T_{B^*}), T_{B^*}) = -1.$$

If those values are written in Gauss Equation, then, $B^* = N$. And if the equation of (3.55) is considered, we can say that;

$$\bar{\bar{D}}_{T_{B^*}} T_{B^*} = \frac{\theta'}{\|W\|}(-\sinh\theta T + \cosh\theta B) \tag{3.63}$$

If we get norm of (3.63), we can write that;

$$\gamma_{B^*} = \frac{\theta'}{\|W\|}$$

$\bar{\bar{D}}_{T_{B^*}} T_{B^*} = 0$ if and only if $\left(\overline{B^*}\right)$ curve geodesic spray is an integral curve. In this case, from the equation of (3.63), we can write that;

$$\begin{cases} \dfrac{-\theta' \sinh\theta}{\|W\|} = 0, \\ \\ \dfrac{\theta' \cosh\theta}{\|W\|} = 0 \end{cases}$$

where $\theta' = 0$. In this case, from the equations of (3.13) and (2.18), $\dfrac{\kappa}{\tau} =$ constant. This means α Mannheim curve is a helix.

Result 3.9 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. If α Mannheim curve is a helix, Mannheim partner of α is a straight line.

Let γ_{C^*} be geodesic curve in S_1^2 for $\alpha^*_{C^*}(s) = C^*(s)$. In this case, we can write that;

$$\gamma_{C^*} = \left\| \bar{D}_{T_{C^*}} T_{C^*} \right\|$$

From Gauss Equation, it can be written that;

$$D_{T_{C^*}} T_{C^*} = \bar{D}_{T_{C^*}} T_{C^*} + \varepsilon g(S(T_{C^*}), T_{C*}) C^*$$

where

$$\varepsilon = g(C^*, C^*) = +1, \; S(T_{C^*}) = -T_{C^*} \text{ and } \; \mathrm{g}(S(T_{C^*}), T_{C^*}) = -1.$$

If those values are written in Gauss equation, then,

$$C^* = -\sin\varphi^* T^* + \cos\varphi^* B^*$$

And if the equation of (3.57) is considered, we can say that;

$$\bar{D}_{T_{C^*}} T_{C^*} = -\frac{\|W^*\|}{(\varphi^*)'} N^* \tag{3.64}$$

If the equation of (3.60) is considered, geodesic curve is,

$$\gamma_{C^*} = \frac{\kappa (k_C)^3}{\lambda\tau\left(\sqrt{k_C^2 - 1}\right)'}$$

$\overline{D}_{T_{C^*}} T_{C^*} = 0$ if and only if $\left(\overline{C^*}\right)$ curve geodesic spray is an integral curve. In this case, from the equation of (3.64), we can write that; $\kappa^* = \tau^* = 0$. And from the equation of (3.9), $\kappa = 0$.

Result 3.10 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. α Mannheim curve does not have any partner curve because α Mannheim curveis a straight line.

Result 3.11 Let (α, α^*) be timelike-spacelike Mannheim curve pairs. In terms of S_1^2 or H_0^2, α^* curve's (T^*), (N^*) and (B^*) spherical indicator curves' and (C^*) fixed pol curve's geodesic curvatures are respectively;

$$\gamma_{T^*} = \gamma_C = \frac{\|W\|}{\theta'}$$

$$\begin{aligned} \gamma_{N^*} \quad = \quad & \left(\left[\frac{\left(\frac{-\sinh\theta}{k_C}\right)' + \left(\frac{\kappa}{k_N}\right)}{\sqrt{|(\theta')^2 + \|W\|^2|}} + \cosh\theta \right]^2 + \left[\frac{\left(\frac{1}{k_N}\right)'}{\sqrt{|(\theta')^2 + \|W\|^2|}} \right]^2 \right. \\ & \left. + \left[\frac{\left(\frac{\cosh\theta}{k_C}\right)' - \frac{\tau}{k_N}}{\sqrt{|(\theta')^2 + \|W\|^2|}} - \sinh\theta \right]^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\gamma_{B^*} = \frac{\theta'}{\|W\|}, \gamma_{C^*} = \frac{\kappa (k_C)^3}{\lambda\tau k_C'}.$$

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Smarandache-R-Module and Mcrita Context

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Abstract: In this paper we introduced Smarandache-2-algebraic structure of R-module namely Smarandache-R-module. A Smarandache-2-algebraic structure on a set N means a weak algebraic structure $A0$ on N such that there exist a proper subset M of N, which is embedded with a stronger algebraic structure $A1$, stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset from the empty set, from the unit element if any, from the whole set. We define Smarandache-R-module and obtain some of its characterization through S-algebra and Morita context. For basic concept we refer to Raul Padilla.

Key Words: R-module, Smarandache-R-module, S-algebras, Morita context and Cauchy modules.

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§1. Preliminaries

Definition 1.1 *Let S be any field. An S-algebra A is an (R, R)-bimodule together with module morphisms $\mu : A\otimes_R A \to A$ and $\eta : R \to A$ called multiplication and unit linear maps respectively such that*

$$A \otimes_R A \otimes_R A \underset{1_A\otimes\mu}{\overset{\mu\otimes 1_A}{\Longrightarrow}} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu) \text{ and}$$

$$R \underset{1_A\otimes\eta}{\overset{\eta\otimes 1_A}{\Longrightarrow}} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\eta \otimes 1_A) = \mu \circ (1_A \otimes \eta).$$

Definition 1.2 *Let A and B be S-algebras. Then $f : A \to B$ is an S-algebra homomorphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.*

Definition 1.3 *Let S be a commutative field with 1_R and A an S-algebra M is said to be a left A-module if for a natural map $\pi : A \otimes_R M \to M$, we have $\pi \circ (1_A \otimes \pi) = \pi \circ (\mu \otimes 1_M)$.*

Definition 1.4 *Let S be a commutative field. An S-coalgebra is an (R, R)-bimodule C with R-linear maps $\Delta : C \to C\otimes_R C$ and $\varepsilon : C \to R$, called comultiplication and counit respectively such*

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that $C \xrightarrow{\Delta} C \otimes_R C \underset{\Delta\otimes 1_C}{\overset{1_C\otimes\Delta}{\rightrightarrows}} C \otimes_R C \otimes_R C$ *with* $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ *and* $C \xrightarrow{\Delta} C \otimes_R C \underset{\varepsilon\otimes 1_C}{\overset{1_C\otimes\varepsilon}{\rightrightarrows}} R$ *with* $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\varepsilon \otimes 1_C) \circ \Delta$.

Definition 1.5 *Let C and D be S-coalgebras. A coalgebra morphism $f : C \to D$ is a module morphism if it satisfies $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$.*

Definition 1.6 *Let A be an S-algebra and C an S-coalgebra. Then the convolution product is defined by $f * g = \mu \circ (f \otimes g) \circ \Delta$ with $1Hom_R(C, A) = \eta \circ \in (1_R)$ for all $f, g \in Hom_R(C, A)$.*

Definition 1.6 *For a commutative field S, an S-bialgebra B is an R-module which is an algebra (B, μ, η) and a coalgebra (B, Δ, ε) such that Δ and ε are algebra morphisms or equivalently μ and η are coalgebra morphisms.*

Definition 1.7 *Let R, S be fields and M an (R, S)-bimodule. Then, $M^* = Hom_R(M, R)$ is an (S, R)-bimodule and for every left R-module L, there is a canonical module morphism $\alpha_L^M : M^* \otimes_R L \to Hom_R(M, L)$ defined by $\alpha_L^M(m^* \otimes l)(m) = m^*(m)l$ for all $m \in M, m^* \in M^*$ and $l \in L$. If α_L^M is an isomorphism for each left R-module L, then ${}_RM_S$ is called a Cauchy module.*

Definition 1.8 *Let R, S be fields with multiplicative identities M, an (S, R)-bimodule and N, an (R, S)-bimodule. Then the six-tuple datum $K = [R, S, M, N, \langle,\rangle_R, \langle,\rangle_S]$ is said to be a Morita context if the maps $\langle,\rangle_R : N \otimes_S M \to R$ and $\langle,\rangle_S : M \otimes_R N \to S$ are binmodule morphisms satisfying the following associativity conditions:*

$$m'\langle n, m\rangle_R = \langle m', n\rangle s\ m \ \text{and}\ \langle n, m\rangle Rn' = n\langle m, n'\rangle s$$

$\langle,\rangle_R$ and $\langle,\rangle_S$ are called the Morita maps.

§2. Smarandache-R-Modules

Definition 2.1 *A Smarandache-R-module is defined to be such an R-module that there exists a proper subset A of R which is an S-Algebra with respect to the same induced operations of R.*

§3. Results

Theorem 3.1 *Let R be a R-module. There exists a proper subset A of R which is an S-coalgebra iff A^* is an S-algebra.*

Proof Let us assume A^* is an S-algebra. For proving that A is an S-coalgebra we check the counit conditions as follows:

$$\varepsilon : A \approxeq A \otimes_R S \xrightarrow{1_A \otimes \mu} A \otimes_R A^* \xrightarrow{\psi_A} R.$$

Next, we check the counit condition as follows:

$$\begin{aligned}\Delta : A \quad &\approxeq \quad A \otimes_R S \otimes_R S \overset{1_A \otimes_\eta End_S(A)}{\longrightarrow} A \otimes_R (A^* \otimes_R A) \otimes_R A^* \\ &\overset{1_A \otimes_A \otimes_A^*}{\longrightarrow} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \overset{1_A \otimes_R A}{\longrightarrow} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \\ &\overset{1_A \otimes_A}{\longrightarrow} (A \otimes_R A) \otimes_R A^* \overset{\approxeq}{\longrightarrow} (A \otimes_R A) \otimes_R A \overset{\approxeq}{\longrightarrow} A \otimes_R A \overset{\approxeq}{\longrightarrow} A.\end{aligned}$$

Thus, A is an S-coalgebra.

Conversely, Let us assume A is an S-coalgebra. Now to prove that A^* is an S-algebra, we check the unit conditions as follows

$$\eta : R \overset{\eta End_S(A)}{\longrightarrow} A \otimes_R A^* \to 1_A \otimes A \overset{\approxeq}{\longrightarrow} A.$$

We check the multiplication conditions as follows A is a Cauchy module. Notice that

$$\begin{aligned}&A \otimes_R A \to R, \\ &A \approxeq A \otimes_R A \otimes_R R \overset{1_A \otimes_\eta End_S(A)}{\longrightarrow} A \otimes_R A \otimes_R A^* \to R \otimes_R A^* \overset{\approxeq}{\longrightarrow} A^*, \\ &\mu : A \otimes_R A \overset{\varepsilon \otimes 1_A}{\longrightarrow} A^* \otimes_R A \overset{\approxeq}{\longrightarrow} R \otimes_R A^* \overset{\approxeq}{\longrightarrow} A^*.\end{aligned}$$

Thus, A^* is an S-algebra. By definition, R is a smarandache R-module. □

Theorem 3.2 *Let R be an R-module. Then there exists a proper subset $End_S(M)^*$ of R which is an S-algebra.*

Proof Let us assume that R be an R-module. For proving that $End_S(M)$ is an S-coalgebra which satisfies multiplication and unit conditions $\mu : End_S(M) \otimes_R End_S(M) \to End_S(M)$ and $\eta : R \to End_S(M)$, we check the comultiplication condition as follows:

$$\Delta : End_S(M) \approxeq End_S(M) \otimes_R \overset{1_{End(M)} \otimes_n}{\longrightarrow} End_S(M) \otimes_R End_S(M).$$

Next, we check the counit conditions as follows:

$$\begin{aligned}\varepsilon : End_S(M) \quad &\approxeq \quad End_S(M) \otimes_R R \overset{1_{End(M)} \otimes_\eta}{\longrightarrow} End_S(M) \otimes_R End_S(M) \\ &\overset{\approxeq \otimes \approxeq}{\longrightarrow} Hom_R(M, M) \otimes_R Hom_R(M, M) \\ &\overset{\approxeq \otimes \approxeq}{\longrightarrow} (M' \otimes_R M) \otimes_R (M' \otimes_R M) \overset{\psi M \otimes_\psi M}{\longrightarrow} R \otimes_R R \overset{\approxeq}{\longrightarrow} R.\end{aligned}$$

Thus $End_S(M)$ is an S-coalgebra. By Theorem 3.1, $End_S(M)^*$ is an S-algebra. Hence, R is a Smarandache R-module. □

Theorem 3.3 *Let R be an R-module. Then there exists a proper subset $M \otimes_R M^*$ of R which is an S-algebra.*

Proof For proving that $M \otimes_R M^*$ is an S-algebra, we check the multiplication and unit conditions as follows:

$$\begin{aligned}\mu : (M \otimes_R M^*) \otimes (M \otimes_R M^*) &\xrightarrow{\cong} M \otimes_R (M^* \otimes_R M) \otimes_R M^* \\ &\xrightarrow{1_M \psi_M \otimes 1_M} M \otimes_R R \otimes_R M^* \\ &\xrightarrow{1_M \otimes \psi_M} M \otimes_R M^*.\end{aligned}$$

As M is a Cauchy module, we have

$$\eta : R \to End_S(M) \xrightarrow{\cong} M \otimes_R M^*,$$

which implies that $M \otimes_R M^*$ is an S-algebra. Hence, R is a Smarandache R-module. □

Theorem 3.4 *Let R be an R-module. Then there exists a proper subset the datum $[R, M, N, \langle, \rangle_R]$ a morita context $(M \otimes_R N)^*$ of R which is an S-algebra.*

Proof Let us assume that R be an R-module. For proving that $M \otimes_R N$ is an S-algebra, we have

$$\begin{aligned}\mu : (M \otimes_R N) \otimes_R (M \otimes_R N) &\to M \otimes_R (N \otimes_R M) \otimes_R N \\ &\xrightarrow{1_M \otimes \langle, \rangle \otimes 1_N} M \otimes_R R \otimes_R N \xrightarrow{\cong} M \otimes_R N,\end{aligned}$$

which shows that the multiplication condition is satisfied.

Also, since M and N are Cauchy R-modules, there exist maps

$$\eta End_R(M) : R \to M^* \otimes_R M \quad \text{and} \quad \eta End_S(N) : R \to N^* \otimes_R N$$

that can be used to prove the unit condition as follows:

$$\begin{aligned}\eta : R \cong R \otimes_R R &\xrightarrow{\eta End_S(M) \otimes \eta End_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M^* \otimes_R N^*) \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M \otimes_R N)^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} R^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong} R \otimes_R (M \otimes_R N) \xrightarrow{\cong} (M \otimes_R N),\end{aligned}$$

which implies that $M \otimes_R N$ is an S-algebra. By definition, R is a Smarandache R-module.

Theorem 3.5 *Let R be an R-module. Then there exists a proper subset the datum $[R, M, N, \langle, \rangle R]$ a morita context $M \otimes_R N$ of R which is an S-coalgebra.*

Proof Let us assume that R be an R-module. For proving that $(M \otimes_R N)$ is an S-coalgebra, we have

$$\begin{aligned}
\Delta : M\otimes_R N &= (M\otimes_R N)\otimes_R (R\otimes_R) \\
&\stackrel{1_{M\otimes N}\otimes\eta End_S(M)\otimes\eta End_S(N)}{\longrightarrow} (M^*\otimes_R M)\otimes_R (N^*\otimes_R N) \\
&\stackrel{1_{M\otimes N}\otimes\cong}{\longrightarrow} (M\otimes_R N)\otimes_R (M\otimes_R N)\otimes_R (M^*\otimes_R N^*) \\
&\stackrel{1_{M\otimes N}\otimes\cong}{\longrightarrow} (M\otimes_R N)\otimes_R (M\otimes_R N)\otimes_R (M\otimes_R N)^* \\
&\stackrel{1_{M\otimes N}\otimes\langle,\rangle R^*}{\longrightarrow} (M\otimes_R N)\otimes_R (M\otimes_R N)\otimes_R R \\
&\stackrel{\cong}{\longrightarrow} (M\otimes_R N)\otimes_R (M\otimes_R N).
\end{aligned}$$

Also, we have the counit condition as follows:

$$\begin{aligned}
\varepsilon : M\otimes_R N &\cong (M\otimes_R N)\otimes_R R \stackrel{1_{M\otimes N}\otimes\eta End_S(M)}{\longrightarrow} (M\otimes_R N)\otimes_R (M^*\otimes_R M) \\
&\stackrel{\langle .\rangle_R\otimes 1_{M^*\otimes M}}{\longrightarrow} R\otimes_R M^*\otimes_R M \stackrel{\cong}{\longrightarrow} M^*\otimes_R M \stackrel{\psi_M}{\longrightarrow} R,
\end{aligned}$$

which implies that $\Longrightarrow$ $M\otimes_R N$ is an S-coalgebra. Hence, R is a Smarandache R-module. □

Theorem 3.6 *Let R be an R-module. Then there exists a proper subset the datum $[R, M, N, \langle,\rangle R]$ a Morita context iff $M\otimes_R N$ is an S-bialgebra.*

Proof First, if $M\otimes_R N$ is an S-bialgebra by Theorem 3.5, we know that $M\otimes_R N$ is an S-algebra and $M\otimes_R N$ is an S-coalgebra. Hence by definition, R is a Smarandache R-module.

If $M\otimes_R N$ is an S-bialgebra, we have the map

$$\varepsilon = \langle .\rangle_R : M\otimes_R N \to R.$$

Associativity of the map $\varepsilon = \langle,\rangle_R$ holds because the diagram

$$\begin{array}{ccc}
(M\otimes_R N)\otimes_R M & \stackrel{\cong}{\longrightarrow} & M\otimes_R (N\otimes_R M) \\
\varepsilon\otimes 1_M \searrow & & \swarrow 1_M\otimes\varepsilon \\
& \mathrm{M} &
\end{array}$$

is commutative. Hence the datum $[R, M, N, \langle,\rangle R]$ is a Morita context. □

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Generalized Vertex Induced Connected Subsets of a Graph

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Abstract: Let k be a positive integer. A graph $G = (V, E)$ is said to be a Π_k - connected graph if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected. In this paper, we explore some properties of Π_k - connectedness and its minimality conditions with respect to other graph theoretic parameters.

Key Words: Graph, subgraph, Π_k - connected graph, minimal Π_k-connected graph.

AMS(2010): 05C15, 05C69.

§1. Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $p =| V |$ and $q =| E |$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices $X \subseteq V$. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. A non-trivial graph G is called connected if any two of its vertices are linked by a path in G. A graph G is called n-connected (for $n \in N$) if $|V(G)| > n$ and $G - X$(the graph that results from removing all vertices in X and all edges incident with these vertices) is connected for any vertex set $X \subseteq V(G)$ with $|X| < n$. The greatest integer n such that G is n - connected is called the connectivity $\kappa(G)$ of G. A cut-edge or cut-vertex of G is an edge or a vertex whose deletion increases the number of components. Unless mentioned otherwise for terminology and notation the reader may refer [1] and [5].

The general problem consists of selecting a set of land parcels for conservation to ensure species availability. This problem is also related to site selection, reserve network design and corridor design. Biologists have highlighted the importance of addressing negative ecological impacts of habitats fragmentation when selecting parcels for conservation. Ways to increase the spatial coherence among the set of parcels selected for conservation have been investigated. Conservation planning via Π_k -connected graph model is an important conservation method, in this model they increase the genetic diversity and allow greater mobility for better response to predation and stochastic events such as fire, as well as long term climate change. This motivated us to study Π_k - connectedness in the following manner:

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For any positive integer k. A graph G is said to be a Π_k - connected graph if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected.

For more detail, we refer [2]-[4], [6]– [10] and [13].

§2. Π_k-Connected Graphs

Proposition 2.1 *For any graph G with $p \geq 1$ vertices is a Π_1 - connected graph.*

Proposition 2.2 *For a given $p \geq 3$ vertices, there exist a Π_3 - connected graph of a graph G.*

Proof Removal of t independent edges from a complete graph on p vertices results into a Π_3 - connected graph, where $0 \leq t \leq \frac{p}{2}$ if p is even and $0 \leq t \leq \frac{p-1}{2}$ if p is odd. □

Proposition 2.3 *Let ξ be the number of edges required to make a totally disconnected graph which is a Π_3 - connected graph and hence*

$$\xi(\overline{K}_p) = \begin{cases} \frac{p^2-2p}{2} & \textit{if } p \textit{ is even} \\ \frac{p^2-2p+1}{2} & \textit{if } p \textit{ is odd} \end{cases}$$

For a complete bipartite graph $K_{m,n}$, the number of edges to be added to make it a Π_3 - connected graph is given by $\xi(\overline{K}_m) + \xi(\overline{K}_n)$.

Proposition 2.4 *In general the number of edges required to make complete n- partite graph $K_{x_1,x_2,x_3,\ldots,x_n}$ as a Π_3 - connected graph is given by*

$$\xi(\overline{K}_{x_1}) + \xi(\overline{K}_{x_2}) + \cdots + \xi(\overline{K}_{x_n}).$$

Proposition 2.5 *The complete bipartite graph $K_{m,n}$ is a Π_3 - connected graph if $m = 1, 2$ and $n = 1, 2$.*

Theorem 2.1 *For any graph G with $p \geq 3$ vertices is a Π_3 - connected graph if and only if $deg(v_i) \geq p - 2$ for all $v_i \in V(G)$.*

Proof Let G be a Π_3 - connected graph. Suppose on contrary, there exists v_j such that $deg(v_j) \leq p-3$. Let v_1 and v_2 be any two vertices which are not adjacent to v_j, thus the graph induced by v_1, v_2 and v_j is not connected, which is a contradiction. Hence $deg(v_i) \geq p - 2$ for all $v_i \in G$.

Conversely, suppose $deg(v_i) \geq p-2$ for all $v_i \in G$, let v_1, v_2, v_3 be any three vertices. Then v_1 is adjacent to at least one of v_2 and v_3. v_2 is adjacent to at least one of v_1 and v_3. Also v_3 is adjacent to at least one of v_1 and v_2. Therefore G is a Π_3 - connected graph. □

Theorem 2.2 *If a graph G is a Π_k- connected graph, then*

$$\delta(G) \geq \begin{cases} t & \textit{if} \;\; p = t(k-1)+1 \\ t+1 & \textit{if} \;\; p = t(k-1)+r+1, 1 \leq r \leq k-2 \end{cases}$$

Proof Let G be a Π_k connected graph with $p = t(k-1)+1$ vertices. Suppose on contrary that $\delta(G) = s < t$. Let v be a vertex with $deg(v) = s$. Now we partition the remaining $p-1$ vertices into t vertex disjoint sets such that each set contains a vertex adjacent to v. Since $s < t$, there exists at least one set N which has no vertex adjacent to v. Then $\langle N \cup \{v\} \rangle$ is a disconnected subgraph on k vertices, which is a contradiction.

Now, let $p-1 = t(k-1)+r$. Suppose on contrary that $\delta(G) = s < t+1$, let v be a vertex such that $deg(v) = s$. Now partition the remaining $p-1$ vertices excluding the vertex v in $t+1$ number of vertex disjoint subsets having t subsets with cardinality $k-1$ and one with cardinality r in such a way that each subset contains exactly one vertex adjacent to v. Since $deg(v) < t+1$, there exist at least one subset having no vertex adjacent to v. If the cardinality of such a subset, say T is $k-1$ then $\langle T \cup \{v\} \rangle$ is a disconnected subgraph on k - vertices, again a contradiction. If the cardinality of such a subset, say D is r, then take a vertex which is adjacent to v from a subset A with cardinality $k-1$ to D and any one vertex from D to A, then $\langle A \cup \{v\} \rangle$ is a disconnected subgraph which is induced by k - vertices, a contradiction. Hence

$$\delta(G) \geq \begin{cases} t & \text{if} \;\; p-1 = t(k-1) \\ t+1 & \text{if} \;\; p-1 = t(k-1)+r, 1 \leq r \leq k-2 \end{cases} \qquad \square$$

Theorem 2.3 *Let G be a Π_k - connectedHence the result follows. Then G is a Π_{k+1} - connected graph with $2 \leq k \leq p-1$.*

Proof On contrary, suppose G is a Π_k - connected graph but not a Π_{k+1} - connected one. Let S be set of $k+1$ vertices on which the graph induced has more than one component. Clearly a subset T consisting of k vertices on which the graph induced is disconnected which is a subgraph of G, which is a contradiction. Hence G is also a Π_{k+1} - connected graph with $2 \leq k \leq p-1$. □

Theorem 2.4 *A graph G is a Π_k-connected graph for all k, $1 \leq k \leq p$ if and only if G is isomorphic to K_p.*

Proof Let G be a Π_k - connected graph for all k, $1 \leq k \leq p$. Clearly G is isomorphic to a complete graph K_p as G is a Π_2 - connected graph.

On the other hand, let G be a complete graph on p vertices. In K_p, every pair of vertices are adjacent. Hence G is a Π_2 - connected graph. As we have proved, every Π_2 - connected graph is Π_t - connected graph for all t, $3 \leq t \leq p$. Therefore G is a Π_k - connected graph with $1 \leq k \leq p$. □

Theorem 2.5 *For any Square tree T^2 of a tree T with diameter $d(T) \geq 5$ is a Π_{p-1} - connected* *graph.*

Proof Let T be any tree with diameter $d(T) \geq 5$. Consider any four non pendent vertices $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ in T such that v_{i+1} is adjacent to v_{i+2}, v_{i+2} is adjacent to v_{i+3}, v_{i+3} is adjacent to v_{i+4}, then in square of T, v_{i+2} is adjacent to v_{i+1}, v_{i+3}, v_{i+4} and v_{i+3} is adjacent to v_{i+1}, v_{i+2}, v_{i+4}. Removal of v_{i+2} and v_{i+3} disconnects the square and since no vertex is a cut vertex, removal of any one vertex does not disconnect T^2. Hence square of any tree T with diameter $d(T) \geq 5$ is a Π_{p-1} - connected graph. $\square$

Proposition 2.6 *For any integer $m \geq 1$, $K_{1,m}$, $K_{1,m}^2$ are Π_2 - connected graphs.*

Proof Let $K_{1,m}$ be any star. Then the square of a star is a complete graph on $m+1$ vertices and hence a Π_2 - connected graph. $\square$

§3. Minimality Conditions on Π_k-Connected Graphs

A Π_k - connected graph G is said to be a vertex minimal Π_k - connected graph if G is not a Π_{k-1} - connected graph. A vertex minimal Π_k - connected graph G is said to be a partially vertex - edge minimal Π_k - connected graph if $G - e$ is not a Π_k - connected graph for some $e \in E(G)$. A vertex minimal Π_k - connected graph G is said to be a totally vertex - edge minimal Π_k - connected graph if $G - e$ is not a Π_k - connected graph for every $e \in E(G)$. For more details, refer [12].

