Theory of Abel Grassmann's Groupoids

Madad Khan ▌Florentin Smarandache ▌Saima Anis

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Preface

It is common knowledge that common models with their limited boundaries of truth and falsehood are not sufficient to detect the reality so there is a need to discover other systems which are able to address the daily life problems. In every branch of science problems arise which abound with uncertainties and impaction. Some of these problems are related to human life, some others are subjective while others are objective and classical methods are not sufficient to solve such problems because they can not handle various ambiguities involved. To overcome this problem, Zadeh [67] introduced the concept of a fuzzy set which provides a useful mathematical tool for describing the behavior of systems that are either too complex or are ill-defined to admit precise mathematical analysis by classical methods. The literature in fuzzy set and neutrosophic set theories is rapidly expanding and application of this concept can be seen in a variety of disciplines such as artificial intelligence, computer science, control engineering, expert systems, operating research, management science, and robotics.

Zadeh introduced the degree of membership of an element with respect to a set in 1965, Atanassov introduced the degree of non-membership in 1986, and Smarandache introduced the degree of indeterminacy (i.e. neither membership, nor non-membership) as independent component in 1995 and defined the neutrosophic set. In 2003 W. B. Vasantha Kandasamy and Florentin Smarandache introduced for the first time the I-neutrosophic algebraic structures (such as neutrosophic semigroup, neutrosophic ring, neutrosophic vector space, etc.) based on neutrosophic numbers of the form $a + bI$, where 'I' is the literal indeterminacy such that $I^2 = I$, while a, b are real (or complex) numbers. In 2013 Smarandache introduced the refined neutrosophic set, and in 2015 the refined neutrosophic algebraic structures built on sets on refined neutrosophic numbers of the form $a + b_1I_1 + b_2I_2 + \ldots + b_nI_n$, where I_1, I_2, . . . , I_n are types of sub-indeterminacies; in the same year he also introduced the (t, i, f)-neutrosophic structures.

In 1971, Rosenfeld [53] first applied fuzzy sets to the study of algebraic structures, and he initiated a novel notion called fuzzy groups. This pioneer work started a burst of studies on various fuzzy algebras. Kuroki [28] studied fuzzy bi-ideals in semigroups and he examined some fundamental properties of fuzzy semigroups in [28]. Mordesen [37] has demonstrated a theoretical exposition of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines and fuzzy languages. It is worth noting that these fuzzy structures may give rise to more useful models in some practical applications. The role of fuzzy theory in automata and formal languages has extensively been discussed by Mordesen [37].

Pu and Liu [49] initiated the concept of fuzzy points and they also proposed some inspiring ideas such as *belongingness to* (denoted by $\in$) and *quasi-coincidence* (denoted by q) of a fuzzy point with a fuzzy set. Murali [42] proposed the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on fuzzy subsets. These ideas played a vital role to generate various types of fuzzy subsets and fuzzy algebraic structures. Bhakat and Das [1, 2] applied these notions to introducing (α, β)-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Among (α, β)-fuzzy subgroups, it should be noted that the concept of $(\in, \in\vee q)$-fuzzy subgroups is of vital importance since it is the most viable generalization of the conventional fuzzy subgroups in Rosenfeld's sense. Then it is natural to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. In fact, many authors have studied $(\in, \in \vee q)$-fuzzy algebraic structures in different contexts [19, 22, 55]. Recently, Shabir et al. [55] introduced $(\in, \in \vee q_k)$-fuzzy ideals (quasi-ideals and bi-ideals) of semigroups and gave various characterizations of particular classes of semigroups in terms of these fuzzy ideals. M. Khan introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals in AG-groupoids

An AG-groupoid is an algebraic structure that lies in between a groupoid and a commutative semigroup. It has many characteristics similar to that of a commutative semigroup. If we consider $x^2y^2 = y^2x^2$, which holds for all x, y in a commutative semigroup, on the other hand one can easily see that it holds in an AG-groupoid with left identity e and in AG**-groupoids. In addition to this $xy = (yx)e$ holds for any subset $\{x, y\}$ of an AG-groupoid. This simply gives that how an AG-groupoid has closed connections with commutative algebras.

We extend now for the first time the AG-Groupoid to the Neutrosophic AG-Groupoid. A neutrosophic AG-groupoid is a neutrosophic algebraic structure that lies between a neutrosophic groupoid and a neutrosophic commutative semigroup.

Let M be an AG-groupoid under the law "." One has $(ab)c = (cb)a$ for all a, b, c in M. Then $MUI = \{a + bI$, where a, b are in M, and I is literal indeterminacy such that $I^2 = I\}$ is called a neutrosophic AG-groupoid. A neutrosophic AG-groupoid in general is not an AG-groupoid.

If on MUI one defines the operation "$*$" as: $(a+bI)*(c+dI) = ac+bdI$, then the neutrosophic AG-groupoid $(MUI, *)$ is also an AG-groupoid since:

$$\begin{aligned}
[(a_1 + b_1I) * (a_2 + b_2I)] * (a_3 + b_3I) &= [a_1a_2 + b_1b_2I] * (a_3 + b_3I) \\
&= (a_1a_2)a_3 + (b_1b_2)b_3I \\
&= (a_3a_2)a_1 + (b_3b_2)b_1I.
\end{aligned}$$

Also

$$\begin{aligned}[(a_3+b_3I)*(a_2+b_2I)]*(a_1+b_1I) &= [a_3a_2+b_3b_2I]*(a_1+b_1I) \\ &= (a_3a_2)a_1+(b_3b_2)b_1I.\end{aligned}$$

In chapter one we discuss congruences in an AG-groupoid. In this chapter we discuss idempotent separating congruence μ defined as: $a\mu b$ if and only if $(a^{-1}e)a=(b^{-1}e)b$, in an inverse AG**-groupoid S. We characterize μ in two ways and show (a) that $S/\mu \simeq E$, (E is the set of all idempotents of S) if and only if E is contained in the centre of S, also it is shown; (b) that μ is identical congruence on S if and only if E is self-centralizing. We show that the relations $\tau_{\min}$ and $\tau_{\max}$ show are smallest and largest congruences on S. Moreover we show that the relation ρ defined as: $a\rho b$ if only if $a^{-1}(ea)=b^{-1}(eb)$, is a maximum idempotent separating congruence.

In chapter two we discuss gamma ideals in Γ-AG**-groupoid. Moreover we show that a locally associative Γ-AG**-groupoid S has associative powers and S/ρ_Γ, where $a\rho_\Gamma b$ implies that $a\Gamma b_\Gamma^n=b_\Gamma^{n+1}$, $b\Gamma a_\Gamma^n=a_\Gamma^{n+1}\ \forall\, a,b\in S$, is a maximal separative homomorphic image of S. The relation η_Γ is the least left zero semilattice congruence on S, where η_Γ is define on S as $a\eta_\Gamma b$ if and only if there exists some positive integers m, n such that $b_\Gamma^m\in a\Gamma S$ and $a_\Gamma^n\in b\Gamma S$.

In chapter three we discuss embedding and direct products in AG-groupoids.

In chapter four we introduce the concept of left, right, bi, quasi, prime (quasi-prime) semiprime (quasi-semiprime) ideals in AG-groupoids. We introduce m system in AG-groupoids. We characterize quasi-prime and quasi-semiprime ideals and find their links with m systems. We characterize ideals in intra-regular AG-groupoids. Then we characterize intra-regular AG-groupoids using the properties of these ideals.

In chapter five we introduce a new class of AG-groupoids namely strongly regular and characterize it using its ideals.

In chapter six we introduce the fuzzy ideals in AG-groupoids and discuss their related properties.

In chapter seven we characterize intra-regular AG-groupoids by the properties of the lower part of $(\in,\in\vee q)$-fuzzy bi-ideals. Moreover we characterize AG-groupoids using $(\in,\in\vee q_k)$-fuzzy.

In chapter eight we discuss interval valued fuzzy ideals of AG-groupoids.

In chapter nine we characterize a Abel-Grassmann's groupoid in terms of its $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideals.

In chapter ten we characterize intra-regular AG-groupoids in terms of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft ideals.

1

Congruences on Inverse AG-groupoids

In this chapter we discuss idempotent separating congruence μ defined as: $a\mu b$ if and only if $(a^{-1}e)a = (b^{-1}e)b$, in an inverse AG**-groupoid S. We characterize μ in two ways and show (a) that $S/\mu \simeq E$, (E is the set of all idempotents of S) if and only if E is contained in the centre of S, also it is shown; (b) that μ is identical congruence on S if and only if E is self-centralizing. We show that the relations $\tau_{\min}$ and $\tau_{\max}$ are smallest and largest congruences on S. Also we show that the relation ρ defined as: $a\rho b$ if only if $a^{-1}(ea) = b^{-1}(eb)$, is a maximum idempotent separating congruence.

1.1 AG-groupoids

The idea of generalization of a commutative semigroup was first introduced by Kazim and Naseeruddin in 1972 (see [24]). They named it as a left almost semigroup (LA-semigroup). It is also called an Abel-Grassmann's groupoid (AG-groupoid) [47].

An AG-groupoid is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related with a commutative semigroup, because if an AG-groupoid contains a right identity, then it becomes a commutative semigroup [43]. The connection of a commutative inverse semigroup with an AG-groupoid has been given in [39] as: a commutative inverse semigroup $(S, \circ)$ becomes an AG-groupoid $(S, \cdot)$ under $a \cdot b = b \circ a^{-1}$, for all $a, b \in S$. An AG-groupoid $(S, .)$ with left identity becomes a semigroup $(S, \circ)$ defined as: for all x, $y \in S$, there exists $a \in S$ such that $x \circ y = (xa)y$ [47].

An AG-groupoid is a groupoid S whose elements satisfy the left invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. In an AG-groupoid, the medial law [24] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in S$. An AG-groupoid may or may not contains a left identity. If an AG-groupoid contains a left identity, then it is unique [43]. In an AG-groupoid S with left identity, the paramedial law $(ab)(cd) = (db)(ca)$ holds for all $a, b, c, d \in S$. If an AG-groupoid contains a left identity, then it satisfies the following law

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

Note that a commutative AG-groupoid S with left identity becomes a commutative semigroup because if a, b and $c \in S$. Then using left invertive law and commutative law, we get

$$(ab)c = (cb)a = a(cb) = a(bc).$$

In [15] J. M. Howie defined a relation μ as $(a, b) \in \mu$ if and only if $a^{-1}ea = b^{-1}eb$ on an inverse semigroup and show it maximum idempotent separating congruence and characterize it in two ways. Also it is shown that $S/\mu \simeq E$ if and only if E is central in S and that $\mu = 1_S$, the identical congruence on S, if and only if E is self centralizing in S. Moreover, J. M. Howie in [14] defined a relations $\tau_{\min}$ and $\tau_{\max}$ as $a\tau_{\min}b$ if and only if $aa^{-1}\tau bb^{-1}$ and $\exists e \in E$ such that $e\tau aa^{-1}$ $ea = eb$ and $a\tau_{\max}b$ if and only if $a^{-1}ea\tau b^{-1}eb$ for all $e \in E$ and shown these as the smallest and largest congruences on an inverse semigroup with trace τ. In this chapter, we defined these congruences for inverse AG**-groupoid and also characterize it. An AG-groupoid S is called an inverse AG-groupoid if for every $a \in S$ there exists $a' \in S$ such that $(aa')a = a$, $(a'a)a' = a'$ where a' is an inverse for a. We will write a^{-1} instead of a'. If S is an inverse AG-groupoid, then $(ab)^{-1} = a^{-1}b^{-1}$ and $(a^{-1})^{-1} = a$ for all $a, b \in S$.

Example 1 *Let $S = \{1, 2, 3\}$ and the binary operation "$\cdot$" defined on S as follows:*

$\cdot$	1	2	3
1	2	2	3
2	2	2	2
3	2	2	2

Clearly S is non-associative and non-commutative because $2 = (1 \cdot 1) \cdot 3 \neq 1 \cdot (1 \cdot 3) = 3$ and $1 \cdot 3 \neq 3 \cdot 1$. $(S, \cdot)$ is an AG**-groupoid without left identity.

Lemma 2 *Let S be an inverse AG**-groupoid and δ defined by $a \ \delta \ b$, if and only if $aa^{-1} = bb^{-1}$, is a congruence relation.*

Proof. Clearly δ is reflexive, symmetric and transitive, so δ is an equivalence relation. Let $a \ \delta \ b$ which implies that $aa^{-1} = bb^{-1}$, then we get.

$$\begin{aligned} (ac)(ac)^{-1} &= (ac)(a^{-1}c^{-1}) = (aa^{-1})(cc^{-1}) = (bb^{-1})(cc^{-1}) \\ &= (bc)(b^{-1}c^{-1}) = (bc)(bc)^{-1}. \end{aligned}$$

Similarly we can show that $(ca)(ca)^{-1} = (cb)(cb)^{-1}$. ■

Lemma 3 *Let S be an inverse AG**-groupoid, then the relation $\mu = \{(a, b) \in S \times S : a^{-1}a = b^{-1}b\}$ is a congruence on S.*

Proof. It is available in [47]. ■

Lemma 4 *The congruence relation δ is equivalent to μ.*

Proof. Let $a\mu b$, this implies that $a^{-1}a = b^{-1}b$. Then we have

$$\begin{aligned} aa^{-1} &= ((aa^{-1})a)a^{-1} = (a^{-1}a)(aa^{-1}) = (b^{-1}b)(aa^{-1}) \\ &= (a^{-1}a)(bb^{-1}) = (b^{-1}b)(bb^{-1}) \\ &= ((bb^{-1})b)b^{-1} = bb^{-1}. \end{aligned}$$

Thus $a\delta b$.

Conversely, If $a\delta b$, then $aa^{-1} = bb^{-1}$. Then

$$\begin{aligned} a^{-1}a &= a^{-1}((aa^{-1})a) = (aa^{-1})(a^{-1}a) = (bb^{-1})(a^{-1}a) \\ &= (aa^{-1})(b^{-1}b) = (bb^{-1})(b^{-1}b) \\ &= ((b^{-1}b)b^{-1})b = b^{-1}b. \end{aligned}$$

Hence $a\mu b$. ■

Corollary 5 *If μ is congruence on an inverse AG^{**}-groupoid, then $(a,b) \in \mu$, if and only if $(a^{-1}, b^{-1}) \in \mu$.*

Proof. It is same as in [15]. ■

Example 6 *Let $S = \{1,2,3,4\}$ and the binary operation "$\cdot$" defined on S as follows:*

$\cdot$	1	2	3	4
1	4	1	2	3
2	3	4	1	2
3	2	3	4	1
4	1	2	3	4

Clearly $(S, \cdot)$ is non-associative, non-commutative and it is an AG^{**}-groupoid with left identity 4. Every element is an inverse of itself and so $a^{-1}a = aa^{-1}$, for all a in S.

The following lemma is available in [47].

Lemma 7 *The set E of all idempotents in an AG^{**}-groupoid forms a semilattice structure.*

1.2 Inverse AG**-groupoids

In the rest, by S we shall mean an inverse AG**-groupoid in which $aa^{-1} = a^{-1}a$, holds for every $a \in S$.

Let ρ be a congruence on S. The restriction of ρ to E, is congruence on E, which we call trace of ρ and is denoted by $\tau = tr\rho$. The set $ker\rho = \{a \in S/(\exists e \in E)a\rho e\}$ is the kernel of ρ.

Theorem 8 *Let E be the set of all idempotents of S and let τ be a congruence on E, then the relation $\tau_{\min}= \{(a,b) \in S \times S : aa^{-1}\tau bb^{-1}$ and there exist $e \in E, e\ \tau\ aa^{-1}$ and $ea = eb\}$ is the smallest congruence on S with trace τ.*

Proof. Clearly τ is reflexive. Now let $a\tau_{\min}b$, this implies that $aa^{-1}\tau bb^{-1}$ and there exist $e \in E$ such that $e\tau aa^{-1}$ and $ea = eb$. As $e\ \tau\ aa^{-1}$ and $aa^{-1}\tau bb^{-1}$ which implies that $e\tau bb^{-1}$ also $eb = ea$ which implies that $b\tau_{\min}a$, which shows that $\tau_{\min}$ is symmetric. Again let $a\tau_{\min}b$ and $b\tau_{\min}c$ which implies that $aa^{-1}\tau bb^{-1}\tau cc^{-1}$ this implies that $aa^{-1}\tau\ cc^{-1}$. Also $e\tau aa^{-1}$ and $f\tau bb^{-1}$ for $e, f \in E$. Since τ is compatible so, $ef\tau(aa^{-1})(aa^{-1}) = aa^{-1}$which implies that $ef\tau aa^{-1}$. Now $ea = eb$ implies that$f(ea) = f(eb)$ so we have

$$\begin{aligned} f(ea) &= (ff)(ea) = (ae)(ff) = (ae)f = (fe)a, \text{ and} \\ f(eb) &= (ff)(eb) = (be)(ff) = (be)f = (fe)b \end{aligned}$$

Also $fb = fc$ implies that $e(fb) = e(fc)$. Now

$$\begin{aligned} e(fb) &= (ee)(fb) = (bf)(ee) = (bf)e = (ef)b = (fe)b \\ e(fc) &= (ee)(fc) = (cf)(ee) = (cf)e = (ef)c = (fe)c \end{aligned}$$

Hence $(fe)a = (fe)c$ which shows that $\tau_{\min}$ is transitive.
Now let $a\tau_{\min}b$, then

$$\begin{aligned} (ca)(ca)^{-1} &= (ca)(c^{-1}a^{-1}) = (cc^{-1})(aa^{-1})\tau(cc^{-1})(bb^{-1}) \\ &= (cb)(c^{-1}b^{-1}) = (cb)(cb)^{-1}, \text{ and} \end{aligned}$$

$$\begin{aligned} (cc^{-1})e\tau(cc^{-1})(aa^{-1}) &= (ca)(c^{-1}a^{-1}) \\ &= (ca)(ca)^{-1}, \text{ where } (cc^{-1})e \in E, \text{ and} \end{aligned}$$

$$\begin{aligned} ((cc^{-1})e)(ca) &= ((cc^{-1})c)(ea) \\ &= ((cc^{-1})c)(eb) = ((cc^{-1})e)(cb). \end{aligned}$$

Therefore $ca\tau_{\min}ca$.
Again let $a\tau_{\min}b$ then by definition $aa^{-1}\tau bb^{-1}, e\tau aa^{-1}$ and $ea = eb$
Now

$$\begin{aligned} (ac)(ac)^{-1} &= (ac)(a^{-1}c^{-1}) \\ &= (aa^{-1})(cc^{-1})\tau(bb^{-1})(cc^{-1}) = (bc)(bc)^{-1} \text{ and} \\ e(cc^{-1})\tau(aa^{-1})(cc^{-1}) &= (ac)(a^{-1}c^{-1}) \\ &= (ac)(ac)^{-1}, \text{ where } e(cc^{-1}) \in E \end{aligned}$$

Also

$$(e(cc^{-1}))(ac) = (ea)((cc^{-1})c) = (eb)((cc^{-1})c) = (e(cc^{-1}))(bc).$$

Thus $ac\tau_{\min}bc$. Therefore $\tau_{\min}$ is a congruence relation.

The remaining proof is same as in [14]. ■

Theorem 9 *Let E be the set of all idempotents of S and let τ be a congruence on E, then the relation $\tau_{\max} = \{(a,b) \in S \times S : (\forall e \in E)\ a^{-1}(ea)\tau b^{-1}(eb)\}$ is the largest congruence on S with trace τ.*

Proof. Clearly $\tau_{\max}$ is an equivalence relation as τ is an equivalence relation on E.

Let us suppose that $a\tau_{\max}b$, then $a^{-1}(ea)\tau b^{-1}(eb)$ so

$$\begin{aligned}(ac)^{-1}(e(ac)) &= (a^{-1}c^{-1})((ee)(ac)) = (a^{-1}c^{-1})((ea)(ec)) \\ &= (a^{-1}(ea))(c^{-1}(ea))\tau(b^{-1}(eb))(c^{-1}(ec)) \\ &= (b^{-1}c^{-1})((eb)(ec)) = (bc)^{-1}(e(bc)).\end{aligned}$$

Thus $ac\tau_{\max}bc$. Similarly $ca\tau_{\max}cb$. Therefore $\tau_{\max}$ is congruence on S.

Remaining proof is same as in [14]. ■

The relation $1_s = \{(x,x) : x \in S\}$ is a congruence relation which we call the identical congruence. A congruence whose trace is the identical congruence 1 is called idempotent separating.

Theorem 10 *Let E be the set of all idempotents of S and let the relation μ defined as $a\mu b$ if and only if $(a^{-1}e)a = (b^{-1}e)b$, for any e in E, is an idempotent separating congruence on S.*

Proof. It is easy to prove that μ is an equivalence relation. Now let $a\mu b$, then $(a^{-1}e)a = (b^{-1}e)b$, for every idempotent e in E, now we get

$$\begin{aligned}((ac)^{-1}e)(ac) &= ((a^{-1}c^{-1})(ee))(ac) = ((a^{-1}e)(c^{-1}e))(ac) \\ &= ((a^{-1}e)a)((c^{-1}e)c) = ((b^{-1}e)b)(c^{-1}e)c \\ &= ((b^{-1}e)(c^{-1}e))(bc) = ((bc)^{-1}e)(bc).\end{aligned}$$

Thus $ac\mu bc$. Similarly $ca\mu cb$. Hence μ is a congruence relation on S.

Now let $e\mu f$ for e, f in E. Then for every g in E, $(e^{-1}g)e = (f^{-1}g)f$ so by (1), we have $eg = fg$. The equality holds in particular when $g = e$. Hence $e = fg$. Similarly for $g = f$, we obtain $ef = f$. Since $ef = fe$, so $e = f$. Thus μ is idempotent separating. ■

If E is the semilattice of an inverse semigroup S, we define $E\zeta$, the centralizer of E in S, by

$$E\zeta = \{z \in S : ez = ze \text{ for every } e \text{ in } E\}.$$

Clearly $E \subseteq E\zeta$ If $E\zeta = S$, then the idempotents are central. If $E\zeta = E$, we shall say that E is self-centralizing.

Theorem 11 *Let E be the set of all idempotents of S and let μ be the idempotent separating congruence on S. Then $Ker\mu = E\zeta$ where $E\zeta$ be the centralizer of E in S.*

Proof. Let S be an inverse AG**-groupoid and let μ be the idempotent separating congruence on S. Let $a \in Ker\mu$, so $a\mu f$ for some $f \in E$. also $a^{-1}\mu f^{-1} = f$, so $a^{-1}a\mu f$, implies that $a\mu aa^{-1}$. So for all e in E $(a^{-1}e)a = ((a^{-1}a)^{-1}e)(a^{-1}a)$, then we get

$$\begin{aligned} ((a^{-1}a)^{-1}e)(a^{-1}a) &= ((aa^{-1})e)(a^{-1}a) \\ &= (e(a^{-1}a))(a^{-1}a) = (a^{-1}a)e, \text{ that is} \end{aligned}$$

$$(a^{-1}e)a = (a^{-1}a)e. \tag{5}$$

Also we have,

$$\begin{aligned} ea &= e((aa^{-1})a) = (aa^{-1})(ea) = ((ea)a^{-1})a \\ &= ((a^{-1}a)e)a = ((a^{-1}e)a)a = (aa)(a^{-1}e) \\ &= (ea^{-1})(aa) = ((aa)a^{-1})e = ((aa^{-1})a)e = ae. \end{aligned}$$

Thus $a \in E\zeta$.

Conversely, assume that $a \in E\zeta$. Then for all e in E, $ae = ea$, so

$$\begin{aligned} (a^{-1}e)a &= (ae)a^{-1} = (ea)a^{-1} = (ea)((a^{-1}a)a^{-1}) \\ &= (a^{-1}a)((ea)a^{-1}) = (a^{-1}a)((a^{-1}a)e) \\ &= (e(a^{-1}a))(aa^{-1}) = ((aa^{-1})^{-1}e)(aa^{-1}). \end{aligned}$$

Thus $a\mu aa^{-1}$ and so $a \in Ker\mu$. Hence $E\zeta = Ker\mu$. ■

Theorem 12 *Let E be the set of all idempotents of S and let μ be the idempotent separating congruence on S. Then $(a, b) \in \mu$ if and only if $a^{-1}a = b^{-1}b$, and $ab^{-1} \in E\zeta$. Dually $(a, b) \in \mu$ if and only if $aa^{-1} = bb^{-1}$ and $a^{-1}b \in E\zeta$.*

Proof. Let $(a, b) \in \mu$, then $(a^{-1}e)a = (b^{-1}e)b$ which implies that $(ae)a^{-1} = (be)b^{-1}$ for all e in E.

Now $((a^{-1}e)a)((ae)a^{-1}) = ((b^{-1}e)b)((be)b^{-1})$ which implies that

$$((a^{-1}a)e)(aa^{-1}) = ((b^{-1}b)e)(bb^{-1}). \tag{6}$$

Therefore we get

$$\begin{aligned} a^{-1}a &= ((a^{-1}a)a^{-1})((aa^{-1})a) = ((a^{-1}a)(aa^{-1}))(a^{-1}a) \\ &= ((a^{-1}a)(aa^{-1}))(aa^{-1}) = ((b^{-1}b)(aa^{-1}))(bb^{-1}) \\ &= ((aa^{-1})(b^{-1}b))(bb^{-1}) = ((bb^{-1})(b^{-1}b))(aa^{-1}) \\ &= ((b^{-1}b)(b^{-1}b))(aa^{-1}) = (b^{-1}b)(aa^{-1}) = (a^{-1}a)(b^{-1}b). \end{aligned}$$

Similarly we can show that $b^{-1}b = (a^{-1}a)(b^{-1}b)$. Therefore $a^{-1}a = b^{-1}b$.

Now let $(a, b) \in \mu$, then $(a^{-1}e)a = (b^{-1}e)b$ which implies that $(ae)a^{-1} = (be)b^{-1}$, which implies that $(a((ae)a^{-1}))b^{-1} = (a((be)b^{-1}))b^{-1}$.