Proposition 3.1 *For any cycle C_p; $p \geq 4$ vertices, the number of edges to be added to make it a Π_3 - connected graph is given by*

$$q \geq \begin{cases} \frac{p^2-4p}{2} & if\ \ p\ \ is\ even \\ \frac{p^2-4p+1}{2} & if\ \ p\ \ is\ odd \end{cases}$$

where, the equality holds when the resulting graph is a partially vertex - edge minimal Π_3 - connected graph.

Proof Let C_p; $p \geq 4$ vertices be a cycle. Then we have the following cases.

Case 1. Suppose p is even. In any Π_3 - connected graph, the degree each vertex is at least $p-2$. Hence the number of edges in any Π_3 - connected graph is always greater than or equal to $\frac{p(p-2)}{2}$. Therefore the number of edges to be added to C_p is greater than or equal to $\frac{p^2-2p}{2} - p = \frac{p^2-4p}{2}$.

Case 2. Suppose p is odd. In any Π_3 - connected graph on odd number of vertices, the degree of each of the $p-1$ vertices is at least $p-2$ and the degree of one vertex is $p-1$. Hence the number of edges in any Π_3 - connected graph on odd number of vertices is always greater than or equal to $\frac{(p-1)^2}{2}$. Hence the number of edges to be added to C_p is greater than or equal to $\frac{p^2-2p+1}{2} - p = \frac{p^2-4p+1}{2}$. $\square$

Proposition 3.2 *For any path P_p; $p \geq 4$ vertices, the number of edges to be added to make it a Π_3 - connected graph is given by*

$$q \geq \begin{cases} \frac{p^2-4p}{2}+1 & \text{if } p \text{ is even} \\ \frac{p^2-4p+1}{2}+1 & \text{if } p \text{ is odd} \end{cases}$$

where the equality holds when the resulting graph is a partially vertex - edge minimal Π_3 - connected graph.

Proof The proof follows on the same lines as in the above proposition. □

Theorem 3.1 *A connected graph G is a vertex minimal Π_p - connected graph if and only if it has at least one cut vertex.*

Proof Let a connected graph G be a Π_p - connected graph, that is, G is not a Π_{p-1} - connected graph. There exist a vertex v such that the graph induced by $V(G)-v$ is disconnected. Hence v is cut vertex. Conversely, let G be a connected graph with a cut vertex, say v, therefore the subgraph induced by the vertices $V(G)-v$ is disconnected. Hence the graph G is a vertex minimal Π_p - connected graph. □

Theorem 3.2 *For a given $k = 2l+1,\ l \geq 3$, there exists Π_k - connected graph.*

Proof Let $k = 2l+1,\ l \geq 3$ and G be a Π_3 - connected graph on $k-3$ vertices with $V(G) = \{1, 2, 3, \cdots, k-3\}$ and let G' be a graph with $V(G') = \{1, 2, 3, \cdots, k-3, k-2\}$ obtained by adding a vertex $k-2$ and making it adjacent to all the vertices of G. Now take prism of G', label the vertices of second copy of G' in the prism as $\{f(1), f(2), f(3), \cdots, f(k-3), f(k-2)\}$ such that $f(i)$ is the mirror image of i and remove the edge $(k-2, f(k-2))$ from the prism. In the resulting graph H (say), the subgraph induced by any subset $S \subseteq V(H)/\{k-2,\ f(k-2)\}$ containing $k-1$ vertices is connected. The subgraph induced by $V(G) \cup \{k-2,\ f(k-2)\}$ disconnected on $k-3+1+1 = k-1$ vertices and hence every subgraph induced by k vertices is connected. Hence H is Π_k - connected. □

Observation 3.1 The graph obtained in the above theorem is regular when G is a partially vertex - edge minimal Π_3 - connected graph, which is having even order.

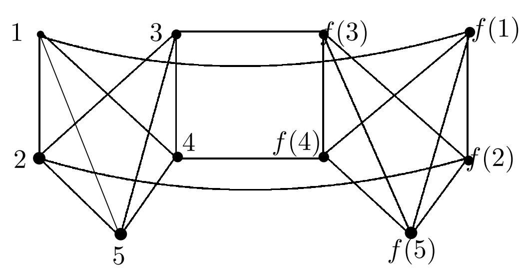

Figure 1. Prism of a Π_3 - connected graph having odd order

For illustration, we construct the above prism of a Π_3 - connected graph having odd order, where prism of a graph G is defined as the cartesian product $G \times K_2$; that is, take two disjoint copies of G and add a matching joining the corresponding vertices in the two copies, [8].

Theorem 3.3 *For any vertex minimal Π_k - connected graph can be embedded in a vertex minimal Π_{k+i} - connected graph, where $i \geq 0$.*

Proof Let G_1 and G_2 be two vertex minimal Π_k and Π_{k+i} - connected graphs. Now we construct a vertex minimal Π_{k+i} - connected graph in which G_1 is an induced vertex minimal Π_k - connected subgraph. Make each vertex of G_1 adjacent to each vertex in G_2 and let the resulting graph be G. Let S be any set of $k+i$ vertices from G. In the following cases we prove the graph induced by S is connected.

Case 1. Suppose $S \cap V(G_1) \cap V(G_2) \neq \varnothing$, then the graph induced by S is connected since each vertex in $S \cap V(G_1)$ is adjacent to every vertex of $S \cap V(G_2)$.

Case 2. Suppose $S \cap V(G_1) \neq \varnothing$ and $S \cap V(G_2) = \varnothing$, then $|S| \geq k$ and $S \subseteq V(G_1)$, the graph induced by S is connected as G_1 is vertex minimal Π_k - connected.

Case 3. Suppose $S \cap V(G_1) = \varnothing$ and $S \cap V(G_2) \neq \varnothing$, then the graph induced by S is connected since S is completely contained in $V(G_2)$ and G_2 is a vertex minimal Π_{k+i} - connected graph. In all the three cases the graph induced by S is connected and G is not Π_{k+i-1} - connected graph since G_2 is a vertex minimal Π_{k+i} - connected graph. Hence the graph G is a vertex minimal Π_{k+i} - connected graph having vertex minimal Π_k - connected graph G_1 as its induced subgraph.

Thus the result follows. □

Theorem 3.4 *A connected graph G is a partially vertex - edge minimal Π_p - connected graph if and only if it has at least one cut edge.*

Proof Let G be a connected graph with a cut edge say $e = uv$. Here u is a cut vertex and also $G - e$ is disconnected, hence G is a partially vertex - edge minimal Π_p - connected graph. Conversely, let G be a partially vertex - edge minimal Π_p - connected graph then $G - e$ is not a Π_p - connected graph. Hence e is a cut edge. □

To prove our next result we make use of the following observations.

Observation 3.2 Removal of $\frac{p}{2}$ independent edges from a complete graph K_p on even number of vertices results into a partially vertex - edge minimal Π_3 - connected graph having $\frac{p(p-2)}{2}$ edges.

Observation 3.3 Removal of $\frac{p-1}{2}$ independent edges from a complete graph on odd number of vertices results into a partially vertex - edge minimal Π_3 - connected graph having $\frac{(p-1)^2}{2}$ edges.

Observation 3.4 Partially vertex - edge minimal Π_3 - connected graph having even order is a regular graph with regularity $p - 2$.

Theorem 3.5 *Let G_1 and G_2 be two partially vertex - edge minimal Π_3 - connected graph. If $V(G_1) = V(G_2)$, then G_1 is isomorphic to G_2.*

Proof Let G_1 and G_2 be partially vertex - edge minimal Π_3 - connected graph having same order. Then, there are following cases:

Case 1. Suppose p is even. As the graphs G_1 and G_2 are partially vertex - edge minimal Π_3 connected graphs, $deg(v) = p-2$, for all $v \in V(G_1)$ and $deg(v) = p-2$ for all $v \in V(G_2)$. Clearly G_1 is isomorphic to G_2.

Case 2. Suppose p is odd. As the graphs G_1 and G_2 are partially vertex - edge minimal Π_3 - connected graphs, in the graphs G_1 and G_2 degree of each of $p-1$ vertices is $p-2$ and degree of one vertex is $p-1$. Hence in this case also G_1 is isomorphic to G_2. □

Theorem 3.6 *Any graph G with order p is a totally vertex - edge minimal Π_p - connected graph if and only if G is isomorphic to a tree on p vertices.*

Proof Let G be a graph of order p which is a strongly critical Π_p - connected graph, that is, $G-e$ is not Π_p - connected for all $e \in V(G)$ and G is not a Π_{p-1} - connected graph. The first condition in a totally vertex - edge minimal Π_p - connected graph which implies that every edge in G is a bridge and the second condition in a totally vertex - edge minimal Π_p - connected graph which again implies that every vertex in G is a cut vertex, clearly G is isomorphic to a tree on p vertices.

Conversely, let G is isomorphic to a tree on p vertices. Since every internal vertex is a cut vertex, we have a disconnected induced subgraph on $p-1$ number of vertices and since every edge is a bridge, $G-e$ is not a Π_p- connected graph for all $e \in V(G)$. Hence G is a totally vertex - edge minimal Π_p - connected graph. □

Theorem 3.7 *Any graph G having even number of vertices is a totally vertex - edge minimal Π_3 - connected graph if and only if $deg(v_i) = p-2$ for all $v_i \in V(G)$.*

Proof Let G be a totally vertex - edge minimal Π_3 - connected graph on even number of vertices, implies $deg(v) \geq p-2$ for all $v \in V(G)$ from the Theorem 2.1. Suppose $deg(v) > p-2$ for some $v \in V(G)$, i.e., $deg(v) = p-1$. There exist a vertex w adjacent to v such that $deg(w) = p-1$. The graph $G - vw$ is still a Π_3 - connected graph. Hence G is not a totally vertex - edge minimal Π_3 - connected graph, which is a contradiction. Hence $deg(v) = p-2$ for all $v \in V(G)$.

Conversely, let G be a graph such that $deg(v) = p-2$ for all $v \in V(G)$. For every vertex in G there exist an unique non adjacent vertex in G. Hence in $G-e$ there exist two vertices say v and w non adjacent to some vertex say u. The graph induced by these three vertices is disconnected and hence the graph is a totally vertex - edge minimal Π_3 - connected graph. □

Observation 3.5 Any complete graph on $p \geq 3$ vertices is a Π_3 - connected graph but not a partially vertex - edge minimal and totally vertex - edge minimal Π_3 - connected graph.

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b-Chromatic Number of Splitting Graph of Wheel

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Abstract: A proper k-coloring is called a b-coloring if there exits a vertex (b-vertex) that has neighbour(s) in all other $k-1$ color classes. The largest integer k for which G admits a b-coloring is called the b-chromatic number denoted as $\varphi(G)$. If b-coloring exists for every integer k satisfying $\chi(G) \leqslant k \leqslant \varphi(G)$ then G is called b-continuous. The b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. We investigate b-chromatic number of the splitting graph of wheel and also discuss its b-continuity and b-spectrum.

Key Words: b-Coloring, b-continuity, b-spectrum.

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§1. Introduction

A proper k-coloring of a graph $G = (V(G), E(G))$ is a mapping $f : V(G) \to \{1, 2, \cdots, k\}$ such that every two adjacent vertices receives different colors. The chromatic number of a graph G is denoted by $\chi(G)$, is the minimum number for which G has a proper k-coloring. The set of vertices with a specific color is called a color class. A b-coloring of a graph G is a variant of proper k-coloring such that every color class has a vertex which is adjacent to at least one vertex in every other color classes and such a vertex is called a color dominating vertex. If v is a color dominating vertex of color class c then we denote it as $cdv(c) = v$. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. The concept of b-coloring was originated by Irving and Manlove [1] and they also observed that every coloring of a graph G with $\chi(G)$ colors is obviously a b-coloring. In the same paper they have introduced the concepts of b-continuity and b-spectrum. If the b-coloring exists for every integer k satisfying $\chi(G) \leqslant k \leqslant \varphi(G)$ then G is called b-continuous and the b-spectrum $S_b(G)$ of a graph G is the set of k integers(colors) for which G has a b-coloring. Kouider and Maheö [2] have obtained lower and upper bounds for the b-chromatic number of the cartesian products of two graphs while Vaidya and Shukla [3,4,5,6] have investigated b-chromatic numbers for various

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graph families. The concept of b-coloring has been extensively studied by Faik [7], Kratochvil *et al.*[8], Alkhateeb [9] and Balakrishnan *et al.* [10].

Definition 1.1 *The splitting graph $S'(G)$ of a graph G is obtained by adding new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively in $S'(G)$.*

Here we investigate b-chromatic number for splitting graph of wheel.

Definition 1.2([1]) *The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m-1$.*

Proposition 1.3([1]) *If graph G admits a b-coloring with m-colors, then G must have at least m vertices with degree at least $m-1$.*

Proposition 1.4 *Let $W_n = C_n + K_1$. Then* $\chi(W_n) = \begin{cases} 3, & n \text{ is even} \\ 4, & n \text{ is odd.} \end{cases}$

Proposition 1.5([11]) $\chi(G) \leqslant \varphi(G) \leqslant m(G)$.

Proposition 1.6([12]) *For any graph G, $\chi(G) \geqslant 3$ if and only if G has an odd cycle.*

§2. Main Results

Lemma 2.1 *For a wheel W_n,*

$$\chi[S'(W_n)] = \begin{cases} 4, & n \text{ is odd} \\ 3, & n \text{ is even} \end{cases}$$

Proof Let $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel W_n which are duplicated by the vertices $v'_1,\ v'_2, \ldots, v'_n$ respectively and let v denotes the apex vertex of W_n which is duplicated by the vertex v'. Let $e_1, e_2, \ldots, e_n$ be the rim edges of W_n. Then the resultant graph $S'[W_n]$ will have order $2(n+1)$ and size $6n$.

Case 1. n is odd

In this case $S'[W_n]$ contains odd W_n as an induced subgraph. Since $\chi(W_n) = 4 \Rightarrow \chi[S'(W_n)] = 4$.

Case 2. n is even

In this case $S'[W_n]$ contains even W_n as an induced subgraph. Since $\chi(W_n) = 3 \Rightarrow \chi[S'(W_n)] = 3$. □

Theorem 2.2 *For a wheel W_n,*

$$\varphi[S'(W_n)] = \begin{cases} 4, & n = 3 \\ 3, & n = 4 \\ 5, & n = 5, 6, 8 \\ 6, & n = 7 \\ 6, & n \geqslant 9 \end{cases}$$

Proof To prove the result we continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1. $n = 3$

In this case the graph $S'(W_3)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_3)] \geqslant 3$. As $m[S'(W_3)] = 4$ and by Lemma 2.1, $\chi[S'(W_3)] = 4$. We have $4 \leqslant \varphi[S'(W_3)] \leqslant 4$ by Proposition 1.5. Thus, $\varphi[S'(W_3)] = 4$.

Case 2. $n = 4$

In this case the graph $S'(W_4)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_4)] \geqslant 3$. As $m[S'(W_4)] = 5$ and by Lemma 2.1 $\chi[S'(W_4)] = 3$. Then by Proposition 1.5 we have $3 \leqslant \varphi[S'(W_4)] \leqslant 5$.

If $\varphi[S'(W_4)] = 5$ then by Proposition 1.3, the graph $S'(W_4)$ must have five vertices of degree at least 4 which is possible. But due to the adjacency of vertices of the graph $S'(W_4)$ any proper coloring with five colors have at least one color class which does not have color dominating vertices hence it will not be b-coloring for the graph $S'(W_4)$. Thus, $\varphi[S'(W_4)] \neq 5$.

Suppose $\varphi[S'(W_4)] = 4$. Now consider the color class $c = \{1, 2, 3, 4\}$ and define the color function as $f : V \to \{1, 2, 3, 4\}$ as $f(v) = 4 = f(v')$, $f(v_1) = 1$, $f(v_2) = 2$, $f(v_1') = 1$, $f(v_2') = 2$, $f(v_3') = 3$, $f(v_4') = 3$ which in turn forces to assign $f(v_3) = 1$, $f(v_4) = 2$. This proper coloring gives the color dominating vertices for color classes $1, 2$ and 4 but not for 3 which is contradiction to our assumption. Thus, $\varphi[S'(W_4)] \neq 4$. Hence, we can color the graph by three colors. For b-coloring, consider the color class $c = \{1, 2, 3\}$ and define the color function as $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1 = f(v_1'), f(v_2) = 2 = f(v_2'), f(v_3) = 1 = f(v_3'), f(v_4) = 2 = f(v_4'), f(v) = 3 = f(v')$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2$, $cdv(3) = v$. Thus $\varphi[S'\,(W_4)] = 3$.

Case 3. $n = 5, 6, 8$

Subcase 3.1 $n = 5$

In this case the graph $S'(W_5)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_5)] \geqslant 3$. As $m[S'(W_5)] = 6$ and by Lemma 2.1, $\chi[S'(W_5)] = 4$. Then by Proposition 1.5 we have $4 \leqslant \varphi(S'(W_5) \leqslant 6$.

If $\varphi(S'(W_5) = 6$ then by Proposition 1.3, the graph $S'(W_5)$ must have six vertices of degree at least five which is possible. But due to the adjacency of vertices of the graph $S'(W_5)$ any proper coloring with six colors have at least one color class which does not have color dominating vertices. Hence it will not be b-coloring for the graph $S'(W_5)$. Thus, $\varphi(S'(W_5) \neq 6$.

Suppose $\varphi(S'(W_5) = 5$. Now consider the color class $=\{1, 2, 3, 4, 5\}$ and define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v'_1) = 2, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 1, f(v'_5) = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus, $\varphi(S'(W_5) = 5$.

Subcase 3.2 $n = 6, 8$

In this case the graph $S'(W_n)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_n)] \geqslant 3$. As $m[S'(W_n)] = 7$ and by Lemma 2.1, $\chi[S'(W_n)] = 3$. Then by Proposition 1.5 we have $3 \leqslant \varphi[S'(W_n)] \leqslant 7$.

If $\varphi[S'(W_n)] = 7$ then by Proposition 1.3, the graph $S'(W_n)$ must have seven vertices of degree at least six which is possible. But due to the adjacency of the vertices of graph $S'(W_n)$ any proper coloring with seven colors have at least one color class which does not have color dominating vertices. Hence it will not be b-coloring for the graph $S'(W_n)$. Thus, $\varphi[S'(W_n)] \neq 7$.

Suppose $\varphi[S'(W_n)] = 6$. Now consider the color class $=\{1, 2, 3, 4, 5, 6\}$ and define the color function as $f : V \to \{1, 2, 3, 4, 5, 6\}$ as $f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1$ which in turn forces to assign $f(v_6) = 2, f(v'_6) = 1$. This proper coloring gives the color dominating vertices for color classes $1, 2, 3, 4$ and 6 but not for 5 which is contradiction to our assumption. Thus, $\varphi[S'(W_n)] \neq 6$.

Suppose that $S'(W_n)$ has b-coloring with 5 colors. Now consider the color class $=\{1, 2, 3, 4, 5\}$ and define the color function as $f : V(G) \to \{1, 2, 3, 4, 5\}$ as $f(v) = 5 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2 = f(v'_3), f(v_4) = 3 = f(v'_4), f(v_5) = 4, f(v_6) = 2, f(v'_1) = 4, f(v'_2) = 4, f(v'_5) = 1, f(v'_6) = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus, $\varphi[S'(W_n)] = 5$.

Case 4. $n = 7$

In this case the graph $S'(W_7)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_7)] \geqslant 3$. As $m[S'(W_7)] = 7$ and by Lemma 2.1, $\chi[S'(W_7)] = 4$. Then by Proposition 1.5 we have $4 \leqslant \varphi[S'(W_7)] \leqslant 7$.

Suppose $\varphi[S'(W_7)] = 7$. Now consider the color class $=\{1, 2, 3, 4, 5, 6, 7\}$ and define the color function as $f : V \to \{1, 2, 3, 4, 5, 6, 7\}$ as $f(v) = 7, f(v') = 6, f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 1, f(v_6) = 4, f(v'_1) = 1, f(v'_2) = 4, f(v'_3) = 4, f(v'_4) = 5, f(v'_5) = 5, f(v'_6) = 4$ which in turn forces to assign $f(v_7) = 2, f(v'_7) = 3$. This proper coloring gives the color dominating vertices for color classes $1, 2, 3, 4$ and 5 but not for 6 and 7 which is contradiction to our assumption. Thus, $\varphi[S'(W_7)] \neq 7$.

Suppose that $S'(W_7)$ has b-coloring with 6 colors. Now consider the color class $=\{1, 2, 3, 4, 5, 6\}$ and define the color function $f : V \to \{1, 2, 3, 4, 5, 6\}$ as $f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 5, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 5$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_7, cdv(6) = v$. Thus, $\varphi[S'(W_7)] = 6$.

Case 5. $n \geqslant 9$

For $n = 9$, the graph $S'(W_9)$ contains an odd cycle. Then by Proposition 1.6, $\chi[S'(W_9)] \geqslant 3$. As $m[S'(W_9)] = 7$ and by Lemma 2.1, $\chi[S'(W_7)] = 4$. Then by Proposition 1.5 we have $4 \leqslant \varphi[S'(W_7)] \leqslant 7$.

Suppose $\varphi[S'(W_9)] = 7$. Consider the color class $\underset{s}{=}\{1, 2, 3, 4, 5, 6, 7\}$ and define the color function $f : V \to \{1, 2, 3, 4, 5, 6, 7\}$ as $f(v) = 6, f(v') = 7, f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 1, f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 2, f(v_7) = 5, f(v'_7) = 5, f(v_8) = 3, f(v'_8) = 4$ which in turn forces to assign $f(v_9) = 2 = f(v'_9)$. This proper coloring gives the color dominating vertices for color classes $1, 2, 3, 4$ and 5 but not for 6 and 7 which is contradiction to our assumption. Thus, $\varphi[S'(W_9)] \neq 7$.

Suppose that $S'(W_9)$ has b-coloring with 6 colors. Consider the color class $\underset{s}{=}\{1, 2, 3, 4, 5, 6\}$ and define the color function $f : V \to \{1, 2, 3, 4, 5, 6\}$ as $f(v) = 6 = f(v'), f(v_1) = 3, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 2, f(v_7) = 5, f(v_8) = 3, f(v_9) = 1 = f(v'_9), f(v'_1) = 4, f(v'_2) = 4, f(v'_3) = 5, f(v'_4) = 5, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 5, f(v'_8) = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_7, cdv(6) = v$. Thus, $\varphi[S'(W_9)] = 6$.

For $n > 9$, we repeat the colors as in the above graph $S'(W_9)$for the vertices $\{v_1, v_2, \ldots, v_9, v'_1, v'_2, \ldots, v'_9, v, v'\}$ and for the remaining vertices assign the colors as $f(v) = 6 = f(v'), f(v_{3k+7}) = 1 = f(v'_{3k+7}), f(v_{3k+8}) = 2 = f(v'_{3k+8})$ where $k \in N$. Hence, $\varphi[S'(W_9)] = 6$, for all $n \geqslant 9$. $\square$

Theorem 2.3 *Let W_n be a wheel. Then, $S'(W_n)$ is b-continuous.*

Proof To prove this result we continue with the terminology and notations used in Lemma 2.1 and consider the following cases.

Case 1. $n = 3$

In this case the graph $S'(W_3)$ is b-continuous as $\chi[S'(W_3)] = \varphi[S'(W_3)] = 4$.

Case 2. $n = 4$

In this case the graph $S'(W_4)$ is b-continuous as $\chi[S'(W_4)] = \varphi[S'(W_4)] = 3$.

Case 3. $n = 5$

In this case by Lemma 2.1, $\chi[S'(W_5)] = 4$ and by Theorem-2.2, $\varphi[S'(W_5)] = 5$. Hence, b-coloring exists for every integer satisfying $\chi[S'(W_5)] \leqslant k \leqslant \varphi[S'(W_5)]$(Here $k = 4, 5$). Thus, $S'(W_5)$ is b-continuous.

Case 4. $n = 6$

In this case by Lemma 2.1, $\chi[S'(W_6)] = 3$ and by Theorem-2.2, $\varphi[S'(W_6)] = 5$. It is obvious that b-coloring for the graph $S'(W_6)$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph $S'(W_6)$ is as follows.

Consider the color class $\underset{s}{=}\{1, 2, 3, 4\}$ and define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v) = f(v') = 4, f(v_1) = f(v'_1) = 3, f(v_2) = f(v'_2) = 1, f(v_3) = f(v'_3) = 2, f(v_4) = f(v'_4) = 3, f(v_5) = f(v'_5) = 1, f(v_6) = f(v'_6) = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v$. Thus, $S'(W_6)$ is four colorable. Hence b-coloring exists for every integer k satisfy $\chi[S'(W_6)] \leqslant k \leqslant \varphi[S'(W_6)]$ (Here $k = 3, 4, 5$). Consequently $S'(W_6)$ is b-continuous.

Case 5. $n = 7$

By Lemma 2.1, $\chi[S'(W_7)] = 4$ and by Theorem 2.2, $\varphi[S'(W_7)] = 6$. It is obvious that b-coloring for the graph $S'(W_7)$ is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph $S'(W_7)$ is as follows.

Consider the color class $\xi\{1, 2, 3, 4, 5\}$ and define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = f(v') = 5, f(v_1) = 3, f(v_1') = 4, f(v_2) = 1, f(v_2') = 4, f(v_3) = 2, f(v_3') = 2, f(v_4) = 3, f(v_4') = 1, f(v_5) = 4, f(v_5') = 1, f(v_6) = 2, f(v_6') = 2, f(v_7) = 1 = f(v_7')$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus, $S'(W_7)$ is five colorable. Hence, b-coloring exists for every integer k satisfy $\chi[S'(W_7)] \leqslant k \leqslant \varphi[S'(W_7)]$(Here $k = 4, 5, 6$). Hence $S'(W_7)$ is b-continuous.

Case 6. $n = 8$

By Lemma 2.1, $\chi[S'(W_8)] = 3$ and by Theorem 2.2,$\varphi[S'(W_8)] = 5$. It is obvious that b-coloring for the graph $S'(W_8)$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph $S'(W_8)$ is as follows.

Consider the color class $\xi\{1, 2, 3, 4\}$ and define the color function as $f : V \to \{1, 2, 3, 4\}$ as $f(v) = f(v') = 4, f(v_1) = 3 = f(v_1'), f(v_2) = 1 = f(v_2'), f(v_3) = 2 = f(v_3'), f(v_4) = 3 = f(v_4'), f(v_5) = 1 = f(v_5'), f(v_6) = 2 = f(v_6'), f(v_7) = 1 = f(v_7'), f(v_8) = 2 = f(v_8')$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v$. Thus, $S'(W_8)$ is four colorable. Hence, b-coloring exists for every integer k satisfy $\chi[S'(W_8)] \leqslant k \leqslant \varphi[S'(W_8)]$(Here $k = 3, 4, 5$). Thus, $S'(W_8)$ is b-continuous.

Case 7. $n \geqslant 9$

For $n = 9$, by Lemma 2.1, $\chi[S'(W_9)] = 4$ and by Theorem 2.2,$\varphi[S'(W_9)] = 6$. It is obvious that b-coloring for the graph $S'(W_9)$ is possible using the number of colors $k = 4, 6$. Now for $k = 5$ the b-coloring for the graph $S'(W_9)$ is as follows.

Consider the color class $\xi\{1, 2, 3, 4, 5\}$ and define the color function as $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v) = f(v') = 5, f(v_1) = 3, f(v_1') = 4, f(v_2) = 1, f(v_2') = 4, f(v_3) = 2, f(v_3') = 2, f(v_4) = 3, f(v_4') = 1, f(v_5) = 4, f(v_5') = 1, f(v_6) = 2, f(v_6') = 2, f(v_7) = 1 = f(v_7'), f(v_8) = 2, f(v_8') = 2, f(v_9) = f(v_9') = 1$. This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v$. Thus, $S'(W_9)$ is five colorable. Hence, b-coloring exists for every integer k satisfy $\chi[S'(W_9)] \leqslant k \leqslant \varphi[S'(W_9)]$(Here $k = 4, 5, 6$). Hence, $S'(W_9)$ is b-continuous.

For odd $n \geqslant 9$, we repeat the colors as in $S'(W_9)$ for the vertices $\{v_1,\ v_2,\ v_9, \ldots, v_1',\ v_2', \ldots, v'_9,\ v,\ v'\}$ and for the remaining vertices gives the colors as follows:

When $k = 5,\ f(v') = f(v) = 5,\ f(v_{3k+7}) = f(v'_{3k+7}) = 1,\ f(v_{3k+8}) = f(v'_{3k+8}) = 2,\ k \in N$.

For even $n > 9$, we repeat the color assignment as in case $n = 8$ discussed above for the vertices $\{v,\ v',\ v_1, \ldots, v_8, v_1', v_2', \ldots, v_8'\}$ and for remaining vertices gives the colors as follows:

When $k = 4,\ f(v') = f(v) = 4,\ f(v_{2k+7}) = 1 = f({v'}_{2k+7}),\ f(v_{2k+8}) = 2 = f({v'}_{2k+8}), k \in N$ and when $k = 5,\ f(v') = f(v) = 5,\ f(v_{2k+8}) = 1 = f({v'}_{2k+8}),\ f(v_{2k+9}) = 2 = f({v'}_{2k+9}), k \in N$. □

Any coloring with $\chi(G)$ is a b-coloring, we state the following obvious result.

Corollary 2.4 *Let W_n be a wheel. Then*

$$S_b[S'(W_n)] = \begin{cases} \{4\}, & n = 3 \\ \{3\}, & n = 4 \\ \{4,5\} & n = 5 \\ \{3,4,5\}, & n = 6, 8 \\ \{4,5,6\}, & n = 7 \\ \{4,5,6\} & for\ odd\ n \geqslant 9 \\ \{3,4,5\} & for\ even\ n > 9 \end{cases}$$

§3. Concluding Remarks

A discussion about b-coloring of wheel is carried out by Alkhateeb [9] while we investigate b-chromatic number of splitting graph of wheel. We also obtain b-spectrum and show that splitting graph of wheel is b-continuous.

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Eccentric Connectivity and Connective Eccentric Indices Of Generalized Complementary Prisms and Some Duplicating Graphs

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Abstract: The connected eccentricity index and the eccentric connectivity index of a graph G is respectively defined by

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{deg(v)}{ecc(v)} \text{ and } \xi^{c}(G) = \sum_{v \in V(G)} deg(v)\ ecc(v),$$

where $ecc(v)$ is the eccentricity of a vertex v in G. In this paper, we have obtained the bounds for connective eccentricity index of those generalized complementary prisms and eccentric connective index of duplication of some graphs.

Key Words: Eccentricity, radius, diameter, complementary prism.

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§1. Introduction

Throughout this paper all graphs we considered are simple and connected. For a vertex $v \in V(G)$, $deg(v)$ denotes the degree of $v, \delta(G)$ and $\Delta(G)$ represent the minimum and maximum degree of G respectively. For vertices $u, v \in V(G)$, the distance $d(u, v)$ is defined as the length of the shortest path between u and v in G. The eccentricity $ecc(v)$ of a vertex v is the maximum among the distance from v to the remaining vertices of G. The diameter $d(G)$ of the graph G is the maximum eccentricity of the vertices of G, while the radius $r(G)$ of the graph G is the minimum eccentricity of the vertices of G. The total eccentricity of the graph G, denoted by $\xi(G)$ is defined as the sum of eccentricities of all the vertices of the graph G. That is

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$\xi(G) = \sum_{v\in V(G)} ecc(v)$. The eccentric connectivity index of G, denoted by $\xi^c(G)$ is defined as

$$\xi^c(G) = \sum_{v\in V(G)} deg(v)\ ecc(v).$$

In [4], the connective eccentricity index (CEI) of a graph G was defined as

$$\xi^{ce}(G) = \sum_{v\in V(G)} \frac{deg(v)}{ecc(v)}.$$

Kathiresan and Arockiaraj introduced some generalization of complementary prisms and studied the Wiener index of those generalized complementary prisms ([8]).

Let G and H be any two graphs on p_1 and p_2 vertices, respectively and let R and S be subsets of $V(G) = \{u_1, u_2, \ldots, u_{p_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{p_2}\}$ respectively. The *complementary product* $G(R)\square H(S)$ has the vertex set $\{(u_i, v_j) : 1 \le i \le p_1, 1 \le j \le p_2\}$ and (u_i, v_j) and (u_h, v_k) are adjacent in $G(R)\square H(S)$

(1) if $i = h, u_i \in R$ and $v_jv_k \in E(H)$, or if $i = h, u_i \notin R$ and $v_jv_k \notin E(H)$ or
(2) if $j = k, v_j \in S$ and $u_iu_h \in E(G)$, or if $j = k$, $v_j \notin S$ and $u_iu_h \notin E(G)$.

In other words, $G(R)\square H(S)$ is the graph formed by replacing each vertex $u_i \in R$ of G by a copy of H, each vertex $u_i \notin R$ of G by a copy of $\overline{H}$, each vertex $v_j \in S$ of H by a copy of G and each vertex $v_j \notin S$ of H by a copy of $\overline{G}$. If $R = V(G)$ (respectively, $S = V(H)$), the complementary product can be written as $G\square H(S)$ (respectively, $G(R)\square H$). The *complementary prism* $G\overline{G}$ obtained from G is $G\square K_2(S)$ with $|S| = 1$. That is, $G\overline{G}$ has a copy of G and a copy of $\overline{G}$ with a matching between the corresponding vertices. In $G\overline{G}$, we have an edge $v\overline{v}$ for each vertex v in G. The authors of [**?**] consider this edge as K_2 or $K_{1,1}$ or P_2. By taking m copies of G and n copies of $\overline{G}$, they generalize the complementary prism as a graph $G\square H(S)$, where $H = K_{m+n}$ (or $K_{m,n}$) and S is a subset of $V(H)$ having m vertices and $H = C_{2m}$ (or P_{2m}) whose vertex set is $\{v_1, v_2, \ldots, v_{2m}\}$ and $S = \{v_1, v_3, \ldots, v_{2m-1}\}$ ([8]).

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The graph $EV(G)$ obtained by duplicating each edge by a vertex of a graph G is defined as follows. The vertex set of $EV(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $EV(G)$ are adjacent in $EV(G)$ in case one of the following holds:

(1) x and y are adjacent vertices in G;
(2) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Motivated by these works, we have obtained the bounds for connective eccentric index of those generalized complimentary prisms and eccentric connective index of duplication of some graphs.

Theorem 1.1([8]) *For the complementary prism* $G\overline{G}, r(G\overline{G}) = 2$ *and*

$$d(G\overline{G}) = \begin{cases} 2 & \text{if } d(G) = d(\overline{G}) = 2 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 1.2([8]) *For any connected graph G with $p \geq 2$,*

$$d(G_{m+n}) = \begin{cases} 2 & \text{if } d(G) = d(\overline{G}) = 2 \text{ and } m = n = 1 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 1.3([8]) *For any connected graph G with $p \geq 2$,*

$$d(G_{m,n}) = \begin{cases} 2 & \text{if } d(G) = d(\overline{G}) = 2 \text{ and } m = n = 1 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 1.4([8]) *For any connected graph G with $p \geq 2$,*

$$d(G^p_{m,m}) = \begin{cases} 2m & \text{if } m > 1 \\ 2 & \text{if } m = 1 \text{ and } d(G) = d(\overline{G}) = 2 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 1.5([8]) *For any connected graph G with $p \geq 2$ $d(G^c_{m,m}) = 2r + 1$ if $m = 2r \geq 2$ and r is a positive integer.*

§2. Main Results

Theorem 2.1 *For any connected graph G on p vertices,*

$$\frac{p(p+1)}{3} \leq \xi^{ce}(G\overline{G}) \leq \frac{p(p+1)}{2}.$$

Proof For any connected graph G on p vertices, by Theorem 1.1, $G\overline{G} \in F_{22}$ while $G \in F_{22}$ and $G\overline{G} \in F_{23}$ while $G \notin F_{22}$. When $G\overline{G} \in F_{22}, ecc(v) = 2$ for all $v \in V(G\overline{G})$. So

$$\xi^{ce}(G\overline{G}) = \sum_{v \in V(G\overline{G})} \frac{deg(v)}{ecc(v)} = \frac{1}{2} \sum_{v \in V(G\overline{G})} deg(v) = \frac{p(p+1)}{2}.$$

When $G\overline{G} \in F_{23}, 2 \leq ecc(v) \leq 3$ for all $v \in V(G\overline{G})$. This implies that

$$\frac{1}{3} \leq \frac{1}{ecc(v)} \leq \frac{1}{2}$$

for all $v \in V(G\overline{G})$ and hence

$$\frac{deg(v)}{3} \leq \frac{deg(v)}{ecc(v)} \leq \frac{deg(v)}{2}$$

for all $v \in V(G\overline{G})$. Therefore

$$\sum_{v\in V(G\overline{G})} \frac{deg(v)}{3} \leq \sum_{v\in V(G\overline{G})} \frac{deg(v)}{ecc(v)} \leq \sum_{v\in V(G\overline{G})} \frac{deg(v)}{2}.$$

Thus

$$\frac{p(p+1)}{3} \leq \xi^{ce}(G\overline{G}) \leq \frac{p(p+1)}{2}. \qquad \square$$

Theorem 2.2 *For any connected graph G on p vertices and q edges with $m, n \geq 1$*

$$\frac{2}{3}\left[(m-n)q + n\binom{p}{2} + \binom{m+n}{2}\right] \leq \xi^{ce}(G_{m+n}) \leq (m-n)q + n\binom{p}{2} + \binom{m+n}{2}.$$

Proof The number of edges in G_{m+n} is $(m-n)q + n\binom{p}{2} + \binom{m+n}{2}$. By Theorem 1.2, $G_{m+n} \in F_{22}$ when $m = 1, n = 1$ and $G_{m+n} \in F_{23}$ otherwise. Hence by Theorem 1.2, the result follows. $\square$

Theorem 2.3 *For any connected graph G on p vertices and q edges,*

$$\frac{2}{3}\left[(m-n)q + n\binom{p}{2} + mn\right] \leq \xi^{ce}(G_{m,n}) \leq (m-n)q + n\binom{p}{2} + mn.$$

Proof The number of edges in $G_{m,n}$ is

$$(m-n)q + n\binom{p}{2} + mn.$$

By Theorem 1.3, $G_{m,n} \in F_{22}$ when $m = n = 1$ and $G_{m,n} \in F_{23}$ otherwise. Hence by Theorem 2.1, the result follows. $\square$

Theorem 2.4 *For any connected graph G on p vertices and q edges,*

$$\binom{p}{2} + p\left(2 - \frac{1}{m}\right) \leq \xi^{ce}(G^p_{m,m}) \leq 2\left[\binom{p}{2} + p\left(2 - \frac{1}{m}\right)\right].$$

Proof In $G^p_{m,m}$, the number of edges is $m\binom{p}{2} + p(2m-1)$. By Theorem 1.4, $r(G^p_{m,m}) = m$ and $d(G^p_{m,m}) = 2m$. So

$$\frac{deg(v)}{2m} \leq \frac{deg(v)}{ecc(v)} \leq \frac{deg(v)}{m}$$

and hence

$$\frac{1}{m}\left[m\binom{p}{2} + p(2m-1)\right] \leq \xi^{ce}(G^p_{m,m}) \leq \frac{2}{m}\left[m\binom{p}{2} + p(2m-1)\right]. \qquad \square$$

Theorem 2.5 *For any connected graph G on p vertices and for any even integer $m \geq 2$,*

$$\frac{2mp(p+1)}{m+1} \leq \xi^{ce}(G^c_{m,m}) \leq \frac{4mp(p+1)}{m+2}.$$

Proof For even integer $m \geq 2$, by Theorem 1.5,

$$\frac{m}{2} + 1 \leq ecc(v) \leq m+1$$

for all $v \in V(G^c_{m,m})$. Also the number of edges in $G^c_{m,m}$ is $m\binom{p}{2} + 2mp$. Therefore

$$\frac{2}{m+1}\left[m\binom{p}{2} + 2mp\right] \leq \xi^{ce}(G^c_{m,m}) \leq \frac{2}{\frac{m}{2}+1}\left[m\binom{p}{2} + 2mp\right]$$

and hence

$$\frac{2mp}{m+1}[p+1] \leq \xi^{ce}(G^c_{m,m}) \leq \frac{4mp}{m+2}[p+1]. \qquad \square$$

Now we determine the exact value of $\xi^c(G)$ for some graph families.

Proposition 2.6 *For any $n \geq 3$,*

$$\xi^c(EV(P_n)) = \begin{cases} \frac{1}{2}(9n^2 - 16n + 7), & n \text{ is odd} \\ \frac{1}{2}(9n^2 - 16n + 8), & n \text{ is even.} \end{cases}$$

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices on the path and $x_1, x_2, \cdots, x_{n-1}$ be the vertices corresponding to the edges of the path P_n. Then,

$$ecc(v_i) = \begin{cases} n-i, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ i-1, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n \end{cases} \quad \text{and} \quad ecc(x_i) = \begin{cases} n-i, & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ i, & \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n-1. \end{cases}$$

Also,

$$deg(v_i) = \begin{cases} 2, & i = 1, n \\ 4, & 2 \leq i \leq n-1 \end{cases} \quad \text{and} \quad deg(x_i) = 2, 1 \leq i \leq n-1.$$

Therefore

$$\begin{aligned}\xi^c(EV(P_n)) &= \sum_{v \in V(EV(P_n))} deg(v)\ ecc(v) \\ &= 4\sum_{i=2}^{n-1} ecc(v_i) + 4(n-1) + 2\sum_{i=1}^{n-1} ecc(x_i)\end{aligned}$$

$$= 4\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(n-i) + 4\sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1}(i-1) + 4(n-1)$$

$$+ 2(n-1) + 2\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(n-i) + 2\sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} i$$

$$= 6\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor}(n-i) + 6\sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} i - 4\left(n-1-\left\lfloor \frac{n}{2} \right\rfloor\right) + 6(n-1)$$

$$= 6n\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) - 6\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} i + 6\sum_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} i - 4n + 4 + 4\left\lfloor \frac{n}{2} \right\rfloor + 6n - 6$$

$$= 6n\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) - 12\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} i + 6\sum_{i=2}^{n-1} i + 2n - 2 + 4\left\lfloor \frac{n}{2} \right\rfloor.$$

$$= 6n\left\lfloor \frac{n}{2} \right\rfloor - 6n - 12\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} i + 12 + 6\sum_{i=1}^{n-1} i - 6 + 2n - 2 + 4\left\lfloor \frac{n}{2} \right\rfloor$$

$$= (6n+4)\left\lfloor \frac{n}{2} \right\rfloor - 4n + 4 - 12\left(\frac{\lfloor \frac{n}{2} \rfloor\left(\lfloor \frac{n}{2} \rfloor + 1\right)}{2}\right) + 6\left(\frac{n(n-1)}{2}\right)$$

$$= (6n+4)\left\lfloor \frac{n}{2} \right\rfloor - 4n + 4 - 6\left\lfloor \frac{n}{2} \right\rfloor\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + 3n^2 - 3n$$

$$= (6n-2)\left\lfloor \frac{n}{2} \right\rfloor - 6\left\lfloor \frac{n}{2} \right\rfloor^2 + 3n^2 - 7n + 4$$

$$= \begin{cases} \frac{1}{2}(9n^2 - 16n + 7), & n \text{ is odd} \\ \frac{1}{2}(9n^2 - 16n + 8), & n \text{ is even.} \end{cases}$$

□

Proposition 2.7 *For any $n \geq 3$,*

$$\xi^c(EV(C_n)) = \begin{cases} 3n^2 - n, & n \textit{ is odd} \\ 3n^2 + 2n, & n \textit{ is even.} \end{cases}$$

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices on the cycle of length n and $x_1, x_2, \cdots, x_n$ be the vertices corresponding to the edges of C_n so that $v_i x_i, x_i v_{i+1} \in E(EV(C_n)), 1 \leq i \leq n$, where $v_{n+1} = v_1$. Then,

$$ecc(v_i) = \begin{cases} \frac{n-1}{2}, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases} \text{ and } ecc(x_i) = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{n+2}{2}, & n \text{ is even, for } 1 \leq i \leq n. \end{cases}$$

Also, $deg(v_i) = 4, 1 \leq i \leq n$ and $deg(x_i) = 2, 1 \leq i \leq n$. Therefore,

$$
\begin{aligned}
\xi^c(EV(C_n)) &= \sum_{v \in V(EV(C_n))} deg(v)\ ecc(v) \\
&= 4\sum_{i=1}^{n} ecc(v_i) + 2\sum_{i=1}^{n} ecc(x_i) \\
&= \begin{cases} 4n\left(\frac{n-1}{2}\right) + 2n\left(\frac{n+1}{2}\right), & n \text{ is odd} \\ 4n\left(\frac{n}{2}\right) + 2n\left(\frac{n+2}{2}\right), & n \text{ is even} \end{cases} \\
&= \begin{cases} 3n^2 - n, & n \text{ is odd} \\ 3n^2 + 2n, & n \text{ is even.} \end{cases}
\end{aligned}
$$

□

Proposition 2.8 *For any $n \geq 2$,*

$$
\xi^c(EV(K_n)) = \begin{cases} 6, & n = 2 \\ 36, & n = 3 \\ 7n(n-1), & n \geq 4. \end{cases}
$$

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and $x_1, x_2, \ldots, x_n, m = \binom{n}{2}$ be the vertices corresponding to the edges of K_n. Then $ecc(v_i) = 2$ for $1 \leq i \leq n$ and $ecc(x_i) = 3$ for $1 \leq i \leq m$. Also $deg(v_i) = 2n - 2, 1 \leq i \leq n$ and $deg(x_i) = 2, 1 \leq i \leq m$. Therefore,

$$
\begin{aligned}
\xi^c(EV(K_n)) &= \sum_{v \in V(EV(K_n))} deg(v)\ ecc(v) \\
&= \sum_{v_i \in V(EV(K_n))} deg(v_i)\ ecc(v_i) + \sum_{x_i \in V(EV(K_n))} deg(x_i)\ ecc(x_i) \\
&= 2n(2n-2) + 6m \\
&= 4n(n-1) + 6\frac{n(n-1)}{2} = 7n(n-1).
\end{aligned}
$$

When $n = 3, ecc(v_i) = ecc(x_i) = 2$. Therefore $\xi^c(EV(K_3)) = 36$. When $n = 2, ecc(v_i) = ecc(x_i) = 1$ and $deg(v_i) = 2$ for $i = 1, 2$ and $deg(x_1) = 2$. So $\xi^c(EV(K_2)) = 6$. □

Proposition 2.9 *The eccentric connectivity index of $K_{1,n}$ is $6n$.*

Proof Let v_0 be the central vertex and $v_1, v_2, \ldots, v_n$ be the pendent vertex of $K_{1,n}$. Let $x_1, x_2, \ldots, x_n$ be the vertices corresponding to the edges of $K_{1,n}$ in $EV(K_{1,n})$. Then $ecc(v_0) = 1$, $ecc(v_i) = 2, 1 \leq i \leq n$ and $ecc(x_i) = 2, 1 \leq i \leq n$. Also $deg(v_0) = 2n$ and $deg(v_i) = deg(x_i) = 2$

for $1 \le i \le n$. Therefore,

$$\begin{aligned}\xi^c(EV(K_{1,n})) &= \sum_{v \in V(EV(K_{1,n}))} deg(v)\ ecc(v) \\ &= 2n + \sum_{i=1}^{n} 4 + \sum_{i=1}^{n} 4 = 6n.\end{aligned}$$

□

Proposition 2.10 *For any $n \ge 3$,*

$$\xi^c(EV(W_n)) = \begin{cases} 28n, & n = 3, 4 \\ 34n, & n = 5 \\ 36n, & n \ge 6. \end{cases}$$

Proof Let v_0 be the central vertex and $v_1, v_2, \ldots, v_n$ be the vertices on the cycle of W_n. Let $x_i, 1 \le i \le n$ be the vertices corresponding to the edges on the cycle and $x_{n+i}, 1 \le i \le n$ be the vertices corresponding to the edges $v_0v_i, 1 \le i \le n$.

Assume that $n \ge 6$. In $EV(W_n), ecc(v_0) = 2, ecc(v_i) = ecc(x_i) = 3, 1 \le i \le n$ and $ecc(x_i) = 4, n+1 \le i \le 2n$. Also $deg(v_0) = 2n, deg(v_i) = 6, 1 \le i \le n$ and $deg(x_i) = 2, 1 \le i \le 2n$. Therefore

$$\begin{aligned}\xi^c(EV(W_n)) &= \sum_{v \in V(EV(W_n))} deg(v)\ ecc(v) \\ &= deg(v_0)\ ecc(v_0) + \sum_{i=1}^{n} deg(v_i)\ ecc(v_i) \\ &\quad + \sum_{i=1}^{n} deg(x_i) ecc(x_i) + \sum_{i=n+1}^{2n} deg(x_i)\ ecc(x_i) \\ &= 4n + 18n + 6n + 8n = 36n.\end{aligned}$$

When $n = 5, ecc(v_0) = 2, ecc(v_i) = 3, 1 \le i \le n$, $ecc(x_i) = 3, 1 \le i \le 2n$ and hence $\xi^c(EV(W_n)) = 34n$. When $n = 3$ and 4, $ecc(v_0) = 2, ecc(v_i) = 2, 1 \le i \le n, ecc(x_i) = 3, 1 \le i \le 2n$ and hence $\xi^c(EV(W_n)) = 28n$. □

Proposition 2.11 *For any $n \ge 2$,*

$$\xi^c(EV(L_n)) = \begin{cases} \frac{27}{2}n^2 + n - \frac{13}{2}, & \text{when } n \text{ is odd} \\ \frac{27}{2}n^2 + n - 4, & \text{when } n \text{ is even.} \end{cases}$$

Proof Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$. Let x_i and y_i be the duplicating vertices of the edges u_iu_{i+1} and v_iv_{i+1} respectively, $1 \le i \le n-1$

and z_i be the duplicating vertex of the edge $u_iv_i, 1 \leq i \leq n$. In $EV(L_n)$,

$$ecc(u_i) = \begin{cases} n+1-i, & 1 \leq i \leq \left[\frac{n}{2}\right] \\ ecc(u_{n+1-i}), & \left[\frac{n}{2}\right]+1 \leq i \leq n, \end{cases}$$

$$ecc(v_i) = ecc(u_i), 1 \leq i \leq n,$$

$$ecc(x_i) = \begin{cases} n+1-i, & 1 \leq i \leq \left[\frac{n}{2}\right] \\ ecc(x_{n-i}), & \left[\frac{n}{2}\right]+1 \leq i \leq n-1, \end{cases}$$

$$ecc(y_i) = ecc(x_i), 1 \leq i \leq n-1 \text{ and}$$

$$ecc(z_i) = \begin{cases} n+1-i, & 1 \leq i \leq \left\lceil\frac{n}{2}\right\rceil \\ ecc(z_{n+1-i}), & \left\lceil\frac{n}{2}\right\rceil+1 \leq i \leq n. \end{cases}$$

Also,

$$deg(u_i) = deg(v_i) = \begin{cases} 4, & i = 1, n \\ 6, & 2 \leq i \leq n-1 \end{cases} \quad \text{and} \quad deg(x_i) = deg(y_i) = deg(z_i) = 2.$$

Therefore,

$$\begin{aligned} \xi^c(EV(L_n)) &= \sum_{v \in V(EV(L_n))} deg(v)\ ecc(v) \\ &= 2\sum_{i=1}^{n} deg(u_i)ecc(u_i) + 2\sum_{i=1}^{n-1} deg(x_i)ecc(x_i) \\ &\quad + \sum_{i=1}^{n} deg(z_i)ecc(z_i) \\ &= 16n + 12\sum_{i=2}^{n-1} ecc(u_i) + 4\sum_{i=1}^{n-1} ecc(x_i) + 2\sum_{i=1}^{n} ecc(z_i). \end{aligned}$$

When n is odd,

$$\begin{aligned} \xi^c(EV(L_n)) &= 16n + 12\left[(n-2)\left(\frac{n+1}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n+3}{2}\right)\right)\right] \\ &\quad + 4\left[(n-1)\left(\frac{n+3}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n-3}{2}\right)\right)\right] \\ &\quad + 2\left[n\left(\frac{n+3}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n-1}{2}\right)\right)\right] \\ &= 16n + 6(n-2)(n+1) + 4(n-3)(n-1) \\ &\quad + 2(n-1)(n+3) + n(n+3) + 2\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) = \frac{27}{2}n^2 + n - \frac{13}{2}. \end{aligned}$$

When n is even,

$$\begin{aligned}
\xi^c(EV(L_n)) &= 16n + 12\left[(n-2)\left(\frac{n-2}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n-4}{2}\right)\right)\right] \\
&\quad + 4\left[(n-1)\left(\frac{n+2}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n-2}{2}\right)\right)\right] \\
&\quad + 2\left[n\left(\frac{n+4}{2}\right) + 2\left(1+2+\cdots+\left(\frac{n-2}{2}\right)\right)\right] \\
&= 16n + 6(n^2-4) + 2(n-1)(n+2) + n(n+4) \\
&\quad + 3(n-4)(n-2) + \frac{3}{2}(n(n-2)) = \frac{27}{2}n^2 + n - 4. \qquad \square
\end{aligned}$$

Proposition 2.12 *For any $n \geq 3$,*

$$\xi^c(EV(C_n \circ K_1)) = \begin{cases} 6n^2 + 12n, & n \text{ is odd} \\ 6n^2 + 16n, & n \text{ is even.} \end{cases}$$

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices in the cycle C_n and u_i be the pendent vertex attached at $v_i, 1 \leq i \leq n$, in $C_n \circ K_1$. Let $x_i, 1 \leq i \leq n$ be the vertices corresponding to the edges of the cycle C_n and $y_i, 1 \leq i \leq n$ be the vertices corresponding to the pendent edges of $C_n \circ K_1$ in $EV(C_n \circ K_1)$. In $EV(C_n \circ K_n)$, for $1 \leq i \leq n$,

$$ecc(v_i) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n+2}{2}, & \text{if } n \text{ is even,} \end{cases}$$

$$ecc(u_i) = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{n+4}{2}, & \text{if } n \text{ is even,} \end{cases}$$

$$ecc(x_i) = \begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{n+2}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{and}$$

$$ecc(y_i) = ecc(u_i).$$

Also $deg(v_i) = 6$ and $deg(u_i) = deg(x_i) = deg(y_i) = 2, 1 \leq i \leq n$. When n is odd,

$$\begin{aligned}
\xi^c(EV(C_n \circ K_1)) &= \sum_{v \in V(EV(C_n \circ K_1))} deg(v)\ ecc(v) \\
&= 6\sum_{i=1}^{n} ecc(v_i) + 4\sum_{i=1}^{n} ecc(u_i) + 2\sum_{i=1}^{n} ecc(x_i) \\
&= 6n\left(\frac{n+1}{2}\right) + 4n\left(\frac{n+3}{2}\right) + 2n\left(\frac{n+3}{2}\right) \\
&= 6n^2 + 12n.
\end{aligned}$$

When n is even,

$$\begin{aligned}\xi^c(EV(C_n \circ K_1)) &= \sum_{v \in V(EV(C_n \circ K_1))} deg(v)\ ecc(v) \\ &= 6\sum_{i=1}^{n} ecc(v_i) + 4\sum_{i=1}^{n} ecc(u_i) + 2\sum_{i=1}^{n} ecc(x_i) \\ &= 6n\left(\frac{n+2}{2}\right) + 4n\left(\frac{n+4}{2}\right) + 2n\left(\frac{n+2}{2}\right) \\ &= 6n^2 + 16n. \qquad \square\end{aligned}$$

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The Moving Coordinate System and Euler-Savary's Formula for the One Parameter Motions On Galilean (Isotropic) Plane

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Abstract: In this article, one Galilean (or called Isotropic) plane moving relative to two other Galilean planes (or Isotropic Planes), one moving and the other fixed, was taken into consideration and the relation between the absolute, relative and sliding velocities of this movement and pole points were obtained. Also a canonical relative system for one-parameter Galilean planar motion was defined. In addition, Euler-Savary formula, which gives the relationship between the curvature of trajectory curves, was obtained with the help of this relative system.