Now we obtain

$$\begin{aligned}(a((ae)a^{-1}))b^{-1} &= ((ae)(aa^{-1}))b^{-1} = (b^{-1}(aa^{-1}))(ae) \\ &= (a(b^{-1}a^{-1}))(ae) = (ea)((b^{-1}a^{-1})a) \\ &= (ea)((aa^{-1})b^{-1}) = (e(aa^{-1}))(ab^{-1}), \text{ and}\end{aligned}$$

$$\begin{aligned}(a((be)b^{-1}))b^{-1} &= ((be)(ab^{-1}))b^{-1} = (b^{-1}(ab^{-1}))(be) \\ &= (b^{-1}b)((ab^{-1})e) = (aa^{-1})((ab^{-1})e) \\ &= (ab^{-1})((aa^{-1})e) = (ab^{-1})(e(aa^{-1})).\end{aligned}$$

Hence $ab^{-1} \in E\zeta$.

Conversely, let $a^{-1}a = b^{-1}b$ and $ab^{-1} \in E\zeta$, then $e(ab^{-1}) = (ab^{-1})e$ for all $e \in E$, which implies that $(a^{-1}(e(ab^{-1})))b = (a^{-1}((ab^{-1})e))b$. Now we get

$$\begin{aligned}(a^{-1}(e(ab^{-1})))b &= (b(e(ab^{-1})))a^{-1} = (e(b(ab^{-1})))a^{-1} \\ &= (e(a(bb^{-1})))a^{-1} = ((ee)(a(aa^{-1})))a^{-1} \\ &= (((aa^{-1})a)(ee))a^{-1} = (ae)a^{-1} = (a^{-1}e)a,\end{aligned}$$

Now

$$\begin{aligned}(a^{-1}((ab^{-1})e))b &= (a^{-1}((ab^{-1})(ee)))b = (a^{-1}((ae)(b^{-1}e)))b \\ &= ((ae)(a^{-1}(b^{-1}e)))b = (((b^{-1}e)a^{-1})(ea))b \\ &= (((b^{-1}e)e)(a^{-1}a))b = ((eb^{-1})(a^{-1}a))b \\ &= ((eb^{-1})(b^{-1}b))b = (b(b^{-1}b))(eb^{-1}) \\ &= (b^{-1}e)((b^{-1}b)b) = (b^{-1}e)((bb^{-1})b) \\ &= (b^{-1}e)b.\end{aligned}$$

Therefore $(a^{-1}e)a = (b^{-1}e)b$. Hence $a\mu b$.

Let $a\mu b$ then by definition $\left(a^{-1}e\right)a = \left(b^{-1}e\right)b$. Now as $a^{-1}a = b^{-1}b$ so $aa^{-1} = bb^{-1}$.

Now as $\left(a^{-1}e\right)a = \left(b^{-1}e\right)b$ which implies that $(a^{-1}\left(\left(a^{-1}e\right)a\right))b = (a^{-1}(\left(b^{-1}e\right)b))b$.

So we get

$$\begin{aligned}(a^{-1}\left(\left(a^{-1}e\right)a\right))b &= \left(\left(a^{-1}e\right)\left(a^{-1}a\right)\right)b = \left(\left(a^{-1}e\right)\left(b^{-1}b\right)\right)b \\ &= \left(b\left(b^{-1}b\right)\right)\left(a^{-1}e\right) = \left(ea^{-1}\right)\left(\left(bb^{-1}\right)b\right) \\ &= \left(ea^{-1}\right)b = \left(ba^{-1}\right)e = \left(ba^{-1}\right)(ee) \\ &= (ee)\left(a^{-1}b\right) = e\left(a^{-1}b\right), \text{ and}\end{aligned}$$

Now we get

$$\begin{aligned}(a^{-1}(\left(b^{-1}e\right)b))b &= (b(\left(b^{-1}e\right)b))a^{-1} = (\left(b^{-1}e\right)(bb))a^{-1} \\ &= (\left(b^{-1}b\right)(eb))a^{-1} = (e\left(\left(bb^{-1}\right)b\right))a^{-1} \\ &= (eb)a^{-1} = (a^{-1}b)e.\end{aligned}$$

Hence $a^{-1}b \in E\zeta$. ■

Theorem 13 *Let E be the set of all idempotents of S and let μ be the idempotent separating congruence on S. Then $S/\mu \simeq E$ if and only if E is central in S.*

Proof. Since μ is idempotent separating congruence so S/μ is a semilattice if each -class contains atmost one idempotent. Thus if S/μ is semilattice then $S/\mu = E$. Let us suppose that each μ class contains an idempotent that is for every $x \in S$, there exist an $f \in E$ such that $f\mu x$ which implies that $ff^{-1} = xx^{-1}$ and $f^{-1}x \in E\zeta$, thus

$$x = (xx^{-1})x = (ff^{-1})x = f^{-1}x \in E\zeta,$$

but this holds for any x in S, so $E\zeta = S$.

Conversely, suppose that $E\zeta = S$, then $xf^{-1} \in S = E\zeta$ and

$$xx^{-1} = (xx^{-1})(xx^{-1}) = (xx^{-1})(xx^{-1})^{-1} = ff^{-1},$$

Then by theorem 5, $x\mu f$, that is, $x\mu xx^{-1}$, which shows that every μ class contains an idempotent. ■

Theorem 14 *Let E be the set of all idempotents of S and let μ be the idempotent separating congruence on S. Then $\mu = 1_S$, the identical congruence on S, if and only if E is self centralizing in S.*

Proof. Let $\mu = 1_S$, Then for $z \in E\zeta$ implies that $ze = ez$, for all $e \in E$ if we write f for zz^{-1} then $zz^{-1} = f = ff = ff^{-1}$ also we get

$$\left(zf^{-1}\right)e = \left(ef^{-1}\right)z = (ef)\,z = z(ef) = z\left(ef^{-1}\right) = e\left(zf^{-1}\right).$$

Therefore $zf^{-1} \in E\zeta$. Then by theorem 5, $z\mu zz^{-1}$, but $\mu = 1_S$, so $z = zz^{-1} \in E$. Thus $E\zeta = E$.

Conversely, assume that $E\zeta = E$. Let $x\mu y$ then $x^{-1}x = y^{-1}y$ and $xy^{-1} \in E\zeta = E$, since xy^{-1} is idempotent so $(xy^{-1})^{-1} = xy^{-1}$, implies that $x^{-1}y = xy^{-1}$, also $(x^{-1}, y^{-1}) \in \mu$ so

$$\begin{aligned} xx^{-1} &= ((xx^{-1})x)x^{-1} = ((yy^{-1})x)x^{-1} \\ &= ((xy^{-1})y)x^{-1} = ((x^{-1}y)y)x^{-1} \\ &= (x^{-1}y)(x^{-1}y) = x^{-1}y. \end{aligned}$$

Also we get

$$\begin{aligned} x &= (xx^{-1})x = (yy^{-1})x = (xy^{-1})y \\ &= (x^{-1}y)y = (xx^{-1})y = (yy^{-1})y = y. \end{aligned}$$

Hence $\mu = 1_S$. ■

Theorem 15 *Let E be the set of all idempotents of S then the relation defined by $a\rho b$ if only if $a^{-1}(ea) = b^{-1}(eb)$ is a maximum idempotent separating congruence on S.*

Proof. Clearly ρ is an equivalence relation. Let $a\rho b$, which implies that $a^{-1}(ea) = b^{-1}(eb)$. Now

$$\begin{aligned}(ac)^{-1}(e(ac)) &= (a^{-1}c^{-1})(e(ac)) = (a^{-1}c^{-1})((ee)(ac))\\ &= (a^{-1}c^{-1})((ea)(ec)) = (a^{-1}(ea))(c^{-1}(ec))\\ &= (b^{-1}(eb))(c^{-1}(ec)) = (b^{-1}c^{-1})((eb)(ec))\\ &= (bc)^{-1}(e(bc)).\end{aligned}$$

Therefore $ac\rho bc$. Similarly $ca\rho cb$. Hence ρ is a congruence relation. Now suppose that $e\rho f$, where e, $f \in E$, then for every idempotent g we have $e^{-1}(ge) = f^{-1}(gf)$, which implies that $ge = gf$. In particular when $g = e$, then $ee = ef$, implies that $e = ef$ and for $g = f$, $fe = ff$ implies that $fe = f$, but since $ef = fe$ implies that $e = f$. Thus ρ is idempotent separating congruence. Now let η be any other idempotent separating congruence. We shall show that $\eta \subseteq \rho$. Let $(x, y) \in \eta$ then $(x^{-1}, y^{-1}) \in \eta$, since η is congruence, it follows that $x\eta y$ which implies that $ex\eta ey$, also $x^{-1}(ex)\eta y^{-1}(ey)$, but both $x^{-1}(ex)$ and $y^{-1}(ey)$ are idempotents, and so it follows that $x^{-1}(ex) = y^{-1}(ey)$. Thus $x\rho y$. Hence ρ is maximum. ■

Theorem 16 *Let E be the set of all idempotents of S then the relation defined on S with $\sigma = \{(a, b) \in S\times S\ (\forall e \in E) : ((a^{-1})^2e)a^2 = ((b^{-1})^2e)b^2\}$ is a congruence relation on S.*

Proof. It is clear that σ is an equivalence relation. Now suppose that $a\sigma b$ and c is an arbitrary element of S, then

$$\begin{aligned}(((ac)^{-1})^2e)\,(ac)^2 &= ((a^{-1}c^{-1})^2e)\,(ac)^2\\ &= \left(\left((a^{-1})^2(c^{-1})^2\right)e\right)(a^2c^2)\\ &= (((a^{-1})^2e)(c^{-1})^2)(a^2c^2)\\ &= (((a^{-1})^2e)a^2)((c^{-1})^2c^2)\\ &= (((b^{-1})^2e)b^2)((c^{-1})^2c^2)\\ &= (((b^{-1})^2e)(c^{-1})^2)(b^2c^2)\\ &= ((b^{-1})^2(c^{-1})^2e)(b^2c^2)\\ &= ((b^{-1}c^{-1})^2e)\,(bc)^2\\ &= (((bc)^{-1})^2e)\,(bc)^2\,.\end{aligned}$$

Thus $(ac, bc) \in \sigma$. Similarly $(ca, cb) \in \sigma$. Hence σ is congruence relation. ■

Lemma 17 *Let E be the set of all idempotents of S then the centralizer $E\zeta$ of E in S, is an inverse subgroupoid of S.*

Proof. Let $a, b \in E\zeta$, then $ae = ea$ and $be = eb$, for all $e \in E$, so

$$(ab)\,e = (ab)\,(ee) = (ae)\,(be) = (ea)\,(eb) = (ee)\,(ab) = e\,(ab)\,.$$

Therefore $E\zeta$ is a subgroupoid of S.

Now let $a \in E\zeta$ then $ae = ea$ implies that $(ae)^{-1} = (ea)^{-1}$ or $a^{-1}e = ea^{-1}$, so $a^{-1} \in E\zeta$. Hence $E\zeta$ is an inverse subgroupoid. ■

Theorem 18 *Let S be an inverse AG^{**}-groupoid with semilattice E and let ρ be the maximum idempotent separating congruence on S then S/ρ is fundamental.*

Proof. Every idempotent in S/ρ has the form $e\rho$. Let us suppose that $(a\rho, b\rho) \in \rho_{S/\rho}$ then for every e in E $(a\rho)^{-1}\,((e\rho)\,(a\rho)) = (b\rho)^{-1}\,((e\rho)\,(b\rho))$ which implies that $\left(a^{-1}\,(ea)\right)\rho = \left(b^{-1}\,(eb)\right)\rho$, consequently$a^{-1}\,(ea)\,\rho b^{-1}\,(eb)$ but ρ is idempotent separating so $a^{-1}\,(ea) = b^{-1}\,(eb)$ that is $a\rho b$ implies that $a\rho = b\rho$ so $\rho_{S/\rho}$ is identical. Thus S/ρ is fundamental. ■

2

Structural Properties of Γ-AG**-groupoids

In this chapter we discuss gamma ideals in Γ-AG**-groupoids. We show that a locally associative Γ-AG**-groupoid S has associative powers and S/ρ_Γ, where $a\rho_\Gamma b$ implies that $a\Gamma b_\Gamma^n = b_\Gamma^{n+1}$, $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$ $\forall$ $a, b \in S$, is a maximal separative homomorphic image of S. The relation η_Γ is the least left zero semilattice congruence on S, where η_Γ is define on S as $a\eta_\Gamma b$ if and only if there exists some positive integers m, n such that $b_\Gamma^m \in a\Gamma S$ and $a_\Gamma^n \in b\Gamma S$.

2.1 Gamma Ideals in Γ-AG-groupoids

Let S and Γ be any non-empty sets. If there exists a mapping $S\times\Gamma\times S \rightarrow S$ written as (x, α, y) by $x\alpha y$, then S is called a Γ-AG-groupoid if $x\alpha y \in S$ such that the following Γ-left invertive law holds for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$

$$(x\alpha y)\beta z = (z\alpha y)\beta x. \tag{1}$$

A Γ-AG-groupoid also satisfies the Γ-medial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (w\alpha y)\beta(x\gamma z). \tag{2}$$

Note that if a Γ-AG-groupoid contains a left identity, then it becomes an AG-groupoid with left identity.

A Γ-AG-groupoid is called a Γ-AG**-groupoid if it satisfies the following law for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$

$$x\alpha(y\beta z) = y\alpha(x\beta z). \tag{3}$$

A Γ-AG**-groupoid also satisfies the Γ-paramedial law for all $w, x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$

$$(w\alpha x)\beta(y\gamma z) = (z\alpha y)\beta(x\gamma w). \tag{4}$$

Definition 19 *Let S be a Γ-AG-groupoid, a non-empty subset A of S is called Γ-AG-subgroupoid if $a\gamma b \in A$ for all a, $b \in A$ and $\gamma \in \Gamma$ or A is called Γ-AG-subgroupoid if $A\Gamma A \subseteq A$.*

Definition 20 *A subset A of a Γ-AG-groupoid S is called Γ-left (right) ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and A is called Γ-two-sided ideal of S if it is both Γ-left and Γ-right ideal.*

Definition 21 *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-bi-ideal of S if $(A\Gamma S)\,\Gamma A \subseteq A$.*

Definition 22 *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-interior ideal of S if $(S\Gamma A)\,\Gamma S \subseteq A$.*

Definition 23 *A Γ-AG-groupoid A of a Γ-AG-groupoid S is called a Γ-quasi-ideal of S if $S\Gamma A \cap A\Gamma S \subseteq A$.*

Definition 24 *A Γ-AG-subgroupoid A of a Γ-AG-groupoid S is called a Γ-$(1,2)$-ideal of S if $(A\Gamma S)\,\Gamma(A\Gamma A) \subseteq A$.*

Definition 25 *A Γ-two-sided ideal P of a Γ-AG-groupoid S is called Γ-prime (Γ-semiprime) if for any Γ-two-sided ideals A and B of S, $A\Gamma B \subseteq P\,(A\Gamma A \subseteq P)$ implies either $A \subseteq P$ or $B \subseteq P\,(A \subseteq P)$.*

Definition 26 *An element a of an Γ-AG-groupoid S is called an intra-regular if there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$ and S is called an intra-regular Γ-AG-groupoid S, if every element of S is an intra-regular.*

Example 27 *Let $S = \{1,2,3,4,5,6,7,8,9\}$. The following multiplication table shows that S is an AG-groupoid and also an AG-band.*

$\cdot$	1	2	3	4	5	6	7	8	9
1	1	4	7	3	6	8	2	9	5
2	9	2	5	7	1	4	8	6	3
3	6	8	3	5	9	2	4	1	7
4	5	9	2	4	7	1	6	3	8
5	3	6	8	2	5	9	1	7	4
6	7	1	4	8	3	6	9	5	2
7	8	3	6	9	2	5	7	4	1
8	2	5	9	1	4	7	3	8	6
9	4	7	1	6	8	3	5	2	9

It is easy to observe that S is a simple AG-groupoid that is there is no left or right ideal of S. Now let $\Gamma = \{\alpha,\ \beta,\ \gamma\}$ defined as follows.

α	1	2	3	4	5	6	7	8	9
1	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2	2	2
6	2	2	2	2	2	2	2	2	2
7	2	2	2	2	2	2	2	2	2
8	2	2	2	2	2	2	2	2	2
9	2	2	2	2	2	2	2	2	2

β	1	2	3	4	5	6	7	8	9
1	8	8	8	8	8	8	8	8	8
2	8	8	8	8	8	8	8	8	8
3	8	8	8	8	8	8	8	8	8
4	8	8	8	8	8	8	8	8	8
5	8	8	8	8	8	8	8	8	8
6	8	8	8	8	8	8	8	8	8
7	8	8	8	8	8	8	8	8	8
8	8	8	8	8	8	8	8	8	8
9	8	8	8	8	8	8	8	8	8

γ	1	2	3	4	5	6	7	8	9
1	8	8	8	8	8	8	8	8	8
2	8	8	8	8	8	8	8	8	8
3	8	8	8	8	8	8	8	8	8
4	8	8	8	8	8	8	8	8	8
5	8	8	8	8	8	8	8	8	8
6	8	8	8	8	8	8	8	8	8
7	8	8	8	8	8	8	8	8	8
8	8	8	8	8	8	8	8	8	8
9	8	8	8	8	8	8	8	9	9

It is easy to prove that S is a Γ-AG-groupoid because $(a\pi b)\,\psi c = (c\pi b)\,\psi a$ for all $a,\ b,\ c \in S$ and $\pi,\ \psi \in \Gamma$. Clearly S is non-commutative and non-associative because $8\gamma 9 \neq 9\gamma 8$ and $(1\alpha 2)\,\beta 3 \neq 1\alpha\,(2\beta 3)$.

Example 28 *Let $S = \{1, 2, 3, \}$. The following Cayley's table shows that S is an AG-groupoid.*

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

Let us define $\Gamma = \{\alpha,\ \beta,\ \gamma\}$ as follows.

α	1	2	3
1	1	1	1
2	1	1	1
3	1	1	1

β	1	2	3
1	2	2	2
2	2	2	2
3	2	2	3

γ	1	2	3
1	1	1	1
2	1	1	1
3	1	1	3

Clearly S is an intra-regular Γ-AG-groupoid because $1 = (2\beta(1\alpha 1))\gamma 3$, $2 = (1\alpha(2\beta 2))\beta 3$, $3 = (3\beta(3\gamma 3))\beta 3$.

Theorem 29 *A Γ-AG^{**}-groupoid S is an intra-regular Γ-AG^{**}-groupoid if $S\Gamma a = S$ or $a\Gamma S = S$ holds for all $a \in S$.*

Proof. Let S be a Γ-AG**-groupoid such that $S\Gamma a = S$ holds for all $a \in S$, then $S = S\Gamma S$. Let $a \in S$ and therefore, by using (2), we have

$$\begin{aligned} a &\in S = (S\Gamma S)\Gamma S = ((S\Gamma a)\Gamma(S\Gamma a))\Gamma S = ((S\Gamma S)\Gamma(a\Gamma a))\Gamma S \\ &= (S\Gamma(a\Gamma a))\Gamma S. \end{aligned}$$

Which shows that S is an intra-regular Γ-AG**-groupoid.

Let $a \in S$ and assume that $a\Gamma S = S$ holds for all $a \in S$, then by using (1), we have

$$a \in S = S\Gamma S = (a\Gamma S)\Gamma S = (S\Gamma S)\Gamma a = S\Gamma a.$$

Thus $S\Gamma a = S$ holds for all $a \in S$ and therefore it follows from above that S is an intra-regular. ■

Corollary 30 *If S is a Γ-AG^{**}-groupoid such that $a\Gamma S = S$ holds for all $a \in S$, then $S\Gamma a = S$ holds for all $a \in S$.*

Theorem 31 *If S is an intra-regular Γ-AG^{**}-groupoid, then $(B\Gamma S)\Gamma B = B \cap S$, where B is a Γ-bi-(Γ-generalized bi-) ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then clearly $(B\Gamma S)\Gamma B \subseteq B \cap S$. Now let $b \in B \cap S$ which implies that $b \in B$ and $b \in S$, then since S is an intra-regular Γ-AG**-groupoid so there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $b = (x\alpha(b\beta b))\gamma y$. Now we have

$$\begin{aligned}
b &= (x\alpha(b\beta b))\gamma y = (b\alpha(x\beta b))\gamma y = (y\alpha(x\beta b))\gamma b \\
&= (y\alpha(x\beta((x\alpha(b\beta b))\gamma y)))\gamma b = (y\alpha((x\alpha(b\beta b))\beta(x\gamma y)))\gamma b \\
&= ((x\alpha(b\beta b))\alpha(y\beta(x\gamma y)))\gamma b = (((x\gamma y)\alpha y)\alpha((b\beta b)\beta x))\gamma b \\
&= ((b\beta b)\alpha(((x\gamma y)\alpha y)\beta x))\gamma b = ((b\beta b)\alpha((x\alpha y)\beta(x\gamma y)))\gamma b \\
&= ((b\beta b)\alpha((x\alpha x)\beta(y\gamma y)))\gamma b = (((y\gamma y)\beta(x\alpha x))\alpha(b\beta b))\gamma b \\
&= (b\alpha(((y\gamma y)\beta(x\alpha x))\beta b))\gamma b \in (B\Gamma S)\Gamma B.
\end{aligned}$$

Which shows that $(B\Gamma S)\Gamma B = B \cap S$. ■

Corollary 32 *If S is an intra-regular Γ-AG^{**}-groupoid, then $(B\Gamma S)\Gamma B = B$, where B is a Γ-bi-(Γ-generalized bi-) ideal of S.*

Theorem 33 *If S is an intra-regular Γ-AG^{**}-groupoid, then $(S\Gamma I)\Gamma S = S \cap I$, where I is a Γ-interior ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then clearly $(S\Gamma I)\Gamma S \subseteq S \cap I$. Now let $i \in S \cap I$ which implies that $i \in S$ and $i \in I$, then since S is an intra- regular Γ-AG**-groupoid so there exist $x, y \in S$ and $\alpha, \gamma, \delta \in \Gamma$ such that $i = (x\alpha(i\delta i))\gamma y$. Now we have

$$\begin{aligned}
i &= (x\alpha(i\delta i))\gamma y = (i\alpha(x\delta i))\gamma y = (y\alpha(x\delta i))\gamma i \\
&= (y\alpha(x\delta i))\gamma((x\alpha(i\delta i))\gamma y) = (((x\alpha(i\delta i))\gamma y)\alpha(x\delta i))\gamma y \\
&= ((i\gamma x)\alpha(y\delta(x\alpha(i\delta i))))\gamma y = (((y\delta(x\alpha(i\delta i)))\gamma x)\alpha i)\gamma y \in (S\Gamma I)\Gamma S.
\end{aligned}$$

Which shows that $(S\Gamma I)\Gamma S = S \cap I$. ■

Corollary 34 *If S is an intra-regular Γ-AG^{**}-groupoid, then $(S\Gamma I)\Gamma S = I$, where I is a Γ-interior ideal of S.*

Lemma 35 *If S is an intra-regular regular Γ-AG^{**}-groupoid, then $S = S\Gamma S$.*

Proof. It is simple. ■

Lemma 36 *A subset A of an intra-regular Γ-AG^{**}-groupoid S is a Γ-left ideal if and only if it is a Γ-right ideal of S.*

Proof. Let S be an intra-regular Γ-AG**-groupoid and let A be a Γ-right ideal of S, then $A\Gamma S \subseteq A$. Let $a \in A$ and since S is an intra-regular Γ-AG**-groupoid so there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Let $p \in S\Gamma A$ and $\psi \in \Gamma$, then by we have

$$\begin{aligned} p &= s\psi a = s\psi((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\psi(s\gamma y) = (a\beta(x\delta a))\psi(s\gamma y) \\ &= ((s\gamma y)\beta(x\delta a))\psi a = ((a\gamma x)\beta(y\delta s))\psi a = (((y\delta s)\gamma x)\beta a)\psi a \\ &= (a\beta a)\psi((y\delta s)\gamma x) = (x\beta(y\delta s))\psi(a\gamma a) = a\psi((x\beta(y\delta s))\gamma a) \in A\Gamma S \subseteq A. \end{aligned}$$

Which shows that A is a Γ-left ideal of S.

Let A be a Γ-left ideal of S, then $S\Gamma A \subseteq A$. Let $a \in A$ and since S is an intra-regular Γ-AG**-groupoid so there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Let $p \in A\Gamma S$ and $\psi \in \Gamma$, then we have

$$\begin{aligned} p &= a\psi s = ((x\beta(a\delta a)\gamma y)\psi s = (s\gamma y)\psi(x\beta(a\delta a)) = ((a\delta a)\gamma x)\psi(y\beta s) \\ &= ((y\beta s)\gamma x)\psi(a\delta a) = (a\gamma a)\psi(x\delta(y\beta s)) = ((x\delta(y\beta s))\gamma a)\psi a \in S\Gamma A \subseteq A. \end{aligned}$$

Which shows that A is a Γ-right ideal of S. ■

Theorem 37 *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-bi-(Γ-generalized bi-) ideal of S.
(ii) $(A\Gamma S)\Gamma A = A$ and $A\Gamma A = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A \subseteq A$. Let $a \in A$, then since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\ &= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\ &= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = ((a\beta(x\delta a))\beta(y\delta(x\gamma y)))\gamma a \end{aligned}$$

$$\begin{aligned} &= ((a\beta y)\beta((x\delta a)\delta(x\gamma y)))\gamma a = ((x\delta a)\beta((a\beta y)\delta(x\gamma y)))\gamma a \\ &= ((x\delta a)\beta((a\beta x)\delta(y\gamma y)))\gamma a = (((y\gamma y)\delta(a\beta x))\beta(a\delta x))\gamma a \\ &= (a\beta(((y\gamma y)\delta(a\beta x))\delta x))\gamma a \in (A\Gamma S)\Gamma A. \end{aligned}$$

Thus $(A\Gamma S)\Gamma A = A$ holds. Now we have

$$
\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\
&= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = ((a\beta(x\delta a))\beta(y\delta(x\gamma y)))\gamma a \\
&= (((y\delta(x\gamma y))\beta(x\delta a))\beta a)\gamma a = (((a\delta x)\beta((x\gamma y)\delta y))\beta a)\gamma a \\
&= (((a\delta x)\beta((y\gamma y)\delta x))\beta a)\gamma a = (((a\delta(y\gamma y))\beta(x\delta x))\beta a)\gamma a \\
&= ((((x\delta x)\delta(y\gamma y))\beta a)\beta a)\gamma a \\
&= ((((x\delta x)\delta(y\gamma y))\beta((x\beta(a\delta a))\gamma y))\beta a)\gamma a \\
&= ((((x\delta x)\delta(y\gamma y))\beta((a\beta(x\delta a))\gamma y))\beta a)\gamma a \\
&= ((((x\delta x)\delta(a\beta(x\delta a)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((a\delta((x\delta x)\beta(x\delta a)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((a\delta((a\delta x)\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= ((((a\delta x)\delta(a\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= ((((a\delta a)\delta(x\beta(x\delta x)))\beta((y\gamma y)\gamma y))\beta a)\gamma a \\
&= (((((y\gamma y)\gamma y)\delta(x\beta(x\delta x)))\beta(a\delta a))\beta a)\gamma a \\
&= ((a\beta((((y\gamma y)\gamma y)\delta(x\beta(x\delta x)))\delta a))\beta a)\gamma a \\
&\subseteq ((A\Gamma S)\Gamma A)\Gamma A \subseteq A\Gamma A.
\end{aligned}
$$

Hence $A = A\Gamma A$ holds.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 38 *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) $(A\Gamma S)\Gamma(A\Gamma A) = A$ and $A\Gamma A = A$.
Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-$(1,2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)(A\Gamma A) \subseteq A$ and $A\Gamma A \subseteq A$. Let $a \in A$, then since S is an intra- regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a)\gamma y$. Now

$$
\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta((x\beta(a\delta a))\gamma y)))\gamma a = (y\beta((x\beta(a\delta a))\delta(x\gamma y)))\gamma a \\
&= ((x\beta(a\delta a))\beta(y\delta(x\gamma y)))\gamma a = (((x\gamma y)\beta y)\beta((a\delta a)\delta x))\gamma a \\
&= (((y\gamma y)\beta x)\beta((a\delta a)\delta x))\gamma a = ((a\delta a)\beta(((y\gamma y)\beta x)\delta x))\gamma a \\
&= ((a\delta a)\beta((x\beta x)\delta(y\gamma y)))\gamma a = (a\beta((x\beta x)\delta(y\gamma y)))\gamma(a\delta a) \in (A\Gamma S)\Gamma(A\Gamma A).
\end{aligned}
$$

Thus $(A\Gamma S)\Gamma(A\Gamma A) = A$. Now we have

$$
\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y)
\end{aligned}
$$

$$
\begin{aligned}
&= (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) \\
&= (a\beta(x\delta a))\gamma((y\beta(x\delta a))\gamma y) \\
&= (((y\beta(x\delta a))\gamma y)\beta(x\delta a))\gamma a \\
&= ((a\gamma x)\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= ((((x\beta(a\delta a))\gamma y)\gamma x)\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= (((x\gamma y)\gamma(x\beta(a\delta a)))\beta(y\delta(y\beta(x\delta a))))\gamma a \\
&= (((x\gamma y)\gamma y)\beta((x\beta(a\delta a))\delta(y\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((x\beta(a\delta a))\delta(y\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((x\beta y)\delta((a\delta a)\beta(x\delta a))))\gamma a \\
&= (((y\gamma y)\gamma x)\beta((a\delta a)\delta((x\beta y)\beta(x\delta a))))\gamma a \\
&= ((a\delta a)\beta(((y\gamma y)\gamma x)\delta((x\beta y)\beta(x\delta a))))\gamma a \\
&= ((a\delta a)\beta(((y\gamma y)\gamma x)\delta((x\beta x)\beta(y\delta a))))\gamma a \\
&= ((((x\beta x)\beta(y\delta a))\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= ((((a\beta y)\beta(x\delta x))\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= (((((x\delta x)\beta y)\beta a)\delta((y\gamma y)\gamma x))\beta(a\delta a))\gamma a \\
&= (((x\beta(y\gamma y))\delta(a\gamma((x\delta x)\beta y)))\beta(a\delta a))\gamma a \\
&= ((a\delta((x\beta(y\gamma y))\gamma((x\delta x)\beta y)))\beta(a\delta a))\gamma a \\
&= ((a\delta((x\beta(x\delta x))\gamma((y\gamma y)\beta y)))\beta(a\delta a))\gamma a \\
&\in ((A\Gamma S)\Gamma(A\Gamma A))\Gamma A \subseteq A\Gamma A.
\end{aligned}
$$

Hence $A\Gamma A = A$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 39 *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-interior ideal of S.
(ii) $(S\Gamma A)\Gamma S = A$.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-interior ideal of an intra-regular Γ-AG**-groupoid S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $a \in A$, then since S is an intra- regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$
\begin{aligned}
a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\
&= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) \\
&= (((x\beta(a\delta a))\gamma y)\beta(x\delta a))\gamma y \\
&= ((a\gamma x)\beta(y\delta(x\beta(a\delta a))))\gamma y \\
&= (((y\delta(x\beta(a\delta a)))\gamma x)\beta a)\delta y \in (S\Gamma A)\Gamma S.
\end{aligned}
$$

Thus $(S\Gamma A)\Gamma S = A$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 40 *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-quasi ideal of S.
(ii) $S\Gamma Q \cap Q\Gamma S = Q$.

Proof. $(i) \Longrightarrow (ii)$: Let Q be a Γ-quasi ideal of an intra-regular Γ-AG**-groupoid S, then $S\Gamma Q \cap Q\Gamma S \subseteq Q$. Let $q \in Q$, then since S is an intra-regular so there exist $x,\, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $q = (x\alpha(q\gamma q))\beta y$. Let $p\delta q \in S\Gamma Q$, for some $\delta \in \Gamma$, then

$$\begin{aligned} p\delta q &= p\delta((x\alpha(q\gamma q))\beta y) = (x\alpha(q\gamma q))\delta(p\beta y) = (q\alpha(x\gamma q))\delta(p\beta y) \\ &= (q\alpha p)\delta((x\gamma q)\beta y) = (x\gamma q)\delta((q\alpha p)\beta y) = (y\gamma(q\alpha p))\delta(q\beta x) \\ &= q\delta((y\gamma(q\alpha p))\beta x) \in Q\Gamma S. \end{aligned}$$

Now let $q\delta y \in Q\Gamma S$, then we have

$$\begin{aligned} q\delta p &= ((x\alpha(q\gamma q))\beta y)\delta p = (p\beta y)\delta(x\alpha(q\gamma q)) \\ &= x\delta((p\beta y)\alpha(q\gamma q)) = x\delta((q\beta q)\alpha(y\gamma p)) \\ &= (q\beta q)\delta(x\alpha(y\gamma p)) = ((x\alpha(y\gamma p))\beta q)\delta q \in S\Gamma Q. \end{aligned}$$

Hence $Q\Gamma S = S\Gamma Q$. Then we have

$$q = (x\alpha(q\gamma q))\beta y = (q\alpha(x\gamma q))\beta y = (y\alpha(x\gamma q))\beta q \in S\Gamma Q.$$

Thus $q \in S\Gamma Q \cap Q\Gamma S$ implies that $S\Gamma Q \cap Q\Gamma S = Q$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 41 *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) A is a Γ-two-sided two-sided ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let S be an intra-regular Γ-AG**-groupoid and let A be a Γ-$(1,2)$-ideal of S, then $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Let $a \in A$, then since S is an intra-regular so there exist $x,\, y \in S$ and $\beta, \gamma, \delta \in \Gamma$, such that

$a = (x\beta(a\delta a))\gamma y$. Now let $\psi \in \Gamma$, then

$$\begin{aligned}
s\psi a &= s\psi((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\psi(s\gamma y) \\
&= (a\beta(x\delta a))\psi(s\gamma y) = ((s\gamma y)\beta(x\delta a))\psi a \\
&= ((s\gamma y)\beta(x\delta a))\psi((x\beta(a\delta a))\gamma y) \\
&= (x\beta(a\delta a))\psi(((s\gamma y)\beta(x\delta a))\gamma y) \\
&= (y\beta((s\gamma y)\beta(x\delta a)))\psi((a\delta a)\gamma x) \\
&= (a\delta a)\psi((y\beta((s\gamma y)\beta(x\delta a)))\gamma x) \\
&= (x\delta(y\beta((s\gamma y)\beta(x\delta a))))\psi(a\gamma a) \\
&= (x\delta(y\beta((a\gamma x)\beta(y\delta s))))\psi(a\gamma a) \\
&= (x\delta((a\gamma x)\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= ((a\gamma x)\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= ((((x\beta(a\delta a))\gamma y)\gamma x)\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= (((x\gamma y)\gamma(x\beta(a\delta a)))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= ((((a\delta a)\gamma x)\gamma(y\beta x))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= ((((y\beta x)\gamma x)\gamma(a\delta a))\delta(x\beta(y\beta(y\delta s))))\psi(a\gamma a) \\
&= (((y\beta(y\delta s))\gamma x)\delta((a\delta a)\beta((y\beta x)\gamma x)))\psi(a\gamma a) \\
&= (((y\beta(y\delta s))\gamma x)\delta((a\delta a)\beta((x\beta x)\gamma y)))\psi(a\gamma a)
\end{aligned}$$

$$\begin{aligned}
&= ((a\delta a)\delta(((y\beta(y\delta s))\gamma x)\beta((x\beta x)\gamma y)))\psi(a\gamma a) \\
&= ((((x\beta x)\gamma y)\delta((y\beta(y\delta s))\gamma x))\delta(a\beta a))\psi(a\gamma a) \\
&= (a\delta(((x\beta x)\gamma y)\delta(((y\beta(y\delta s))\gamma x)\beta a)))\psi(a\gamma a) \in (A\Gamma S)\Gamma\,(A\Gamma A) \subseteq A.
\end{aligned}$$

Hence A is a Γ-left ideal of S and so A is a Γ-two-sided ideal of S.

$(ii) \Longrightarrow (i)$: Let A be a Γ-two-sided ideal of S. Let $y \in (A\Gamma S)\Gamma\,(A\Gamma A)$, then $y = (a\beta s)\gamma(b\delta b)$ for some $a, b \in A$, $s \in S$ and $\beta, \gamma, \delta \in \Gamma$. Now we have

$$y = (a\beta s)\gamma(b\delta b) = b\gamma((a\beta s)\delta b) \in A\Gamma S \subseteq A.$$

Hence $(A\Gamma S)\Gamma\,(A\Gamma A) \subseteq A$ and therefore A is a Γ-$(1, 2)$-ideal of S. ■

Theorem 42 *In an intra-regular Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) A is a Γ-$(1, 2)$-ideal of S.

(ii) A is a Γ-interior ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-$(1, 2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Let $p \in (S\Gamma A)\Gamma S$, then $p = (s\mu a)\psi s'$ for some $a \in A$, $s, s' \in S$ and $\mu, \psi \in \Gamma$. Since S is intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$\begin{aligned}
p &= (s\mu a)\psi s' = (s\mu((x\beta(a\delta a))\gamma y))\psi s' \\
&= ((x\beta(a\delta a))\mu(s\gamma y))\psi s' = (s'\mu(s\gamma y))\psi(x\beta(a\delta a)) \\
&= (s'\mu(s\gamma y))\psi(a\beta(x\delta a)) = a\psi((s'\mu(s\gamma y))\beta(x\delta a)) \\
&= ((x\beta(a\delta a))\gamma y)\psi((s'\mu(s\gamma y))\beta(x\delta a)) \\
&= ((a\beta(x\delta a))\gamma y)\psi((s'\mu(s\gamma y))\beta(x\delta a)) \\
&= ((a\beta(x\delta a))\gamma(s'\mu(s\gamma y)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma((x\delta a)\mu(s\gamma y)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma((y\delta s)\mu(a\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta s')\gamma(a\mu((y\delta s)\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta a)\gamma(s'\mu((y\delta s)\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((a\beta a)\gamma((y\delta s)\mu(s'\gamma x)))\psi(y\beta(x\delta a)) \\
&= ((y\beta(x\delta a))\gamma((y\delta s)\mu(s'\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma((x\delta a)\mu(s'\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma((x\delta s')\mu(a\gamma x)))\psi(a\beta a) \\
&= ((y\beta(y\delta s))\gamma(a\mu((x\delta s')\gamma x)))\psi(a\beta a) \\
&= (a\gamma((y\beta(y\delta s))\mu((x\delta s')\gamma x)))\psi(a\beta a) \\
&\in (A\Gamma S)\Gamma(A\Gamma A) \subseteq A.
\end{aligned}$$

Thus $(S\Gamma A)\Gamma S \subseteq A$. Which shows that A is a Γ-interior ideal of S.

$(ii) \implies (i)$: Let A be a Γ-interior ideal of S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $p \in (A\Gamma S)\Gamma(A\Gamma A)$, then $p = (a\mu s)\psi(b\alpha b)$, for some $a, b \in A$, $s \in S$ and $\mu, \psi, \alpha \in \Gamma$. Since S is intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$\begin{aligned}
p &= (a\mu s)\psi(b\alpha b) = ((b\alpha b)\mu s)\psi a \\
&= ((b\alpha b)\mu s)\psi((x\beta(a\gamma a))\gamma y) \\
&= (x\beta(a\gamma a))\psi(((b\alpha b)\mu s)\gamma y) \\
&= ((((b\alpha b)\mu s)\gamma y)\beta(a\gamma a))\psi x \\
&= ((a\gamma a)\beta(y\delta((b\alpha b)\mu s)))\psi x \\
&= (((y\delta((b\alpha b)\mu s))\gamma a)\beta a)\psi x \in (S\Gamma A)\Gamma S \subseteq A.
\end{aligned}$$

Thus $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$.

Now by using (3) and (4), we have

$$\begin{aligned}
A\Gamma A &\subseteq A\Gamma S = A\Gamma(S\Gamma S) = S\Gamma(A\Gamma S) = (S\Gamma S)\Gamma(A\Gamma S) \\
&= (S\Gamma A)\Gamma(S\Gamma S) = (S\Gamma A)\Gamma S \subseteq A.
\end{aligned}$$

Which shows that A is a Γ-$(1, 2)$-ideal of S. ■

Theorem 43 *In an intra-regular* Γ-AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-bi-ideal of S.
(ii) A is a Γ-interior ideal of S.
Proof. $(i) \implies (ii)$: Let A be a Γ-bi-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma A \subseteq A$. Let $p \in (S\Gamma A)\Gamma S$, then $p = (s\mu a)\psi s^{'}$ for some $a \in A$, $s, s^{'} \in S$ and $\mu, \psi \in \Gamma$. Since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$\begin{aligned}
p &= (s\mu a)\psi s^{'} = (s\mu((x\beta(a\delta a))\gamma y))\psi s^{'} \\
&= ((x\beta(a\delta a))\mu(s\gamma y))\psi s^{'} = (s^{'}\mu(s\gamma y))\psi(x\beta(a\delta a)) \\
&= ((a\delta a)\mu x)\psi((s\gamma y)\beta s^{'}) \\
&= (((s\gamma y)\beta s^{'})\mu x)\psi(a\delta a) = ((x\beta s^{'})\mu(s\gamma y))\psi(a\delta a) \\
&= (a\mu a)\psi((s\gamma y)\delta(x\beta s^{'})) = (((s\gamma y)\delta(x\beta s^{'}))\mu a)\psi a \\
&= (((s\gamma y)\delta(x\beta s^{'}))\mu((x\beta(a\delta a))\gamma y))\psi a \\
&= (((s\gamma y)\delta(x\beta(a\delta a)))\mu((x\beta s^{'})\gamma y))\psi a \\
&= ((((a\delta a)\gamma x)\delta(y\beta s))\mu((x\beta s^{'})\gamma y))\psi a \\
&= ((((x\beta s^{'})\gamma y)\delta(y\beta s))\mu((a\delta a)\gamma x))\psi a \\
&= ((a\delta a)\mu((((x\beta s^{'})\gamma y)\delta(y\beta s))\gamma x))\psi a \\
&= ((x\delta(((x\beta s^{'})\gamma y)\delta(y\beta s)))\mu(a\gamma a))\psi a \\
&= (a\mu((x\delta(((x\beta s^{'})\gamma y)\delta(y\beta s)))\gamma a))\psi a \\
&\in (A\Gamma S)\Gamma A \subseteq A.
\end{aligned}$$

Thus $(S\Gamma A)\Gamma S \subseteq A$. Which shows that A is a Γ-interior ideal of S.
$(ii) \implies (i)$: Let A be a Γ-interior ideal of S, then $(S\Gamma A)\Gamma S \subseteq A$. Let $p \in (A\Gamma S)\Gamma A$, then $p = (a\mu s)\psi b$ for some $a, b \in A$, $s \in S$ and $\mu, \psi \in \Gamma$. Since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $b = (x\beta(b\delta b))\gamma y$. Now

$$\begin{aligned}
p &= (a\mu s)\psi b = (a\mu s)\psi((x\beta(b\delta b))\gamma y) = (x\beta(b\delta b))\psi((a\mu s)\gamma y) \\
&= (((a\mu s)\gamma y)\beta(b\delta b))\psi x = ((b\gamma b)\beta(y\delta(a\mu s)))\psi x \\
&= (((y\delta(a\mu s))\gamma b)\beta b)\psi x \in (S\Gamma A)\Gamma S \subseteq A.
\end{aligned}$$

Thus $(A\Gamma S)\Gamma A \subseteq A$.
Now

$$\begin{aligned}
A\Gamma A &\subseteq A\Gamma S = A\Gamma(S\Gamma S) = S\Gamma(A\Gamma S) = (S\Gamma S)\Gamma(A\Gamma S) \\
&= (S\Gamma A)\Gamma(S\Gamma S) = (S\Gamma A)\Gamma S \subseteq A.
\end{aligned}$$

Which shows that A is a Γ-bi-ideal of S. ■

Theorem 44 *In an intra-regular* Γ*-*AG^{**}*-groupoid* S*, the following conditions are equivalent.*

(i) A is a Γ-$(1,2)$-ideal of S.
(ii) A is a Γ-quasi ideal of S.

Proof. $(i) \implies (ii)$: Let A be a Γ-$(1,2)$-ideal of intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Now we have

$$\begin{aligned} S\Gamma A &= S\Gamma(A\Gamma A) = S\Gamma((A\Gamma A)\Gamma A) \\ &= (A\Gamma A)\Gamma(S\Gamma A) = (A\Gamma S)\Gamma(A\Gamma A) \subseteq A. \end{aligned}$$

and by using (1) and (3), we have

$$\begin{aligned} A\Gamma S &= (A\Gamma A)\Gamma S = ((A\Gamma A)\Gamma A)\Gamma S = (S\Gamma A)\Gamma(A\Gamma A) = (S\Gamma(A\Gamma A))\Gamma(A\Gamma A) \\ &= ((S\Gamma S)\Gamma(A\Gamma A))\Gamma(A\Gamma A) = ((A\Gamma A)\Gamma(S\Gamma S))\Gamma(A\Gamma A) \\ &= (A\Gamma S)\Gamma(A\Gamma A) \subseteq A. \end{aligned}$$

Hence $(A\Gamma S) \cap (S\Gamma A) \subseteq A$. Which shows that A is a Γ-quasi ideal of S.

$(ii) \implies (i)$: Let A be a Γ-quasi ideal of S, then $(A\Gamma S) \cap (S\Gamma A) \subseteq A$. Now $A\Gamma A \subseteq A\Gamma S$ and $A\Gamma A \subseteq S\Gamma A$. Thus $A\Gamma A \subseteq (A\Gamma S) \cap (S\Gamma A) \subseteq A$. Then

$$(A\Gamma S)\Gamma(A\Gamma A) = (A\Gamma A)\Gamma(S\Gamma A) \subseteq A\Gamma(S\Gamma A) = S\Gamma(A\Gamma A) \subseteq S\Gamma A.$$

and

$$\begin{aligned} (A\Gamma S)\Gamma(A\Gamma A) &= (A\Gamma A)\Gamma(S\Gamma A) \subseteq A\Gamma(S\Gamma A) = S\Gamma(A\Gamma A) \\ &= (S\Gamma S)\Gamma(A\Gamma A) = (A\Gamma A)\Gamma(S\Gamma S) \subseteq A\Gamma S. \end{aligned}$$

Thus $(A\Gamma S)\Gamma(A\Gamma A) \subseteq (A\Gamma S) \cap (S\Gamma A) \subseteq A$. Which shows that A is a Γ-$(1,2)$-ideal of S. ■

Lemma 45 *Let* A *be a subset of an intra-regular* Γ*-*AG^{**}*-groupoid* S*, then* A *is a* Γ*-two-sided ideal of* S *if and only if* $A\Gamma S = A$ *and* $S\Gamma A = A$*.*

Proof. It is simple. ■

Theorem 46 *For an intra-regular* Γ*-*AG^{**}*-groupoid* S *the following statements are equivalent.*

(i) A is a Γ-left two-sided ideal of S.
(ii) A is a Γ-right two-sided ideal of S.
(iii) A is a Γ-two-sided ideal of S.
(iv) $A\Gamma S = A$ and $S\Gamma A = A$.
(v) A is a Γ-quasi ideal of S.
(vi) A is a Γ-$(1,2)$-ideal of S.
(vii) A is a Γ-generalized bi-ideal of S.

$(viii)$ A is a Γ-bi-ideal of S.
(ix) A is a Γ-interior ideal of S.

Proof. $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ are easy.
$(iii) \Longrightarrow (iv)$ is followed by above Lemma and $(iv) \Longrightarrow (v)$ is obvious.
$(v) \Longrightarrow (vi)$ It is easy.
$(vi) \Longrightarrow (vii)$: Let A be a Γ-$(1,2)$-ideal of an intra-regular Γ-AG**-groupoid S, then $(A\Gamma S)\Gamma(A\Gamma A) \subseteq A$. Let $p \in (A\Gamma S)\Gamma A$, then $p = (a\mu s)\psi b$ for some $a, b \in A$, $s \in S$ and $\mu, \psi \in \Gamma$. Now since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that such that $b = (x\beta(b\delta b))\gamma y$ then we have

$$\begin{aligned} p &= (a\mu s)\psi b = (a\mu s)\psi((x\beta(b\delta b))\gamma y) \\ &= (x\beta(b\delta b))\psi((a\mu s)\gamma y) = (y\beta(a\mu s))\psi((b\delta b)\gamma x) \\ &= (b\delta b)\psi((y\beta(a\mu s))\gamma x) = (x\delta(y\beta(a\mu s)))\psi(b\gamma b) \\ &= (x\delta(a\beta(y\mu s)))\psi(b\delta b) \\ &= (a\delta(x\beta(y\mu s)))\psi(b\delta b) \in (A\Gamma S)\Gamma(A\Gamma A) \subseteq A. \end{aligned}$$

Which shows that A is a Γ-generalized bi-ideal of S.
$(vii) \Longrightarrow (viii)$ is simple.
$(viii) \Longrightarrow (ix)$ is followed easily.
$(ix) \Longrightarrow (i)$ is followed by previous results . ■

Theorem 47 *In a Γ-AG^{**}-groupoid S, the following conditions are equivalent.*

(i) S is intra-regular.
(ii) Every Γ-bi-ideal of S is Γ-idempotent.

Proof. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$: Since $S\Gamma a$ is a Γ-bi-ideal of S, and by assumption $S\Gamma a$ is Γ-idempotent, so we have

$$\begin{aligned} a &\in (S\Gamma a)\,\Gamma\,(S\Gamma a) = ((S\Gamma a)\,\Gamma\,(S\Gamma a))\,\Gamma\,(S\Gamma a) \\ &= ((S\Gamma S)\,\Gamma\,(a\Gamma a))\,\Gamma\,(S\Gamma a) \subseteq (S\Gamma\,(a\Gamma a))\,\Gamma\,(S\Gamma S) \\ &= (S\Gamma\,(a\Gamma a))\,\Gamma S. \end{aligned}$$

Hence S is intra-regular. ■

Lemma 48 *If I and J are Γ-two-sided ideals of an intra-regular Γ-AG^{**}-groupoid S , then $I \cap J$ is a Γ-two-sided ideal of S.*

Proof. It is simple. ■

Lemma 49 *In an intra-regular Γ-AG^{**}-groupoid $I\Gamma J = I \cap J$, for every Γ-two-sided ideals I and J in S.*

Proof. Let I and J be any Γ-two-sided ideals of S, then obviously $I\Gamma J \subseteq I \cap J$. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, then $(I \cap J)\,(I \cap J) \subseteq I\Gamma J$, also,

$I \cap J$ is a Γ-two-sided ideal of S, so we have $I \cap J = (I \cap J)(I \cap J) \subseteq I\Gamma J$. Hence $I\Gamma J = I \cap J$. ■

Lemma 50 *Let S be a Γ-AG^{**}-groupoid, then S is an intra-regular if and only if every Γ-left ideal of S is Γ-idempotent.*

Proof. Let S be an intra-regular Γ-AG**-groupoid, then every Γ-two-sided ideal of S is Γ-idempotent.