Key Words: Kinematics, moving coordinate system, Euler Savary formula, Galilean plane (isotropic plane).

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§1. Introduction

Galilean Geometry, is described by Yaglom, [1]. So far, many researcher has done a lot of studies as [2-4], etc in the Galilean Plane (or Isotropic Plane) and Galilean Space. Also, Euler-Savary's formula is very famous theorem. It gives relation between curvature of roulette and curvatures of these base curve and rolling curve, [14]. It takes place in a lot of studies of engineering and mathematics. A few of them are studies worked by Alexander and Maddocks,[5], Buckley and Whitfield, [6], Dooner and Griffis, [7], Ito and Takahaski, [8], Pennock and Raje, [9].

In 1959, Müller, [10]; defined one-parameter planar motion in the Euclidean plane E^2. He studied the moving coordinate system and Euler-Savary's formula during one parameter planar motions. Then, Ergin in 1991 and 1992, [11-[12]; considering the Lorentzian Plane L^2, instead of the Euclidean plane E^2, introduced the one parameter planar motion in the Lorentzian plane L^2 and gave the relations between both the velocities and accelerations and also defined the moving coordinate system. Furthermore, in 2002 Aytun [13] studied the Euler Savary formula for the one parameter Lorentzian motions as using Müller's Method [10]. And in 2003, Ikawa [14] gave the Euler-Savary formula on Minkowski without using Müller's Method [10].

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In 1983, Otto Röschel, [15]; studied kinematics in the isotropic plane. He investigated fundamental properties of the point-paths, developed a formula analog to the wellknown formula of Euler-Savary and studied special motions: An isotropic elliptic motion and an isotropic four-bar-motion. And in 1985, he [16]; studied motions $\sum / \sum_0$ in the isotropic plane. Given C^2 -curve k in the moving frame $\sum$ he found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondance between the isotropic curvatures A and A_0 of k and k_0. Then he investigated third - order properties of the point-paths. And then in 2013, Yüce, [17], considering the Galilean Plane G^2, instead of the Euclidean Plane E^2 or instead of the Lorentzian Plane L^2, defined one parameter planar Galilean motion in Galilean Plane G^2 analog [10] or [11]. Moreover, they analyzed the relationships between the absolute, relative and sliding velocities of one-parameter Galilean Planar motion as well as the related pole lines.

Now we investigate the moving coordinate system and Euler Savary's Formula during the one parameter planar Galilean motion in Galilean Plane G^2 analog [10] or [11] by using Müller's method.

§2. Preliminaries

In this section, the basic information about Galilean geometry which is described by Yaglom, [1], will be given.

Let $\{x\}$ and $\{x'\}$ be two relative frames and origin point O with velocity v on a line o move according to relative frame $\{x'\}$, that is, $b\,(t) = b + vt$ where t is time and b is coordinate of point O with respect to coordinate system $\{x'\}$ at the moment $t = 0$ (see, Figure 1).

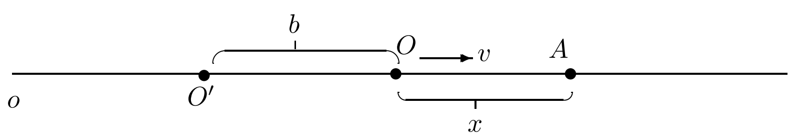

Figure 1 The rectilinear motion

Then, relation between coordinates of x and x' is

$$\begin{aligned} x' &= x + b\,(t) && (1) \\ &= x + b + vt. && (2) \end{aligned}$$

Also, since time would be $t' = t + a$ (example, t is Gregorian calendar, t' is Hijra calendar), we can write

$$\begin{cases} x' = x + vt + b \\ t' = t + a. \end{cases} \qquad (3)$$

This transformations (1.1) are called *Galilean Transformations for rectilinear motions.* If point $A\,(x,t)$ with coordinate x and t of a (two-dimensional) plane xOt (see, Figure 2) represents position of point $A\,(x)$ on a line o at time t, then two-dimensional Geometry which is invariant under the Galilean Transformations for rectilinear motions is obtained.

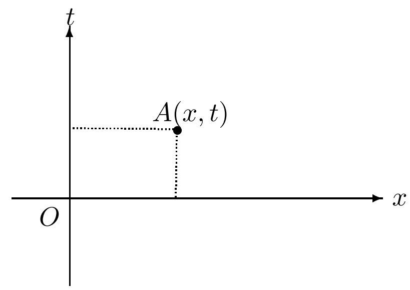

Figure 2 xOt plane

So, this geometry is called *the geometry of Galileo's principle of relativity for rectilinear motions* or *two-dimensional Galilean geometry* and is represented by G^2. Since we shall only talk about the two-dimensional Galilean geometry in this work, we shall shortly call *Galilean plane*. If transformation (3) is arranged as x instead of t and y instead of x, we get

$$\begin{aligned} x' &= x + a \\ y' &= y + vx + b. \end{aligned} \tag{4}$$

This transformation (4) composed of the shear transformation

$$\begin{aligned} x_1 &= x \\ y_1 &= y + vx \end{aligned} \tag{5}$$

and the translation transformation

$$\begin{aligned} x' &= x_1 + a \\ y' &= y_1 + b. \end{aligned} \tag{6}$$

Theorem 2.1([1]) *Transformation* (4) *maps*

(1) *lines onto lines;*

(2) *parallel lines onto parallel lines;*

(3) *collinear segments* AB, CD *onto collinear segments* $A'B'$,$C'D'$ *with* $\frac{C'D'}{A'B'} = \frac{CD}{AB}$*;*

(4) *a figure* F *onto a figure* F' *of the same area.*

In the Galilean plane, the vectors $\{\mathbf{g}_1 = (1, 0), \mathbf{g}_2 = (0, 1)\}$ are called *orthogonal basis vectors* of G^2, and also a vector which is parallel to vector $\mathbf{g}_2$ is called *special vector*. If $\{\mathbf{g}_{1,}\mathbf{g}_2\}$ are orthogonal basis vectors and $\mathbf{a}, \mathbf{b} \in G^2$ whose coordinates are (x_1, x_2) and (y_1, y_2) according to this basis vectors $\{\mathbf{g}_{1,}\mathbf{g}_2\}$, respectively, then the Galilean inner product of vectors $\mathbf{a}, \mathbf{b} \in G^2$ with respect to bases $\{\mathbf{g}_{1,}\mathbf{g}_2\}$ is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle_G = x_1 y_1 \tag{7}$$

(also you can see in [4]). If $\mathbf{a}, \mathbf{b}$ are special vectors, then the Galilean special inner product of special vectors is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle_\delta = x_2 y_2. \tag{8}$$

Hence, the norm of every vector $\mathbf{a} = (x_1, x_2) \in G^2$ on the Galilean plane is denoted by $\|\mathbf{a}\|_G$ and is defined by

$$\|\mathbf{a}\|_G = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_G} = |x_1| \tag{9}$$

and the norm of every special vector $\mathbf{a} = (0, x_2) \in G^2$ on the Galilean Plane is denoted by $\|\mathbf{a}\|_\delta$ and is defined by

$$\|\mathbf{a}\|_\delta = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_\delta} = |x_2|\,. \tag{10}$$

The distance between points $A\,(x_1, x_2)$ and $B\,(y_1, y_2)$ on the Galilean Plane is denoted by d_{AB} and is defined by

$$d_{AB} = \sqrt{\langle \mathbf{AB}, \mathbf{AB} \rangle_G} = y_1 - x_1, \tag{11}$$

where $y_1 > x_1$ (see, Figure 3).

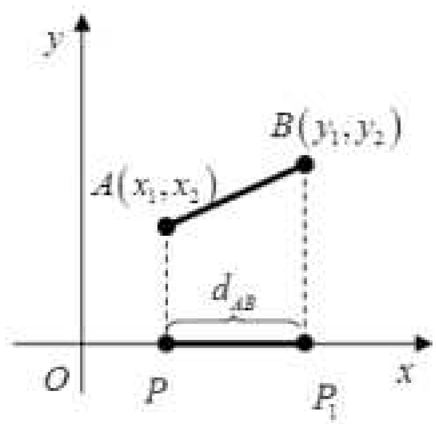

Figure 3 The distance between two points in G^2

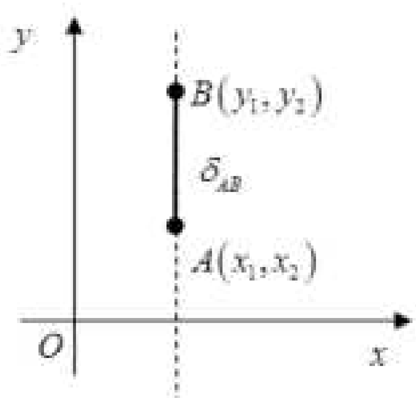

Figure 4 The special distance between two points

That is, d_{AB} is equal to $\|\mathbf{PP}_1\|$ in the sense of Euclidean Geometry. If the distance between

$A(x_1, x_2)$ and $B(y_1, y_2)$ is equal to zero ($x_1 = y_1$), then special distance of the points $A(x_1, x_2)$ and $B(y_1, y_2)$ is denoted by δ_{AB} and is defined by

$$\delta_{AB} = \sqrt{\langle \mathbf{AB}, \mathbf{AB} \rangle_\delta} = y_2 - x_2 \tag{12}$$

here $y_2 > x_2$ (see, Figure 4). The set of points $M(x, y)$ whose distances from a fixed point $Q(a, b)$ have constant absolute value r is called a *Galilean circle*, and is denoted by S. Thus, the circle S in the Galilean Plane is defined by

$$(x - a)^2 = r^2 \tag{13}$$

or

$$x^2 + 2px + q = 0 \tag{14}$$

where $p = -a$, $q = a^2 - r^2$. Also in the Galilean Plane, lines are parallel to y-axis are separable from class of lines and these lines are called *special lines* and others *are called ordinary lines.* Therefore, the circle S in the Galilean Plane consists of two special lines whose distance from Q is r (see, Figure 5).

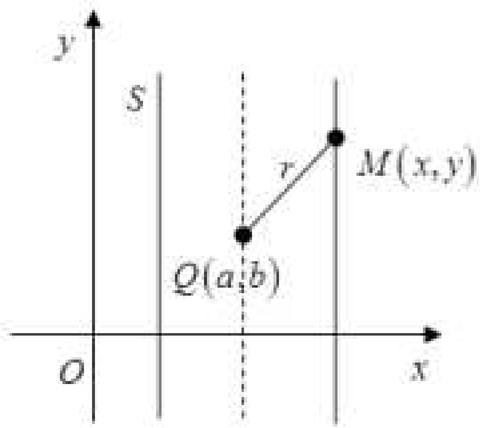

Figure 5 The circle in G^2

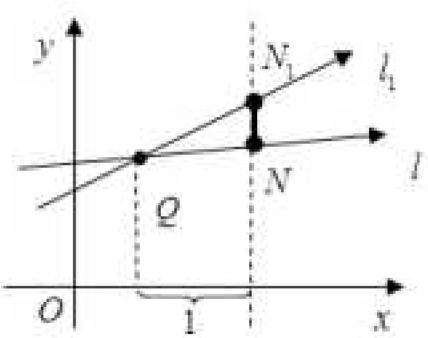

Figure 6 The angle between two intersecting lines

However, the angle between two ordinary lines $y = kx + s$ and $y = k_1x + s_1$ intersecting at a point $Q = (x_0, y_0)$ (see, Figure 6) is defined by

$$\delta_{ll_1} = k_1 - k. \tag{15}$$

But the *right angle* is defined by angle between ordinary line and special line in the Galilean Plane. So, the special lines are perpendicular to ordinary lines and also special vectors are perpendicular to ordinary vectors. Consequently, let S be a unit circle with centered at O and $M\,(x, y)$ be a point on S. Assume that l denotes line OM and α denotes δ_{Ol} (see, Figure 7).

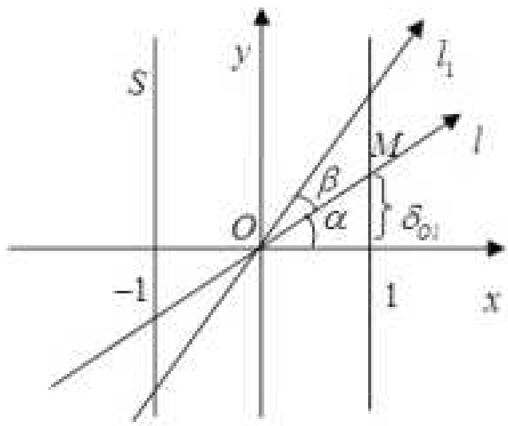

Figure 7 The trigonometry in G^2

Then, we have

$$cosg\alpha = 1 \tag{16}$$

and

$$sing\alpha = \alpha. \tag{17}$$

Also, suppose that l_1 be another ordinary line and $\delta_{ll_1} = \beta$. Then we get

$$\cos g\,(\alpha + \beta) = 1 \tag{18}$$

and

$$\sin g\,(\alpha + \beta) = \sin g\alpha \cos g\beta + \cos g\alpha \sin g\beta. \tag{19}$$

We can define a circle by another definition in Euclidean Geometry that the set of points M from which a given ordinary segment AB (i.e., a segment on an ordinary line) is seen at a constant directed angle α. If we use this definition in the Galilean Plane, we have equation

$$ax^2 + 2b_1x + 2b_2y + c = 0 \tag{20}$$

which are (Euclidean) parabolas and this set is called a *Galilean cycle* and denoted by Z. Here each of lines which are parallel to y-axis, is a *diameter* of cycle Z and it is denoted by d (see, Figure 8).

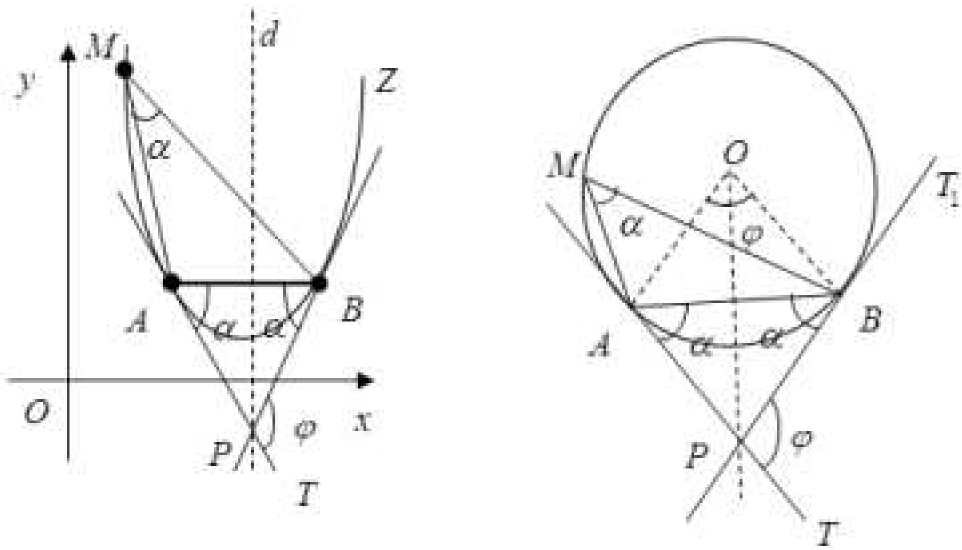

Figure 8 The cycle in G^2 and circle in E^2

Also, the length of an arc AB of a curve Γ is equal to the length $s = d_{AB}$ of the cord AB (see, Figure 9).

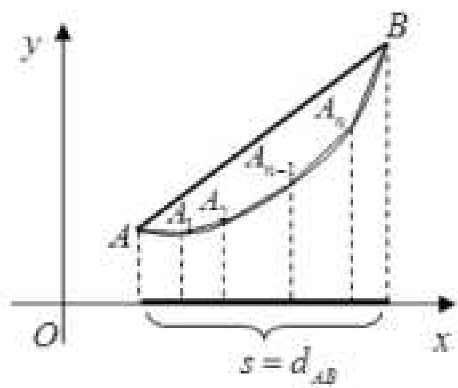

Figure 9 The length of an arc in G^2

Thus, the radius of cycle Z is defined by

$$r = \frac{1}{2a}. \tag{21}$$

Furthermore, the curvature ρ of Γ at A is defined as the rate change of the tangent at A, that is, the curvature of Γ at A is

$$\rho = \lim_{\triangle s \to 0} \frac{\triangle \varphi}{\triangle s} \tag{22}$$

where $\triangle \varphi = TAT_0'$ is angle between the two neighboring tangents, $\triangle s = arcAM$ is the scalar arc element of Γ such that M is a point of curve Γ (see, Figure 10).

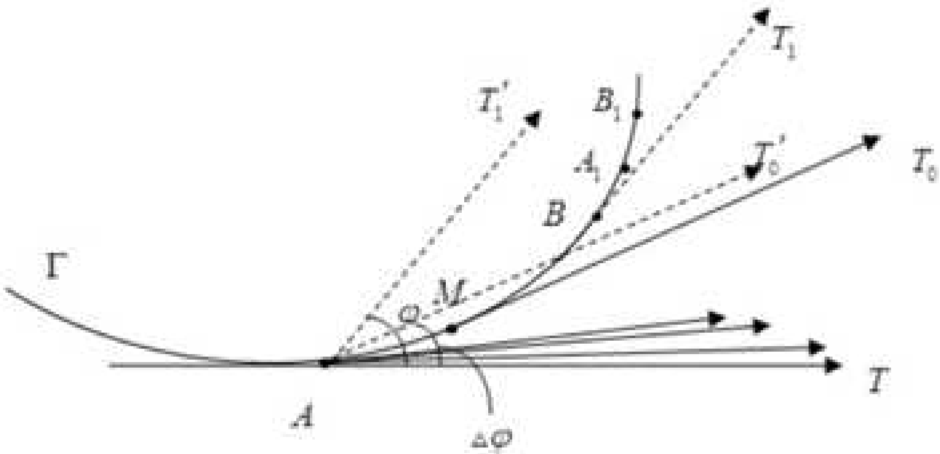

Figure 10 The curvature in G^2

Therefore, the radius of curvature of Γ at A, denoted r, is

$$r = \frac{1}{\rho}. \tag{23}$$

Now, let consider all cycles passing through the points of Γ and having the same tangent $AT = l$ at A as Γ. From these cycles, we select that for any other cycle Z_0 which is closest distance MM' between points on Γ and Z sufficiently close to A which project to the same point N on l is larger than the distance between the corresponding points M and M_0' of Γ and Z_0. This cycle Z_0 is called the *osculating cycle of curve* Γ at A (see, Figure 11), [1].

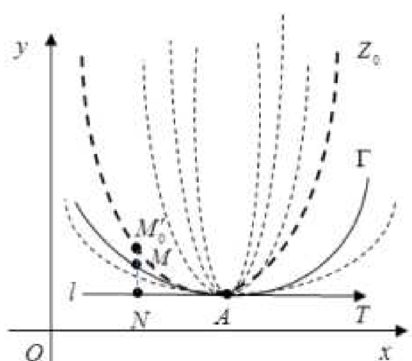

Figure 11 The cycle in G^2

§3. One Parameter Planar Galilean Motion

Let G and G' $\left(G = G' = G^2\right)$ be moving and fixed Galilean planes and $\{O; \mathbf{g}_1, \mathbf{g}_2\}$ and $\{O'; \mathbf{g}_1', \mathbf{g}_2'\}$ be their coordinate systems, respectively. The motion defined by the transformation

$$\mathbf{x} = \mathbf{x}' - \mathbf{u} \tag{24}$$

is called as *one parameter planar Galilean motion* and denoted by $B = G/G'$ where

$$\mathbf{OO'} = \mathbf{u} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2 \tag{25}$$

for $u_1, u_2 \in \mathbb{R}$, $\mathbf{x}, \mathbf{x'}$ are the coordinate vectors with respect to the moving and fixed rectangular coordinate systems of a point $X = (x_1, x_2) \in G$ respectively. Also, these vectors $\mathbf{x}, \mathbf{x'}$ and $\mathbf{u}$ and shear rotation angle φ between $\mathbf{g}_1$ and $\mathbf{g}'_1$ are continuously differentiable functions of a time parameter t (see, Figure 12).

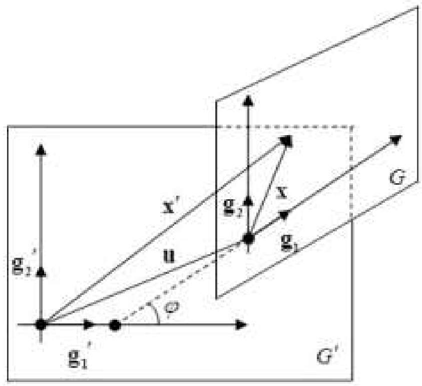

Figure 12 The motion $B = G/G'$

We can write

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{g}'_1 + \varphi\mathbf{g}'_2 \\ \mathbf{g}_2 &= \mathbf{g}'_2 \end{aligned} \tag{26}$$

for the shear rotation angle $\varphi = \varphi(t)$ between $\mathbf{g}_1$ and $\mathbf{g}'_1$. In this study, we suppose that

$$\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0 \tag{27}$$

where "." denotes the derivation with respect to "t". By differentiating the equations (25) and (26), the derivative formulae of the motion $B = G/G'$ are

$$\begin{aligned} \dot{\mathbf{g}_1} &= \dot{\varphi}\mathbf{g}_2 \\ \dot{\mathbf{g}_2} &= \mathbf{0} \\ \dot{\mathbf{u}} &= u_1\mathbf{g}_1 + (u_2 + u_1\dot{\varphi})\,\mathbf{g}_2. \end{aligned} \tag{28}$$

The velocity of the point X with respect to G is defined as the relative velocity $\mathbf{V}_r$ and is founded by

$$\mathbf{V}_r\dot{x_1}\mathbf{g}_1 + \dot{x_2}\mathbf{g}_2. \tag{29}$$

Furthermore, velocity of the point $X \in G$ according to G' is known as the absolute velocity, and is found as

$$\mathbf{V_a} = -\dot{u_1}\mathbf{g}_1 + (-\dot{u_2} - u_1\dot{\varphi} + x_1\dot{\varphi})\,\mathbf{g}_2 + \mathbf{V}_r. \tag{30}$$

Thus, we get the sliding velocity

$$\mathbf{V}_f = -\dot{u_1}\mathbf{g}_1 + \left(-\dot{u_2} - u_1\dot{\varphi} + x_1\dot{\varphi}\right)\mathbf{g}_2. \tag{31}$$

In the general one parameter motions, the points whose sliding velocity is zero, i.e.,$\mathbf{V}_f = \mathbf{0}$ are called the *pole point* or *instantaneous shear rotation pole point* and in the Galilean Plane G, the pole point $P = (p_1, p_2) \in G$ of the motion $B = G/G'$ is defined by

$$p... \begin{cases} p_1 = u_1 + \frac{\dot{u_2}(t)}{\dot{\varphi}(t)} \\ p_2 = p_2\left(t\left(\lambda\right)\right) \end{cases} \tag{32}$$

for $\lambda \in \mathbb{R}$ (see, Figure 13) [17].

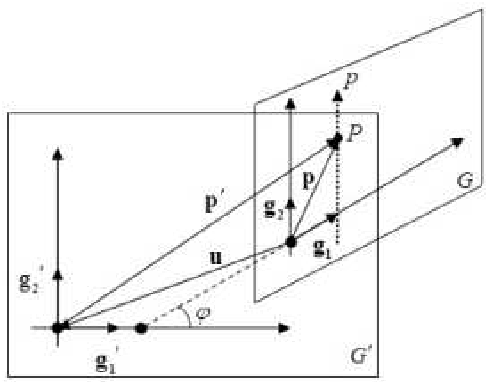

Figure 13 The pole line in Galilean plane

Corollary 3.1([17]) *During one parameter planar motion $B = G/G'$ invariants points in both planes at any instant t have been on a special line in the plane G. That is, there only exists pole line in the Galilean Plane G at any instant t. For all $t \in I$, this pole lines are parallel to y-axis and these pole lines form bundles of parallel lines. Using equations* (31)*and* (32)*, for sliding velocity, we can write*

$$\mathbf{V}_f = \left\{0\mathbf{g}_1 + \left(x_1 - p_1\right)\mathbf{g}_2\right\}\dot{\varphi}. \tag{33}$$

Corollary 3.2([17]) *During one parameter planar motion $B = G/G'$, the pole ray $\mathbf{PX} = (x_1 - p_1)\mathbf{g}_1 + (x_2 - p_2)\mathbf{g}_2$ and $\mathbf{V}_f = \{0\mathbf{g}_1 + (x_1 - p_1)\mathbf{g}_2\}\dot{\varphi}$ are perpendicular vectors, i.e., $\left\langle \mathbf{PX}, \mathbf{V}_f \right\rangle_G = 0$. Thus, under the motion $B = G/G'$, the focus of the points $X \in G$ is an orbit curve that is normal pass through the shear rotation pole P.*

Corollary 3.3([17]) *Under the motion $B = G/G'$, the norm of the sliding velocity $\mathbf{V}_f$ is*

$$\left\|\mathbf{V}_f\right\|_\delta = \left\|\mathbf{PX}\right\|_G \left|\dot{\varphi}\right|. \tag{34}$$

That is, during the motion $B = G/G'$, all of the orbits of the points $X \in G$ are such curves *whose normal lines pass thoroughly the pole point P. At any instant t, the motion $B = G/G'$ is a Galilean instantaneous shear rotation with the angular velocity $\dot{\varphi}$ about the pole point P.*

Since there exist pole points in every moment t, during the one-parameter plane motion $B = G/G'$ any pole point P is situated various positions on the plane G and G'. The position of the pole point P on the moving plane G is usually a curve and this curve is called *moving pole curve* and is denoted by (P). Also the position of this pole point P on the fixed plane G' is usually a curve and this curve is called *fixed pole curve* and is denoted by (P').

§4. The Moving Coordinate System on the Galilean Planes

In this section, we study on three Galilean planes, and investigate relative, sliding and absolute velocity, a point of X on a plane according to the other two plane and relations between the pole points. Let A and G be moving and G' be fixed Galilean plane and $\{B, \mathbf{a}_1, \mathbf{a}_2\}$, $\{O; \mathbf{g}_1, \mathbf{g}_2\}$ and $\{O'; \mathbf{g}'_1, \mathbf{g}'_2\}$ their coordinate systems, respectively (see, Figure 14).

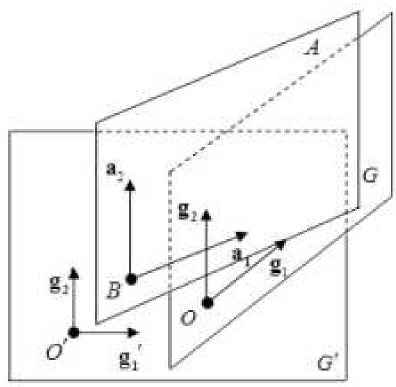

Figure 14 The two moving and one fixed coordinate system

Assume that φ and ψ are rotation angles of one parameter planar motions A/G and A/G', respectively. Let us consider a point X with the coordinates of (x_1, x_2) in moving plane A. Since

$$\mathbf{BX} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{35}$$

$$\mathbf{O}B = \mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 \tag{36}$$

$$\mathbf{O'B} = \mathbf{b}' = b'_1\mathbf{a}_1 + b'_2\mathbf{a}_2 \tag{37}$$

are vectors on the moving system of A, we have

$$\mathbf{x} = \mathbf{OX} = \mathbf{OB} + \mathbf{BX} = \mathbf{b} + x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{38}$$

$$\mathbf{x}' = \mathbf{O'X} = \mathbf{O'B} + \mathbf{BX} = \mathbf{b}' + x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{39}$$

where vector $\mathbf{x}$ and $\mathbf{x}'$ denote the point X with respect to the coordinate systems of G and G', respectively. Let's find the velocities of one parameter motion with the help of the differentiation the equations (38) and (39). Assume that "$d...$" denotes the differential with respect to G and "$d'^{...}$" denotes the differential with respect to G'.

The derivative equations in motion $B = A/G$, are

$$d\mathbf{a}_1 = d\varphi \mathbf{a}_2 \tag{40}$$

$$d\mathbf{a}_2 = 0 \tag{41}$$

$$db = db_1\mathbf{a}_1 + (b_1 d\varphi + db_2)\,\mathbf{a}_2 \tag{42}$$

and the derivative equations in motion $B = A/G'$ taking $d'b = d'b'$, are

$$d'\mathbf{a}_1 = d'\psi \mathbf{a}_2 \tag{43}$$

$$d'\mathbf{a}_2 = 0 \tag{44}$$

$$d'b = db'_1\mathbf{a}_1 + (b'_1 d\psi + db'_2)\,\mathbf{a}_2. \tag{45}$$

So differential of X with respect to G is

$$d\mathbf{x} = (\sigma_1 + dx_1)\,\mathbf{a}_1 + (\sigma_2 + \tau x_1 + dx_2)\,\mathbf{a}_2 \tag{46}$$

where $\sigma_1 = db_1, \sigma_2 = db_2 + b_1 d\varphi, \tau = d\varphi$. Therefore the relative velocity vector of X with respect to G is

$$\mathbf{V}_r = \frac{d\mathbf{x}}{dt} \tag{47}$$

and also differential of X with respect to G' is

$$d'\mathbf{x} = (\sigma'_1 + dx_1)\,\mathbf{a}_1 + (\sigma'_2 + \tau' x_1 + dx_2)\,\mathbf{a}_2 \tag{48}$$

where $\sigma'_1 = db'_1, \sigma'_2 = db'_2 + b'_1 d\psi, \tau' = d\psi$. Thus, the absolute velocity vector of X with respect to G' is

$$\mathbf{V}_a = \frac{d'\mathbf{x}}{dt}. \tag{49}$$

Here $\sigma_1, \sigma_2, \sigma'_1, \sigma'_2, \tau$ and τ' are the *Pfaffian forms* of one parameter motion with respect to t. If $V_r = 0$ and $V_a = 0$ then the point X is fixed in the planes G and G', respectively. Thus, the conditions that the point is fixed in planes G and G' become

$$dx_1 = -\sigma_1, dx_2 = -\sigma_1 - \tau x_1 \tag{50}$$

and

$$dx_1 = -\sigma'_1, dx_2 = -\sigma'_2 - \tau' x_1, \tag{51}$$

respectively. Substituting equation (50) into equation (48) and considering that the sliding velocity of the point X is $\mathbf{V}_f = \frac{d_f\mathbf{x}}{dt}$, we have

$$\mathbf{d}_f\mathbf{x} = \{\sigma'_1 - \sigma_1\}\,\mathbf{a}_1 + \{(\sigma'_2 - \sigma_2) + (\tau' - \tau)\}\,\mathbf{a}_2. \tag{52}$$

Therefore from (46), (48) and (52) we may give the following theorem.

Theorem 4.1 *If X is a fixed point on G, then we have*

$$\mathbf{d}'\mathbf{x} = \mathbf{d}_f\mathbf{x} + \mathbf{dx}, \tag{53}$$

that is, $\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r$. *Thus, velocities law is preserved.*

Remark 4.2 In the motion A/G, the absolute velocity $\widetilde{\mathbf{V}}a$ corresponds the differential of

$$d\mathbf{x} = \sigma_1\mathbf{a}_1 + \{\sigma_2 + \tau x_1\}\,\mathbf{a}_2 + dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{54}$$

according to plane G of the point X, and the relative velocity $\widetilde{\mathbf{V}}_r$ which is the velocity of X according to the plane A, is equal to the differential of

$$dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{55}$$

with respect to A of the point X. Thus the sliding velocity $\widetilde{\mathbf{V}}_f$ with respect to motion A/G is the differential of

$$\sigma_1\mathbf{a}_1 + \{\sigma_2 + \tau x_1\}\,\mathbf{a}_2 \tag{56}$$

according to G of the point X. Similarly, in the motion A/G', the absolute velocity $\widetilde{\mathbf{V}}'_a$ is equal to the differential of

$$d'\mathbf{x} = \sigma'_1\mathbf{a}_1 + \{\sigma'_2 + \tau' x_1\}\,\mathbf{a}_2 + dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{57}$$

with respect to G' of the point X, and the relative velocity $\widetilde{\mathbf{V}}'_r$ is the differential of

$$dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{58}$$

with respect to A of the point X. So the sliding velocity $\widetilde{\mathbf{V}}'_f$ corresponds the differential of

$$\sigma'_1\mathbf{a}_1 + \{\sigma'_2 + \tau' x_1\}\,\mathbf{a}_2 \tag{59}$$

with respect to G' of the point X.Since the motion G/G' is characterized by the inverse motion of A/G and the motion A/G', we have the sliding motion $\mathbf{d}_f\mathbf{x}$ when we subtract from the sliding velocity of the motion A/G' to the sliding velocity of the motion A/G. So we can write

$$\mathbf{V}_f = \widetilde{\mathbf{V}}'_f - \widetilde{\mathbf{V}}_f. \tag{60}$$

To avoid the cases of pure translation we assume that $\dot{\varphi} \neq 0, \dot{\psi} \neq 0$.

In one parameter planar Galilean motions the pole point is characterised by vanishing sliding velocity, i.e., $d_f x = 0$. So, the pole point P of the one parameter planar motion G/G' is obtained as

$$p_1 = -\frac{\sigma'_2 - \sigma_2}{\tau' - \tau} \tag{61}$$

$$p_2 = p_2(\lambda), \lambda \in \mathbb{R} \tag{62}$$

where $\mathbf{BP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2$. Note that here we find

$$\sigma_1' - \sigma_1 = 0. \tag{63}$$

§5. The Shear Rotation Poles for Moving Galilean Planes with Respect to the Other

Let us have three planes such as A, G, G' moving with respect to together and also occur in two one parameter planar Galilean motion with respect to each other. In the determined time t, pairs of plane $(A, G), (A, G')$ and (G, G') have a determined shear rotation pole line, and instantaneous shear rotation motions arise with angular velocity about the pole line. Accordingly, three planes moving with respect to together constitute a three-member kinematic chain.

Motion A/G of plane A with respect to plane G is formulated by equation system (41). Here $d\varphi = \tau$ is infinitesimal shear rotation angle, that is, $\frac{\tau}{dt}$ is an angular velocity. Differential of point X with respect to plane A

$$d\mathbf{BX} = dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2. \tag{64}$$

The differential corresponds to relative velocity with respect to plane A. If point X is fixed, then we can write $d\mathbf{BX} = \mathbf{0}$. In the equation (46), the differential of point X is given with respect to plane G. From here, sliding velocity of point X with respect to motion A/G is

$$\sigma_1\mathbf{a}_1 + (\sigma_2 + \tau x_1)\,\mathbf{a}_2. \tag{65}$$

However the shear rotation pole of motion is characterized by vanishing the sliding velocity. So, for the shear rotation pole line q of motion A/G, we have

$$q ... \left\{ q_1 = -\frac{\sigma_2}{\tau}, q_2 = q_2(\xi)\ \ \xi \in \mathbb{R}. \right. \tag{66}$$

Similarly for the shear rotation pole line q' of motion A/G', we get

$$q' ... \left\{ q_1' = -\frac{\sigma_2'}{\tau'}, q_2' = q_2'(\mu),\ \ \mu \in \mathbb{R}. \right. \tag{67}$$

And also, the angular velocity of motion G/G' is

$$\frac{d(\psi - \varphi)}{dt} = \frac{\tau' - \tau}{dt} \tag{68}$$

and for the shear rotation pole line p, from equations (61) and (62) we can rewrite

$$p ... \left\{ \begin{array}{l} p_1 = -\frac{\sigma_2' - \sigma_2}{\tau' - \tau} \\ p_2 = p_2(\lambda), \lambda \in \mathbb{R} \end{array} \right. . \tag{69}$$

So, we give following theorem.

Theorem 5.1 *If three Galilean planes form one parameter planar Galilean motions pairwisely, there exist three shear rotation pole lines at every moment t, and each of these three lines is parallel to the others.*

Corollary 5.2 *Generally, if there are $n-$Galilean planes which form one parameter planar Galilean motions pairwisely, then we tell of $n-$member kinematic chain. If the each motions is connected time (real) parameter t , there exit $\begin{pmatrix} n \\ 2 \end{pmatrix}$ relative shear rotation pole lines at every moment t and every each line is parallel to each others.*

Theorem 5.3 *The rate of the distance of three shear rotation poles is as the rate of their angular velocities.*

Proof Since $q_1 = -\frac{\sigma_2}{\tau}, q_1' = -\frac{\sigma_2'}{\tau'}$ and $p_1 = -\frac{\sigma_2' - \sigma_2}{\tau' - \tau}$, it is hold

$$(q_1 - q_1) : (p_1 - q_1') : (q_1 - p_1) = (\tau' - \tau) : \tau : -\tau'. \tag{70}$$

Thus, we can write

$$\left\|\overrightarrow{QQ'}\right\|_G : \left\|\overrightarrow{Q'P}\right\|_G : \left\|\overrightarrow{PQ}\right\|_G = (\tau' - \tau) : \tau : -\tau'. \tag{71}$$

§6. Euler-Savary's Formula for One Parameter Motions in the Galilean Plane

We studied one parameter Galilean motion adding $\{B, \mathbf{a}_1, \mathbf{a}_2\}$ moving system to the motion of G with respect to G'. Now, in this section, we choose a special relative system $\{B, \mathbf{a}_1, \mathbf{a}_2\}$ satisfying the following conditions:

(i) The initial point B of the system is a pole point P on the pole line that coordinates are p_1 and p_2.

(ii) The axes $\{B; \mathbf{a}_1\}$ coincides with the common tangent of the pole curves (P) and (P').

If we consider the condition i), then from equations (61) and (62) we have

$$p_1 = p_2 = 0. \tag{72}$$

Thus, we can write

$$\sigma_1' = \sigma_1, \sigma_2' = \sigma_2. \tag{73}$$

Therefore we get

$$\mathbf{db} = \mathbf{dp} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 = \mathbf{d'p} = \mathbf{db}'. \tag{74}$$

Considering the condition ii), then we have $\sigma_2' = \sigma_2 = 0$. So if we choose $\sigma_1' = \sigma_1 = \sigma$, then the derivative equations for the canonical relative system $\{P, \mathbf{a}_1, \mathbf{a}_2\}$ are

$$\begin{aligned} d\mathbf{a}_1 &= \tau\mathbf{a}_2, \quad d'\mathbf{a}_1 = \tau'\mathbf{a}_2 \\ d\mathbf{a}_2 &= \mathbf{0}, \qquad d'\mathbf{a}_2 = \mathbf{0} \\ d\mathbf{p} &= \sigma\mathbf{a}_1, \quad d'\mathbf{p} = \sigma\mathbf{a}_1. \end{aligned} \tag{75}$$

Moreover, τ is the cotangent angle, that is, two neighboring tangets angle of curve (P), and τ' is also the cotangent angle of curve (P') where, $\sigma = ds$ is the scalar arc element of the pole curves (P) and (P'). And so $\tau : \sigma$ is the curvature of the moving pole curve (P). Similarly, $\tau' : \sigma$ is the curvature of the fixed pole curve (P'). Hence from (23) the radius of curvature of the pole curves (P) and (P') are

$$r = \frac{\sigma}{\tau} \tag{76}$$

and

$$r' = \frac{\sigma}{\tau'} \tag{77}$$

respectively. Moving plane G rotates the infinitesimal instantaneous angle of the $d\phi = \tau' - \tau$ around the shear rotation pole P within the time scale dt with respect to fixed plane G'. Therefore the angular velocity of shear rotational motion of G with respect to G' is

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \dot{\phi}. \tag{78}$$

Hence we get

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \frac{1}{r'} - \frac{1}{r} \tag{79}$$

from equations (76), (77) and (78). We accept that for the direction of unit tangent vector $\mathbf{a}_1$, pole curves (P) and (P') are drawn to the positive x-axis direction that is, $\frac{ds}{dt} > 0$, and so we have $r > 0$. Similarly we can write $r' > 0$.

Now we will investigate case of the point X' which is on the diameter d of osculating cycle of trajectory curve which is drawn in the fixed plane G' by a point X of moving plane G in the movement G/G'. In the canonical relative system, let coordinates of points X at plane G and X' at plane G' be the (x_1, x_2) and (x_1', x_2'), respectively. In the movement $\mathbb{D}/\mathbb{D}'$, there is a point X' which is on center of curvature of osculating cycle of trajectory curve of X are situated together with the instantaneous rotation pole P in every moment t such that

$$\mathbf{PX} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

and

$$\mathbf{PX}' = x_1'\mathbf{a}_1 + x_2'\mathbf{a}_2$$

have same direction which passes the pole point P. So we can write

$$x_1 : x_2 = x_1' : x_2'$$

or

$$x_1x_2' - x_1'x_2 = 0. \tag{80}$$

Considering the condition $ii)$ we obtain the condition that the point X to be the fixed in the moving plane G is

$$dx_1 = -\sigma, dx_2 = -\tau x_1 \tag{81}$$

and the point X' to be the fixed in the moving plane G' is

$$dx_1' = -\sigma, dx_2' = -\tau' x_1'. \tag{82}$$

Differentiating the equation(80) and from the conditions (81) and (82) we have

$$(x_2 - x_2')\,\sigma + x_1x_1'\,(\tau - \tau') = 0. \tag{83}$$

If the polar coordinates are passed, then we get

$$x_1 = a\cos g\alpha = a,\ x_2 = a\sin g\alpha = a\alpha \tag{84}$$

$$x_1' = a'\cos g\alpha = a',\ x_2' = a'\sin g\alpha = a'\alpha. \tag{85}$$

Thus, we can write

$$(a\sin g\alpha - a'\sin g\alpha)\,\sigma + aa'\,(\tau - \tau') = 0. \tag{86}$$

So from last equation and equation (79) we have

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sin g\alpha = \frac{1}{r'} - \frac{1}{r} = \frac{d\phi}{ds}. \tag{87}$$

Here, r and r' are the radii of curvature of the pole curves P and P', respectively. ds represents the scalar arc element and $d\phi$ represents the infinitesimal Galilean angle of the motion of the pole curves. The equation (87) is called the *Euler-Savary formula* for one-parameter motion in Galilean plane G.

Consequently, the following theorem can be given.

Theorem 6.1 *Let G and G' be the moving and fixed Galilean planes, respectively. A point $X \in G$, draws a trajectory whose a point at the normal axis of curvature is X' on the plane G' in one-parameter planar motion G/G'. In the inverse motion of G/G', a point X' assumed on G' draws a trajectory whose a point at the normal axis of curvature is X on the plane G. The relation between the points X and X' which is given by the Euler-Savary formula given in the equation (87).*

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Laplacian Energy of Binary Labeled Graph

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Abstract: Let G be a binary labeled graph and $A_l(G) = (l_{ij})$ be its label adjacency matrix. For a vertex v_i, we define label degree as $L_i = \sum\limits_{j=1}^{n} l_{ij}$. In this paper, we define label Laplacian energy $LE_l(G)$. It depends on the underlying graph G and labels of the vertices. We compute label Laplacian spectrum of families of graph. We also obtain some bounds for label Laplacian energy.

Key Words: Label Laplacian matrix, label Laplacian eigenvalues, label Laplacian energy.

AMS(2010): 05C50, 15A15, 15A18.

§1. Introduction

For an n-vertex graph G with adjacency matrix A whose eigenvalues are $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, the energy of the graph G is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. The concept of Energy of graph was introduced by Ivan Gutman, in connection with the π-molecular energy. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of (n, m) graph G. If $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n = 0$ are the eigenvalues of $L(G)$, then the Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \tfrac{2m}{n}|$$

However, in the last few years, research on graph energy has much intensified, resulting in a very large number of publications which can be found in the literature [4, 5, 6, 7, 9, 16]. In spectral graph theory , the eigenvalues of several matrices like adjacency matrix, Laplacian matrix [8], distance matrix [10] etc. are studied extensively for more than 50 years. Recently minimum covering matrix, color matrix, maximum degree etc are introduced and studied in [1,2,3].

Motivated by this, P.G. Bhat and S. D'Souza have introduced a new matrix $A_l(G)$ called label matrix [14] of a binary labeled graph $G = (V, X)$, whose elements are defined as follows:

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$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 1, \\ c, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = 0, l(v_j) = 1 \text{ or vice-versa}, \\ 0, & \text{otherwise.} \end{cases}$$

where a, b, and c are distinct non zero real numbers. The eigenvalues λ_1, λ_2, $\ldots$, λ_n of $A_l(G)$ are said to be label eigenvalues of the graph G and form its label spectrum. The label eigenvalues satisfy the following simple relations:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad and \quad \sum_{i=1}^{n} \lambda_i^2 = 2Q \tag{1.1}$$

where

$$Q = n_1a^2 + n_2b^2 + n_3c^2 \tag{1.2}$$

Where n_1, n_2 and n_3 denotes number of edges with $(0,0)$, $(1,1)$ and $(0,1)$ as end vertex labels respectively.

The *label degree* of the vertex v_i, denoted by L_i, is given by $L_i = \sum\limits_{i=1}^{n} l_{ij}$. A Graph G is said to be *k-label regular* if $L_i = k$ for all i. The *label Laplacian matrix* of a binary labeled graph G is defined as

$$L_l(G) = Diag(L_i) - A_l(G)$$

where $Diag(L_i)$ denotes the diagonal matrix of the label degrees. Since $L_l(G)$ is real symmetric, all its eigenvalues μ_i, $i = 1, 2, \ldots, n$, are real and can be labeled as $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$. These form the *label Laplacian spectrum* of G. Several results on Laplacian of Graph G are reported in the Literature ([6, 11, 12, 13, 16]).

This paper is organized as follows. In section 2, we establish relationship between λ_i and μ_i and some general results on Laplacian label eigenvalues μ_i. In section 3, lower bound and upper bounds for $LE_l(G)$ are obtained. In the last section label Laplacian spectrum is derived for family of graphs.

§2. Label Laplacian Energy

The following Lemma 2.1 shows the similarities between the spectra of label matrix and label Laplacian matrix. For a labeled graph, let $P_A(x)$ and $P_L(x)$ denote the label and label Laplacian characteristic polynomials respectively.

Lemma 2.1 *If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the label spectrum of k-label regular graph G, then $\{k - \lambda_n, k - \lambda_{n-1}, \ldots, k - \lambda_1\}$ is the label Laplacian spectrum of G.*

Proof The label Laplacian characteristic polynomial for k-label regular graph G is given by

$$P_L(x) = \det(L_l(G) - xI) = (-1)^n \det(A_l(G) - (k-x)I) = (-1)^n P_A(k-x) \tag{2.1}$$

Thus, if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ is the label spectrum of k-label regular graph G, then from equation (2.1), it follows that $k - \lambda_n \geqslant k - \lambda_{n-1} \geqslant \ldots \geqslant k - \lambda_1$ is the label Laplacian spectrum of G.□

We first introduce the auxiliary eigenvalues γ_i, defined as

$$\gamma_i = \mu_i - \tfrac{1}{n}\sum_{j=1}^{n} L_j$$

Lemma 2.2 *If $\mu_1, \mu_2, \cdots, \mu_n$ are the label Laplacian eigenvalues of $L_l(G)$, then*

$$\sum_{i=1}^{n} \mu_i^2 = 2Q + \sum_{i=1}^{n} L_i^2.$$

Proof We have

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2 &= trace\ \ (L_l(G))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji} = 2\sum_{i<j} l_{ij}^2 + \sum_{i=1}^{n} l_{ii}^2 \\ &= 2[n_1(a)^2 + n_2(b)^2 + n_3(c)^2] + \sum_{i=1}^{n} L_i^2 = 2Q + \sum_{i=1}^{n} L_i^2\end{aligned}$$

□

Lemma 2.3 *Let G be a binary labeled graph of order n. Then $\sum_{i=1}^{n} \gamma_i = 0$ and $\sum_{i=1}^{n} \gamma_i^2 = 2R$, where*

$$R = Q + \tfrac{1}{2}\sum_{i=1}^{n}\left(L_i - \tfrac{1}{n}\sum_{j=1}^{n} L_j\right)^2$$

and Q is given by equation (1.2).

Proof Note that

$$\sum_{i=1}^{n} \mu_i = tr(L_l(G)) = \sum_{i=1}^{n} L_i \ \text{ and } \ \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} L_i^2 + 2Q$$

From which we have,

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n}\left(\mu_i - \tfrac{1}{n}\sum_{j=1}^{n} L_j\right) = \sum_{i=1}^{n} \mu_i - \sum_{j=1}^{n} L_j = 0$$

and

$$\begin{aligned}\sum_{i=1}^{n}\gamma_i^2 &= \sum_{i=1}^{n}\left(\mu_i-\frac{1}{n}\sum_{j=1}^{n}L_j\right)^2\\ &= \sum_{i=1}^{n}\mu_i^2-\frac{2}{n}\sum_{i=1}^{n}L_j\sum_{i=1}^{n}\mu_i+\left(\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2\\ &= \sum_{i=1}^{n}L_i^2+2Q-\frac{2}{n}\left(\sum_{i=1}^{n}L_j\right)^2+\left(\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2\\ &= 2Q+\sum_{i=1}^{n}\left(L_i-\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2=2R\end{aligned}$$

□

Definition 2.1 *Let G be a binary labeled graph of order n. Then the label Laplacian energy of G, denoted by $LE_l(G)$, is defined as $\sum\limits_{i=1}^{n}|\gamma_i|$, i.e.*

$$LE_l(G)=\sum_{i=1}^{n}|\mu_i-\tfrac{1}{n}\sum_{j=1}^{n}L_j|$$

In 2006, I.Gutman and B.Zhou defined Laplacian energy $LE(G)$ of a graph G. More on Laplacian energy reader can refer ([8], [15], [17], [18]). In Chemistry, there are situations where chemists use labeled graphs, such as vertices represent two distinct chemical species and the edges represent a particular reaction between two corresponding species. We mention that this paper deals only the mathematical aspects of label Laplacian energy of a graph and it is a new concept in the literature.