Conversely, assume that every Γ-left ideal of S is Γ-idempotent. Since $S\Gamma a$ is a Γ-left ideal of S, so we have

$$\begin{aligned} a &\in S\Gamma a = (S\Gamma a)\Gamma(S\Gamma a) = ((S\Gamma a)\Gamma(S\Gamma a))\Gamma(S\Gamma a) \\ &= ((S\Gamma S)\Gamma(a\Gamma a))\Gamma(S\Gamma a) \subseteq (S\Gamma(a\Gamma a))\Gamma(S\Gamma S) \\ &= (S\Gamma(a\Gamma a))\Gamma S. \end{aligned}$$

Hence S is intra-regular. ■

Lemma 51 *In an AG^{**}-groupoid S, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $A = (S\Gamma A)(S\Gamma A)$, where A is any Γ-left ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Let A be a Γ-left ideal of an intra-regular Γ-AG**-groupoid S, then $S\Gamma A \subseteq A$ and then, $(S\Gamma A)(S\Gamma A) = S\Gamma A \subseteq A$. Now $A = A\Gamma A \subseteq S\Gamma A = (S\Gamma A)(S\Gamma A)$, which implies that $A = (S\Gamma A)(S\Gamma A)$.

$(ii) \Longrightarrow (i)$: Let A be a Γ-left ideal of S, then $A = (S\Gamma A)(S\Gamma A) \subseteq A\Gamma A$, which implies that A is Γ-idempotent and so S is an intra-regular. ■

Theorem 52 *A Γ-AG^{**}-groupoid S is called Γ-totally ordered under inclusion if P and Q are any Γ-two-sided ideals of S such that either $P \subseteq Q$ or $Q \subseteq P$.*

A Γ-two-sided ideal P of a Γ-AG**-groupoid S is called Γ-strongly irreducible if $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, for all Γ-two-sided ideals A, B and P of S.

Lemma 53 *Every Γ-two-sided ideal of an intra-regular Γ-AG^{**}-groupoid S is Γ-prime if and only if it is Γ-strongly irreducible.*

Proof. It is an easy. ■

Theorem 54 *Every Γ-two-sided ideal of an intra-regular Γ-AG^{**}-groupoid S is Γ-prime if and only if S is Γ-totally ordered under inclusion.*

Proof. Assume that every Γ-two-sided ideal of S is Γ-prime. Let P and Q be any Γ-two-sided ideals of S, so , $P\Gamma Q = P \cap Q$, and $P \cap Q$ is a Γ-two-sided ideal of S, so is prime, therefore $P\Gamma Q \subseteq P \cap Q$, which implies that $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$, which implies that $P \subseteq Q$ or $Q \subseteq P$. Hence S is Γ-totally ordered under inclusion.

Conversely, assume that S is Γ-totally ordered under inclusion. Let I, J and P be any Γ-two-sided ideals of S such that $I\Gamma J \subseteq P$. Now without loss of generality assume that $I \subseteq J$ then

$$I = I\Gamma I \subseteq I\Gamma J \subseteq P.$$

Therefore either $I \subseteq P$ or $J \subseteq P$, which implies that P is Γ-prime. ■

Theorem 55 *The set of all* Γ*-two-sided ideals of an intra-regular* Γ*-*AG^{**}*-groupoid* S*, forms a* Γ*-semilattice structure.*

Proof. Assume that $\Gamma_{\mathcal{I}}$ be the set of all Γ-two-sided ideals of an intra-regular Γ-AG**-groupoid S and let A, $B \in \Gamma_{\mathcal{I}}$, since A and B are Γ-two-sided ideals of S, then by using (2), we have

$$\begin{aligned} (A\Gamma B)\Gamma S &= (A\Gamma B)\,\Gamma\,(S\Gamma S) = (A\Gamma S)\,\Gamma\,(B\Gamma S) \subseteq A\Gamma B. \\ \text{Also } S\Gamma(A\Gamma B) &= (S\Gamma S)\,\Gamma\,(A\Gamma B) = (S\Gamma A)\,\Gamma(S\Gamma B) \subseteq A\Gamma B. \end{aligned}$$

Thus $A\Gamma B$ is a Γ-two-sided ideal of S. Hence $\Gamma_{\mathcal{I}}$ is closed. Also we have, $A\Gamma B = A \cap B = B \cap A = B\Gamma A$, which implies that $\Gamma_{\mathcal{I}}$ is commutative, so is associative. Now $A\Gamma A = A$, for all $A \in \Gamma_{\mathcal{I}}$. Hence $\Gamma_{\mathcal{I}}$ is Γ-semilattice. ■

Theorem 56 *For an intra-regular* Γ*-*AG^{**}*-groupoid* S*, the following statements holds.*

(i) Every Γ-right ideal of S is Γ-semiprime.
(ii) Every Γ-left ideal of S is Γ-semiprime.
(iii) Every Γ-two-sided ideal of S is Γ-semiprime

Proof. (i) : Let R be a Γ-right ideal of an intra-regular Γ-AG**-groupoid S. Let $a\delta a \in R$ for some $\delta \in \Gamma$ and let $a \in S$. Now since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$. Now we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\ &= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) \\ &= (x\beta(y\beta(x\delta a)))\gamma((a\delta a)\gamma y) \\ &= (a\delta a)\gamma((x\beta(y\beta(x\delta a)))\gamma y) \in R\Gamma(S\Gamma S) = R\Gamma S \subseteq R. \end{aligned}$$

Which shows that R is Γ-semiprime.

(ii) : Let L be a Γ-left ideal of S. Let $a\delta a \in L$ for some $\delta \in \Gamma$ and let $a \in S$ now since S is an intra-regular so there exist x, $y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a = (x\beta(a\delta a))\gamma y$, then we have

$$\begin{aligned} a &= (x\beta(a\delta a))\gamma y = (a\beta(x\delta a))\gamma y = (y\beta(x\delta a))\gamma a \\ &= (y\beta(x\delta a))\gamma((x\beta(a\delta a))\gamma y) = (x\beta(a\delta a))\gamma((y\beta(x\delta a))\gamma y) \\ &= (y\beta(y\beta(x\delta a)))\gamma((a\delta a)\gamma x) = (a\delta a)\gamma((y\beta(y\beta(x\delta a)))\gamma x) \\ &= (x\delta(y\beta(y\beta(x\delta a))))\gamma(a\gamma a) \in S\Gamma L \subseteq L. \end{aligned}$$

Which shows that L is Γ-semiprime.
(iii) is obvious. ■

Theorem 57 *A Γ-two-sided ideal of an intra-regular Γ-AG^{**}-groupoid S is minimal if and only if it is the intersection of two minimal Γ-two-sided ideals.*

Proof. *Let S be an intra-regular Γ-AG^{**}-groupoid and Q be a minimal Γ-two-sided ideal of S, let $a \in Q$. As $S\Gamma(S\Gamma a) \subseteq S\Gamma a$ and $S\Gamma(a\Gamma S) \subseteq a\Gamma(S\Gamma S) = a\Gamma S$, which shows that $S\Gamma a$ and $a\Gamma S$ are Γ-left ideals of S so $S\Gamma a$ and $a\Gamma S$ are Γ-two-sided ideals of S.*

Now

$$\begin{aligned}
& S\Gamma(S\Gamma a \cap a\Gamma S) \cap (S\Gamma a \cap a\Gamma S)\Gamma S \\
= \;& S\Gamma(S\Gamma a) \cap S\Gamma(a\Gamma S) \cap (S\Gamma a)\Gamma S \cap (a\Gamma S)\Gamma S \\
\subseteq \;& (S\Gamma a \cap a\Gamma S) \cap (S\Gamma a)\Gamma S \cap S\Gamma a \subseteq S\Gamma a \cap a\Gamma S.
\end{aligned}$$

Which implies that $S\Gamma a \cap a\Gamma S$ is a Γ-quasi ideal so $S\Gamma a \cap a\Gamma S$ is a Γ-two-sided ideal.

Also since $a \in Q$, we have

$$S\Gamma a \cap a\Gamma S \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q \cap Q \subseteq Q.$$

Now since Q is minimal so $S\Gamma a \cap a\Gamma S = Q$, where $S\Gamma a$ and $a\Gamma S$ are minimal Γ-two-sided ideals of S, because let I be a Γ-two-sided ideal of S such that $I \subseteq S\Gamma a$, then

$$I \cap a\Gamma S \subseteq S\Gamma a \cap a\Gamma S \subseteq Q,$$

which implies that

$$I \cap a\Gamma S = Q. \text{ Thus } Q \subseteq I.$$

So we have

$$\begin{aligned}
S\Gamma a &\subseteq S\Gamma Q \subseteq S\Gamma I \subseteq I, \text{ gives} \\
S\Gamma a &= I.
\end{aligned}$$

Thus $S\Gamma a$ is a minimal Γ-two-sided ideal of S. Similarly $a\Gamma S$ is a minimal Γ-two-sided ideal of S.

Conversely, let $Q = I \cap J$ be a Γ-two-sided ideal of S, where I and J are minimal Γ-two-sided ideals of S, then Q is a Γ-quasi ideal of S, that is $S\Gamma Q \cap Q\Gamma S \subseteq Q$.

Let $Q^{'}$ be a Γ-two-sided ideal of S such that $Q^{'} \subseteq Q$, then

$$\begin{aligned}
S\Gamma Q^{'} \cap Q^{'}\Gamma S &\subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q, \text{ also } S\Gamma Q^{'} \subseteq S\Gamma I \subseteq I \\
\text{and } Q^{'}\Gamma S &\subseteq J\Gamma S \subseteq J.
\end{aligned}$$

Now

$$\begin{aligned} S\Gamma\left(S\Gamma Q^{'}\right) &= (S\Gamma S)\,\Gamma\left(S\Gamma Q^{'}\right) = \left(Q^{'}\Gamma S\right)\Gamma\,(S\Gamma S) \\ &= \left(Q^{'}\Gamma S\right)\Gamma S = (S\Gamma S)\,\Gamma Q^{'} = S\Gamma Q^{'} \end{aligned}$$

implies that $S\Gamma Q^{'}$ *is a* Γ*-left ideal and hence a* Γ*-two-sided ideal. Similarly* $Q^{'}\Gamma S$ *is a* Γ*-two-sided ideal of* S*.*

But since I *and* J *are minimal* Γ*-two-sided ideals of* S*, so*

$$S\Gamma Q^{'} = I \text{ and } Q^{'}\Gamma S = J.$$

But $Q = I \cap J$*, which implies that,*

$$Q = S\Gamma Q^{'} \cap Q^{'}\Gamma S \subseteq Q^{'}.$$

Which give us $Q = Q^{'}$*. Hence* Q *is minimal.* ■

2.2 Locally Associative Γ-AG**-groupoids

In this section we introduce a new non-associative algebraic structure namely locally associative Γ-AG**-groupoids and decompose it using Γ-congruences. An AG-groupoid S is called a locally associative Γ-AG-groupoid if $(a\alpha a)\beta a = a\alpha(a\beta a)$, holds for all a in S and $\alpha, \beta \in \Gamma$. If S is a locally associative AG-groupoid then it is easy to see that $(S\Gamma a)\Gamma S = S\Gamma(a\Gamma S)$ or $(S\Gamma S)\Gamma S = S\Gamma(S\Gamma S)$. For particular $\alpha \in \Gamma$, let us denote $a\alpha a = a_{\alpha}^{2}$ for some $\alpha \in \Gamma$ and $a\alpha a = a_{\Gamma}^{2}$, $\forall\ \alpha \in \Gamma$ i.e. $a\Gamma a = a_{\Gamma}^{2}$ and generally $a\Gamma a\Gamma a...a\Gamma a = a_{\Gamma}^{n}$(n times.)

Let S be an Γ-AG**-groupoid and a relation ρ_{Γ} be defined on S as follows: $a\rho_{\Gamma}b$ if and only if there exists a positive integer n such that $a\Gamma b_{\Gamma}^{n} = b_{\Gamma}^{n+1}$ and $b\Gamma a_{\Gamma}^{n} = a_{\Gamma}^{n+1}$, for all a and b in S.

Proposition 58 *If* S *is a locally associative* Γ*-*AG^{**}*-groupoid, then* $a\Gamma a_{\Gamma}^{n+1} = (a_{\Gamma}^{n+1})\Gamma a$*, for all* a *in* S *and positive integer* n*.*

Proof.

$$\begin{aligned} a\Gamma a_{\Gamma}^{n+1} &= a\Gamma(a_{\Gamma}^{n}\Gamma a) = a_{\Gamma}^{n}\Gamma(a\Gamma a) = (a_{\Gamma}^{n-1}\Gamma a)\Gamma(a\Gamma a) = (a\Gamma a)\Gamma(a\Gamma a_{\Gamma}^{n-1}) \\ &= (a\Gamma a)\Gamma a_{\Gamma}^{n} = (a_{\Gamma}^{n}\Gamma a)\Gamma a = (a_{\Gamma}^{n+1})\Gamma a. \end{aligned}$$

■

Proposition 59 *In a locally associative* Γ*-*AG^{**}*-groupoid* S*,* $a_{\Gamma}^{m}a_{\Gamma}^{n} = a_{\Gamma}^{m+n}$ $\forall\ a \in S$ *and positive integers* m*,* n*.*

Proof.

$$\begin{aligned} a_\Gamma^{m+1}a_\Gamma^n &= (a_\Gamma^m\Gamma a)\Gamma a_\Gamma^n = (a_\Gamma^n\Gamma a)\Gamma a_\Gamma^m = (a\Gamma a_\Gamma^n)\Gamma a_\Gamma^m \\ &= (a_\Gamma^m\Gamma a_\Gamma^n)\Gamma a = a_\Gamma^{m+n}\Gamma a = a_\Gamma^{m+n+1}. \end{aligned}$$

■

Proposition 60 *If S is a locally associative Γ-AG^{**}-groupoid, then for all a, b in S, $(a\Gamma b)_\Gamma^n = a_\Gamma^n\Gamma b_\Gamma^n$ and positive integer $n \geq 1$ and $(a\Gamma b)_\Gamma^n = b_\Gamma^n\Gamma a_\Gamma^n$, for $n \geq 2$.*

Proof.

$$(a\Gamma b)_\Gamma^2 = (a\Gamma b)\Gamma(a\Gamma b) = (a\Gamma a)\Gamma(b\Gamma b) = a^2\Gamma b^2.$$

$$(a\Gamma b)_\Gamma^{k+1} = (a\Gamma b)_\Gamma^k\Gamma(a\Gamma b) = (a_\Gamma^k\Gamma b_\Gamma^k)\Gamma(a\Gamma b) = (a_\Gamma^k\Gamma a)\Gamma(b_\Gamma^k\Gamma b) = a_\Gamma^{k+1}\Gamma b_\Gamma^{k+1}.$$

Let $n \geq 2$. Then by (3) and (1), we get

$$\begin{aligned} (a\Gamma b)_\Gamma^n &= a_\Gamma^n\Gamma b_\Gamma^n = (a\Gamma a_\Gamma^{n-1})\Gamma(b\Gamma b_\Gamma^{n-1}) = b\Gamma((a\Gamma a_\Gamma^{n-1})\Gamma b_\Gamma^{n-1})) \\ &= b\Gamma((b_\Gamma^{n-1}\Gamma a_\Gamma^{n-1})\Gamma a) = b\Gamma((b\Gamma a)_\Gamma^{n-1}\Gamma a) = (b\Gamma a)_\Gamma^{n-1}\Gamma(b\Gamma a) \\ &= (b\Gamma a)_\Gamma^n = b_\Gamma^n\Gamma a_\Gamma^n. \end{aligned}$$

■

Proposition 61 *In a locally associative Γ-AG^{**}-groupoid S, $(a_\Gamma^m)_\Gamma^n = a_\Gamma^{mn}$ for all $a \in S$ and positive integers m, n.*

Proof.

$$(a_\Gamma^{m+1})_\Gamma^n = (a_\Gamma^m\Gamma a)_\Gamma^n = (a_\Gamma^m)_\Gamma^n\Gamma a_\Gamma^n = a_\Gamma^{mn}\Gamma a_\Gamma^n = a_\Gamma^{mn+n} = a_\Gamma^{n(m+1)}.$$

■

Theorem 62 *Let S be a locally associative Γ-AG^{**}-groupoid. If $a\Gamma b_\Gamma^m = b_\Gamma^{m+1}$ and $b\Gamma a_\Gamma^n = a_\Gamma^{n+1}$ for $a, b \in S$ and positive integers m, n, then $a\rho_\Gamma b$.*

Proof. If $n > m$, then

$$\begin{aligned} b_\Gamma^{n-m}\Gamma(a\Gamma b_\Gamma^m) &= b_\Gamma^{n-m}\Gamma b_\Gamma^{m+1} \\ a\Gamma(b_\Gamma^{n-m}\Gamma b_\Gamma^m) &= b_\Gamma^{n-m+m+1} \\ a\Gamma b_\Gamma^{n-m+m} &= b_\Gamma^{n+1} \\ a\Gamma b_\Gamma^n &= b_\Gamma^{n+1}. \end{aligned}$$

■

Theorem 63 *The relation ρ_Γ on a locally associative Γ-AG^{**}-groupoid is a congruence relation.*

Proof. Evidently ρ_Γ is reflexive and symmetric. For transitivity we may proceed as follows.

Let $a\rho_\Gamma b$ and $b\rho_\Gamma c$ so that there exist positive integers n, m such that

$$a\Gamma b_\Gamma^n = b_\Gamma^{n+1},\ b\Gamma a_\Gamma^n = a_\Gamma^{n+1}, \text{ and}$$

$$b\Gamma c_\Gamma^m = c_\Gamma^{m+1},\ c\Gamma b_\Gamma^m = b_\Gamma^{m+1}.$$

Let $k = (n+1)(m+1) - 1$, that is, $k = n(m+1) + m$. Thus we get,

$$\begin{aligned} a\Gamma c_\Gamma^k &= a\Gamma c_\Gamma^{n(m+1)+m} = a\Gamma(c_\Gamma^{n(m+1)}\Gamma c_\Gamma^m) = a\Gamma\{(c_\Gamma^{m+1})_\Gamma^n\Gamma c_\Gamma^m\} \\ &= a\Gamma\{(b\Gamma c_\Gamma^m)_\Gamma^n\Gamma c_\Gamma^m\} = a\Gamma\{(b_\Gamma^n\Gamma c_\Gamma^{mn})\Gamma c^m\} = a\Gamma(c_\Gamma^{m(n+1)}\Gamma b^n) \\ &= c_\Gamma^{m(n+1)}\Gamma(a\Gamma b_\Gamma^n) = c_\Gamma^{m(n+1)}\Gamma b_\Gamma^{n+1} = (c_\Gamma^m\Gamma b)_\Gamma^{n+1} = b_\Gamma^{n+1}\Gamma c_\Gamma^{m(n+1)} \\ &= (b\Gamma c_\Gamma^m)_\Gamma^{n+1} = c_\Gamma^{k+1}. \end{aligned}$$

Similarly, $c\Gamma a^k = a_\Gamma^{k+1}$. Thus ρ_Γ is an equivalence relation. To show that ρ_Γ is compatible, assume that $a\rho_\Gamma b$ such that for some positive integer n,

$$a\Gamma b_\Gamma^n = b_\Gamma^{n+1} \text{ and } b\Gamma a_\Gamma^n = a_\Gamma^{n+1}.$$

Let $c \in S$, then, we get

$$(a\Gamma c)\Gamma(b\Gamma c)_\Gamma^n = (a\Gamma c)\Gamma(b_\Gamma^n\Gamma c_\Gamma^n) = (a\Gamma b_\Gamma^n)\Gamma(c\Gamma c_\Gamma^n) = b_\Gamma^{n+1}\Gamma c_\Gamma^{n+1} = (b\Gamma c)_\Gamma^{n+1}.$$

Similarly, $(b\Gamma c)\Gamma(a\Gamma c)_\Gamma^n = (a\Gamma c)_\Gamma^{n+1}$. Hence ρ_Γ is a congruence relation on S. ■

Lemma 64 *Let S be a locally associative Γ-AG^{**}-groupoid, then $a\Gamma b\rho_\Gamma b\Gamma a$, for all a, b in S.*

Proof.

$$\begin{aligned} (a\Gamma b)\Gamma(b\Gamma a)_\Gamma^{n+1} &= (a\Gamma b)\Gamma(a_\Gamma^{n+1}\Gamma b_\Gamma^{n+1}) = (a\Gamma a_\Gamma^{n+1})\Gamma(b\Gamma b_\Gamma^{n+1}) \\ &= a_\Gamma^{n+2}\Gamma b_\Gamma^{n+2} = (b\Gamma a)_\Gamma^{n+2}. \end{aligned}$$

Similarly, $(b\Gamma a)\Gamma(a\Gamma b)_\Gamma^{n+1} = (a\Gamma b)_\Gamma^{n+2}$. Hence $a\Gamma b\rho b\Gamma a$, for all a, b in S. ■

A relation ρ on an AG-groupoid S is called separative if $a\Gamma b\rho a_\Gamma^2$ and $a\Gamma b\rho_\Gamma b_\Gamma^2$ implies that $a\rho_\Gamma b$.

Theorem 65 *The relation ρ_Γ is separative.*

Proof. Let $a, b \in S$, $a\Gamma b\rho_\Gamma a_\Gamma^2$, and $a\Gamma b\rho_\Gamma b_\Gamma^2$. Then by definition of ρ_Γ there exist positive integers m and n such that,

$$\begin{aligned} (a\Gamma b)\Gamma(a_\Gamma^2)_\Gamma^m &= (a_\Gamma^2)_\Gamma^{m+1},\ a_\Gamma^2\Gamma(a\Gamma b)_\Gamma^m = (a\Gamma b)_\Gamma^{m+1} \text{ and} \\ (a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n &= (b_\Gamma^2)_\Gamma^{n+1},\ b_\Gamma^2\Gamma(a\Gamma b)_\Gamma^n = (a\Gamma b)_\Gamma^{n+1}. \end{aligned}$$

Then

$$\begin{aligned}(a\Gamma b)\Gamma a_\Gamma^{2m} &= (a\Gamma b)\,\Gamma\,(a_\Gamma^m \Gamma a_\Gamma^m) = (a\Gamma a_\Gamma^m)\,\Gamma\,(b\Gamma a_\Gamma^m) = (a_\Gamma^{m+1})\Gamma\,(b\Gamma a_\Gamma^m)\\ &= b\Gamma(a_\Gamma^{m+1}\Gamma a_\Gamma^m) = b\Gamma a_\Gamma^{2m+1}, \text{ but } (a\Gamma b)\Gamma a_\Gamma^{2m} = (a_\Gamma^2)_\Gamma^{m+1} = a_\Gamma^{2m+2},\end{aligned}$$

which implies that $b\Gamma a_\Gamma^{2m+1} = a_\Gamma^{2m+2}$. Also $(a\Gamma b)\Gamma(b_\Gamma^2)_\Gamma^n = (b_\Gamma^2)_\Gamma^{n+1}$, implies that $b_\Gamma^{2n+1}\Gamma a = b_\Gamma^{2n+2}$. Also, we get

$$b_\Gamma^{2n+2}\Gamma b_\Gamma^2 = (b_\Gamma^{2n+1}\Gamma a)\Gamma b_\Gamma^2,$$

this implies that

$$b_\Gamma^{2n+4} = b_\Gamma^2\Gamma(a\Gamma b_\Gamma^{2n+1}) = a\Gamma(b_\Gamma^2\Gamma b_\Gamma^{2n+1}) = a\Gamma b_\Gamma^{2n+3}.$$

Hence, $a\rho_\Gamma b$. ■

Theorem 66 *Let S be a locally associative Γ-AG^{**}-groupoid. Then S/ρ_Γ is a maximal separative commutative image of S.*

Proof. ρ_Γ is separative, and hence S/ρ_Γ is separative. We now show that ρ_Γ is contained in every separative congruence relation σ_Γ on S. Let $a\rho_\Gamma b$ so that there exists a positive integer n such that,

$$a\Gamma b_\Gamma^n = b_\Gamma^{n+1} \text{ and } b\Gamma a_\Gamma^n = a_\Gamma^{n+1}.$$

We need to show that $a\sigma_\Gamma b$, where σ_Γ is a separative congruence on S. Let k be any positive integer such that,

$$a\Gamma b^k\Gamma\sigma b_\Gamma^{k+1} \text{ and } b\Gamma a_\Gamma^k\sigma a_\Gamma^{k+1}. \tag{5}$$

Suppose $k \geqslant 3$.

$$\begin{aligned}(a\Gamma b_\Gamma^{k-1})_\Gamma^2 &= (a\Gamma b_\Gamma^{k-1})\Gamma(a\Gamma b_\Gamma^{k-1}) = a_\Gamma^2\Gamma b_\Gamma^{2k-2} = (a\Gamma a)\Gamma(b_\Gamma^{k-2}\Gamma b_\Gamma^k)\\ &= (a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k) = (a\Gamma b_\Gamma^{k-2})\Gamma b^{k+1}\end{aligned}$$

Therefore

$$(a\Gamma b_\Gamma^{k-2})\Gamma(a\Gamma b_\Gamma^k)\sigma_\Gamma(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1}.$$

Thus we get

$$\begin{aligned}(a\Gamma b_\Gamma^{k-2})\Gamma b_\Gamma^{k+1} &= (b_\Gamma^{k+1}\Gamma b_\Gamma^{k-2})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = (a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k.\\ \text{Also } (a\Gamma b_\Gamma^{k-1})\Gamma b_\Gamma^k &= (b_\Gamma^k\Gamma b_\Gamma^{k-1})\Gamma a = b_\Gamma^{2k-1}\Gamma a = (b_\Gamma^{k-1}\Gamma b_\Gamma^k)\Gamma a\\ &= (a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1},\end{aligned}$$

implies that

$$(a\Gamma b_\Gamma^{k-1})_\Gamma^2\sigma_\Gamma(a\Gamma b_\Gamma^k)\Gamma b_\Gamma^{k-1}.$$

Since $a\Gamma b_{\Gamma}^{k}\sigma_{\Gamma}b_{\Gamma}^{k+1}$ and $(a\Gamma b_{\Gamma}^{k})\Gamma b_{\Gamma}^{k-1}\sigma_{\Gamma}b_{\Gamma}^{k+1}\Gamma b_{\Gamma}^{k-1}$, hence $(a\Gamma b_{\Gamma}^{k-1})_{\Gamma}^{2}\sigma_{\Gamma}(b_{\Gamma}^{k})_{\Gamma}^{2}$. It further implies that,

$$(a\Gamma b_{\Gamma}^{k-1})_{\Gamma}^{2}\sigma_{\Gamma}(a\Gamma b_{\Gamma}^{k-1})b_{\Gamma}^{k}\sigma_{\Gamma}(b_{\Gamma}^{k})_{\Gamma}^{2}.$$

Thus

$$a\Gamma b_{\Gamma}^{k-1}\sigma_{\Gamma}b_{\Gamma}^{k}.$$

Similarly,

$$b\Gamma a_{\Gamma}^{k-1}\sigma_{\Gamma}a_{\Gamma}^{k}.$$

Thus if (13) holds for k, it holds for $k-1$.

Now obviously (13) yields

$$a\Gamma b_{\Gamma}^{3}\sigma_{\Gamma}'b_{\Gamma}^{4} \text{ and } b\Gamma a_{\Gamma}^{3}\sigma_{\Gamma}'a_{\Gamma}^{4}.$$

Also, we get

$$\begin{aligned}
&(a\Gamma b_{\Gamma}^{3})\Gamma a_{\Gamma}^{2}\sigma_{\Gamma}'b_{\Gamma}^{4}a_{\Gamma}^{2} \text{ and } (b\Gamma a_{\Gamma}^{3})\Gamma b_{\Gamma}^{2}\sigma_{\Gamma}'a_{\Gamma}^{4}\Gamma b_{\Gamma}^{2}\\
&(a_{\Gamma}^{2}\Gamma b_{\Gamma}^{3})\Gamma a\sigma'\Gamma b_{\Gamma}^{4}\Gamma a_{\Gamma}^{2} \text{ and } (b_{\Gamma}^{2}\Gamma a_{\Gamma}^{3})\Gamma b\sigma_{\Gamma}'a_{\Gamma}^{4}\Gamma b_{\Gamma}^{2}\\
&(b_{\Gamma}^{3}\Gamma a_{\Gamma}^{2})\Gamma a\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{4} \text{ and } (a_{\Gamma}^{3}\Gamma b_{\Gamma}^{2})\Gamma b\sigma_{\Gamma}'b_{\Gamma}^{2}\Gamma a_{\Gamma}^{4}\\
&a_{\Gamma}^{3}\Gamma b_{\Gamma}^{3}\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{4} \text{ and } b_{\Gamma}^{3}\Gamma a_{\Gamma}^{3}\sigma_{\Gamma}'b_{\Gamma}^{2}\Gamma a_{\Gamma}^{4}\\
&a_{\Gamma}^{3}\Gamma b_{\Gamma}^{3}\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{4} \text{ and } a_{\Gamma}^{3}\Gamma b_{\Gamma}^{3}\sigma_{\Gamma}'b_{\Gamma}^{2}\Gamma a_{\Gamma}^{4},
\end{aligned}$$

which implies that $(b_{\Gamma}^{2}\Gamma a)_{\Gamma}^{2}\sigma_{\Gamma}'a_{\Gamma}^{3}\Gamma b_{\Gamma}^{3}\sigma_{\Gamma}'(a_{\Gamma}^{2}\Gamma b)_{\Gamma}^{2}$, and as σ_{Γ}' is separative and $(b_{\Gamma}^{2}\Gamma a)\Gamma(a_{\Gamma}^{2}\Gamma b)=(b_{\Gamma}^{2}\Gamma a_{\Gamma}^{2})\Gamma(a\Gamma b)=(a_{\Gamma}^{2}\Gamma b_{\Gamma}^{2})\Gamma(a\Gamma b)=a_{\Gamma}^{3}\Gamma b_{\Gamma}^{3}$, so $a_{\Gamma}^{2}\Gamma b\sigma'\Gamma b_{\Gamma}^{2}\Gamma a$. Now we get

$$\begin{aligned}
&(a_{\Gamma}^{2}\Gamma b)\Gamma a\sigma'\Gamma(b_{\Gamma}^{2}\Gamma a)\Gamma a\\
&(a\Gamma b)\Gamma a_{\Gamma}^{2}\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{2}\\
&a_{\Gamma}^{2}\Gamma(b\Gamma a)\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{2}\\
&b\Gamma a_{\Gamma}^{3}\sigma_{\Gamma}'a_{\Gamma}^{2}\Gamma b_{\Gamma}^{2} \text{ but } b\Gamma a_{\Gamma}^{3}\sigma_{\Gamma}'a_{\Gamma}^{4},
\end{aligned}$$

Thus $(b\Gamma a)_{\Gamma}^{2}\sigma_{\Gamma}'b\Gamma a_{\Gamma}^{3}\sigma_{\Gamma}'(a_{\Gamma}^{2})_{\Gamma}^{2}$, now since σ_{Γ}' is separative and $a_{\Gamma}^{2}\Gamma(b\Gamma a)=b\Gamma a_{\Gamma}^{3}$, so we get $b\Gamma a\sigma_{\Gamma}'a_{\Gamma}^{2}$.

Similarly we can obtain $a\Gamma b\sigma_{\Gamma}'b_{\Gamma}^{2}$.

Also it is easy to show that (13) holds for $k=2$.

Thus if (5) holds for k, it holds for $k=1$. By induction down from k, it follows that (5) holds for $k=1$, $a\Gamma b\sigma_{\Gamma}b_{\Gamma}^{2}$ and $b\Gamma a\sigma_{\Gamma}a_{\Gamma}^{2}$. Now it is easy to see that $a\Gamma b\sigma_{\Gamma}b_{\Gamma}^{2}$, we get $(b\Gamma a)_{\Gamma}^{2}\sigma_{\Gamma}b_{\Gamma}^{3}\Gamma a$, and again from $a\Gamma b\sigma_{\Gamma}b_{\Gamma}^{2}$ we get $b_{\Gamma}^{3}\Gamma a\sigma_{\Gamma}b_{\Gamma}^{4}$. So $(b\Gamma a)_{\Gamma}^{2}\sigma_{\Gamma}b_{\Gamma}^{3}\Gamma a\sigma b_{\Gamma}^{4}$ implies that $b\Gamma a\sigma_{\Gamma}b_{\Gamma}^{2}$ which further implies that $a\Gamma b\sigma_{\Gamma}b\Gamma a$. Thus we obtain $a\sigma_{\Gamma}b$. Hence $\rho_{\Gamma}\subseteq\sigma_{\Gamma}$ and so S/ρ_{Γ} is the maximal separative commutative image of S. ■

Lemma 67 *If $x\Gamma a=x$ $(a=a_{\Gamma}^{2})$ for some x in a locally associative Γ-AG^{**}-groupoid then $x_{\Gamma}^{n}\Gamma a=x_{\Gamma}^{n}$ for some positive integer n.*

Proof. Let $n = 2$, then using (2), we get

$$x^2_\Gamma \Gamma a = (x\Gamma x)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x\Gamma a) = x\Gamma x = x^2_\Gamma.$$

Let the result be true for k, that is, $x^k_\Gamma \Gamma a = x^k_\Gamma$. Then by (2) and Proposition 1, we get

$$x^{k+1}_\Gamma \Gamma a = (x\Gamma x^k_\Gamma)\Gamma(a\Gamma a) = (x\Gamma a)\Gamma(x^k_\Gamma \Gamma a) = x\Gamma x^k_\Gamma = x^{k+1}_\Gamma.$$

Hence $x^n_\Gamma \Gamma a = x^n_\Gamma$ for all positive integers n. ∎

Lemma 68 *If S is a Γ-AG-groupoid, then $Q = \{x \mid x \in S,\ x\Gamma a = x$ and $a = a^2_\Gamma\}$ is a commutative subsemigroup.*

Proof. As $a\Gamma a = a$, we have $a \in Q$. Now if x, $y \in Q$, then by identity (2),

$$x\Gamma y = (x\Gamma a)\Gamma(y\Gamma a) = (x\Gamma y)\Gamma(a\Gamma a) = (x\Gamma y)\Gamma a.$$

To prove that Q is commutative and associative, assume that x, y and z belong to Q. Then by using (1), we get

$$x\Gamma y = (x\Gamma a)\Gamma y = (y\Gamma a)\Gamma x = y\Gamma x. \text{ Also}$$

$$(x\Gamma y)\Gamma z = (z\Gamma y)\Gamma x = x\Gamma(y\Gamma z).$$

Hence Q is a commutative subsemigroup of S. ∎

Theorem 69 *Let ρ_Γ and σ_Γ be separative congruences on locally associative Γ-AG^{**}-groupoid S and $x^2_\Gamma a = x^2_\Gamma (a = a^2_\Gamma)$ for all x in S. If $\rho_\Gamma \cap (Q_\Gamma \times Q_\Gamma) \subseteq \sigma_\Gamma \cap (Q_\Gamma \times Q_\Gamma)$, then $\rho_\Gamma \subseteq \sigma_\Gamma$.*

Proof. If $x\rho_\Gamma y$ then,

$$(x^2_\Gamma \Gamma(x\Gamma y))^2_\Gamma \rho_\Gamma (x^2_\Gamma \Gamma(x\Gamma y)\Gamma(x^2_\Gamma \Gamma y^2_\Gamma)\rho_\Gamma (x^2_\Gamma y^2_\Gamma)^2_\Gamma.$$

It follows that $(x^2_\Gamma \Gamma(x\Gamma y))^2_\Gamma$, $(x^2_\Gamma y^2_\Gamma)^2_\Gamma \in Q_\Gamma$. Now by (2), (1), (3), respectively, we get,

$$\begin{aligned}
(x^2_\Gamma (x\Gamma y))\Gamma(x^2_\Gamma \Gamma y^2_\Gamma) &= (x^2_\Gamma \Gamma x^2_\Gamma)\Gamma((x\Gamma y)\Gamma y^2_\Gamma) = (x^2_\Gamma \Gamma x^2_\Gamma)\Gamma(y^3_\Gamma \Gamma x) \\
&= x^4_\Gamma \Gamma(y^3_\Gamma \Gamma x) = y^3_\Gamma \Gamma(x^4_\Gamma \Gamma x) = y^3_\Gamma \Gamma x^5_\Gamma \quad \text{and} \\
(y^3_\Gamma \Gamma x^5_\Gamma)\Gamma a &= (y^3_\Gamma \Gamma x^5_\Gamma)\Gamma(a\Gamma a) = (y^3_\Gamma \Gamma a)\Gamma(x^5_\Gamma \Gamma a) = y^3_\Gamma \Gamma x^5_\Gamma.
\end{aligned}$$

So $x^2_\Gamma \Gamma(x\Gamma y)\Gamma(x^2_\Gamma \Gamma y^2_\Gamma) \in Q$. Hence $(x^2_\Gamma \Gamma(x\Gamma y))^2_\Gamma \sigma_\Gamma (x^2_\Gamma (x\Gamma y)\Gamma(x^2_\Gamma \Gamma y^2_\Gamma)\sigma_\Gamma (x^2_\Gamma y^2_\Gamma)^2_\Gamma$ implies that

$$x^2_\Gamma \Gamma(x\Gamma y)\sigma x^2_\Gamma \Gamma y^2_\Gamma.$$

Since $x^2_\Gamma \Gamma y^2_\Gamma \rho_\Gamma x^4_\Gamma$ and $(x^2_\Gamma \Gamma y^2_\Gamma)$, $x^4_\Gamma \in Q$. Thus $x^2_\Gamma \Gamma y^2_\Gamma \sigma_\Gamma x^4_\Gamma$ and we get $(x^2_\Gamma)^2_\Gamma \sigma_\Gamma x^2_\Gamma (x\Gamma y)\sigma_\Gamma (x\Gamma y)^2_\Gamma$ which implies that $x^2_\Gamma \sigma_\Gamma x\Gamma y$. Finally, $x^2_\Gamma \rho_\Gamma y^2_\Gamma$ and x^2_Γ, $y^2_\Gamma \in Q$, implying that $x^2_\Gamma \sigma_\Gamma y^2_\Gamma$, $x^2_\Gamma \sigma_\Gamma x\Gamma y\sigma_\Gamma y^2_\Gamma$. Thus $x\sigma_\Gamma y$ because σ_Γ is separative. ∎

Lemma 70 *Every left zero congruence is commutative.*

Proof. Let $a\sigma_\Gamma a$ and $b\sigma_\Gamma b$ which implies that $a\Gamma b\sigma_\Gamma a\Gamma b$, $(a\Gamma b)\Gamma(a\Gamma b)\sigma(a\Gamma b)^2_\Gamma = (b_\Gamma a)^2_\Gamma$ and so we obtain $a\Gamma b\sigma_\Gamma b\Gamma a$. ■

The relation η_Γ define on S by $a\eta_\Gamma b$ if and only if there exists some positive integers m, n such that $b^m_\Gamma \in a\Gamma S$ and $a^n_\Gamma \in b\Gamma S$.

Theorem 71 *Let S be a locally associative Γ-AG^{**}-groupoid. Then the relation η_Γ is the least semilattice congruence on S.*

Proof. The relation η_Γ is obviously reflexive and symmetric. To show transitivity, let $a\eta_\Gamma b$ and $b\eta_\Gamma c$, where $a, b, c \in S$. Then $a\Gamma x = b^m_\Gamma$ for some x and $b\Gamma y = c^n_\Gamma$, for some x and $y \in S$. Then we get

$$c^{mn}_\Gamma = (c^n_\Gamma)^m_\Gamma = (b_\Gamma y)^m_\Gamma = y^m_\Gamma \Gamma b^m_\Gamma = y^m_\Gamma \Gamma(a\Gamma x) = a\Gamma(y^m_\Gamma \Gamma x),$$

implies that $c^k_\Gamma = a\Gamma z$, where $k = mn$ and $z = (y^m_\Gamma \Gamma x)$. Similarly, $b\Gamma x' = a^{m\prime}_\Gamma$ and $c\Gamma y' = b^{n'}_\Gamma$ implies that $a^{k'}_\Gamma = c\Gamma z'$.

Let $a, b, c \in S$ and $a\eta_\Gamma b \Leftrightarrow (\exists\ m, n \in Z^+)(\exists\ x,\ y \in S)\ b^m_\Gamma = a\Gamma x, a^n_\Gamma = b\Gamma y$. If $m = 1$, $n > 1$, that is $b = a\Gamma x$, $a^n_\Gamma = b\Gamma y$ for some $x, y \in S$, then

$$b^3_\Gamma = (b\Gamma b)\Gamma(a\Gamma x) = a\Gamma(b^2_\Gamma \Gamma x) \in a\Gamma S.$$

Similarly we can consider the case $m = n = 1$. Suppose that m, $n > 1$. Then we obtain

$$\begin{aligned}(b\Gamma c)^m_\Gamma &= b^m_\Gamma \Gamma c^m_\Gamma = (a\Gamma x)\Gamma c^m_\Gamma = (a\Gamma x)\Gamma(c\Gamma c^{m-1}_\Gamma) \\ &= (a\Gamma c)\Gamma(x\Gamma c^{m-1}) = (a\Gamma c)\Gamma y, \text{ where } y = x\Gamma c^{m-1}_\Gamma.\end{aligned}$$

Thus $a\Gamma c\eta b\Gamma c$ and $c\Gamma a\eta c\Gamma b$.

Now to show that η_Γ is a semilattice congruence on S, first we need to show that $a\eta_\Gamma b$ implies $a\Gamma b\eta_\Gamma a$.

Let $a\eta_\Gamma b$, then $b^m_\Gamma = a\Gamma x$ and $a^n_\Gamma = b\Gamma y$ for some x and $y \in S$. So

$$(a\Gamma b)^m_\Gamma = a^m_\Gamma \Gamma b^m_\Gamma = a^m_\Gamma \Gamma(a\Gamma x) = a\Gamma(a^m_\Gamma \Gamma x).$$

Also $a^n_\Gamma = b\Gamma y$ implies that $a^{n+2}_\Gamma = a^2_\Gamma \Gamma a^n_\Gamma = (a\Gamma a)\Gamma(b\Gamma y) = (a\Gamma b)\Gamma(a\Gamma y)$. Hence $a\Gamma b\eta_\Gamma a$ which implies that $a^2_\Gamma \eta_\Gamma a$, $(a^2_\eta)_\Gamma = (a_\eta)_\Gamma$ and so S/η_Γ is idempotent.

Next we show that η_Γ is commutative. By Proposition 4, $(a\Gamma b)^2_\Gamma = (b\Gamma a)^2_\Gamma$, which shows that $a\Gamma b\eta b\Gamma a$ that is $a_\eta \Gamma b_\eta = b_\eta \Gamma a_\eta$, that is S/η_Γ is a commutative AG-groupoid and so is left zero commutative semigroup of idempotents. Therefore η_Γ is a semilattice congruence on S. Next we will show that η_Γ is contained in any other left zero semilattice congruence ρ_Γ on S. Let $a\eta_\Gamma b$, then $b^m_\Gamma = a\Gamma x$ and $a^n_\Gamma = b\Gamma y$. Now since $a\rho_\Gamma a^2_\Gamma$ and $b\rho_\Gamma b^2_\Gamma$, it implies that $a\Gamma x\rho_\Gamma a^2_\Gamma \Gamma x$, $a\rho_\Gamma a^n_\Gamma$ and $b\rho_\Gamma b^m_\Gamma$ which further implies that $a\rho_\Gamma b\Gamma y$ and $b\rho_\Gamma a\Gamma x$. It is easy fact that $a\Gamma b\rho b\Gamma a$, for some $\Gamma \in \Gamma$. Also since $b\rho_\Gamma b_\Gamma^2$ and ρ_Γ is compatable, so we get $b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$. We can easily see that $b\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma y$ which implies that $b\Gamma a\rho_\Gamma b_\Gamma^2\Gamma y$. Similarly we can show that $a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x$. So $a\rho_\Gamma b\Gamma y\rho_\Gamma b_\Gamma^2\Gamma yb\Gamma a\rho_\Gamma a\Gamma b\rho_\Gamma a_\Gamma^2\Gamma x\rho_\Gamma a\Gamma x\rho_\Gamma b$ implies that $a\rho_\Gamma b$. Thus η_Γ is a least semilattice congruence on S. ■

Theorem 72 *η_Γ is separative.*

Proof. Let $a_\Gamma^2\eta_\Gamma a\Gamma b$ and $a\Gamma b\eta_\Gamma b_\Gamma^2$, then there exist positive integers m, $m^{'}$ and $n^{'}, n$ such that:

$$\begin{aligned} (a_\Gamma^2)_\Gamma^m &= (a\Gamma b)_\Gamma^2\Gamma x, (a\Gamma b)_\Gamma^{m^{'}} = (a_\Gamma^2)_\Gamma^2\Gamma x^{'} \text{ and} \\ (a\Gamma b)_\Gamma^{n^{'}} &= (b_\Gamma^2)_\Gamma^2\Gamma y^{'}, (b_\Gamma^2)_\Gamma^n = (a\Gamma b)_\Gamma^2\Gamma y. \end{aligned}$$

Now we get,

$$\begin{aligned} a_\Gamma^{2m+2} &= a_\Gamma^{2m}\Gamma a_\Gamma^2 = (a_\Gamma^2)_\Gamma^m\Gamma a_\Gamma^2 = ((a_\Gamma\Gamma b)_\Gamma^2\Gamma x)\Gamma a_\Gamma^2 \\ &= (a^2\Gamma x)\Gamma(a\Gamma b)_\Gamma^2 = (a^2\Gamma x)\Gamma(a_\Gamma^2\Gamma b_\Gamma^2) = (a^2\Gamma x)\Gamma(b_\Gamma^2\Gamma a_\Gamma^2) \\ &= b_\Gamma^2\Gamma((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2) = b_\Gamma^2\Gamma t_6, \text{ where } t_6 = ((a_\Gamma^2\Gamma x)\Gamma a_\Gamma^2). \end{aligned}$$

Similarly,

$$\begin{aligned} b_\Gamma^{2n+2} &= b_\Gamma^{2n}\Gamma b_\Gamma^2 = ((a\Gamma b)_\Gamma^2\Gamma y)\Gamma b_\Gamma^2 = (b_\Gamma^2\Gamma y)\Gamma(a_\Gamma^2\Gamma b_\Gamma^2) = a_\Gamma^2\Gamma((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2) \\ &= a^2\Gamma t_7, \text{ where } t_7 = ((b_\Gamma^2\Gamma y)\Gamma b_\Gamma^2). \end{aligned}$$

Hence η_Γ is separative. ■

Theorem 73 *Let S be a locally associative Γ-AG^{**}-groupoid. Then S/η_Γ is a maximal semilattice separative image of S.*

Proof. By Theorem 6, η_Γ is the least semilattice congruence on S and S/η_Γ is a semilattice. Hence S/η_Γ is a maximal semilattice separative image of S. ■

2.3 Decomposition to Archimedean Locally Associative AG-subgroupoids

Theorem 74 *Every locally associative Γ-AG^{**}-groupoid S is uniquely expressible as a semilattice Y of Archimedean locally associative Γ-AG^{**}-groupoids $(S_\pi)_\Gamma$ $(\pi \in Y)$. The semilattice Y is isomorphic with the maximal semilattics separative image S/η_Γ of S and $(S_\pi)_\Gamma$ $(\pi \in Y)$ are the equivalence classes of S modη_Γ.*

Proof. η_Γ is least semilattice congruence on S. Next we will prove that equivalence classes modη_Γ are Archimedean locally associative Γ-AG**-groupoids and the semilattice Y is isomorphic to S/η_Γ. Let $a, b \in (S_\pi)_\Gamma$, where $\pi \in Y$, then $a\eta_\Gamma b$ implies that $a_\Gamma^m \in b\Gamma S$, $b_\Gamma^n \in a\Gamma S$, so $a_\Gamma^m = b\Gamma x$ and $b_\Gamma^n = a\Gamma y$, where $x, y \in S$. If $x \in S_\theta$, $\theta \neq \pi$ then $\pi = \pi\theta$, then we get $a_\Gamma^{m+1} = a\Gamma a_\Gamma^m = a\Gamma(b\Gamma x) = b\Gamma(a\Gamma x) \in b\Gamma(S_{\pi\theta})_\Gamma = b\Gamma(S_\pi)_\Gamma$. Similarly one can show that $b_\Gamma^{n+1} \in a\Gamma(S_\pi)_\Gamma$. This shows that $(S_\pi)_\Gamma$ is right Archimedean and so is locally associative Archimedean Γ-AG**-groupoid S. Next we show the uniqueness. Let S be a semilattice Y of Archimedean AG**-groupoid $(S_\pi)_\Gamma$, $\pi \in Y$. We need to show that $(S_\pi)_\Gamma$ are equivalence classes of S mod η_Γ. Let $a, b \in S$.Then we show that $a\eta_\Gamma b$ if and only if a and b belong to the same $(S_\pi)_\Gamma$. If a and b both belong to the same $(S_\pi)_\Gamma$, then each divides the power of the other. Since $(S_\pi)_\Gamma$ is Archimedean, $a\eta_\Gamma b$ by definition. Conversely, if $a\eta_\Gamma b$ then $a\Gamma x = b_\Gamma^m$ and $b\Gamma y = a_\Gamma^n$ for some $x, y \in S$ and some $m, n \in Z^+$. If $x \in (S_\partial)_\Gamma$, then $a\Gamma x \in (S_{\pi\partial})_\Gamma$ and $b_\Gamma^m \in (S_\theta)_\Gamma$, so that $\pi\partial = \theta$. Hence $\theta \leq \pi$, in the semilattice Y. By symmetry, it follows that $\pi \leq \theta$ that is $\pi = \theta$. $\blacksquare$

3

Embedding and Direct Product of AG-groupoids

3.1 Embedding in AG-groupoids

In this chapter we prove that under certain conditions a right cancellative AG**-groupoid can be embedded in a cancellative commutative monoid whose special type of elements form an abelian group and the identity of this group coincides with the identity of the commutative monoid.

An element a in an AG-groupoid S is called left cancellative, if $ab = ac$ implies that $b = c$. Similarly, c is right cancellative, if $ac = bc$ implies that $a = b$.

In this chapter we shall consider that S is a right cancellative AG**-groupoid with left identity and T is a subgroupoid of S such that elements of S commute with elements of T^2. A relation ρ has been introduced on the subset N of $S \times T^2$, so that we obtain an AG-groupoid with right identity. We have proved that N/ρ is a cancellative commutative monoid. A mapping from S to N/ρ has been defined to show that it is in fact an epimorphism from S to a commutative sub-monoid A, of N/ρ. At the end it has been shown that special type of elements of N/ρ form an Abelian group.

Lemma 75 *If S is an AG**-groupoid, then $(ab)^2 = a^2b^2 = b^2a^2$, for all a, b in S.*

Proof. By (2) and (4), we get $(ab)^2 = (ab)(ab) = (aa)(bb) = a^2b^2$, also $(ab)^2 = (ab)(ab) = (ba)(ba) = b^2a^2$. ■

Example 76 *Let $S = \{a, b, c\}$, and the binary operation $(\cdot)$ be defined on S as follows:*

$\cdot$	a	b	c
a	c	a	b
b	b	c	a
c	a	b	c

Then $(S, \cdot)$ is an AG**-groupoid with left identity c. Clearly it is cancellative.