Lemma 2.4 *If G is k- label regular, then $LE_l(G)=E_l(G)$.*

If G is k- label regular, then $k=L_i=\frac{1}{n}\sum\limits_{j=1}^{n}L_j$ for $i=1,2,\cdots,n$. Using Lemma 2.1,

$$\gamma_i=\mu_i-\tfrac{1}{n}\sum_{j=1}^{n}L_j=(k-\lambda_{n+1-i})-k=-\lambda_{n+1-i}$$

for $i=1,2,\cdots,n$. Hence, the lemma follows from the definitions of the label energy and label Laplacian energy.

§2. Bounds for the Label Laplacian Energy

Lemma 3.1([17]) *Let $a_1,a_2,\ldots,a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right]\leqslant n\sum_{i=1}^{n}a_i-\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2\leqslant n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right].$$

Theorem 3.1 *Let G be a binary labeled graph with n vertices and m edges. Then*

$$\sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}} \leqslant LE_l(G) \leqslant \sqrt{2(n-1)R+n\Delta^{\frac{2}{n}}},$$

where $\Delta = \left|\det\left(L_l(G) - \frac{1}{n}\sum\limits_{j=1}^{n} L_j I\right)\right|$.

Proof Note that

$$\sum_{i=1}^{n}|\gamma_i| = LE_l(G) \quad \text{and} \quad \sum_{i=1}^{n}\gamma_i^2 = 2R,$$

where $R = \left[n_1(a)^2 + n_2(b)^2 + n_3(c)^2\right] + \dfrac{1}{2}\sum\limits_{i=1}^{n}\left(L_i - \frac{1}{n}\sum\limits_{j=1}^{n} L_j\right)^2$.

Using Lemma 3.1, it can be easily checked that Theorem 3.1 is true if $\Delta = 0$. Now we assume that $\Delta \neq 0$. By setting $a_i = \gamma_i^2$, $\quad i = 1, 2, \ldots, n$, and

$$K = n\left[\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right] \geqslant 0.$$

From Lemma 3.1, we have

$$K \leqslant n\sum_{i=1}^{n}\gamma_i^2 - \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 \leqslant (n-1)K,$$

which can be further expressed as

$$K \leqslant 2nR - (LE_l(G))^2 \leqslant (n-1)K$$

$$2nR - (n-1)K \leqslant (LE_l(G))^2 \leqslant 2nR - K, \tag{3.1}$$

where

$$K = n\left[\frac{1}{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right] = n\left[\frac{1}{n}2R - \Delta^{\frac{2}{n}}\right] = 2R - n\Delta^{\frac{2}{n}}$$

Substituting in equation (3.1), we obtain

$$\sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}} \leqslant LE_l(G) \leqslant \sqrt{2R(n-1)+n\Delta^{\frac{2}{n}}}. \qquad \square$$

Theorem 3.2 *Let G be a binary labeled graph of order $n \geqslant 2$. Then*

$$2\sqrt{R} \leqslant LE_l(G) \leqslant \sqrt{2nR}.$$

Proof Consider the term

$$\begin{aligned} S &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\gamma_i|-|\gamma_j|)^2 = 2n\sum_{i=1}^{n}|\gamma_i|^2 - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{j=1}^{n}|\gamma_j|\right) \\ &= 2n.2R - 2(LE_l(G))^2 = 4nR - 2(LE_l(G))^2 \end{aligned}$$

Note that $S \geqslant 0$, i.e., $4nR - 2(LE_l(G))^2 \geqslant 0$, which implies $LE_l(G) \leqslant \sqrt{2nR}$. We have $\left(\sum\limits_{i=1}^{n}\gamma_i\right)^2 = 0$ and the fact that $R \geqslant 0$,

$$2R = \sum_{i=1}^{n}\gamma_i^2 = \left(\sum_{i=1}^{n}\gamma_i\right)^2 - 2\sum_{1\leqslant i<j\leqslant n}\gamma_i\gamma_j \leqslant 2\left|\sum_{1\leqslant i<j\leqslant n}\gamma_i\gamma_j\right| \leqslant 2\sum_{1\leqslant i<j\leqslant n}|\gamma_i||\gamma_j| \tag{3.2}$$

Thus,

$$\begin{aligned} LE_l(G)^2 &= \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 = \sum_{i=1}^{n}|\gamma_i|^2 + 2\sum_{1\leqslant i<j\leqslant n}|\gamma_i||\gamma_j| \\ &\geqslant 2R + 2R = 4R \end{aligned}$$

from Lemma 2.3 and equation (3.1). Hence, $LE_l(G) \geqslant 2\sqrt{R}$. □

Theorem 3.3 *Let G be a labelled graph of order n. Then*

$$LE_l(G) \leqslant \frac{1}{n}\sum_{i=1}^{n}L_i + \sqrt{(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n}L_i\right)^2\right]}.$$

Proof We have

$$\gamma_n = 0 - \frac{1}{n}\sum_{i=1}^{n}L_i = \frac{1}{n}\sum_{i=1}^{n}L_i.$$

Consider the non-negative term

$$\begin{aligned} S &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(|\gamma_i|-|\gamma_j|)^2 \\ &= 2(n-1)\sum_{i=1}^{n}\gamma_i^2 - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{j=1}^{n}|\gamma_j|\right) \\ &= 2(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n}L_i\right)^2\right] - 2\left(LE_l(G) - \frac{1}{n}\sum_{i=1}^{n}L_i\right)^2 \geqslant 0. \end{aligned}$$

Hence,

$$LE_l(G) \leqslant \frac{1}{n}\sum_{i=1}^{n} L_i + \sqrt{(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)^2\right]}. \qquad \square$$

§4. Label Laplacian Spectrum of Some Graphs

Theorem 4.1 *For $n \geqslant 2$, the label Laplacian spectrum of complete graph K_n is*

$$\left\{ \begin{array}{cccc} 0 & ma + (n-m)c & (n-m)b + mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{array} \right\}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Proof Let $v_1, v_2, \cdots, v_m$ vertices of K_n be labeled zero and $v_{m+1}, v_{m+2}, \cdots, v_n$ be labeled 1. Then the label degree of vertex v_i is $L(v_i) = (m-1)a + (n-m)c$ for $i = 1, 2, \ldots, m$, $L(v_i) = (n-m-1)b + mc$ for $i = m+1, m+2, \ldots, n$,

$$L_l(K_n) = \left[\begin{array}{c|c} [ma + (n-m)c]I_m - aJ_{m\times m} & -cJ_{m\times(n-m)} \\ \hline -cJ_{(n-m)\times m} & [(n-m)b + mc]I_{n-m} - bJ_{(n-m)\times(n-m)} \end{array} \right]$$

Consider

$$\begin{aligned} &\det\left(\mu I - L_l(K_n)\right) \\ &= \left| \begin{array}{c|c} [\mu - \{ma + (n-m)c\}]I_m + aJ_{m\times m} & cJ_{m\times(n-m)} \\ \hline cJ_{(n-m)\times m} & [\mu - \{(n-m)b + mc\}]I_{n-m} + bJ_{(n-m)\times(n-m)} \end{array} \right|. \end{aligned}$$

Step 1 Replacing column C_1 by $C_1' = C_1 + C_2 + \ldots + C_n$, we obtain determinant $\mu \det(B)$.

Step 2 In determinant B, replace the row R_i by $R_i' = R_i - R_{i-1}$ for $i = 2, 3, \ldots, m, m+2, m+3, \ldots, n$, we obtain

$$\det(B) = \left(\mu - \{(m-1)a + (n-m)c\}\right)^{m-1}\left(\mu - \{(n-m-1)b + mc\}\right)^{n-m-1}\det(C).$$

Step 3 By changing C_i by $C_i' = C_i + C_{i+1} + \ldots + C_n$ for $i = m+1$ *to* n in determinant C, we get a new determinant D of order $m+1$, i.e.

$$\det(D) = \begin{vmatrix} 1 & a & a & \ldots & a & (n-m)c \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 1 & c & c & \ldots & c & \mu - mc \end{vmatrix}$$

Step 4 By expanding determinant D over the first column , we obtain $\det(D) = \mu - mc + (-1)^{m+2}(-1)^{m+1}(n-m)c = \mu - nc$.

Step 5 By back substitution,

$$\det\left(\mu I - L_l(G)\right) = \mu\left(\mu - [ma + (n-m)c]\right)^{m-1}\left(\mu - [(n-m)b + mc]\right)^{n-m-1}(\mu - nc).$$

Hence, label Laplacian spectrum of K_n is,

$$\begin{Bmatrix} 0 & ma + (n-m)c & (n-m)b + mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{Bmatrix} \qquad \square$$

Corollary 4.1 *For $n \geqslant 2$, the label Laplacian spectrum of $K_n - \{(0,0)\}$ is*

$$\begin{Bmatrix} 0 & (m-2)a + (n-m)c & ma + (n-m)c & (n-m)b + mc & nc \\ 1 & 1 & m-2 & n-m-1 & 1 \end{Bmatrix}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Corollary 4.2 *For $n \geqslant 2$, the label Laplacian spectrum of $K_n - \{(1,1)\}$ is*

$$\begin{Bmatrix} 0 & ma + (n-m)c & (n-m-2)b + mc & (n-m)b + mc & nc \\ 1 & m-1 & 1 & n-m-2 & 1 \end{Bmatrix}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Theorem 4.2 *The label Laplacian spectrum of star graph S_n is*

$$\begin{Bmatrix} 0 & a & c & \frac{\alpha+\beta}{2} & \frac{\alpha-\beta}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where m denotes the number of vertices including the central vertex labeled zero, remaining vertices labeled one, $m \leq n$, $\alpha = ma + (n-m+1)c$ and $\beta = \sqrt{[ma + (n-m+1)c]^2 - 4acn}$.

Proof Let $v_1, v_2, \cdots, v_m$ be labeled as zero and remaining vertices be labeled as one,where v_1 is the central vertex. Then, $L(v_1) = a(m-1) + c(n-m)$ and

$$L(v_i) = \begin{cases} a, & \text{for} \quad i = 2, 3, \cdots, m \\ c, & \text{for} \quad i = m+1, m+2, \cdots, n \end{cases}$$

$$L_l(S_n) = \begin{bmatrix} a(m-1)+c(n-m) & -a & -a & \cdots & -a & -c & -c & \cdots & -c \\ -a & a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -a & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & 0 & 0 & \cdots & 0 & 0 & 0 & \dots & 0 \\ -a & 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ -c & 0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 \\ -c & 0 & 0 & \cdots & 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -c & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c \end{bmatrix}$$

where rows and columns are denoted by $v_1, v_2, \cdots, v_m, v_{m+1}, v_{m+2}, \cdots, v_n$ for the matrix $L_l(S_n)$. Consider $det(\mu I - L_l(G))$.

Step 1 Replace the column C_1 by $C_1' = C_1+C_2+\cdots+C_n$. Then we see that $det(\mu I - L_l(G))$ is of the form $\mu \det(B)$.

Step 2 In $det(B)$, replace R_i by $R_i' = R_i - R_{i-1}$, $i = 3, 4, \cdots, m, m+2, \cdots, n$. Simplifying we get $\det(B) = (\mu-a)^{m-2}(\mu-c)^{n-m-1}\det(C)$.

Step 3 In $\det(C)$, replace C_i by $C_i' = C_i + C_{i+1} + \cdots + C_n$ for $i = m, m+1, \cdots, n$. Then it reduces to the order $m+1$.

Step 4 In $\det(C)$, Replacing C_i by $C_i' = C_i + C_{i+1} + \cdots + C_m$ for $i = 2, 3, \cdots, (m-1)$, we get

$$\det(C) = \begin{vmatrix} 1 & (m-1)a & (m-2)a & \dots & 2a & a & (n-m)c \\ 1 & (\mu-a) & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & (\mu-c) \end{vmatrix}$$

Expanding over the last column, we get $\det(C) = \mu^2 - [am + (n-m+1)c]\mu + acn$. By back substitution, we obtain

$$\det(\mu I - L_l(G)) = \mu(\mu-a)^{m-2}(\mu-c)^{n-m-1}\left(\mu^2 - [am+(n-m+1)c]\mu + acn\right).$$

Hence, label Laplacian spectrum of S_n is given by

$$\begin{Bmatrix} 0 & a & c & \frac{\alpha+\beta}{2} & \frac{\alpha-\beta}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where $\alpha = ma + (n-m+1)c$ and $\beta = \sqrt{[ma+(n-m+1)c]^2 - 4acn}$. □

Corollary 4.3 *The label Laplacian spectrum of star graph S_n is*

$$\begin{Bmatrix} 0 & b & c & \frac{\delta+\gamma}{2} & \frac{\delta-\gamma}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where m denotes the number of vertices including the central vertex labeled zero, remaining vertices labeled one, $m \leq n$, $\delta = mb + (n-m+1)c$ and $\gamma = \sqrt{[mb+(n-m+1)c] - 4bcn}$.

Proof Let v_1 be the central vertex . Let $v_1, v_2, \cdots, v_m$ be labeled as one and remaining vertices be labeled as zero. Then , $L(v_1) = b(m-1) + c(n-m)$ and

$$L(v_i) = \begin{cases} b, & \text{for} \quad i = 2, 3, \cdots, m \\ c, & \text{for} \quad i = m+1, m+2, \cdots, n \end{cases}$$

The remaining proof of this corollary is similar to Theorem 4.2. □

Corollary 4.4 *If the vertices of cycle C_{2n} are labeled 0 and 1 alternately, then $LE_l(C_{2n}) = cLE(C_{2n}) = cE(C_{2n})$.*

Corollary 4.5 *If the vertices of path P_n are labeled 0 and 1 alternately, then $LE_l(P_n) = cLE(P_n)$.*

Lemma 4.1([10]) *Let M, N, P, Q be matrices, M invertible and*

$$S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

Then $\det(S) = \det(M)\det(Q - PM^{-1}N)$. *Furthermore, if M, P are commute, then* $\det(S) = \det(MQ - PN)$.

Theorem 4.3 *The label Laplacian spectrum of complete bipartite graph $K(r,s)$ with $m_1 \leqslant r, m_2 \leqslant s$, the number of zeros in the vertex set of order r, s respectively, is given by*

$$\begin{Bmatrix} 0 & (am_2 + (s-m_2)c) & (cm_2 + (s-m_2)b) & (am_1 + (r-m_1)c) & (cm_1 + (r-m_1)b) \\ 1 & m_1 - 1 & r - m_1 - 1 & m_2 - 1 & s - m_2 - 1 \end{Bmatrix}$$

and the roots of

$$
\begin{aligned}
&[\mu^3-\mu^2\{(a+c)(m_1+m_2)+(b+c)((r+s)-(m_1+m_2))\}\\
&+\mu\{ac(m_1^2+m_2^2)+(c^2+ab)(m_2(s-m_2)+m_1(r-m_1))\\
&+(ab+bc+ca)(m_1(s-m_2)+m_1(r-m_1))+bc((r-m_1)\\
&+(s-m_2))^2+(b^2+c^2)(r-m_1)(s-m_2)\}-\{(a+c)bc(r-m_1)(s-m_2)(m_1+m_2)\\
&+abc\,(m_1(s-m_2)(s+m_1-m_2)+m_2(r-m_1)(r+m_2-m_1))\\
&+bc^2(r-m_1)(s-m_2)(r+s-m_1-m_2)\\
&+ac(a+c)m_1^2m_2+c(ac+c^2+ab)m_1m_2(s-m_2)+ac(b+c)m_1m_2(r-m_1)\}]=0.
\end{aligned}
$$

Proof Let the labels of $r+s$ vertices of $K(r,s)$ be $\underbrace{000\cdots0}_{m_1}\underbrace{111\cdots1}_{r-m_1}$ and $\underbrace{000\cdots0}_{m_2}\underbrace{111\cdots1}_{s-m_2}$.

$$L_l(K(r,s))=\left[\begin{array}{c|c} A_{r\times r} & -B_{r\times s}\\ \hline -B^T_{s\times r} & C_{s\times s}\end{array}\right]_{(r+s)\mathrm{x}(r+s)} \quad \text{with } B=\left[\begin{array}{c|c} aJ_{m_1\times m_2} & cJ_{m_1\times s-m_2}\\ \hline cJ_{r-m_1\times m_2} & bJ_{r-m_1\times s-m_2}\end{array}\right]_{r\mathrm{x}s}$$

Characteristic polynomial of $L_l(K(r,s))$ is

$$\phi(L_l(K(r,s)),\mu)=\left|\begin{array}{c|c} (\mu I-A)_{r\times r} & B_{r\times s}\\ \hline B^T_{s\times r} & (\mu I-C)_{s\times s}\end{array}\right|=|\mu I-A||(\mu I-C)-B^TA^{-1}B| \tag{4.1}$$

by Lemma 4.1. Let us denote the label degree of m_1, $r-m_1$, m_2 and $s-m_2$ vertices as $W=cm_2+(s-m_2)b$, $X=cm_2+(s-m_2)b$, $Y=am_1+(r-m_1)c$ and $Z=cm_1+(r-m_1)b$ respectively. Then $A=Diag[\mu-W,\mu-W,\ldots,\mu-W,\mu-W,\mu-X,\mu-X,\ldots,\mu-X]$, $C=Diag[\mu-Y,\mu-Y,\ldots,\mu-Y,\mu-Y,\mu-Z,\mu-Z,\ldots,\mu-Z]$. Note that

$$B^TA^{-1}B=\left[\begin{array}{c|c} \left\{\frac{m_1a^2}{\mu-W}+\frac{c^2(r-m_1)}{\mu-X}\right\}J_{m_2\times m_2} & \left\{\frac{acm_1}{\mu-W}+\frac{bc(r-m_1)}{\mu-X}\right\}J_{m_2\times s-m_2}\\ \hline \left\{\frac{acm_1}{\mu-W}+\frac{bc(r-m_1)}{\mu-X}\right\}J_{s-m_2\times m_2} & \left\{\frac{c^2m_1}{\mu-W}+\frac{b^2(r-m_1)}{\mu-X}\right\}J_{s-m_2\times s-m_2}\end{array}\right]$$

By applying elementary transformations, $\det(C-B^TA^{-1}B)$ reduces to order m_2+1. Hence,

$$|C-B^TA^{-1}B|=(\mu-Y)^{m_2-1}(\mu-Z)^{s-m_2-1}\det(E), \tag{4.2}$$

where

$$\det(E)=\begin{vmatrix} (\mu-Y)-m_2G & -(m_2-1)G & (m_2-2)G & \ldots & -G & -(s-m_2)H\\ 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \vdots & & & \ldots & & \vdots\\ -m_2H & -(m_2-1)H & -(m_2-2)H & \ldots & -H & (\mu-Z)-(s-m_2)K\end{vmatrix}$$

and $G = \frac{m_1a^2}{\mu-W} + \frac{c^2(r-m_1)}{\mu-X}$, $H = \frac{acm_1}{\mu-W} + \frac{bc(r-m_1)}{\mu-X}$ and $K = \frac{c^2m_1}{\mu-W} + \frac{b^2(r-m_1)}{\mu-X}$.

Now expression (4.1) becomes

$$\begin{aligned}
&\phi(L_l(K(r,s),\mu) = (\mu-W)^{m_1-1}(\mu-X)^{r-m_1-1}(\mu-Y)^{m_2-1}(\mu-Z)^{s-m_2-1}\\
&\times[(\mu-W)(\mu-X)(\mu-Y)(\mu-Z) - c^2m_1(s-m_2)(\mu-X)(\mu-Y)\\
&-b^2(r-m_1)(s-m_2)(\mu-W)(\mu-Y) - a^2m_1m_2(\mu-Z)(\mu-X)\\
&-c^2(r-m_1)m_2(\mu-Z)(\mu-W) + (r-m_1)m_1m_2(s-m_2)(c^4+a^2b^2-2abc^2)].
\end{aligned}$$

On further simplification, we obtain

$$\begin{aligned}
&\phi(L_l(K(r,s),\mu) = \mu(\mu-W)^{m_1-1}(\mu-X)^{r-m_1-1}(\mu-Y)^{m_2-1}(\mu-Z)^{s-m_2-1}\\
&[\mu^3 - \mu^2(X+Y+Z+W) + \mu(WX+XY+YZ+ZW+WY+XZ)\\
&-(XYZ+XYW+XWZ+WYZ) - c^2m_1(s-m_2)(\mu-(X+Y))\\
&-b^2(r-m_1)(s-m_2)(\mu-(Y+W)) - a^2m_1m_2(\mu-(X+Z))\\
&-c^2(r-m_1)m_2(\mu-(W+Z))].
\end{aligned}$$

Substituting W,X,Y and Z and reducing the terms we get

$$\begin{Bmatrix} 0 & (am_2+(s-m_2)c) & (cm_2+(s-m_2)b) & (am_1+(r-m_1)c) & (cm_1+(r-m_1)b) \\ 1 & m_1-1 & r-m_1-1 & m_2-1 & s-m_2-1 \end{Bmatrix}$$

and the roots of

$$\begin{aligned}
&[\mu^3 - \mu^2\{(a+c)(m_1+m_2) + (b+c)((r+s)-(m_1+m_2))\}\\
&+\mu\{ac(m_1^2+m_2^2) + (c^2+ab)(m_2(s-m_2)+m_1(r-m_1))\\
&+(ab+bc+ca)(m_1(s-m_2)+m_1(r-m_1)) + bc((r-m_1)\\
&+(s-m_2))^2 + (b^2+c^2)(r-m_1)(s-m_2)\} - \{(a+c)bc(r-m_1)(s-m_2)(m_1+m_2)\\
&+abc\,(m_1(s-m_2)(s+m_1-m_2) + m_2(r-m_1)(r+m_2-m_1))\\
&+bc^2(r-m_1)(s-m_2)(r+s-m_1-m_2)\\
&+ac(a+c)m_1^2m_2 + c(ac+c^2+ab)m_1m_2(s-m_2) + ac(b+c)m_1m_2(r-m_1)\}] = 0. \qquad \square
\end{aligned}$$

Theorem 4.4 *Let $S(m,n)$ be a double star graph with central vertices labeled zero and the pendent vertices labeled one. Then the characteristic polynomial of label Laplacian matrix of $S(m,n)$ is*

$$\mu(\mu-c)^{m+n-4}\left(\mu^3 - [(m+n)c+2a]\mu^2 + [ac(m+n)+c^2mn+2ac]\mu - [ac^2(m+n)]\right).$$

Proof Let v_m and v_{m+1} be the central vertices of $S(m,n)$ with zero labels. Remaining $m+n-2$ vertices be given label one. Characteristic polynomial of $L_l(S(m,n))$ is

$$|\mu I - L_l(S(m,n))| = \left|\begin{array}{cc|cc} (\mu-c)I_{m-1} & cJ_{m-1\times 1} & O_{m-1\times 1} & O_{m-1\times n-1} \\ cJ_{1\times m-1} & (\mu-(m-1)c-a)I_1 & aI_1 & O_{1\times n-1} \\ \hline O_{1\times m-1} & aI_1 & (\mu-(n-1)c-a)I_1 & cJ_{1\times n-1} \\ O_{n-1\times m-1} & O_{m-1\times 1} & cJ_{n-1\times 1} & (\mu-c)I_{n-1} \end{array}\right|$$

Using elementary transformations, we get

$$|\mu I - L_l(S(m,n))| = \mu(\mu-c)^{m+n-4}\begin{vmatrix} (\mu-mc-a) & a & 0 \\ a-c & (\mu-(n-1)c-a) & (n-1)c \\ -c & c & (\mu-c) \end{vmatrix}$$

Hence, the characteristic polynomial of $S(m,n)$ is

$$\begin{aligned}\phi(L_l(S(m,n)),\mu) &= \mu(\mu-c)^{m+n-4}\left(\mu^3-[(m+n)c+2a]\mu^2+[ac(m+n)\right.\\ &\quad \left.+c^2mn+2ac]\mu-[ac^2(m+n)]\right). \qquad \square\end{aligned}$$

Corollary 4.6 *Let $S(m,n)$ be a double star graph with central vertices labeled one and the pendent vertices labeled zero. Then the characteristic polynomial of label Laplacian matrix of $S(m,n)$ is*

$$\mu(\mu-c)^{m+n-4}\left(\mu^3-[(m+n)c+2b]\mu^2+[bc(m+n)+c^2mn+2bc]\mu-[bc^2(m+n)]\right).$$

Definition 4.1 *The crown graph S_n^0 for an integer $n \geqslant 3$ is the graph with vertex set $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and edge set $\{u_iv_i : 1 \leqslant i,j \leqslant n, i \neq j\}$.*

Lemma 4.2([10]) *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 block symmetric matrix. Then the eigenvalues of A are the eigenvalues of the matrices A_0+A_1 and A_0-A_1.

Theorem 4.5 *The label Laplacian spectrum of crown graph S_n^0 of order $2n$ is*

$$\left\{\begin{array}{ccccccc} 0 & ma+(n-m)c & (n-m)b+mc & nc & (m-2)a+c(n-m) & \frac{\xi+\eta}{2} & \frac{\xi-\eta}{2} \\ 1 & m-1 & n-m-1 & 1 & m-1 & 1 & 1 \end{array}\right\}$$

where $X = 2a(m-1)+2b(n-m-1)+cn$, $Y = 4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2acm(m-1)$, $\xi = X$, $\eta = \sqrt{X^2-4Y}$ and m denotes the number of vertices labelled zero in each vertex set of the crown graph.