Example 77 *Let $S = \{1, 2, 3, 4\}$, the binary operation $(\cdot)$ be defined on S*

as follows:

$\cdot$	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	2	3	3	3

It is non-commutative and non-associative because $4 = 1 \cdot 4 \neq 4 \cdot 1 = 2$, $2 = (2 \cdot 1) \cdot 1 \neq 2 \cdot (1 \cdot 1) = 4$. $(S, \cdot)$ *is an* AG^{**}*-groupoid. The subset* $A = \{2, 4\}$*, of* S*, is a commutative sub-semigroup of* S*.*

3.2 Main Results

Theorem 78 *If* T *is a subgroupoid of a right cancellative* AG^{**}*-groupoid* S *with left identity and elements of* S *commute with elements of* T^2*, then* S *becomes a commutative monoid.*

Proof. Let $N = \{(s_i t_j^2, t_k^2) : s_i \in S \text{ and } t_j, t_k \in T\}$, clearly N is closed because by (2) and lemma 75, we get $(s_i t_j^2, t_k^2)(s_l t_m^2, t_n^2) = ((s_i s_l)(t_j t_m)^2, (t_k t_n)^2)$, for all $s_i, s_l \in S$ and $t_j, t_m, t_k, t_n \in T$. Define a relation ρ on N as $(s_i t_j^2, t_k^2)\rho(s_l t_m^2, t_n^2)$ if and only if $(s_i t_j^2)t_n^2 = (s_l t_m^2)t_k^2$. It is easy to prove that ρ is reflexive and symmetric. To prove that ρ is transitive, we proceed as follows. Let $(s_i t_j^2, t_k^2)\rho(s_l t_m^2, t_n^2)$ and $(s_l t_m^2, t_n^2)\rho\left(s_p t_q^2, t_r^2\right)$. Then $(s_i t_j^2)t_n^2 = (s_l t_m^2)t_k^2$ and $(s_l t_m^2)t_r^2 = (s_p t_q^2)t_n^2$. Multiply the first equation from left by t_r^2, then by lemma 75, we obtain $t_n^2((s_i t_j^2)t_r^2) = t_n^2((s_p t_q^2)t_k^2)$ which implies that $(s_i t_j^2)t_r^2 = (s_p t_q^2)t_k^2$, thus $(s_i t_j^2, t_k^2)\rho\left(s_p t_q^2, t_r^2\right)$, proving that ρ is transitive.

If $(s_i t_j^2, t_k^2)\rho(s_l t_m^2, t_n^2)$, then $(s_i t_j^2)t_n^2 = (s_l t_m^2)t_k^2$, now we get $(t_n^2 t_j^2)s_i = (t_k^2 t_m^2)s_l$. Multiplying this equation by s_p from left side and we get $(t_n^2 t_j^2)(s_p s_i) = (t_k^2 t_m^2)(s_p s_l)$, now multiply this equation by $t_q^2 t_r^2$ from right side and using lemma 75, we get $((s_i t_j^2)(s_p t_q^2))(t_n^2 t_r^2) = (s_l t_m^2)(s_p t_q^2)(t_k^2 t_r^2)$. Thus

$$((s_i t_j^2)(s_p t_q^2), t_k^2 t_r^2)\rho((s_l t_m^2)(s_p t_q^2), t_n^2 t_r^2)$$

that is,

$$(s_i t_j^2, t_k^2)(s_p t_q^2, t_r^2)\rho(s_l t_m^2, t_n^2)(s_p t_q^2, t_r^2).$$

This shows that ρ is right compatible. Similarly we can show that ρ is left compatible. Hence ρ is a congruence relation on N.

Let $M = N/\rho = \{[(s_i t_j^2, t_k^2)] : s_i \in S \text{ and } t_j, t_k \in T\}$ where $[(s_i t_j^2, t_k^2)]$ represents any class in N/ρ. Then it is easy to see that M is an AG**-groupoid. Clearly $[(t_o^2, t_o^2)]$ is the right identity in M, where t_0 is an arbitrary element of T, because if $[(s_i t_j^2, t_k^2)]$ is an arbitrary element in M, then $((s_i t_j^2)t_o^2)t_k^2 = (s_i t_j^2)(t_k^2 t_o^2)$. Therefore $((s_i t_j^2)t_o^2, t_k^2 t_o^2)\rho(s_i t_j^2, t_k^2)$ which implies that $(s_i t_j^2, t_k^2)(t_o^2, t_o^2)\rho(s_i t_j^2, t_k^2)$ or $[(s_i t_j^2, t_k^2)][(t_o^2, t_o^2)] = [(s_i t_j^2, t_k^2)]$.

Hence $[(t_o^2, t_o^2)]$ is the right identity in M. Since M is an AG**-groupoid with right identity so it will become a commutative monoid.

Let t_x be any fixed element of T. We define a mapping $\Phi : S \longrightarrow M$ by $(s_i)\Phi = [(s_i t_x^2, t_x^2)]$, for all $s_i \in S$ and $t_x \in T$. Suppose $s_i, s_j \in S$ such that $s_i = s_j$. Then clearly $[(s_i t_x^2, t_x^2)] = [(s_j t_x^2, t_x^2)]$ for $t_x \in T$. Thus $(s_i)\Phi = (s_j)\Phi$. This shows that Φ is well defined. Next we show that $(s_i s_j)\,\Phi = (s_i)\Phi(s_j)\Phi$. Since $(s_i)\Phi(s_j)\Phi = [((s_i s_j)(t_x^2 t_x^2), t_x^2 t_x^2)]$. Also using lemma 75, we get $((s_i s_j)(t_x^2 t_x^2))t_x^2 = (t_x^2(t_x^2 t_x^2))(s_i s_j)) = ((t_x^2 t_x^2)t_x^2)(s_i s_j)) = ((s_i s_j)t_x^2)(t_x^2 t_x^2)$, this implies that $((s_i s_j)(t_x^2 t_x^2), t_x^2 t_x^2)\rho((s_i s_j)t_x^2, t_x^2)$ and so $[((s_i s_j)(t_x^2 t_x^2), t_x^2 t_x^2)] = \left[((s_i s_j)t_x^2, t_x^2)\right] = (s_i s_j)\Phi$. Hence $(s_i)\Phi(s_j)\Phi = (s_i s_j)\Phi$. This shows that Φ is a homomorphism.

It is one-to-one, because $(s_i)\Phi = (s_j)\Phi$ implies that $[(s_i t_x^2, t_x^2)] = [(s_j t_x^2, t_x^2)]$, that is, $(s_i t_x^2, t_x^2)\rho(s_j t_x^2, t_x^2)$. Thus $(s_i t_x^2)t_x^2 = \left(s_j t_x^2\right) t_x^2$, which implies that $s_i = s_j$.

If $A = \{[(s_i t_x^2, t_x^2)] : s_i \in S \text{ and } t_x \in T\}$. Then $A \subset M$ and monomorphism $\Phi : S \longrightarrow A$ is onto. As for every $[(s_i t_x^2, t_x^2)]$ in A there exists s_i such that $(s_i)\Phi = [(s_i t_x^2, t_x^2)]$. Clearly $[(t_o^2, t_o^2)]$ belongs to A. ■

Lemma 79 *A right cancellative AG-groupoid with left identity is left cancellative.*

Proof. It is easy. ■

Since S contains the left identity so it is easy to see that $[(t_j^2, t_k^2)] \in M$.

Now we prove the following theorem.

Theorem 80 *M is cancellative and elements of the form $[(t_i^2, t_j^2)]$ in M, form an Abelian group.*

Proof. Let us suppose that $(s_i t_j^2, t_k^2)(s_p t_q^2, t_r^2)\rho(s_l t_m^2, t_n^2)(s_p t_q^2, t_r^2)$, that is,

$$[(s_i t_j^2, t_k^2)][(s_p t_q^2, t_r^2)] = [(s_l t_m^2, t_n^2)][(s_p t_q^2, t_r^2)]$$

which implies that

$$[(s_i t_j^2)(s_p t_q^2), t_k^2 t_r^2)] = [(s_l t_m^2)(s_p t_q^2), t_n^2 t_r^2)],$$

Then we get,

$$[(s_i s_p)(t_j^2 t_q^2), t_k^2 t_r^2)] = [(s_l s_p)(t_m^2 t_q^2), t_n^2 t_r^2)]$$

which implies that

$$((s_i s_p)(t_j^2 t_q^2))(t_n^2 t_r^2) = ((s_l s_p)(t_m^2 t_q^2))(t_k^2 t_r^2).$$

Now lemma 75, we get

$$((s_i s_p)(t_n^2 t_j^2))(t_r^2 t_q^2) = ((s_l s_p)(t_k^2 t_m^2))(t_r^2 t_q^2),$$

now since S is right cancellative so we get $(s_i s_p)(t_n^2 t_j^2) = (s_l s_p)(t_k^2 t_m^2)$ which by lemma 75 implies that $s_p((t_n^2 t_j^2)s_i) = s_p((t_k^2 t_m^2)s_l)$, therefore by lemma 79, we get $(t_n^2 t_j^2)s_i = (t_k^2 t_m^2)s_l$, using we get,$(s_i t_j^2)t_n^2 = (s_l t_m^2)t_k^2$. Thus $(s_i t_j^2, t_k^2)\rho(s_l t_m^2, t_n^2)$. Hence M is right cancellative. Similarly we can show that M is left cancellative. Now using lemma 75, we can easily see that $(t_i^2 t_j^2)t_o^2 = (t_j^2 t_i^2)t_o^2$ which implies that $(t_i^2 t_j^2, t_o^2)\rho(t_j^2 t_i^2, t_o^2)$, that is, $[(t_i^2, t_j^2)][(t_j^2, t_i^2)] = [(t_o^2, t_o^2)]$. Thus $[(t_i^2, t_j^2)]$ is the inverse of $[(t_j^2, t_i^2)]$. Hence all the cancellative elements $[(t_i^2, t_j^2)]$ of M form an Abelian group G in M. We note that the product of two cancellative elements of G, is in G. We have proved in theorem 1, that $[(t_o^2, t_o^2)]$ is the identity element of M, since G contains elements of the form $[(t_x^2, t_y^2)]$, therefore $[(t_o^2, t_o^2)]$ is in G which is unique since G is a group. ■

3.3 Direct Products in AG-groupoids

In this section we show that the direct product of regular $\mathcal{AG}$-groupoids is the most generalized class of the direct product of an $\mathcal{AG}$-groupoids. It has proved that the direct product of weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular and $(2,2)$-regular $\mathcal{AG}$-groupoids with left identity coincide. Also we have proved that the direct product of intra-regular $\mathcal{AG}$-groupoids with left identity ($\mathcal{AG}^{**}$-groupoids) is regular but the converse is not true in general. Further we have shown that non-associative direct product of regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular $\mathcal{AG}^*$-groupoids do not exist.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are AG-groupoids, then $\mathcal{S}_1 \times \mathcal{S}_2 = \{(s_1, s_2) : s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2\}$ is an AG-groupoid under the point-wise multiplication of ordered pairs.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exist $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = ((ax)a, (bm)b)$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called regular if all elements of $\mathcal{S}$ are regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **weakly regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exist $x, y \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_2$ such that $(a,b) = ((ax)(ay), (bl)(bm))$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called weakly regular if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are weakly regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called an **intra-regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exist $x, y \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_2$ such that $(a,b) = ((xa^2)y, (lb^2)m)$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called intra-regular if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are intra-regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **right regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exists $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = (a^2x, b^2m) = ((aa)x, (bb)m)$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called right regular if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are right regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **left regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exists $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = (xa^2, mb^2) = (x(aa), m(bb))$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called left regular if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are left regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **left quasi regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exist $x, y \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_2$ such that $(a,b) = ((xa)(ya), (lb)(mb))$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called left quasi regular if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are left quasi regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **completely regular** element of $\mathcal{S}$ if (a,b) is regular, left regular and right regular. $\mathcal{S}_1 \times \mathcal{S}_2$ is called completely regular if it is regular, left and right regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **(2,2)-regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exists $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = ((a^2x)a^2, (b^2m)b^2)$ and $\mathcal{S}_1 \times \mathcal{S}_2$ is called $(2,2)$-regular $\mathcal{AG}$-groupoid if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are $(2,2)$-regular.

An element (a,b) of an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ is called a **strongly regular** element of $\mathcal{S}_1 \times \mathcal{S}_2$ if there exists $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = ((ax)a, (bm)b)$ and $ax = xa$, $bm = mb$. $\mathcal{S}_1 \times \mathcal{S}_2$ is called strongly regular $\mathcal{AG}$-groupoid if all elements of $\mathcal{S}_1 \times \mathcal{S}_2$ are strongly regular.

Example 81 *Let us consider an $\mathcal{AG}$-groupoid $\mathcal{S} = \{a, b, c\}$ in the following multiplication table.*

.	a	b	c
a	c	c	c
b	c	c	a
c	c	c	a

Clearly $\mathcal{S}$ is non-commutative and non-associative, because $bc \neq cb$ and $(cc)a \neq c(ca)$. Note that $\mathcal{S}$ has no left identity.

Example 82 *Let us consider an $\mathcal{AG}$-groupoid $\mathcal{S}_1 = \{a, b, c, d, e, f\}$ with left identity e and $\mathcal{S}_2 = \{g, h, i, j, k, l\}$ with left identity j in the following Cayley's tables.*

.	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	f	f	d	f
d	a	b	f	f	c	f
e	a	b	c	d	e	f
f	a	b	f	f	f	f

.	g	h	i	j	k	l
g	g	g	g	g	g	g
h	g	h	h	h	h	h
i	g	h	l	l	i	l
j	g	h	i	j	k	l
k	g	h	l	l	l	l
l	g	h	l	l	j	l

Clearly $\mathcal{S}_1 \times \mathcal{S}_2$ is non-commutative and non-associative, because $(ed, ik) \neq (de, ki)$ and $((de)e, (ik)k) \neq (d(ee), i(kk))$.

Lemma 83 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular $\mathcal{AG}$-groupoid, then $\mathcal{S}_1 \times \mathcal{S}_2 = (\mathcal{S}_1 \times \mathcal{S}_2)^2$.*

Proof. Let $\mathcal{S}_1 \times \mathcal{S}_2$ be a regular $\mathcal{AG}$-groupoid, then $(\mathcal{S}_1 \times \mathcal{S}_2)^2 \subseteq \mathcal{S}_1 \times \mathcal{S}_2$ is obvious. Let $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ where $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$, then since $\mathcal{S}_1 \times \mathcal{S}_2$ is regular so there exists $(x,y) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that $(a,b) = ((ax)a, (by)b)$. Now by using (2), we have

$$(a,b) = ((ax)a, (by)b) = ((ax)(by), (ab)) \in (\mathcal{S}_1 \times \mathcal{S}_2)(\mathcal{S}_1 \times \mathcal{S}_2) = (\mathcal{S}_1 \times \mathcal{S}_2)^2.$$

Similarly if $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that $\mathcal{S}_1 \times \mathcal{S}_2 = (\mathcal{S}_1 \times \mathcal{S}_2)^2$. ■

The converse is not true in general, because $\mathcal{S}_1 \times \mathcal{S}_2 = (\mathcal{S}_1 \times \mathcal{S}_2)^2$ holds but $\mathcal{S}_1 \times \mathcal{S}_2$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular, because $(d,k) \in \mathcal{S}_1 \times \mathcal{S}_2$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

Theorem 84 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular if and only if for all $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$, $(a,b) = ((xa)(az), (lb)(bm))$ holds for some $x, z \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_2$.*

Proof. Let $\mathcal{S}_1 \times \mathcal{S}_2$ be an intra-regular $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$, there exist $x, y \in \mathcal{S}_1$ and $l, k \in \mathcal{S}_2$ such that $(a,b) = ((xa^2)y, (lb^2)k)$. Now $y = uv$ and $k = pq$ for some

$u, v \in \mathcal{S}_1$ and $p, q \in \mathcal{S}_2$. Thus we have

$$\begin{aligned}
(a,b) &= ((xa^2)y, (lb^2)k) = ((x(aa))y, (l(bb))k) \\
&= ((a(xa))y, (b(lb))k) = ((y(xa))a, (k(lb))b) \\
&= ((y(xa))((xa^2)y), (k(lb))((lb^2)k)) \\
&= (((uv)(xa))((xa^2)y), ((pq)(lb))((lb^2)k)) \\
&= (((ax)(vu))((xa^2)y), ((bl)(qp))((lb^2)k)) \\
&= (((ax)t)((xa^2)y), ((bl)j)((lb^2)k)) \\
&= ((((xa^2)y)t)(ax), (((lb^2)k)j)(bl)) \\
&= (((ty)(xa^2))(ax), ((jk)(lb^2))(bl)) \\
&= (((a^2x)(yt))(ax), ((b^2l)(kj))(bl)) \\
&= (((a^2x)s)(ax), ((b^2l)r)(bl)) \\
&= (((sx)(aa))(ax), ((rl)(bb))(bl)) \\
&= (((aa)(xs))(ax), ((bb)(lr))(bl)) \\
&= (((aa)w)(ax), ((bb)n)(bl)) \\
&= (((wa)a)(ax), ((nb)b)(bl)) \\
&= ((za)(ax), (mb)(bl)) \\
&= ((xa)(az), (lb)(bm)),
\end{aligned}$$

where $vu = t$, $qp = j$, $yt = s$, $kj = r$, $xs = w$, $lr = n$, $wa = z$ and $nb = m$ for some $t, s, w, z \in \mathcal{S}_1$ and $j, r, n, m \in \mathcal{S}_2$.

Conversely, let for all $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$, $(a,b) = ((xa)(az), (lb)(bm))$ holds for some $x, z \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_2$. Now we have

$$\begin{aligned}
(a,b) &= ((xa)(az), (lb)(bm)) = (a((xa)z), b((lb)m)) \\
&= (((xa)(az))((xa)z), ((lb)(bm))((lb)m)) \\
&= ((a((xa)z))((xa)z), (b((lb)m))((lb)m)) \\
&= ((((xa)z)((xa)z))a, (((lb)m)((lb)m))b) \\
&= ((((xa)(xa))(zz))a, (((lb)(lb))(mm))b) \\
&= ((((ax)(ax))(zz))a, (((bl)(bl))(mm))b) \\
&= (((a((ax)x))(zz))a, ((b((bl)l))(mm))b) \\
&= ((((zz)((ax)x))a)a, (((mm)((bl)l))b)b) \\
&= (((z^2((ax)x))a)a, ((m^2((bl)l))b)b) \\
&= ((((ax)(z^2x))a)a, (((bl)(m^2l))b)b) \\
&= (((((z^2x)x)a)a)a, ((((m^2l)l)b)b)b) \\
&= ((((x^2z^2)a)a)a, (((l^2m^2)b)b)b) \\
&= ((a^2(x^2z^2))a, (b^2(l^2m^2))b) \\
&= ((a(x^2z^2))(aa), (b(l^2m^2))(bb)) \\
&= ((at)(aa), (bs)(bb)),
\end{aligned}$$

where $x^2z^2 = t$ and $l^2m^2 = s$ for some $t \in \mathcal{S}_1$, $s \in \mathcal{S}_2$.
Now we have

$$\begin{aligned}
(a,b) &= ((at)(aa),(bs)(bb)) \\
&= ((((at)(aa))t)(aa),(((bs)(bb))s)(bb)) \\
&= ((((aa)(ta))t)(aa),(((bb)(sb))s)(bb)) \\
&= (((a^2(ta))t)(aa),((b^2(sb))s)(bb)) \\
&= (((t(ta))a^2)(aa),((s(sb))b^2)(bb)) \\
&= ((ua^2)v,(pb^2)q).
\end{aligned}$$

Where $t(ta) = u$, $aa = v$, $s(sb) = p$ and $bb = q$ for some $u,v \in \mathcal{S}_1$, $p,q \in \mathcal{S}_2$. Thus $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular. ■

Theorem 85 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.
(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular.
Proof. $(i) \Longrightarrow (ii)$ Let $\mathcal{S}_1 \times \mathcal{S}_2$ be a weakly regular $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x,y \in \mathcal{S}_1$ and $l,m \in \mathcal{S}_2$ such that $(a,b) = ((ax)(ay),(bl)(bm))$ and $x = uv$, $l = pq$ for some $u,v \in \mathcal{S}_1$, $p,q \in \mathcal{S}_2$. Let $vu = t \in \mathcal{S}_1$ and $qp = n \in \mathcal{S}_2$. Now we have

$$\begin{aligned}
(a,b) &= ((ax)(ay),(bl)(bm)) \\
&= ((ya)(xa),(mb)(lb)) \\
&= ((ya)((uv)a),(mb)((pq)b)) \\
&= ((ya)((av)u),(mb)((bq)p)) \\
&= ((av)((ya)u),(bq)((mb)p)) \\
&= ((a(ya))(vu),(b(mb))(qp)) \\
&= ((a(ya))t,(b(mb))n) \\
&= ((y(aa))t,(m(bb))n) = ((ya^2)t,(mb^2)n).
\end{aligned}$$

Thus $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular.
$(ii) \Longrightarrow (i)$ It is easy. ■

Theorem 86 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.
(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular.
Proof. $(i) \Longrightarrow (ii)$ Let $\mathcal{S}_1 \times \mathcal{S}_2$ be a weakly regular $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then for any $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x,y \in \mathcal{S}_1$ and $m,n \in \mathcal{S}_2$ such that $(a,b) = ((ax)(ay),(bm)(bn))$. Now let $xy = t$ and $mn = s$ for

some $t \in \mathcal{S}$. Now

$$\begin{aligned}(a,b) &= ((ax)(ay),(bm)(bn)) \\ &= ((aa)(xy),(bb)(mn)) = (a^2t, b^2s).\end{aligned}$$

Thus $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular.

$(ii) \Longrightarrow (i)$ It is easy. ■

Theorem 87 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.

(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is left regular.

Proof. $(i) \Longrightarrow (ii)$ Let $\mathcal{S}_1 \times \mathcal{S}_2$ be a weakly regular $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x, y \in \mathcal{S}_1$ and $m, n \in \mathcal{S}_2$ such that $(a,b) = ((ax)(ay),(bm)(bn))$. Now by using (2) and (3), we have

$$\begin{aligned}(a,b) &= ((ax)(ay),(bm)(bn)) = ((aa)(xy),(bb)(mn)) \\ &= ((yx)(aa),(nm)(bb)) = ((yx)a^2,(nm)b^2) \\ &= ((ta^2),(sb^2)) \text{ where } yx = t,\ nm = s \text{ for some } t \in \mathcal{S}_1 \text{ and } s \in \mathcal{S}_2.\end{aligned}$$

Thus $\mathcal{S}_1 \times \mathcal{S}_2$ is left regular.

$(ii) \Longrightarrow (i)$ It follows easily. ■

Theorem 88 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.

(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is left quasi regular.

Proof. The proof of this Lemma is straight forward, so is omitted. ■

Theorem 89 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity, then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is $(2,2)$-regular.

(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular.

Proof. $(i) \Longrightarrow (ii)$ Let $\mathcal{S}_1 \times \mathcal{S}_2$ be a $(2,2)$-regular $\mathcal{AG}$-groupoid with left identity, then for $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x \in \mathcal{S}_1$ and $m \in \mathcal{S}_2$ such that $(a,b) = ((a^2x)a^2,(b^2m)b^2)$. Now

$$(a,b) = ((a^2x)a^2,(b^2m)b^2) = (ya^2, nb^2), \text{ where } a^2x = y \in \mathcal{S}_1 \text{ and } b^2m = n \in \mathcal{S}_2,$$

and by using (3), we have

$$\begin{aligned}(a,b) &= ((a^2x)(aa),(b^2m)(bb)) \\ &= ((aa)(xa^2),(bb)(mb^2)) \\ &= (a^2z, b^2l), \text{ where } xa^2 = z \in \mathcal{S}_1 \text{ and } mb^2 = l \in \mathcal{S}_2.\end{aligned}$$

And we have

$$\begin{aligned}
(a,b) &= ((a^2x)(aa),(b^2m)(bb)) \\
&= ((aa)(xa^2),(bb)(mb^2)) \\
&= ((aa)((ex)(aa)),(bb)((em)(bb))) \\
&= ((aa)((aa)(xe)),(bb)((bb)(me))) \\
&= ((aa)(a^2t),(bb)(b^2s)) \\
&= (((a^2t)a)a,((b^2s)b)b) \\
&= ((((aa)t)a)a,(((bb)s)b)b) \\
&= ((((ta)a)a)a,(((sb)b)b)b) \\
&= (((aa)(ta))a,((bb)(sb))b) \\
&= (((at)(aa))a,((bs)(bb))b) \\
&= ((a((at)a))a,(b((bs)b))b) \\
&= ((ay)a,(bn)b),\ \text{where } xe=t\in\mathcal{S}_1, \\
me &= s\in\mathcal{S}_2 \text{ and } (at)a=y\in\mathcal{S}_1,\ (bs)b=n\in\mathcal{S}_2.
\end{aligned}$$

Thus $\mathcal{S}_1\times\mathcal{S}_2$ is left regular, right regular and regular, so $\mathcal{S}_1\times\mathcal{S}_2$ is completely regular.