Proof Let the labels of n vertices of S_n^0 be $\underbrace{000\ldots0}_{m}\underbrace{111\cdots1}_{n-m}$ in each partite set. Then

$$L_l(S_n^0) = \left[\begin{array}{c|c} A & -B \\ \hline -B & A \end{array}\right]$$

where

$$A = Diag\begin{array}{c} \\ \left[\right. \end{array}\begin{array}{cccccccc} v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+2} & \cdots & v_n \\ W & W & \cdots & W & Z & Z & \cdots & Z \end{array}\begin{array}{c} \\ \left.\right], \end{array}$$

W and Z are the label degrees of the m zero label vertices and $n-m$ one label vertices respectively given by $W = a(m-1) + c(n-m)$ and $Z = b(n-m-1) + cm$. Note that

$$B = \left[\begin{array}{c|c} a(J-I)_{m\times m} & cJ_{m\times n-m} \\ \hline cJ_{n-m\times m} & a(J-I)_{n-m\times n-m} \end{array}\right]$$

From Lemma 4.2, the label Laplacian spectrum of $L_l(S_n^0)$ is the union of spectrum of $A+B$ and $A-B$. Observe that $A+B = L_l(K_n)$. Hence, by Theorem 4.2, we obtain

$$Spec_l(A+B) = \left\{\begin{array}{cccc} 0 & ma+(n-m)c & (n-m)b+mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{array}\right\} \tag{4.3}$$

Also, $A - B = A + A_l(K_n)$. Consider $\det(\mu I - (A + A_l(K_n)))$.

Step 1 Replacing R_i by $R_i' = R_i - R_{i-1}$, for $i = 2, 3, \cdots, m, m+2, m+3, \cdots, n$, we obtain

$$\det(\mu I - (A + A_l(K_n))) = (\mu - (a(m-2) + c(n-m)))^{m-1}(\mu - (b(n-m-2) + cm)))^{n-m-1}\det(E).$$

Step 2 Replacing C_i by $C_i' = C_i + C_{i+1} + \ldots + C_m$, $i = 1, 2, \ldots, m-1$ and replacing C_j by $C_j = C_j + C_{j+1} + \ldots + C_n$, $j = 1, 2, \ldots, n-1$, the $\det(E)$ reduces to a determinant of order $m+1$.

$$\det(E) = \left|\begin{array}{cccccc} \sigma & -a(m-1) & -a(m-2) & \ldots & -a & -c(n-m) \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -cm & -c(m-1) & -c(m-2) & \ldots & -c & \varsigma \end{array}\right|,$$

where $\sigma = \mu - [2a(m-1) + c(n-m)]$ and $\varsigma = \mu - [2b(n-m-1) + cm]$.

Step 3 Expanding over the first column

$$\begin{aligned} \det(E) &= \mu^2 - \mu[2a(m-1) + 2b(n-m-1) + cm] + 4ab(m-1)(n-m-1) \\ &\quad +2bc(n-m-1)(n-m) + 2cam(m-1). \end{aligned}$$

Step 4 Substituting $\det(E)$ in Step 1,

$$\begin{aligned}&\det(\mu I-(A+A_l(K_n)))=(\mu-(a(m-2)+c(n-m)))^{m-1}\\&\times(\mu-(b(n-m-2)+cm)))^{n-m-1}\{\mu^2-\mu[2a(m-1)+2b(n-m-1)+cm]\\&+4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2cam(m-1)\}.\end{aligned}$$

Hence, label Laplacian spectrum of $A-B$ is

$$Spec_l(A-B)=\begin{pmatrix} a(m-2)+c(n-m) & m-1\\ b(n-m-2)+cm & n-m-1\\ \frac{X+\sqrt{X^2-4Y}}{2} & 1\\ \frac{X-\sqrt{X^2-4Y}}{2} & 1\end{pmatrix} \tag{4.4}$$

where

$$\begin{aligned}X&=2a(m-1)+2b(n-m-1)+cn,\\Y&=4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2acm(m-1).\end{aligned}$$

The union of expressions (4.3) and (4.4) is the label Laplacian spectrum of S_n^0. □

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Some Results on Total Mean Cordial Labeling of Graphs

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to be a Total Mean Cordial graph if there exists a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge xy assign the label $\left\lceil \frac{f(x)+f(y)}{2} \right\rceil$ where $x, y \in V(G)$, and the total number of 0, 1 and 2 are balanced. That is $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with x ($x = 0, 1, 2$). In this paper, we investigate the total mean cordial labeling behavior of $L_n \odot K_1$, $S(P_n \odot 2K_1)$, $S(W_n)$ and some more graphs.

Key Words: Smarandachely total mean cordial labeling, cycle, path, wheel, union, corona, ladder.

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§1. Introduction

Throughout this paper we considered finite, undirected and simple graphs. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Labeled graphs serves as a useful mathematical model for a broad range of application such as coding theory, X-ray crystallography analysis, communication network addressing systems, astronomy, radar, circuit design and database management [1]. Ponraj, Ramasamy and Sathish Narayanan [3] introduced the concept of total mean cordial labeling of graphs and studied about the total mean cordial labeling behavior of Path, Cycle, Wheel and some more standard graphs. In [4,6], Ponraj and Sathish Narayanan proved that $K_n^c + 2K_2$ is total mean cordial if and only if $n = 1, 2, 4, 6, 8$ and they investigate the total mean cordial labeling behavior of prism, gear, helms. In [5], Ponraj, Ramasamy and Sathish Narayanan investigate the Total Mean Cordiality of Lotus inside a circle, bistar, flower graph, $K_{2,n}$, Olive tree, P_n^2, $S(P_n \odot K_1)$, $S(K_{1,n})$. In this paper we investigate $L_n \odot K_1$, $S(P_n \odot 2K_1)$, $S(W_n)$ and some more graphs. If x is any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. For basic definitions that are not defined here are used in the sense of Harary [2].

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§2. Preliminaries

Definition 2.1 *A total mean cordial labeling of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge xy assign the label $\left\lceil \frac{f(x)+f(y)}{2} \right\rceil$ where $x, y \in V(G)$, and the total number of 0, 1 and 2 are balanced. That is $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with x $(x = 0, 1, 2)$. If there exists a total mean cordial labeling on a graph G, we will call G is total mean cordial.*

Furthermore, let $H \leq G$ be a subgraph of G. If there is a function f from $V(G) \to \{0, 1, 2\}$ such that $f|_H$ is a total mean cordial labeling but $\left\lceil \dfrac{f(u)+f(v)}{2} \right\rceil$ is a constant for all edges in $G \setminus H$, such a labeling and G are then respectively called Smarandachely total mean cordial labeling and Smarandachely total mean cordial labeling graph respect to H.

The following results are frequently used in the subsequent section.

Definition 2.2 *The product graph $G_1 \times G_2$ is defined as follows: Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$. Note that the graph $L_n = P_n \times P_2$ is called the ladder on n steps.*

Definition 2.3 *Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1$ and $v \in V_2\}$. Also the graph $W_n = C_n + K_1$ is called the wheel.*

Definition 2.4 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 2.5 *The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Definition 2.6 *The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv of G by a path uwv.*

Theorem 2.7([7]) *Let G be a (p, q) Total Mean Cordial graph and $n \neq 3$ then $G \cup P_n$ is also total mean cordial.*

Main Results

Theorem 3.1 *$S(W_n)$ is total mean cordial.*

Proof Let $V(S(W_n)) = \{u, u_i, x_i, y_i : 1 \leq i \leq n\}$, $E(S(W_n)) = \{u_i y_i, y_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n y_n, y_n u_1\} \cup \{u x_i, x_i u_i : 1 \leq i \leq n\}$. Clearly $|V(S(W_n))| + |V(S(W_n))| = 7n + 1$.

Case 1. $n \equiv 0 \pmod{12}$.

Let $n = 12t$ and $t > 0$. Define $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned}
f(x_i) &= 0, \quad 1 \le i \le 12t \\
f(u_i) &= 0, \quad 1 \le i \le 2t \\
f(u_{2t+i}) &= 2, \quad 1 \le i \le 7t \\
f(u_{9t+i}) &= 1, \quad 1 \le i \le 3t \\
f(y_i) &= 1, \quad 1 \le i \le 2t-1 \\
f(y_{2t-1+i}) &= 2, \quad 1 \le i \le 7t \\
f(y_{9t-1+i}) &= 1, \quad 1 \le i \le 3t+1.
\end{aligned}$$

Here $ev_f(0) = 28t + 1$, $ev_f(1) = ev_f(2) = 28t$.

Case 2. $n \equiv 1 \pmod{12}$.

Let $n = 12t + 1$ and $t > 0$. Define a map $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned}
f(x_i) &= 0, \quad 1 \le i \le 12t+1 \\
f(u_i) &= 0, \quad 1 \le i \le 2t \\
f(u_{2t+i}) &= 2, \quad 1 \le i \le 7t \\
f(u_{9t+i}) &= 1, \quad 1 \le i \le 3t+1 \\
f(y_i) &= 1, \quad 1 \le i \le 2t-1 \\
f(y_{2t-1+i}) &= 2, \quad 1 \le i \le 7t+1 \\
f(y_{9t+i}) &= 1, \quad 1 \le i \le 3t+1.
\end{aligned}$$

Here $ev_f(0) = ev_f(1) = 28t + 3$, $ev_f(2) = 28t + 2$.

Case 3. $n \equiv 2 \pmod{12}$.

Let $n = 12t + 2$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned}
f(x_i) &= 0, \quad 1 \le i \le 12t+2 \\
f(u_i) &= 0, \quad 1 \le i \le 2t \\
f(u_{2t+i}) &= 2, \quad 1 \le i \le 7t+1 \\
f(u_{9t+1+i}) &= 1, \quad 1 \le i \le 3t+1 \\
f(y_i) &= 1, \quad 1 \le i \le 2t \\
f(y_{2t+i}) &= 2, \quad 1 \le i \le 7t+1 \\
f(y_{9t+1+i}) &= 1, \quad 1 \le i \le 3t+1.
\end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 28t + 5$.

Case 4. $n \equiv 3 \pmod{12}$.

Let $n = 12t - 9$ and $t > 0$. For $n = 3$, the Figure 1 shows that $S(W_3)$ is total mean cordial.

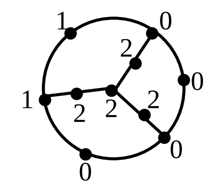

Figure 1

Now assume $t \geq 2$. Define a map $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned}
f(x_i) &= 0, \quad 1 \leq i \leq 12t - 9 \\
f(u_i) &= 0, \quad 1 \leq i \leq 2t - 2 \\
f(u_{2t-2+i}) &= 2, \quad 1 \leq i \leq 7t - 5 \\
f(u_{9t-7+i}) &= 1, \quad 1 \leq i \leq 3t - 2 \\
f(y_i) &= 1, \quad 1 \leq i \leq 2t - 3 \\
f(y_{2t-3+i}) &= 2, \quad 1 \leq i \leq 7t - 5 \\
f(y_{9t-8+i}) &= 1, \quad 1 \leq i \leq 3t - 1.
\end{aligned}$$

In this case $ev_f(0) = ev_f(1) = 28t - 21$, $ev_f(2) = 28t - 20$.

Case 5. $n \equiv 4 \pmod{12}$.

Let $n = 12t - 8$ and $t > 0$. The following Figure 2 shows that $S(W_4)$ is total mean cordial.

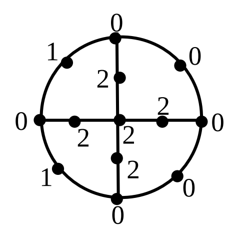

Figure 2

Now assume $t \geq 2$. Define $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned}
f(x_i) &= 0, \quad 1 \leq i \leq 12t - 8 \\
f(u_i) &= 0, \quad 1 \leq i \leq 2t - 2 \\
f(u_{2t-2+i}) &= 2, \quad 1 \leq i \leq 7t - 5 \\
f(u_{9t-7+i}) &= 1, \quad 1 \leq i \leq 3t - 1 \\
f(y_i) &= 1, \quad 1 \leq i \leq 2t - 3 \\
f(y_{2t-3+i}) &= 2, \quad 1 \leq i \leq 7t - 4 \\
f(y_{9t-7+i}) &= 1, \quad 1 \leq i \leq 3t - 1.
\end{aligned}$$

In this case $ev_f(0) = 28t - 19$, $ev_f(1) = ev_f(2) = 28t - 18$.

Case 6. $n \equiv 5 \pmod{12}$.

Let $n = 12t - 7$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 7 \\ f(u_i) &= 1, \quad 1 \le i \le 4t - 3 \\ f(u_{4t-3+i}) &= 2, \quad 1 \le i \le 7t - 4 \\ f(u_{11t-7+i}) &= 1, \quad 1 \le i \le t \\ f(y_i) &= 0, \quad 1 \le i \le 4t - 3 \\ f(y_{4t-3+i}) &= 2, \quad 1 \le i \le 7t - 4 \\ f(y_{11t-7+i}) &= 1, \quad 1 \le i \le t. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 28t - 16$.

Case 7. $n \equiv 6 \pmod{12}$.

Let $n = 12t - 6$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 6 \\ f(u_i) &= 0, \quad 1 \le i \le 2t - 1 \\ f(u_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 4 \\ f(u_{9t-5+i}) &= 1, \quad 1 \le i \le 3t - 1 \\ f(y_i) &= 1, \quad 1 \le i \le 2t - 2 \\ f(y_{2t-2+i}) &= 2, \quad 1 \le i \le 7t - 3 \\ f(y_{9t-5+i}) &= 1, \quad 1 \le i \le 3t - 1. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = 28t - 7$, $ev_f(2) = 28t - 6$.

Case 8. $n \equiv 7 \pmod{12}$.

Let $n = 12t - 5$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 5 \\ f(u_i) &= 0, \quad 1 \le i \le 2t - 1 \\ f(u_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 3 \\ f(u_{9t-4+i}) &= 1, \quad 1 \le i \le 3t - 1 \\ f(y_i) &= 1, \quad 1 \le i \le 2t - 2 \\ f(y_{2t-2+i}) &= 2, \quad 1 \le i \le 7t - 3 \\ f(y_{9t-5+i}) &= 1, \quad 1 \le i \le 3t. \end{aligned}$$

Here $ev_f(0) = ev_f(1) = 28t - 11$, $ev_f(2) = 28t - 12$.

Case 9. $n \equiv 8 \pmod{12}$.

Let $n = 12t - 4$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 4 \\ f(u_i) &= 0, \quad 1 \le i \le 2t - 1 \\ f(u_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 2 \\ f(u_{9t-3+i}) &= 1, \quad 1 \le i \le 3t - 1 \\ f(y_i) &= 1, \quad 1 \le i \le 2t - 1 \\ f(y_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 3 \\ f(y_{9t-4+i}) &= 1, \quad 1 \le i \le 3t. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 28t - 9$.

Case 10. $n \equiv 9 \pmod{12}$.

Let $n = 12t - 3$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 3 \\ f(u_i) &= 0, \quad 1 \le i \le 2t - 1 \\ f(u_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 2 \\ f(u_{9t-3+i}) &= 1, \quad 1 \le i \le 3t \\ f(y_i) &= 1, \quad 1 \le i \le 2t - 2 \\ f(y_{2t-2+i}) &= 2, \quad 1 \le i \le 7t - 1 \\ f(y_{9t-3+i}) &= 1, \quad 1 \le i \le 3t. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = 28t - 7$, $ev_f(2) = 28t - 6$.

Case 11. $n \equiv 10 \pmod{12}$.

Let $n = 12t - 2$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \le i \le 12t - 2 \\ f(u_i) &= 0, \quad 1 \le i \le 2t - 1 \\ f(u_{2t-1+i}) &= 2, \quad 1 \le i \le 7t - 1 \\ f(u_{9t-2+i}) &= 1, \quad 1 \le i \le 3t \\ f(y_i) &= 1, \quad 1 \le i \le 2t - 2 \\ f(y_{2t-2+i}) &= 2, \quad 1 \le i \le 7t - 1 \\ f(y_{9t-3+i}) &= 1, \quad 1 \le i \le 3t + 1. \end{aligned}$$

In this case $ev_f(0) = 28t - 5$, $ev_f(1) = ev_f(2) = 28t - 4$.

Case 12. $n \equiv 11 \pmod{12}$.

Let $n = 12t - 1$ and $t > 0$. Define a function $f : V(S(W_n)) \to \{0, 1, 2\}$ by $f(u) = 0$,

$$\begin{aligned} f(x_i) &= 0, \quad 1 \leq i \leq 12t - 1 \\ f(u_i) &= 1, \quad 1 \leq i \leq 4t - 1 \\ f(u_{4t-1+i}) &= 2, \quad 1 \leq i \leq 7t \\ f(u_{11t-1+i}) &= 1, \quad 1 \leq i \leq t \\ f(y_i) &= 0, \quad 1 \leq i \leq 4t - 1 \\ f(y_{4t-1+i}) &= 2, \quad 1 \leq i \leq 7t - 1 \\ f(y_{11t-2+i}) &= 1, \quad 1 \leq i \leq t + 1. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 28t - 6$.

Hence $S(W_n)$ is total mean cordial. □

Theorem 3.2 *$S(P_n \odot 2K_1)$ is total mean cordial.*

Proof Let $V(S(P_n \odot 2K_1)) = \{u_i, v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{u_i^{'} : 1 \leq i \leq n - 1\}$ and $E(S(P_n \odot 2K_1)) = \{u_i u_i^{'}, u_i^{'} u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i, u_i w_i, v_i x_i, w_i y_i : 1 \leq i \leq n\}$. Clearly $|V(S(P_n \odot 2K_1))| + |V(S(W_n \odot 2K_1))| = 12n - 3$. Now we define a map $f : V(S(P_n \odot 2K_1)) \to \{0, 1, 2\}$ by $f(v_1) = 0$, $f(w_1) = 1$, $f(u_n) = 0$,

$$\begin{aligned} f(u_i) &= f(u_i^{'}) &&= 0, \quad 1 \leq i \leq n - 1 \\ f(v_i) &= f(w_i) &&= 1, \quad 2 \leq i \leq n \\ f(x_i) &= f(y_i) &&= 2, \quad 1 \leq i \leq n. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = ev_f(2) = 4n - 1$.

Hence $S(P_n \odot 2K_1)$ is total mean cordial. □

Theorem 3.3 *$L_n \odot K_1$ is total mean cordial.*

Proof Let $V(L_n \odot K_1) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(L_n \odot K_1) = \{x_i u_i, u_i v_i, v_i y_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\}$. Here $|V(L_n \odot K_1)| + |E(L_n \odot K_1)| = 9n - 2$. Define a map $f : V(L_n \odot K_1) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq n \\ f(x_i) &= 0, \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ f(y_i) &= 1, \quad 1 \leq i \leq n \\ f(x_{\lceil \frac{n}{2} \rceil + i}) &= 1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ f(v_i) &= 2, \quad 1 \leq i \leq n. \end{aligned}$$

The following Table 1 shows that f is a total mean cordial labeling of $L_n \odot K_1$.

Nature of n	$ev_f(0)$	$ev_f(1)$	$ev_f(2)$
$n \equiv 0 \pmod 2$	$\left\lfloor \frac{9n-2}{3} \right\rfloor$	$\left\lceil \frac{9n-2}{3} \right\rceil$	$\left\lfloor \frac{9n-2}{3} \right\rfloor$
$n \equiv 1 \pmod 2$	$\left\lceil \frac{9n-2}{3} \right\rceil$	$\left\lfloor \frac{9n-2}{3} \right\rfloor$	$\left\lfloor \frac{9n-2}{3} \right\rfloor$

Hence $L_n \odot K_1$ is Total Mean Cordial. □

Theorem 3.4 *The graph $P_1 \cup P_2 \cup \ldots \cup P_n$ is total mean cordial.*

Proof We prove this theorem by induction on n. For $n = 1, 2, 3$ the result is true, see Figure 3.

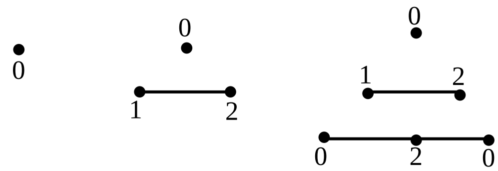

Figure 3

Assume the result is true for $P_1 \cup P_2 \cup \ldots \cup P_{n-1}$. Then by Theorem 2.7, $(P_1 \cup P_2 \cup \ldots \cup P_{n-1}) \cup P_n$ is total mean cordial. □

Theorem 3.5 *Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let GC_n be a graph with $V(GC_n) = V(C_n) \cup \{v_i : 1 \le i \le n\}$ and $E(GC_n) = E(C_n) \cup \{u_iv_i, u_{i+1}v_i : 1 \le i \le n-1\} \cup \{u_nv_n, u_1v_n\}$. Then GC_n is total mean cordial.*

Proof Clearly, $|V(GC_n)| + |E(GC_n)| = 5n$.

Case 1. $n \equiv 0 \pmod 3$.

Let $n = 3t$ and $t > 0$. Define $f : V(GC_n) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= f(v_i) &&= 0, \quad 1 \le i \le t \\ f(u_{t+i}) &= f(v_{t+i}) &&= 2, \quad 1 \le i \le t \\ f(u_{2t+i}) &= f(v_{2t+i}) &&= 1, \quad 1 \le i \le t-1 \end{aligned}$$

$f(u_{3t}) = 1$ and $f(v_{3t}) = 0$. In this case $ev_f(0) = ev_f(1) = ev_f(2) = 5t$.

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$ and $t > 0$. Define $f : V(GC_n) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= f(v_i) &&= 0, \quad 1 \le i \le t \\ f(u_{t+1+i}) &= f(v_{t+i}) &&= 2, \quad 1 \le i \le t \\ f(u_{2t+1+i}) &= f(v_{2t+1+i}) &&= 1, \quad 1 \le i \le t \end{aligned}$$

$f(u_{t+1}) = 0$, $f(v_{2t+1}) = 2$. In this case $ev_f(0) = 5t + 1$, $ev_f(1) = ev_f(2) = 5t + 2$.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t + 2$ and $t > 0$. Construct a vertex labeling $f : V(GC_n) \to \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= f(v_i) &&= 0, \quad 1 \le i \le t+1 \\ f(u_{t+2+i}) &= f(v_{t+1+i}) &&= 2, \quad 1 \le i \le t \\ f(u_{2t+2+i}) &= f(v_{2t+2+i}) &&= 1, \quad 1 \le i \le t \end{aligned}$$

$f(u_{t+1}) = 1$, $f(v_{2t+2}) = 2$. In this case $ev_f(0) = ev_f(1) = 5t + 3$, $ev_f(2) = 5t + 4$. Hence GC_n is total mean cordial. □

Example 3.6 A total mean cordial labeling of GC_8 is given in Figure 4.

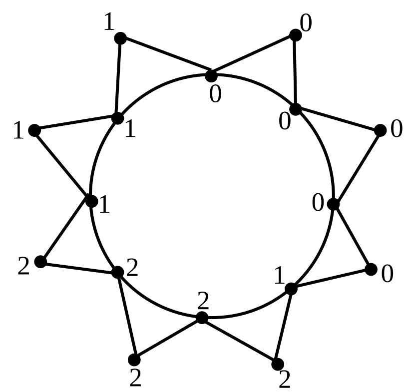

Figure 4

Theorem 3.6 *Let $St(L_n)$ be a graph obtained from a ladder L_n by subdividing each step exactly once. Then $St(L_n)$ is total mean cordial.*

Proof Let $V(St(L_n)) = \{u_i, v_i, w_i : 1 \le i \le n\}$ and $E(St(L_n)) = \{u_iw_i, w_iv_i : 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\}$. It is clear that $|V(St(L_n))| + |E(St(L_n))| = 7n - 2$.

Case 1. $n \equiv 0 \pmod 6$.

Let $n = 6t$. Define a map $f : V(St(L_n)) \to \{0, 1, 2\}$ as follows.

$$\begin{aligned} f(u_i) &= 0, \quad 1 \le i \le 6t \\ f(w_i) &= 0, \quad 1 \le i \le t \\ f(w_{t+i}) &= 1, \quad 1 \le i \le 5t \\ f(v_i) &= 2, \quad 1 \le i \le 5t \\ f(v_{5t+i}) &= 1, \quad 1 \le i \le t. \end{aligned}$$

In this case $ev_f(0) = ev_f(1) = 14t - 1$, $ev_f(2) = 14t$.

Case 2. $n \equiv 1 \pmod 6$.

Let $n = 6t + 1$ and $t \geq 1$. Define a function $f : V(St(L_n)) \rightarrow \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 6t+1 \\ f(w_i) &= 0, \quad 1 \leq i \leq t \\ f(w_{t+i}) &= 2, \quad 1 \leq i \leq 5t+1 \\ f(v_i) &= 1, \quad 1 \leq i \leq 4t+1 \\ f(v_{4t+1+i}) &= 2, \quad 1 \leq i \leq 2t. \end{aligned}$$

Here $ev_f(0) = 14t + 1$, $ev_f(1) = ev_f(2) = 14t + 2$.

Case 3. $n \equiv 2 \pmod 6$.

Let $n = 6t + 2$ and $t \geq 0$. The Figure 5 shows that $St(L_2)$ is total mean cordial.

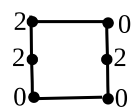

Figure 5

Consider the case for $t \geq 1$. Define $f : V(St(L_n)) \rightarrow \{0, 1, 2\}$ as follows.

$$\begin{aligned} f(u_i) &= 0, \quad 1 \leq i \leq 6t+2 \\ f(w_i) &= 0, \quad 1 \leq i \leq t \\ f(w_{t+i}) &= 1, \quad 1 \leq i \leq 5t+1 \\ f(v_i) &= 2, \quad 1 \leq i \leq 5t+1 \\ f(v_{5t+1+i}) &= 1, \quad 1 \leq i \leq t. \end{aligned}$$

and $f(w_{6t+2}) = 2$, $f(v_{6t+2}) = 0$. Here $ev_f(0) = ev_f(1) = ev_f(2) = 14t + 4$.

Case 4. $n \equiv 3 \pmod 6$.

Let $n = 6t - 3$ and $t \geq 1$. Define a function $f : V(St(L_n)) \rightarrow \{0, 1, 2\}$ by

$$\begin{aligned} f(u_i) &= f(w_i) &&= f(v_i) &&= 0, \quad 1 \leq i \leq 2t-2 \\ f(u_{2t-1+i}) &= f(w_{2t-1+i}) &&= f(v_{2t+i}) &&= 1, \quad 1 \leq i \leq 2t-2 \\ f(u_{4t-2+i}) &= f(w_{4t-1+i}) &&= f(v_{4t-2+i}) &&= 2, \quad 1 \leq i \leq 2t-2 \end{aligned}$$

$f(u_{2t-1}) = f(w_{2t-1}) = 0$, $f(u_{4t-2}) = f(w_{4t-2}) = f(w_{4t-1}) = 1$ and $f(u_{6t-3}) = f(v_{6t-3}) = 2$. In this case $ev_f(0) = 14t - 7$, $ev_f(1) = ev_f(2) = 14t - 8$.

Case 5. $n \equiv 4 \pmod 6$.

Let $n = 6t-2$ and $t > 0$. Define $f : V(St(L_n)) \to \{0,1,2\}$ by

$$\begin{aligned}
f(u_i) &= 0, \quad 1 \le i \le 6t-2\\
f(w_i) &= 0, \quad 1 \le i \le t\\
f(w_{t+i}) &= 1, \quad 1 \le i \le 5t-2\\
f(v_i) &= 2, \quad 1 \le i \le 5t-2\\
f(v_{5t-2+i}) &= 1, \quad 1 \le i \le t.
\end{aligned}$$

In this case $ev_f(0) = ev_f(1) = 14t-5$, $ev_f(2) = 14t-6$.

Case 6. $n \equiv 5 \pmod 6$.

Let $n = 6t-1$ and $t > 0$. Define a function $f : V(St(L_n)) \to \{0,1,2\}$ by

$$\begin{aligned}
f(u_i) &= 0, \quad 1 \le i \le 6t-1\\
f(w_i) &= 0, \quad 1 \le i \le t\\
f(w_{t+i}) &= 1, \quad 1 \le i \le 5t-1\\
f(v_i) &= 2, \quad 1 \le i \le 5t-1\\
f(v_{5t-1+i}) &= 1, \quad 1 \le i \le t.
\end{aligned}$$

Here $ev_f(0) = ev_f(1) = ev_f(2) = 14t-3$.

Hence $St(L_n)$ is total mean cordial. □

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Number of Regions in Any Simple Connected Graph

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Abstract: A graph $G(v,e)$ is simple if it is without self loops and parallel edges and a graph $G(v,e)$ is connected if every vertex of graph is connected with each other. This paper is dealing with the problem of finding the number of regions in any simple connected graph. In other words this paper generalize the Eulers result on number of regions in planer graphs to all simple non planar graphs according to Euler number of regions in planar graphs is given by $f = e - v + 2$. Now we extend Eulers result to all simple graphs. I will prove that the number of regions in any simple connected graph is equal to

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \qquad r \in N$$

The minimum number of regions in any complete graph is

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2 - 3n + 4}{2}$$

Where [ ] represents greatest integer function, and n is the number of vertices of graph.

Key Words: Planar graph, simple graph, non-planar graph, complete graph, regions of a graph, crossing number.

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§1. Introduction

A planar graph is one that can be drawn on a two-dimensional plane such that no two edges cross. A cubic graph is one in which all vertices have degree three. A three connected graph is one that cannot be disconnected by removal of two vertices. A graph is said to be bipartite

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if the vertices can be colored using exactly two colors such that no two adjacent vertices have the same color. [1] pp 16-20 A plane representation of a graph divides the plane into regions also called windows, faces or meshes. A window is characterized by the set of edges or the set of vertices forming its boundary. Note that windows is not defined in a non planer graph or even in a planer graph not embedded in a plane. Thus a window is a property of the specific plane representation of a graph. The window of a graph may be finite or infinite. The portion of a plane lying outside a graph embedded in a plane is infinite in its extend. Since a planar graph may have different plane representation Euler gives formula for number of windows in a planar graph.[2] pp 88-100.

Lemma 1.1([1]) *A graph can be embedded in a surface of a sphere if and only if it can be embedded in a plane.*

Lemma 1.2([1]) *A planer graph may be embedded in a plane such that any specified region can be made the infinite region.*

Lemma 1.3(Euler theorem, [1], [2]) *A connected planer graph with n vertices and e edges has $e - n + 2$ regions.*

Lemma 1.4([2]) *A plane graph is bipartite if and only if each of its faces has an even number of sides.*

Corollary 1.5([2]) *In a simple connected planar graph with f regions n vertices and e edges $(e > 2)$ the following inequalities must hold.*

$$e \geq \frac{3}{2}f \quad and \quad e \leq 3n - 6.$$

Theorem 1.6([2]) *The spherical embedding of every planar 3-connected graph is unique.*

The crossing number (sometimes denoted as $c(G)$ of a graph G is the smallest number of pair wise crossings of edges among all drawings of G in the plane. In the last decade, there has been significant progress on a true theory of crossing numbers. There are now many theorems on the crossing number of a general graph and the structure of crossing critical graphs, whereas in the past, most results were about the crossing numbers of either individual graphs or the members of special families of graphs. The study of crossing numbers began during the Second World War with Paul Turan. In [4], he tells the story of working in a brickyard and wondering about how to design an efficient rail system from the kilns to the storage yards. For each kiln and each storage yard, there was a track directly connecting them. The problem he Consider was how to lay the rails to reduce the number of crossings, where the cars tended to fall off the tracks, requiring the workers to reload the bricks onto the cars. This is the problem of finding the crossing number of the complete bipartite graph. It is also natural to try to compute the crossing number of the complete graph. To date, there are only conjectures for the crossing numbers of these graphs called Guys conjecture which suggest that crossing number of complete graph K_n is given by $V(K_n) = Z(n)$ [5] [6].

$$Z(n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right],$$

where [ ] represents greatest integer function which can also be written as

$$Z(n) = \begin{cases} \frac{1}{64}n(n-2)^2(n-4)^2 & \text{... if n is even} \\ \frac{1}{64}(n-1)^2(n-3)^2 & \text{... if n is odd} \end{cases}$$

Guy prove it for $n \leq 10$ in 1972 in 2007 Richter prove it for $n \leq 12$ For any graph G, we say that the crossing number $c(G)$ is the minimum number of crossings with which it is possible to draw G in the plane. We note that the edges of G need not be straight line segments, and also that the result is the same whether G is drawn in the plane or on the surface of a sphere. Another invariant of G is the rectilinear crossing number, $c(G)$, which is the minimum number of crossings when G is drawn in the plane in such a way that every edge is a straight line segment. We will find by an example that this is not the same number obtained by drawing G on a sphere with the edges as arcs of great circles. In drawing G in the plane, we may locate its vertices wherever it is most convenient. A plane graph is one which is already drawn in the plane in such a way that no two of its edges intersect. A planar graph is one which can be drawn as a plane graph [9]. In terms of the notation introduced above, a graph G is planar if and only if $c(G) = 0$. The earliest result concerning the drawing of graphs in the plane is due to Fary [7] [10], who showed that any planar graph (without loops or multiple edges) can be drawn in the plane in such a way that every Edge is straight. Thus Farys result may be rephrased: if $c(G) = 0$, then $\bar{c}(G) = 0$. In a drawing, the vertices of the graph are mapped into points of a plane, and the arcs into continuous curves of the plane, no three having a point in common. A minimal drawing does not contain an arc which crosses itself, nor two arcs with more than one point in common. [8][11]In general for a set of n line segments, there can be up to $O(n^2)$ intersection points, since if every segment intersects every other segment, there would be

$$\frac{n(n-1)}{2} = O(n^2)$$

intersection points. To compute them all we require is $O(n^2)$ algorithm.

§2. **Main Result**

Before proving the main result we would like to give the detailed purpose of this paper. Euler gives number of regions in planer graphs which is equal to $f = e - v + 2$. But for non planar graphs the number of regions is still unknown. It is obvious that every graph has different representations; there is no particular representation of non planer simple graphs a graph $G(v, e)$ can be represented in different ways. My aim is to find the number of regions in any simple non planer graph in whatever way we draw it. I will prove that number of regions of any simple non planar graph is equal to

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \quad r \in N$$

where $\sum C_2$ are the total number of intersection points where two edges have a common point, $\sum C_3$ are the total number of intersection points where three edges have a common point, and so on,$\sum C_r$ are the total number of intersecting points where r edges have a common point.

In whatever way we draw the graph. And the minimum number of regions in a complete graph is equal to

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}, \quad n = \text{number of vertices}$$

This result is depending upon Guys conjecture which is true for all complete graphs $n \leq 12$ therefore my result is true for all complete graphs $n \leq 12$ if conjecture is true for all n, then my result is also true for all complete graphs.

Theorem 2.1 *The number of regions in any simple graph is given by*

$$f = e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i, \quad r \in N$$

In particular number of regions in any complete graph is given by

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}$$

This result of complete graphs is true for all graphs $n \leq 12$ it is true for all n if Guys conjecture is true for all n.

Proof Let $G(v,e)$ be a graph contains the finite set of vertices v and finite set of edges e. It is obvious that every graph has a planar representation in a certain stage and in that stage according to Euler number of regions are $f = e - v + 2$. Let n edges remaining in the graph by adding a single edge graph becomes non planar that in this stage it has maximum planarity so if we start to add remaining n edges one by one intersecting points occur and number of regions start to increase. Out of remaining n edges let us suppose that there are certain intersecting points where two edges have a common points it is denoted by C_2 and total such points can be represented by $\sum C_2$ similarly let us suppose that there are certain intersecting points where three edges have a common points it is denoted by C_3 and total such points can be represented by $\sum C_3$ and this process goes on and finally let us suppose that there are certain intersecting points where r edges have a common point it is denoted by C_r and total such points can be represented by $\sum C_r$. It must be kept in mind that graphs cannot be defined uniquely and finitely every graph has different representations. Since my result is true for all representations in whatever way you can represent graph. We first show that if we have finite set of n edges in a plane such that each pair of edges have one common point and no three edges have a common point, number of regions is increased by one by each pair of edges. Let f_n be the number of regions created by finite set of n edges. It is not obvious that every finite set of n edges creates the same number of regions, this follows inductively when we establish a recurrence f_0.

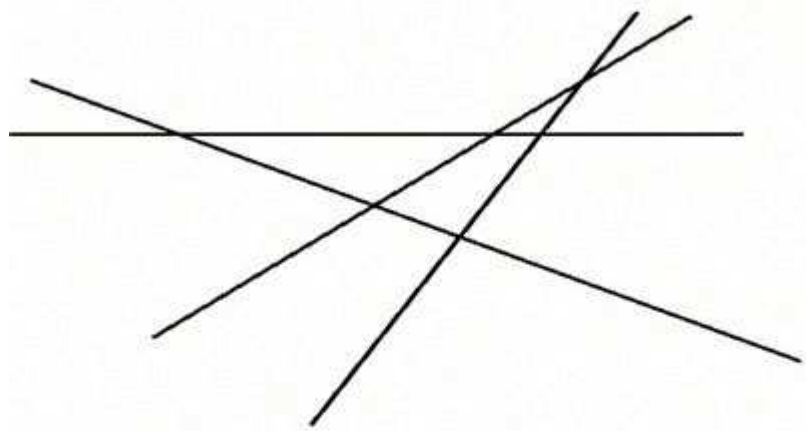

Fig.1

We begin with no edges and one region, so $f_0 = 1$. We prove that

$$f_n = f_{n-1} + n$$

if $n \geq 1$. Consider finite set of n edges, with $n \geq 1$ and let L be one of these edges. The other edges form a finite set of $n-1$ edges. We argue that adding L increases the number of regions by n. The intersection of L with the other edges partition L into n portions. Each of these portions cuts a region into two. Thus adding L increases the number of regions by n .since this holds for all finite set of edges we have

$$f_n = f_{n-1} + n$$

if $n \geq 1$. This determines a unique sequence starting with $f_0 = 1$, and hence every finite set of edges creates same number of regions. Thus it is clear that if two edges have a common point number of region is increased by one we represent it by C_2 and total number of such intersecting point is denoted by $\sum C_2$, similarly if three edges have a common point number of regions is increased by two and we denote it by C_3 and total number of such intersection points is denoted by $\sum C_3$ and number of regions are $2\sum C_3$ this process goes on and finally if r lines have a common point number of regions is increased by r and it is denoted by C_r and total number of such intersection points are denoted by $\sum C_r$ and number of regions increased by $(r-1)\sum C_r$ it must be noted that every graph has different representations any number of intersection points can occur. Thus we conclude that number of regions in any simple graph is given by

$$\begin{aligned} f &= e - v + 2 + \text{sum of all intersecting points where two edges have common point} \\ &\quad +2(\text{sum of all intersecting points where three edges have common point}) \\ &\quad +3(\text{sum of all intersecting points where four edges have common point}) \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad +(r-1)(\text{sum of all intersecting points where r edges have common point}), \end{aligned}$$

written to be

$$f = e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 + ... + (r-1)\sum C_r,$$

which can be expressed as

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \quad r \in N$$

It should be noted that Figures 2-4 below illustrate above result.

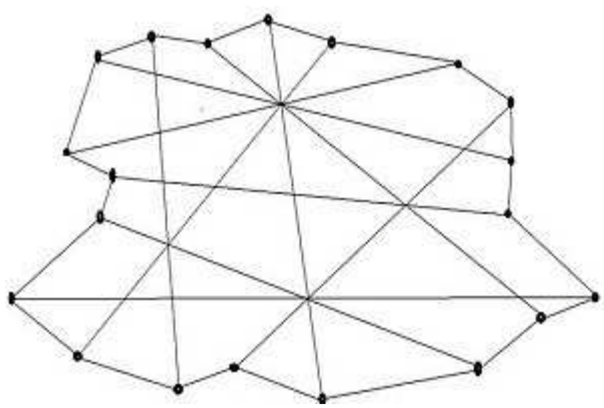

Figure 2

Figure 2 has 20 vertices and 30 edges, there are 9 intersection points where two edges have common point, 2 intersection points where three edges have common point,1 intersection points where four edges have common point, 1 intersection points where five edges have common point, and number of regions is 32 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 + 4\sum C_5. \end{aligned}$$

Substitute above values we get that

$$f = 30 - 20 + 2 + 9 + 4 + 3 + 4 = 32,$$

which verifies that above result.

Figure 3 below has 14 vertices 24 edges 15 intersecting points where two edges have common point. 2 intersection points where three edges have common point,1 intersection points where four edges have common point, and number of regions is 34 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 = 24 - 14 + 2 + 15 + 4 + 3 = 34, \end{aligned}$$

which again verifies that above result.

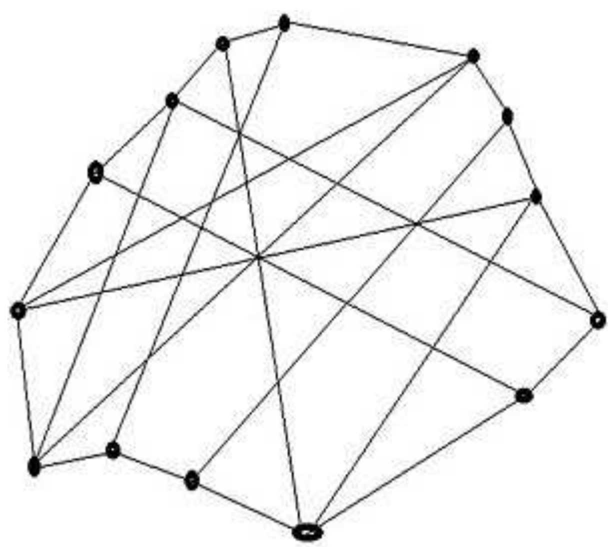

Figure 3

Figure 4 below has 6 vertices 11 edges 2 intersecting points where two edges have common point. 1 intersection points where three edges have common point, and number of regions is 11 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 = 11 - 6 + 2 + 2 + 2 = 11, \end{aligned}$$

verifies that above result again.

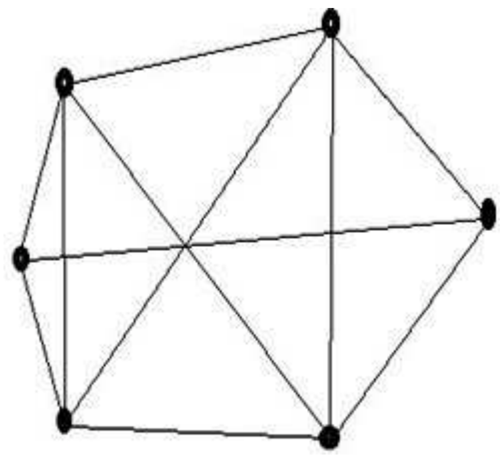

Figure 4

Now if the graph is complete with n vertices then the number of edges in it is $\frac{n(n-1)}{2}$ and minimum number of crossing points are given by Guys conjecture that is

$$Z(n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

which is true for all $n \leq 12$ thus above result is true for all $n \leq 12$, if Guys conjecture is true, then my result is true for all n. We know that every complete graph has a planar representation in a certain stage. When we start to draw any complete graph we add edge one by one and a stage comes when graph has maximum planarity in that stage number of regions according to Euler is $f = e - v + 2$, when we start to add more edges one by one number of crossing numbers occur but according to definition of crossing numbers two edges have a common point and no three edges have a common point it has been shown that if two edges have a common point number of regions is increased by $\sum C_2$, thus the number of regions is given by

$$f = e - v + 2 + \sum C_2,$$

where

$$\sum C_2 = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

is the minimum number of crossing points (Guys conjecture), e the number of edges and v number of vertices.

Let us suppose that graph has n vertices and number of edges is $\frac{n(n-1)}{2}$ substitute these values above we get minimum number of regions in a complete graph is given by

$$\begin{aligned} f &= e - v + 2 + \sum C_2 \\ &= \frac{n(n-1)}{2} - n + 2 + \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] \end{aligned}$$

$$= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}.$$

That proves the result. □

Figures 5-6 below illustrates this result. The Figure 5 below is the complete graph of six vertices and number of regions are as

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{6}{2}\right]\left[\frac{6-1}{2}\right]\left[\frac{6-2}{2}\right]\left[\frac{6-3}{2}\right] + \frac{6^2-3(6)+4}{2} \\ &= \frac{1}{4}\times 3\times 2\times 2\times 1 + \frac{36-18+4}{2} = 14 \end{aligned}$$

This shows that the above result is true.

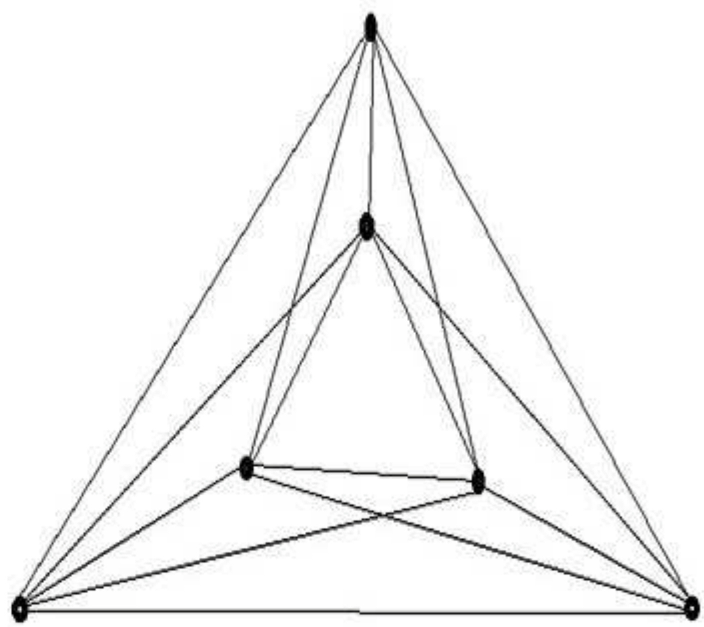

Figure 5

Figure 6 below is the complete graph of 5 vertices and number of regions are as

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{5}{2}\right]\left[\frac{5-1}{2}\right]\left[\frac{5-2}{2}\right]\left[\frac{5-3}{2}\right] + \frac{5^2-3(5)+4}{2} \\ &= \frac{1}{4}\times 2\times 2\times 1\times 1 + \frac{25-15+4}{2} = 8 \end{aligned}$$

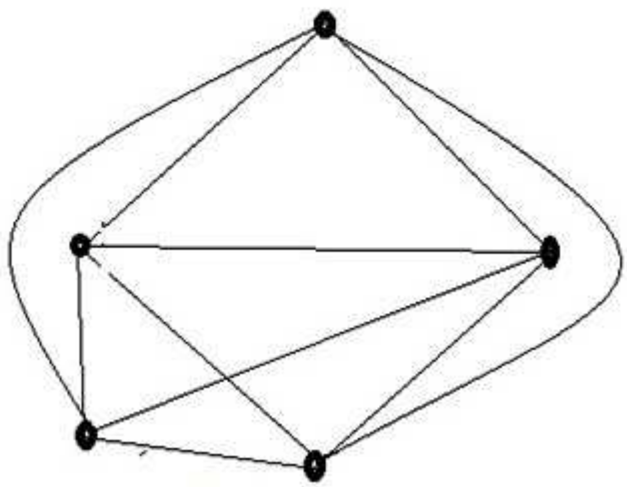

Figure 6

Example 1 Find the number of regions of a complete graph of 8 vertices with minimum crossings. Find number of regions?

Solution Apply the above result we get

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]+\frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{8}{2}\right]\left[\frac{8-1}{2}\right]\left[\frac{8-2}{2}\right]\left[\frac{8-3}{2}\right]+\frac{8^2-3\times 8+4}{2} \\ &= \frac{1}{4}\times 4\times 3\times 3\times 2+\frac{64-24+4}{2}=40 \end{aligned}$$

Example 2 A graph has 10 vertices and 24 edges, there are three points where two edges have a common point, and there is one point where three edges have a common point find the number of regions of a graph?

Solution By applying above formula we get

$$\begin{aligned} f &= e-v+2+\sum_{j=1}^{r} j\sum_{i=2}^{r} C_i \\ &= e-v+2+\sum C_2+2\sum C_3 = 24-10+2+3+2=21 \end{aligned}$$

Thus number of regions is 21.

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A Characterization of Directed Paths

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Abstract: In this note, the non-trivial connected digraphs D with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfying $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2$ are characterized, where $d^-(v_i)$ and $d^+(v_i)$ be the in-degree and out-degree of vertices of D, respectively.

Key Words: Directed path, directed cycle, directed tree, tournament.

AMS(2010): 05C20.

§1. Introduction

Notations and definitions not introduced here can be found in [1]. For a simple graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, V.R.Kulli[2] gave the following characterization. A graph G is a non-empty path if and only if it is connected graph with $n \geq 2$ vertices and $\sum_{i=1}^{n} d_i^2 - 4n + 6 = 0$, where d_i is the degree of vertices of G. In this note, we extend the characterization of paths to directed paths, which is needed to characterize the maximal outer planarity property of some digraph operator(digraph valued function).

We need some concepts and notations on directed graphs. A *directed graph*(or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here, $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. A *directed path* from v_1 to v_n is a collection of distinct vertices $v_1, v_2, v_3, \ldots, v_n$ together with the arcs $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ considered in the following order: $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_{n-1}v_n, v_n$. A directed path is said to be *non-empty* if it has at least one arc. An *arborescence* is a directed graph in which, for a vertex u called the *root*(i.e., a vertex of in-degree zero) and any other vertex v, there is exactly one directed path from u to v. A *directed cycle* is obtained from a nontrivial directed path on adding an arc from the terminal vertex to the initial vertex of the directed path. A *directed tree* is a directed graph which would be a tree if the directions on the arcs are ignored. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out

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from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v.

The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$. A *tournament* is a nontrivial complete asymmetric digraph.

§2. Characterization

Theorem 2.1 *A connected digraph D with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$, $n \geq 2$ is a non-empty directed path if and only if*

$$\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2. \tag{1}$$

Proof Let D be a directed path with n vertices $v_1, v_2, \cdots, v_n$. Then it is easy to verify that the sum of product of in-degree and out-degree of its vertices is $(n-2)$.

To prove the sufficiency part, we are given that D is connected with n vertices $v_1, v_2, \ldots, v_n$ and equation (1) is satisfied. If $n = 2$, then the only connected digraph is a tournament with two vertices(or a directed path with two vertices) and (1) is trivially verified.

Now, suppose that D is connected with $n \geq 3$ vertices. We consider the following two cases:

(i) The total degree of every vertex of D is at most two;

(ii) There exists at least one vertex of D whose total degree is at least three.

In the former case, since D is connected, it is either a directed path or a directed tree or a directed cycle.

Suppose that D is a directed tree with $n \geq 3$ vertices. Then there exists exactly two vertices of total degree one, and $(n-2)$ vertices of total degree two. Thus, $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = \phi < n-2$ violating the condition (1), where ϕ is the number of vertices of D whose in-degree and out-degree are both one. Hence D cannot be a directed tree. On the other hand, if D is a directed cycle with $n \geq 3$ vertices, then $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n > n-2$, again violating the condition (1). Hence D cannot be a directed cycle also. In the latter case, we prove as follows.

Case 1. Suppose that a connected digraph D with $n \geq 3$ vertices has exactly one vertex of total degree three, and remaining vertices of total degree at most two. We consider the following two subcases of Case 1.

Subcase 1. If D is a directed tree, then clearly it has three vertices of total degree one, and $(n-4)$ vertices of total degree two. Thus, $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) \leq \phi' < n-2$, where ϕ' is the number of vertices of D whose in-degree and out-degree are both at least one.

Subcase 2. If D is cyclic, then it has a vertex of total degree one, and $(n-2)$ vertices of total degree two. Thus, $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n > n - 2$.

Case 2. Finally, consider any connected digraph with n vertices having more than one vertex of total degree at least three. Clearly, such a digraph can be obtained by adding new arcs joining pairs of non-adjacent vertices of a digraph described in Case 1. The addition of new arcs increases the total degree of some vertices and there by the above inequality is preserved in this case also. Therefore in all cases, we arrive at a contradiction if we assume that D has some vertices of total degree at least three. Hence we conclude that D is a non-empty directed path. This completes the proof. □

Remark 2.1 It is known that a directed path is a special case of an arborescence. Hence equation (1) is satisfied for an arborescence whose root vertex has out-degree exactly one. For an example, see Fig.1, Fig.2. It is easy to verify that equation (1) is satisfied for an arborescence showed in Fig.1, but not in Fig.2.

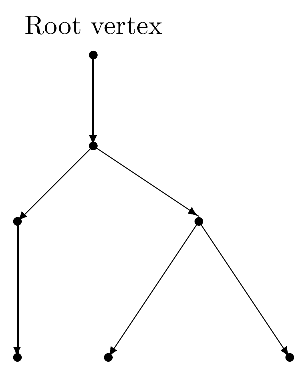

Fig.1

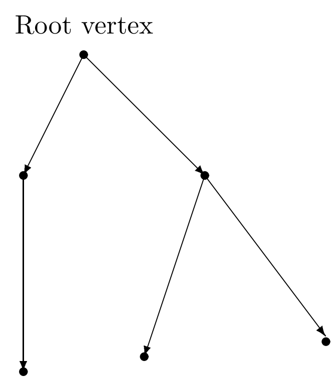

Fig.2

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I want to bring out the secrets of nature and apply them for the happiness of man. I dont know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

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