$(ii)\Longrightarrow(i)$ Assume that $\mathcal{S}_1\times\mathcal{S}_2$ is a completely regular $\mathcal{AG}$-groupoid with left identity, then for any $(a,b)\in\mathcal{S}_1\times\mathcal{S}_2$ there exist $x,y,z\in\mathcal{S}_1$ and $l,m,n\in\mathcal{S}_2$ such that $(a,b)=((ax)a,(bl)b)$, $(a,b)=(a^2y,b^2m)$, $(a,b)=(za^2,nb^2)$. Now

$$\begin{aligned}
(a,b) &= ((ax)a,(bl)b) \\
&= (((a^2y)x)(za^2),((b^2m)l)(nb^2)) \\
&= (((xy)a^2)(za^2),((lm)b^2)(nb^2)) \\
&= (((za^2)a^2)(xy),((nb^2)b^2)(lm)) \\
&= (((a^2a^2)z)(xy),((b^2b^2)n)(lm)) \\
&= (((xy)z)(a^2a^2),((lm)n)(b^2b^2)) \\
&= (a^2(((xy)z)a^2),b^2(((lm)n)b^2)) \\
&= ((ea^2)(((xy)z)a^2),(eb^2)(((lm)n)b^2)) \\
&= ((a^2((xy)z))(a^2e),(b^2((lm)n))(b^2e)) \\
&= ((a^2((xy)z))((aa)e),(b^2((lm)n))((bb)e)) \\
&= ((a^2((xy)z))((ea)a),(b^2((lm)n))((eb)b)) \\
&= ((a^2((xy)z))(aa),(b^2((lm)n))(bb)) \\
&= ((a^2t)a^2,(b^2s)b^2),\ \text{where } (xy)z=t\in\mathcal{S}_1 \text{ and } (lm)n=s\in\mathcal{S}_2.
\end{aligned}$$

This shows that $\mathcal{S}_1\times\mathcal{S}_2$ is $(2,2)$-regular. ■

Lemma 90 *Every weakly regular $\mathcal{AG}$-groupoid $\mathcal{S}_1\times\mathcal{S}_2$ with left identity ($\mathcal{AG}^{**}$-groupoid) is regular.*

Proof. Assume that $\mathcal{S}_1 \times \mathcal{S}_2$ is a weakly regular $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $(a,b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x, y \in \mathcal{S}_1$ and $m, n \in \mathcal{S}_2$ such that $(a,b) = ((ax)(ay), (bm)(bn))$. Let $xy = t \in \mathcal{S}_1$, $t((yx)a) = u \in \mathcal{S}_1$ and $mn = s \in \mathcal{S}_2$, $s((nm)b) = l \in \mathcal{S}_2$. Now by using (1), (2), (3) and (4), we have

$$\begin{aligned}
(a,b) &= ((ax)(ay), (bm)(bn)) = (((ay)x)a, ((bn)m)b) \\
&= (((xy)a)a, ((mn)b)b) = ((ta)a, (sb)b) \\
&= ((t((ax)(ay)))a, (s((bm)(bn)))b) \\
&= ((t((aa)(xy)))a, (s((bb)(mn)))b) \\
&= ((t((yx)(aa)))a, (s((nm)(bb)))b) \\
&= ((t(a((yx)a)))a, (s(b((nm)b)))b) \\
&= ((a(t((yx)a)))a, (b(s((nm)b)))b) \\
&= ((au)a, (bl)b).
\end{aligned}$$

Thus $\mathcal{S}_1 \times \mathcal{S}_2$ is regular. ■

The converse of above Lemma is not true in general, as can be seen from the following example.

Example 91 *[51] Let us consider an $\mathcal{AG}$-groupoid $\mathcal{S}_1 = \{1, 2, 3, 4\}$ with left identity 3 and $\mathcal{S}_2 = \{5, 6, 7, 8\}$ with left identity 6 in the following Cayley's tables.*

.	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

.	5	6	7	8
5	6	6	6	6
6	5	6	7	8
7	5	6	5	6
8	6	6	8	8

Clearly $\mathcal{S}_1 \times \mathcal{S}_2$ is regular, because $(1,5) = ((1.3).1, (5.6).5)$, $(2,6) = ((2.1).2, (6.8).6)$, $(3,7) = ((3.3).3, (7.6).7)$ and $(4,8) = ((4.1).4, (8.6).8)$, but $\mathcal{S}_1 \times \mathcal{S}_2$ is not weakly regular, because $(1,5) \in \mathcal{S}_1 \times \mathcal{S}_2$ is not a weakly regular element of $\mathcal{S}_1 \times \mathcal{S}_2$.

Theorem 92 *If $\mathcal{S}_1 \times \mathcal{S}_2$ is an $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.
(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular.
Proof. $(i) \Longrightarrow (ii)$ It follows easily
$(ii) \Longrightarrow (i)$ It is easy. ■

Lemma 93 *Every strongly regular $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ with left identity ($\mathcal{AG}^{**}$-groupoid) is completely regular.*

Proof. Assume that $\mathcal{S}_1 \times \mathcal{S}_2$ is a strongly regular $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $(a, b) \in \mathcal{S}_1 \times \mathcal{S}_2$ there exist $x \in \mathcal{S}_1$, $y \in \mathcal{S}_2$ such that $(a, b) = ((ax)a, (by)b)$, $ax = xa$ and $by = yb$. Now by using (1), we have

$$\begin{aligned} (a, b) &= ((ax)a, (by)b) = ((xa)a, (yb)b) \\ &= ((aa)x, (bb)y) = (a^2x, b^2y). \end{aligned}$$

This shows that $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular and so $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular. ■

Note that a completely regular $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ need not to be a strongly regular $\mathcal{AG}$-groupoid, as can be seen from the following example.

Example 94 *Let $\mathcal{S} = \{a, b, c, d, e, f, g\}$ be an $\mathcal{AG}$-groupoid with the following multiplication table.*

.	a	b	c	d	e	f	g
a	b	d	f	a	c	e	g
b	e	g	b	d	f	a	c
c	a	c	e	g	b	d	f
d	d	f	a	c	e	g	b
e	g	b	d	f	a	c	e
f	c	e	g	b	d	f	a
g	f	a	c	e	g	b	d

Clearly $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular. Indeed, $\mathcal{S}_1 \times \mathcal{S}_2$ is regular, as $a = (a.e).a$, $b = (b.a).b$, $c = (c.d).c$, $d = (d.g).d$, $e = (e.c).e$, $f = (f.f).f$, $g = (g.b).g$, also $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular, as $a = (a.a).f$, $b = (b.b).f$, $c = (c.c).f$, $d = (d.d).f$, $e = (e.e).f$, $f = (f.f).f$, $g = (g.g).f$, and $\mathcal{S}_1 \times \mathcal{S}_2$ is left regular, as $a = g.(a.a)$, $b = d.(b.b)$, $c = a.(c.c)$, $d = e.(d.d)$, $e = b.(e.e)$, $f = f.(f.f)$, $g = c.(g.g)$, but $\mathcal{S}_1 \times \mathcal{S}_2$ is not strongly regular, because $ax \neq xa$ for all $a \in \mathcal{S}_1 \times \mathcal{S}_2$.

Theorem 95 *In an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ with left identity ($\mathcal{AG}^{**}$-groupoid), the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.
(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular.
(iii) $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular.
(iv) $\mathcal{S}_1 \times \mathcal{S}_2$ is left regular.
(v) $\mathcal{S}_1 \times \mathcal{S}_2$ is left quasi regular.

(vi) $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular.

(vii) For all $(a, b) \in \mathcal{S}_1 \times \mathcal{S}_2$, there exist $x, y \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_1$ such that $(a, b) = ((xa)(az), (lb)(bm))$.

Proof. $(i) \Longrightarrow (ii)$ It follows from above Theorem.

$(ii) \Longrightarrow (iii)$ It follows from above Theorems.

$(iii) \Longrightarrow (iv)$ It follows from above Theorem.

$(iv) \Longrightarrow (v)$ It follows from above Theorem.

$(v) \Longrightarrow (vi)$ It follows from above Theorems.

$(vi) \Longrightarrow (i)$ It follows from above Theorem.

$(ii) \Longleftrightarrow (vii)$ It follows from above Theorem. $\blacksquare$

Remark 96 *Every intra-regular, right regular, left regular, left quasi regular and completely regular $\mathcal{AG}$-groupoids $\mathcal{S}_1 \times \mathcal{S}_2$ with left identity ($\mathcal{AG}^{**}$-groupoids) are regular.*

The converse of above is not true in general. Indeed, from above Example regular $\mathcal{AG}$-groupoid with left identity is not necessarily intra-regular.

Theorem 97 *In an $\mathcal{AG}$-groupoid $\mathcal{S}_1 \times \mathcal{S}_2$ with left identity, the following are equivalent.*

(i) $\mathcal{S}_1 \times \mathcal{S}_2$ is weakly regular.

(ii) $\mathcal{S}_1 \times \mathcal{S}_2$ is intra-regular.

(iii) $\mathcal{S}_1 \times \mathcal{S}_2$ is right regular.

(iv) $\mathcal{S}_1 \times \mathcal{S}_2$ is left regular.

(v) $\mathcal{S}_1 \times \mathcal{S}_2$ is left quasi regular.

(vi) $\mathcal{S}_1 \times \mathcal{S}_2$ is completely regular.

(vii) For all $(a, b) \in \mathcal{S}_1 \times \mathcal{S}_2$, there exist $x, y \in \mathcal{S}_1$ and $l, m \in \mathcal{S}_1$ such that $(a, b) = ((xa)(az), (lb)(bm))$.

$(viii)$ $\mathcal{S}_1 \times \mathcal{S}_2$ is $(2, 2)$-regular.

Proof. It is easy. $\blacksquare$

Remark 98 $(2, 2)$*-regular and strongly regular $\mathcal{AG}$-groupoids $\mathcal{S}_1 \times \mathcal{S}_2$ with left identity are regular.*

The converse of above is not true in general, as can be seen from above Example.

Theorem 99 *Direct product of regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular,* $(2, 2)$*-regular and strongly regular $\mathcal{AG}^*$-groupoids $\mathcal{S}_1 \times \mathcal{S}_2$ becomes semigroups.*

4

Ideals in Abel-Grassmann's Groupoids

In this chapter we introduce the concept of left, right, bi, quasi, prime (quasi-prime) semiprime (quasi-semiprime) ideals in AG-groupoids. We introduce m-system in AG-groupoids. We characterize quasi-prime and quasi-semiprime ideals and find their links with m systems. We characterize ideals in intra-regular AG-groupoids. Then we characterize intra-regular AG-groupoids using the properties of these ideals.

4.1 Preliminaries

Let S be an AG-groupoid. By an **AG-subgroupoid** of S, we means a non-empty subset A of S such that $A^2 \subseteq A$.

A non-empty subset A of an AG-groupoid S is called a **left (right) ideal** of S if $SA \subseteq A$ ($AS \subseteq A$) and it is called a **two-sided ideal** if it is both left and a right ideal of S.

A non-empty subset A of an AG-groupoid S is called a **generalized bi-ideal** of S if $(AS)A \subseteq A$ and an AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A \subseteq A$.

A non-empty subset A of an AG-groupoid S is called a **quasi-ideal** of S if $SA \cap AS \subseteq A$.

Note that every one sided ideal of an AG-groupoid S is a quasi-ideal and right ideal of S is bi-ideal of S.

A non-empty subset A of an AG-groupoid S is called **semiprime** if $a^2 \in A$ implies $a \in A$.

An AG-subgroupoid A of an AG-groupoid S is called a interior ideal of S if $(SA)S \subseteq A$.

An ideal P of an AG-groupoid S is said to be **prime** if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, where A and B are ideals of S. A left ideal P of an AG-groupoid S is said to be a **quasi-prime** if for left ideals A and B of S such that $AB \subseteq P$, we have either $A \subseteq P$ or $B \subseteq P$.

An ideal P of an AG-groupoid S is called **strongly irreducible** if $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, for all ideals A, B and P of S.

If S is an AG-groupoid with left identity e then the **principal left ideal** generated by a fixed element "a" is defined as $\langle a \rangle = Sa = \{sa : s \in S\}$. Clearly, $\langle a \rangle$ is a left ideal of S contains a. Note that if A is an ideal of S, then A^2 is an ideal of S. Also it is easy to verify that $A = \langle A \rangle$ and

$A^2 = \langle A^2 \rangle$.

If an AG-groupoid S contains left identity e then $S = eS \subseteq S^2$. Therefore $S = S^2$. Also Sa becomes bi-ideal and quasi-ideal of S. Using paramedial, medial and left invertive law we get

$$((Sa)S)Sa \subseteq (SS)(Sa) = (aS)(SS) = (aS)S = (SS)a = Sa,$$

It is easy to show that $(Sa)(Sa) \subseteq S(Sa)$. Hence Sa is a bi-ideal of S. Also

$$S(Sa) \cap (Sa)S \subseteq S(Sa) \subseteq Sa.$$

Therefore Sa is a quasi-ideal of S. Also using medial and paramedial laws and (1), we get

$$\begin{aligned} (Sa)^2 &= (Sa)(Sa) = (SS)a^2 = (aa)(SS) = S((aa)S) \\ &= (SS)((aa)S) = (Sa^2)SS = (Sa^2)S. \end{aligned}$$

Therefore $Sa^2 = a^2S = (Sa^2)S$.

Example 100 *Let $S = \{1, 2, 3, 4, 5, 6\}$, and the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4	5	6
1	x	x	x	x	x	x
2	x	x	x	x	x	x
3	x	x	x	x	x	x
4	x	x	x	x	x	x
5	x	x	x	x	x	x
6	x	2	x	x	x	x

Where $x \in \{1, 3, 4, 5\}$. Then $(S, .)$ is an AG-groupoid and $\{2, x\}$ is an ideal of S.

A subset M of an AG-groupoid S is called an m-system if for all $a, b \in M$, there exists $a_1 \in \langle a \rangle$, there exists $b_1 \in \langle b \rangle$ such that $a_1b_1 \in M$ [50].

Example 101 *Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	4	4	4	4	4	8
2	8	4	4	4	4	4	4	4
3	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4
5	4	4	4	4	4	4	4	4
6	4	4	4	4	4	4	4	4
7	4	4	4	4	4	4	4	4
8	2	4	4	4	4	4	4	4

Then $(S, \cdot)$ is an AG-groupoid. The set $\{1, 2, 4, 8\}$ is an m-system in S, because if $1, 2 \in M$, then $4 \in < 1 >$, $8 \in < 2 >$ and $4 \cdot 8 = 4 \in M$.

Lemma 102 *Product of two right ideals of an AG-groupoid with left identity is an ideal.*

Proof. Let S be an AG-groupoid with left identity, therefore $S = S^2$. Now using medial law, we get

$$(AB)S = (AB)(SS) = (AS)(BS) \subseteq AB.$$

■

Lemma 103 *Product of two left ideals of an AG-groupoid with left identity is a left ideal.*

Proof. Let S be an AG-groupoid with left identity, therefore $S = S^2$. Now using medial law, we get

$$S(AB) = (SS)(AB) = (SA)(SB) \subseteq AB.$$

■

Lemma 104 *Let P be a left ideal of an AG-groupoid S with left identity e, then the following are equivalent,*
(i) *P is quasi-prime ideal.*
(ii) *For all left ideals A and B of S: $AB = \langle AB \rangle \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$.*
(iii) *For all left ideals A and B of S: $A \nsubseteq P$ and $B \nsubseteq P \Rightarrow AB \nsubseteq P$.*
(iv) *For all $a, b \in S$: $\langle a \rangle \langle b \rangle \subseteq P \Rightarrow a \in P$ or $b \in P$.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ is trivial.
$(i) \Rightarrow (iv)$
Let $\langle a \rangle \langle b \rangle \subseteq P$, then by (i) either $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$, which implies that either $a \in P$ or $b \in P$.
$(iv) \Rightarrow (ii)$
Let $AB \subseteq P$. Let $a \in A$ and $b \in B$, then $\langle a \rangle \langle b \rangle \subseteq P$, now by (iv) either $a \in P$ or $b \in P$, which implies that either $A \subseteq P$ or $B \subseteq P$. ■

Theorem 105 *A left ideal P of an AG-groupoid S with left identity is quasi-prime if and only if $S \backslash P$ is an m-system.*

Proof. Let P is quasi-prime ideal of an AG-groupoid S with left identity and let $a, b \in S \backslash P$ which implies that $a, b \notin P$. Now by lemma 104(iv), we have $\langle a \rangle \langle b \rangle \nsubseteq P$ and so $\langle a \rangle \langle b \rangle \subseteq S \backslash P$. Now let $a_1 \in \langle a \rangle$ and $b_1 \in \langle b \rangle$ which implies that $a_1 b_1 \in S \backslash P$. Hence $S \backslash P$ is an m-system.

Conversely, assume that $S \backslash P$ be an m-system. Let $a \notin P$ and $b \notin P$, then $a, b \in S \backslash P$. Now there exists a_1 in $\langle a \rangle$ and b_1 in $\langle b \rangle$ such that $a_1 b_1 \in S \backslash P$. This implies that $a_1 b_1 \notin P$, which further implies that $\langle a \rangle \langle b \rangle \nsubseteq P$. Hence by lemma 104$(iv)$, P is a quasi-prime ideal. ■

Let P be a left ideal of an AG-groupoid S, P is called quasi-semiprime if for any left ideal A of S such that $A^2 \subseteq P$, we have $A \subseteq P$.

Lemma 106 *Let A be a left ideal of an AG-groupoid S with left identity e, then the following are equivalent,*

(i) *A is quasi-semiprime.*

(ii) *For any left ideals I of S: $I^2 = \langle I^2 \rangle \subseteq A \Rightarrow I \subseteq A$.*

(iii) *For any left ideals I of S: $I \nsubseteq A \Rightarrow I^2 \nsubseteq A$.*

(iv) *For all $a \in S$: $[\langle a \rangle]^2 \subseteq A \Rightarrow a \in A$.*

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are trivial.

$(i) \Rightarrow (iv)$

Let $[\langle a \rangle]^2 \subseteq A$, then by (i) $\langle a \rangle \subseteq A$, which implies that $a \in A$.

$(iv) \Rightarrow (ii)$

Let $I^2 \subseteq A$, if $a \in I$, then $[\langle a \rangle]^2 \subseteq A$, now by (iv) $a \in A$, which implies that $I \subseteq A$. ■

A subset P of an AG-groupoid S with left identity is called an sp-system if for all $a \in P$, there exists a_1, $b_1 \in \langle a \rangle$ such that $a_1 b_1 \in P$ [50].

Lemma 107 *Every right ideal of an AG-groupoid S with left identity e is an sp-system.*

Proof. Let I be a right ideal of an AG-groupoid S with left identity e. Now let $i \in I$ and $s \in S$. Then by left invertive law, we get $si = (es)i = (is)e \in (IS)S \subseteq I$. Therefore I becomes an ideal of S. Also $\langle i \rangle = \text{Si} \subseteq SI \subseteq I$. Now let i_1, $i_2 \in \langle i \rangle$, which implies that $i_1 i_2 \in I$. Hence I is an sp-system. ■

Note that every right ideal of an AG-groupoid S with left identity becomes an ideal of S.

Theorem 108 (a) *Each m-system is an sp-system.*

(b) *A left ideal I of an AG-groupoid S is quasi-semiprime if and only if $S \backslash I$ is an sp-system.*

Proof. (a) Let $a \in M$, then there exists a_1, $b_1 \in \langle a \rangle$, such that $a_1 b_1 \in M$ implying that M is an sp-system.

(b) A left ideal A of an AG-groupoid S with left identity and let $a \in S \backslash A$ which implies that $a \notin A$. Now let $a_1, b_1 \in \langle a \rangle$ which by lemma 106(iv), implies that $a_1 b_1 \in [\langle a \rangle]^2$ but $[\langle a \rangle]^2 \nsubseteq A$. Therefore $a_1 b_1 \notin A$. Hence $a_1 b_1 \in S \backslash A$, which shows that $S \backslash A$ is an sp-system.

Conversely, assume that $S \backslash A$ is an sp-system. Let $a \notin A$, then $a \in S \backslash A$. Now there exists a_1 and b_1 in $\langle a \rangle$, such that $a_1 b_1 \in S \backslash A$ which implies that $a_1 b_1 \notin A$, which further implies that $[\langle a \rangle]^2 \nsubseteq A$. Hence by lemma 106$(iv)$, A is a quasi-semiprime ideal. ■

4.2 Quasi-ideals of Intra-regular Abel-Grassmann's Groupoids

Here we begin with examples of intra-regular AG-groupoids.

Example 109 *Let $S = \{1, 2, 3, 4, 5, 6\}$, then $(S, \cdot)$ is an AG-groupoid with left identity 5 as given in the following multiplication table:*

$\cdot$	1	2	3	4	5	6
1	4	5	6	1	2	3
2	3	4	5	6	1	2
3	2	3	4	5	6	1
4	1	2	3	4	5	6
5	6	1	2	3	4	5
6	5	6	1	2	3	4

Clearly $(S, \cdot)$ is intra-regular because, $1 = \left(3 \cdot 1^2\right) \cdot 2, 2 = \left(1 \cdot 2^2\right) \cdot 5, 3 = \left(2 \cdot 3^2\right) \cdot 5, 4 = \left(4 \cdot 4^2\right) \cdot 4, 5 = \left(3 \cdot 5^2\right) \cdot 6, 6 = \left(2 \cdot 6^2\right) \cdot 2$.

Example 110 *Let $S = \{a, b, c, d, e\}$, and the binary operation "$\cdot$" be defined on S as follows:*

$*$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	1	1	1
3	1	1	3	4	5	6
4	1	1	6	3	4	5
5	1	1	5	6	3	4
6	1	1	4	5	6	1

Then clearly $(S, *)$ is an AG-groupoid. Also $1 = (1*1^2)*1$, $2 = (2*2^2)*2$, $3 = (3*3^2)*3$, $4 = (3*4^2)*4$ and $5 = (4*5^2)*4$, $6 = (3*6^2)*6$. Therefore $(S, \cdot)$ is an intra-regular AG-groupoid. It is easy to see that $\{1\}$ and $\{1, 2\}$ are quasi-ideals of S.

In the rest by S we shall mean AG**-groupoid such that $S = S^2$.

Theorem 111 *For S the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $R \cap L = RL$, for every semiprime right ideal R and every left ideal L.

(iii) $A = (AS)A$, for every quasi-ideal A.

Proof. $(i) \Rightarrow (iii)$: *Let A be a quasi ideal of S then, A is an ideal of S, thus $(AS)A \subseteq A$.*

Now let $a \in A$, and since S is intra-regular so there exist elements x, y in S such that $a = (xa^2)y$. Now by using medial law with left identity, left *invertive law, medial law and paramedial law, we have*

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\ &= ((a(xa))(y(xy)))a = ((ay)((xa)(xy)))a \\ &= ((xa)((ay)(xy)))a = ((xa)((ax)y^2))a \\ &= ((y^2(ax))(ax))a = (a((y^2(ax))x))a \in (AS)A. \end{aligned}$$

Hence $A = (AS)A$.

$(iii) \Rightarrow (ii)$: *Clearly* $RL \subseteq R \cap L$ *holds. Now*

$$\begin{aligned} S(R\cap L)\cap(R\cap L)S &= SR\cap SL\cap RS\cap LS = RS\cap SL\cap SR\cap LS \\ &\subseteq R\cap L\cap(SR\cap LS) \subseteq R\cap L.\ \textit{And} \\ R\cap L &= ((R\cap L)S)(R\cap L) = (RS\cap LS)(R\cap L) \\ &\subseteq (R\cap LS)(R\cap L) \subseteq RL. \end{aligned}$$

Hence $R \cap L = RL$.

$(ii) \Rightarrow (i)$: *Assume that* $R \cap L = RL$ *for every right ideal* R *and every left ideal* L *of* S. *Since* $a^2 \in a^2S$, *which is a right ideal of* S *and as by given assumption* a^2S *is semiprime which implies that* $a \in a^2S$. *Now clearly* Sa *is a left ideal of* S *and* $a \in Sa$, *Therefore by using left invertive law, medial law, paramedial law and medial law with left identity, we have*

$$\begin{aligned} a &\in Sa\cap a^2S = (Sa)(a^2S) = (Sa)((aa)S) = (Sa)((Sa)(ea)) \\ &\subseteq (Sa)((Sa)(Sa)) = (Sa)((SS)(aa)) \subseteq (Sa)((SS)(Sa)) \\ &= (Sa)((aS)(SS)) = (Sa)((aS)S) = (aS)((Sa)S) \\ &= (a(Sa))(SS) = (a(Sa))S = (S(aa))S = (Sa^2)S. \end{aligned}$$

Hence S *is intra-regular.* ■

Theorem 112 *For* S *the following conditions are equivalent.*

(i) S *is intra-regular.*

(ii) *For an ideal* I *and quasi-ideal* Q, $I \cap Q = IQ$ *and* I *is semiprime.*

(iii) *For quasi-ideals* Q_1 *and* Q_2 , $Q_1 \cap Q_2 = Q_1Q_2$ *and* Q_1 *and* Q_2 *are semiprime.*

Proof. $(i) \Longrightarrow (iii)$: Let Q_1 and Q_2 be a quasi-ideal of S. Now Q_1 and Q_2 become ideals of S. Therefore $Q_1Q_2 \subseteq Q_1 \cap Q_2$. Now let $a \in Q_1 \cap Q_2$ which implies that $a \in Q_1$ and $a \in Q_2$. For $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now using (1) and left invertive law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in (S\,(SQ_1))Q_2 \subseteq (SQ_1)\,Q_2 \subseteq Q_1Q_2.$$

This implies that $Q_1 \cap Q_2 \subseteq Q_1Q_2$. Hence $Q_1 \cap Q_2 = Q_1Q_2$. Next we will show that Q_1 and Q_2 are semiprime. For this let $a^2 \in Q_1$. Therefore $a = (xa^2)y \in (SQ_1)S \subseteq Q_1$. Similarly Q_2 is semiprime.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Obviously Sa is a quasi-ideal contains a and Sa^2 is an ideal contains a^2. By (ii) Sa^2 is semiprime so $a \in Sa^2$. Therefore by (ii) we get

$$a \in Sa^2 \cap Sa = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 113 *For S the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *For quasi-ideals Q_1 and Q_2, $Q_1 \cap Q_2 = (Q_1Q_2)Q_1$.*

Proof. $(i) \Longrightarrow (ii)$: Let Q_1 and Q_2 be quasi-ideals of S. Now Q_1 and Q_2 become ideals of S. Therefore $(Q_1Q_2)Q_1 \subseteq (Q_1S)Q_1 \subseteq Q_1$ and $(Q_1Q_2)Q_1 \subseteq (SQ_2)S \subseteq Q_2$. This implies that $(Q_1Q_2)Q_1 \subseteq Q_1 \cap Q_2$. We can easily see that $Q_1 \cap Q_2$ becomes an ideal. Now, we get,

$$\begin{aligned} Q_1 \cap Q_2 &= (Q_1 \cap Q_2)^2 = (Q_1 \cap Q_2)^2 (Q_1 \cap Q_2) \\ &= ((Q_1 \cap Q_2)(Q_1 \cap Q_2))(Q_1 \cap Q_2) \subseteq (Q_1Q_2)Q_1. \end{aligned}$$

Thus $Q_1 \cap Q_2 \subseteq (Q_1Q_2)Q_1$. Hence $Q_1 \cap Q_2 = (Q_1Q_2)Q_1$.

$(ii) \Longrightarrow (i)$: Let Q be a quasi-ideal of S, then by (ii), we get $Q = Q \cap Q = (QQ)Q \subseteq Q^2Q \subseteq QQ = Q^2$. This implies that $Q \subseteq Q^2$ therefore $Q^2 = Q$. Now since Sa is a quasi-ideal, therefore $a \in Sa = (Sa)^2 = Sa^2 = \left(Sa^2\right)S$. Hence S is intra-regular. ■

Theorem 114 *For S the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *For quasi-ideal Q and ideal J, $Q \cap J \subseteq JQ$, and J is semiprime.*

Proof. $(i) \Longrightarrow (ii)$: Assume that Q is a quasi-ideal and J is an ideal of S. Let $a \in Q \cap J$, then $a \in Q$ and $a \in J$. For each $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Then using (1) and left invertive law we get,

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in (S(SJ)Q \subseteq JQ.$$

Therefore $Q \cap J \subseteq JQ$. Next let $a^2 \in J$. Thus $a = (xa^2)y \in (SJ)S \subseteq J$. Hence J is semiprime.

$(ii) \Longrightarrow (i)$: Since Sa is a quasi and a^2S is a an ideal of S containing a and a^2 respectively. Thus by (ii) J is semiprime so $a \in a^2S$. Therefore by hypothesis, paramedial and medial laws, we get

$$a \in Sa \cap a^2S \subseteq (Sa)(a^2S) = (Sa^2)(aS) \subseteq (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 115 *If A is an interior ideal of S, then A^2 is also interior ideal.*

Proof. Using medial law we immediately obtained the following

$$\begin{aligned} \left(SA^2\right) S &= ((SS)(AA))(SS) = ((SA)(SA))(SS) \\ &= ((SA)S)((SA)S) \subseteq AA = A^2. \end{aligned}$$

■

Theorem 116 *For S the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For quasi-ideal Q, right ideal R and two sided ideal I, $(Q \cap R) \cap I \subseteq (QR)I$ and R, I are semiprime.
(iii) For quasi-ideal Q, right ideal R and right ideal I, $(Q \cap R) \cap I \subseteq (QR)I$ and R, I are semiprime.
(iv) For quasi-ideal Q, right ideal R and interior ideal I, $(Q \cap R) \cap I \subseteq (QR)I$ and R, I are semiprime.

Proof. $(i) \Longrightarrow (iv)$: Let $a \in (Q \cap R) \cap I$. This implies that $a \in Q$, $a \in R$, $a \in I$. Since S is intra-regular therefore for each $a \in S$ there exists $x, y \in S$ such that $a = \left(xa^2\right) y$. Now using left invertive law, medial law, paramedial law and (1) we get,

$$\begin{aligned} a &= \left(xa^2\right) y = (x(aa)) y = (a(xa)) y = \left(a\left(x(\left(xa^2\right) y)\right)\right) y \\ &= \left(a\left(\left(xa^2\right)(xy)\right)\right) y = \left(y\left(\left(xa^2\right)(xy)\right)\right) a = (y((x(aa))(xy)) a \\ &= (y((a(xa))(xy))) a = ((a(xa))(y(xy))) a \\ &= (((y(xy))(xa))a)a \in (((S(SS))(SQ))R)I \subseteq (QR)I. \end{aligned}$$

Therefore $(Q \cap R) \cap I \subseteq (QR)I$. Next let $a^2 \in R$. Then using left invertive law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in RT.$$

This implies that $a \in R$. Similarly we can show that I is semiprime.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$: are obvious.

$(ii) \Longrightarrow (i)$: We know that Sa is a quasi and Sa^2 is right as well as two sided ideal of S containing a and a^2 respectively, and by (ii) Sa^2 is semiprime so $a \in Sa^2$. Then by hypothesis and left invertive law, paramedial and medial laws, we get

$$\begin{aligned} a &\in (Sa \cap Sa^2) \cap Sa^2 = ((Sa)(Sa^2))Sa^2 = ((Sa^2)(Sa^2))Sa \\ &\subseteq ((Sa^2)S)S = (SS)(Sa^2) = (a^2S)(SS) = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 117 *For S the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For every bi-ideal B and quasi-ideal Q, $B \cap Q \subseteq BQ$.
(iii) For every generalized bi-ideal B and quasi-ideal Q, $B \cap Q \subseteq BQ$.

Proof. $(i) \Longrightarrow (iii)$: Let B is a bi-ideal and Q is a quasi-ideal of S. Let $a \in B \cap Q$ which implies that $a \in B$ and $a \in Q$. Since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now B and Q become ideals of S. Then using (1) and left invertive law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in (S(SB)Q \subseteq BQ.$$

Hence $B \cap Q \subseteq BQ$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Using (ii) we get

$$a \in Sa \cap Sa \subseteq Sa^2 = \left(Sa^2\right) S.$$

Hence S is intra-regular. ■

Theorem 118 *For S the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For quasi-ideal Q_1, two sided ideal I and quasi-ideal Q_2, $(Q_1 \cap I) \cap Q_2 \subseteq (Q_1 I) Q_2$, and I is semiprime.*

(iii) *For quasi-ideal Q_1, right ideal I and quasi ideal Q_2, $(Q_1 \cap I) \cap Q_2 \subseteq (Q_1 I) Q_2$, and I is semiprime.*

(iv) *For quasi-ideal Q_1, interior ideal I and quasi-ideal Q_2 $(Q_1 \cap I) \cap Q_2 \subseteq (Q_1 I) Q_2$, and I is semiprime.*

Proof. $(i) \Longrightarrow (v)$: Let Q_1 and Q_2 be quasi-ideals and I be an interior ideal of S respectively. Let $a \in (Q_1 \cap I) \cap Q_2$. This implies that $a \in Q_1, a \in I$ and $a \in Q_2$. For $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now Q_1, Q_2 and I become ideals of S. Therefore by left invertive law, medial law and paramedial law we get,

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))((y(xa))a) \\ &= [a\{y(xa)\}][(xa)y] \in [Q_1\{S(SI)\}][(SQ_2)S] \subseteq (Q_1 I) Q_2. \end{aligned}$$

Hence $(Q_1 \cap I) \cap Q_2 \subseteq (Q_1 I) Q_2$. Next let $a^2 \in I$. Then $a = (xa^2)y = I^2 \subseteq I$. This implies that $a \in I$. Hence that I is semiprime.

$(v) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Since Sa is a quasi and Sa^2 is an ideal of S containing a and a^2 respectively. Also by (ii) Sa^2 is semiprime so $a \in Sa^2$. Thus by using paramedial and medial laws, we get

$$\begin{aligned} a &\in (Sa \cap Sa^2) \cap Sa \subseteq ((Sa)(Sa^2))Sa = ((a^2S)(aS))Sa \\ &= ((a^2S)(SS))(SS) = ((a^2S)S)S = ((SS)a^2)S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 119 *For S the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *Every quasi-ideal is idempotent.*

(iii) *For quasi-ideals A, B, $A \cap B = AB \cap BA$.*

Proof. $(i) \Longrightarrow (iii)$: Let A and B be quasi-ideals of S. Thus

$$AB \cap BA \subseteq AB \subseteq SB \subseteq B \text{ and } AB \cap BA \subseteq BA \subseteq SA \subseteq A.$$

Hence $AB \cap BA \subseteq A \cap B$. Now let $a \in A \cap B$. This implies that $a \in A$ and $a \in B$. Since S is intra-regular AG-groupoid so for a in S there exists $x, y \in S$ such that $a = \left(xa^2\right) y$ and $y = uv$ for some u, v in S. Then by (1) and medial law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))(uv) = (au)((xa)v) \in (AS)((SB)S) \subseteq AB.$$

Similarly we can show that $a \in BA$. Thus $A \cap B \subseteq AB \cap BA$. Therefore $A \cap B = AB \cap BA$.

$(iii) \Longrightarrow (ii)$: Let Q be a quasi-ideal of S. Thus by (iii), $Q \cap Q = QQ \cap QQ$. Hence $Q = QQ$.

$(ii) \Longrightarrow (i)$: Since Sa is a quasi-ideal of S contains a and by (ii) it is idempotent therefore by medial law, we have

$$a \in Sa = (Sa)^2 = (Sa)(Sa) = (SS)a^2 = Sa^2 = (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 120 *For S the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For bi-ideal B, two sided ideal I and quasi-ideal Q,$(B \cap I) \cap Q \subseteq (BI)Q$ and I is semiprime.*

(iii) *For bi-ideal B, right ideal I and quasi-ideal Q,$(B \cap I) \cap Q \subseteq (BI)Q$ and I is semiprime.*

(iv) *For generalized bi-ideal B, interior ideal I and quasi-ideal Q,$(B \cap I) \cap Q \subseteq (BI)Q$ and I is semiprime.*

Proof. $(i) \Longrightarrow (iv)$: Let B be a generalized bi-ideal, I be an interior ideal and Q be a quasi-ideal of S respectively. Let $a \in (B \cap I) \cap Q$. This implies that $a \in B, a \in I$ and $a \in Q$. Since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now B, I and Q become ideals of S. Therefore using left invertive law, medial law, paramedial law and (1) we get,

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))((y(xa))a) \\ &= [a\{y(xa)\}][(xa)y] \in [B\{S(SI)\}][(SQ)S] \subseteq (BI)\,Q. \end{aligned}$$

Therefore $(B \cap I) \cap Q \subseteq (BI)Q$. Next let $a^2 \in I$. Then $a = (xa^2)y = I^2 \subseteq I$. This implies that $a \in I$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Clearly Sa is both quasi and bi-ideal containing a and Sa^2 is two sided ideal contains a^2 respectively. Now by (ii) Sa^2 is semiprime so

$a \in Sa^2$. Therefore using paramedial, medial laws and left invertive law we get,

$$\begin{aligned} a &\in \left(Sa \cap Sa^2\right) \cap Sa \subseteq (\left(Sa)(Sa^2)\right)(Sa) \subseteq (\left(a^2S)(aS)\right)(SS) \\ &\subseteq (\left(a^2S)(SS)\right)(SS) = ((a^2S)S)S = ((SS)a^2)S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 121 *For S the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *For quasi-ideals Q and bi-ideal B, $Q \cap B \subseteq QB$.*
(iii) *For quasi-ideal Q and generalized bi-ideal B, $Q \cap B \subseteq QB$.*

Proof. $(i) \Longrightarrow (iii)$: Let Q and B be quasi and generalized bi-ideal of S. Let $a \in Q \cap B$. This implies that $a \in Q$ and $a \in B$. Since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now, Q and B becomes ideals of S. Therefore using and left invertive law, we get,

$$a = \left(xa^2\right) y = (x\,(aa))y = (a\,(xa))y = (y\,(xa))\,a \in (S(SQ)B \subseteq QB.$$

Thus $a \in QB$. Hence $Q \cap B \subseteq QB$.
$(iii) \Longrightarrow (ii)$ is obvious.
$(i) \Longrightarrow (ii)$: Clearly Sa is both quasi and bi-ideal of S containing a. Therefore using (ii), paramedial law, medial law we get

$$a \in Sa \cap Sa \subseteq (Sa)(Sa) = (Sa^2) = (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 122 *For S the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *For every quasi-ideal Q of $S, Q = (SQ)^2 \cap (QS)^2$.*

Proof. $(i) \Longrightarrow (ii)$: Let Q be any quasi-ideal of S. Now it becomes an ideal of S. Now using medial law and paramedial law we get

$$(SQ)^2 \cap (QS)^2 = (SQ)(SQ) \cap (QS)(QS) = QQ \cap QQ \subseteq Q.$$

Now let $a \in Q$ and since S is intra-regular so there exists $x, y \in S$ such that $a = (xa^2)y$. Then using left invertive law, medial law and paramedial law, we get

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) = (xa^2)((y(xa))y) \\ &= (y(y(xa)))((aa)x) = (aa)((y(y(xa)))x) = (x(y(y(xa))))(aa) \\ &\in S(QQ) = (SS)(QQ) = (SQ)(SQ) = (SQ)^2. \end{aligned}$$

Thus $a \in (SQ)^2$. It is easy to see that $(SQ)^2 = (QS)^2$. Therefore $a \in (SQ)^2 \cap (QS)^2$. Thus $Q \subseteq (SQ)^2 \cap (QS)^2$. Hence $(SQ)^2 \cap (QS)^2 = Q$.

$(ii) \Rightarrow (i)$: Clearly Sa is a quasi-ideal containing a. Thus by (ii) and paramedial law, medial law and left invertive law we get,

$$\begin{aligned} a &\in Sa = (S(Sa))^2 = ((SS)(Sa))^2 = ((aS)(SS))^2 = ((SS)a)^2 \\ &= (Sa)^2 = (Sa^2) = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 123 *For S the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For every quasi-ideal of S, $Q = (SQ)^2Q \cap (QS)^2Q$.

Proof. $(i) \Longrightarrow (ii)$: Let Q be a quasi-ideal of an intra-regular AG-groupoid S with left identity. Now it becomes an ideal of S. Then obviously

$$(SQ)^2Q \cap (QS)^2Q \subseteq Q.$$

Now let $a \in Q$ and since S is intra-regular so there exists $x, y \in S$ such that $a = (xa^2)y$. Then using left invertive law, paramedial law and medial law, we have,

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((xy)(a^2(xy)))a \\ &= (a^2((xy)\,(xy)))a = (a\,(xy))^2a. \end{aligned}$$

Therefore $a \in ((Q\,(SS))^2\,Q = (QS)^2\,Q$. This implies that $a \in (QS)^2\,Q$. Hence $Q \subseteq (QS)^2\,Q$. Now since $(QS)^2 = (SQ)^2$, thus $Q \subseteq (SQ)^2Q$. Therefore $Q \subseteq (SQ)^2Q \cap (QS)^2Q$. Hence $Q = (SQ)^2Q \cap (QS)^2Q$.
$(ii) \Rightarrow (i)$: Clearly Sa is a quasi-ideal containing a. Therefore by (ii) we get,

$$a \in Sa = (S(Sa))^2(Sa) \subseteq (Sa)^2(Sa) = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 124 *For S the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For any quasi-ideals Q_1 and Q_2 of S, $Q_1Q_2 \subseteq Q_2Q_1$ and Q_1, Q_2 are semiprime.

Proof. $(i) \Longrightarrow (ii)$: Let Q_1 and Q_2 be any quasi-ideals of an intra-regular AG-groupoid S with left identity. Now Q_1 and Q_2 become ideals of S. Let $a \in Q_1Q_2$. Then $a = uv$ where $u \in Q_1$ and $v \in Q_2$. Now since S in intra-regular therefore for u and v in S there exists $x_1, x_2, y_{1,} y_2 \in S$ such that $a = (((x_1u^2)\,y_1)((x_2v^2)\,y_2))$. Using medial law, paramedial law, medial law and left invertive law, we have

$$\begin{aligned}
a &= (((x_1u^2)\,y_1)((x_2v^2)\,y_2)) = ((x_1u^2)\,(x_2v^2))\,(y_2y_1) \\
&= ((x_1(uu))\,(x_2(vv)))\,(y_2y_1) = ((u(x_1u))\,(v(x_2v)))\,(y_2y_1) \\
&= (((x_2v)(x_1u))\,(vu))(y_2y_1) = (((x_2x_1)(vu))\,(vu))(y_2y_1) \\
&= (((vu)\,(vu))(x_2x_1))(y_2y_1) = ((y_2y_1)(x_2x_1))\,(((vu)\,(vu)) \\
&= ((y_2y_1)(x_2x_1))\,(v^2u^2) = ((y_2y_1)v^2)((x_2x_1)u^2). \\
&\in ((SS)Q_2^2)\,((SS)\,Q_2^2) \subseteq (SQ_2)\,(SQ_1) \subseteq Q_2Q_1.
\end{aligned}$$

Thus $a \in Q_2Q_1$. Hence $Q_1Q_2 \subseteq Q_2Q_1$. Let $a^2 \in Q_1$. Then since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that, $a = (xa^2)y$. Then using left invertive law, we get

$$a = (xa^2)y = (x\,(aa))\,y = (a\,(xa))\,y = (y\,(xa))\,a \in ((SS)Q_1)Q_1 \subseteq Q_1.$$

Similarly we can show that Q_2 semiprime.

$(ii) \Longrightarrow (i)$: Let Sa be a quasi-ideal of S containing a then by (ii) and using medial law we get,

$$a \in Sa \cap Sa = (Sa)(Sa) = (Sa^2) = (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 125 *For S the following conditions are equivalent.*

(i) S is intra-regular.

(ii) For any quasi-ideal A and two sided ideal B of S, $A \cap B = (AB)A$ and B is semiprime.

(iii) For any quasi-ideal A and right ideal B of $S, A \cap B = (AB)A$ and B is semiprime.

(iv) For any quasi-ideal A and interior ideal B of $S, A, B, A\cap B = (AB)A$ and B is semiprime.

Proof. $(i) \Rightarrow (iv)$: Let A and B be a quasi-ideal and an interior ideal of S respectively. Now A and B are ideals of S. Then $(AB)A \subseteq (AS)A \subseteq A$ and $AB)A \subseteq (SB)S \subseteq B$. Thus $(AB)A \subseteq A \cap B$. Next let $a \in A \cap B$, which implies that $a \in A$ and $a \in B$. Since S is intra-regular so for a there exists $x, y \in S$, such that $a = (xa^2)y$. Then using left invertive law, we get,

$$\begin{aligned}
a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))a \\
&= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\
&= ((xa^2)(y(xy)))a = ((a(xa))(y(xy)))a \\
&= (((y(xy))(xa))a)a = (aa)\,((y(xy))(xa)) \\
&\subseteq (AB)\,(S\,(SA)) \subseteq (AB)A.
\end{aligned}$$

Thus $A \cap B = (AB)A$. Next to show that B is semiprime let $a^2 \in B$. Therefore for each $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y \in BB \subseteq B$. Thus $a^2 \in B$. This implies that $a \in B$. Hence B is semiprime.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Since Sa is quasi-ideal and Sa^2 be two sided ideal containing a and a^2 respectively. And by (ii) Sa^2 is semiprime so $a \in Sa^2$. Therefore using (ii), left invertive law, medial law,and paramedial law, we get

$$\begin{aligned} Sa \cap Sa^2 &= ((Sa)(Sa^2))\,(Sa) \subseteq ((SS)\,(Sa^2))\,(SS) = (\left(a^2 S\right)(SS))S \\ &= (\left(a^2 S\right) S)S = ((SS)\,a^2)S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 126 *For S the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For every left ideal A and B of S, $A \cap B = (AB) \cap (BA)$.*

(iii) *For every quasi ideal A and every left ideal B of S, $A \cap B = (AB) \cap (BA)$.*

(iv) *For every quasi ideals A and B of S, $A \cap B = (AB) \cap (BA)$.*

Proof. $(i) \Longrightarrow (iv)$: Let A and B be any generalized bi-ideal of S, then A and B are ideals of S. Clearly $AB \subseteq A \cap B$, now $A \cap B$ is an ideal and $A \cap B = (A \cap B)^2$. Now $A \cap B = (A \cap B)^2 \subseteq AB$. Thus $A \cap B = AB$ and then $A \cap B = B \cap A = BA$. Hence $A \cap B = (AB) \cap (BA)$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$: Since Sa is a left ideal of an AG-groupoid S with left identity containing a. Therefore by (ii) and medial law we get

$$Sa \cap Sa = (Sa)(Sa) = Sa^2 = (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 127 *For S the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For any quasi-ideals Q and two sided ideal I of S, $Q \cap I = (QI)Q$ and I is semiprime.*

(iii) *For any quasi-ideals Q and right ideal I of $S, Q \cap I = (QI)Q$ and I is semiprime.*

(iv) *For any quasi ideals Q and interior ideal I of $S, Q \cap I = (QI)Q$ and I is semiprime.*

Proof. $(i) \Rightarrow (v)$: Let Q and I be a quasi-ideal and an interior ideal of S respectively. Now Q and I are ideals of S. Then $(QI)Q \subseteq (QS)Q \subseteq Q$ and $(QI)Q \subseteq (SI)S \subseteq I$. Thus $(QI)Q \subseteq Q \cap I$. Next let $a \in Q \cap I$, which implies that $a \in Q$ and $a \in I$. Since S is intra-regular so for a there exists $x, y \in S$, such that $a = (xa^2)y$. Then left invertive law, we get,

$$\begin{aligned} a &= \left(xa^2\right) y = (x\,(aa))y = (a(xa))y = \left(a\left(x(\left(xa^2\right) y)\right)\right) y \\ &= \left(a\left(\left(xa^2\right)(xy)\right)\right) y = \left(y\left(\left(xa^2\right)(xy)\right)\right) a = (y\,((x(aa))\,(xy)))\,a \\ &= (y\,((a(xa))\,(xy)))\,a = ((a\,(xa))\,(y\,(xy)))\,a = (((y\,(xy))\,(xa))\,a)a \\ &= (aa)\,((y(xy))\,(xa)) \in (QI)\,(S\,(SS)\,(SQ) \subseteq (QI)Q. \end{aligned}$$

Thus $Q \cap I = (QI)Q$. Next to show that I is semiprime let $a^2 \in I$. Therefore for each $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y \in (SI)\, S \subseteq I$. Thus $a^2 \in I$. This implies that $a \in I$. Hence I is semiprime.

$(v) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$: Since Sa is a quasi-ideal and Sa^2 be a two sided ideal containing a and a^2 respectively. And by (ii) Sa^2 is semiprime so $a \in Sa^2$. Therefore using (ii), left invertive law, medial law and paramedial we get,

$$\begin{aligned} Sa \cap Sa^2 &= ((Sa)(Sa^2))\,(Sa) = ((SS)\,(Sa^2))\,(SS) = (\left(a^2S\right)(SS))S \\ &= (\left(a^2S\right)S)S = ((SS)\,a^2)S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

4.3 Characterizations of Ideals in Intra-regular AG-groupoids

An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.

Example 128 *Let us consider an AG-groupoid $S = \{a, b, c, d, e, f\}$ with left identity e in the following Clayey's table.*

$.$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	b	b	b	b
c	a	b	f	f	d	f
d	a	b	f	f	c	f
e	a	b	c	d	e	f
f	a	b	f	f	f	f

Example 129 *Let us consider the set $(\mathbb{R}, +)$ of all real numbers under the binary operation of addition. If we define $a * b = b - a - r$, where $a, b, r \in R$, then $(\mathbb{R}, *)$ becomes an AG-groupoid as,*

$$(a * b) * c = c - (a * b) - r = c - (b - a - r) - r = c - b + a + r - r = c - b + a$$

and

$$(c * b) * a = a - (c * b) - r = a - (b - c - r) - r = a - b + c + r - r = a - b + c.$$

Since $(\mathbb{R}, +)$ is commutative so $(a * b) * c = (c * b) * a$ and therefore $(\mathbb{R}, *)$ satisfies a left invertive law. It is easy to observe that $(R, *)$ is non-commutative and non-associative. The same is hold for set of integers and rationals. Thus $(\mathbb{R}, *)$ is an AG-groupoid which is the generalization of an

AG-groupoid given in 1988 (*see [39]*). *Similarly if we define* $a*b = ba^{-1}r^{-1}$, *then* $(\mathbb{R}\backslash\{0\}, *)$ *becomes an AG-groupoid and the same holds for the set of integers and rationals. This AG-groupoid is also the generalization of an AG-groupoid given in* 1988 (*see [39]*).

An element a of an AG-groupoid S is called an intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called intra-regular, if every element of S is intra-regular.

Example 130 *Let* $S = \{a, b, c, d, e\}$ *be an AG-groupoid with left identity* b *in the following multiplication table.*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Clearly S is intra-regular because, $a = (aa^2)a$, $b = (cb^2)e$, $c = (dc^2)e$, $d = (cd^2)c$, $e = (be^2)e$.

An element a of an AG-groupoid S with left identity e is called a left (right) invertible if there exits $x \in S$ such that $xa = e$ $(ax = e)$ and a is called invertible if it is both a left and a right invertible. An AG-groupoid S is called a left (right) invertible if every element of S is a left (right) invertible and S is called invertible if it is both a left and a right invertible.

Note that in an AG-groupoid S with left identity, $S = S^2$.

Theorem 131 *Every AG-groupoid* S *with left identity is an intra-regular if* S *is left (right) invertible.*

Proof. Let S be a left invertible AG-groupoid with left identity, then for $a \in S$ there exists $a^{'} \in S$ such that $a^{'}a = e$. Now by using left invertive law, medial law with left identity and medial law, we have

$$\begin{aligned} a &= ea = e(ea) = (a^{'}a)(ea) \in (Sa)(Sa) = (Sa)((SS)a) \\ &= (Sa)((aS)S) = (aS)((Sa)S) = (a(Sa))(SS) \\ &= (a(Sa))S = (S(aa))S = (Sa^2)S. \end{aligned}$$

Which shows that S is intra-regular. Similarly in the case of right invertible. ■

Theorem 132 *An AG-groupoid* S *is intra-regular if* $Sa = S$ *or* $aS = S$ *holds for all* $a \in S$.

Proof. Let S be an AG-groupoid such that $Sa = S$ holds for all $a \in S$, then $S = S^2$. Let $a \in S$, therefore by using medial law, we have

$$a \in S = (SS)S = ((Sa)(Sa))S = ((SS)(aa))S \subseteq (Sa^2)S.$$

Which shows that S is intra-regular.

Let $a \in S$ and assume that $aS = S$ holds for all $a \in S$, then by using left invertive law, we have

$$a \in S = SS = (aS)S = (SS)a = Sa.$$

Thus $Sa = S$ holds for all $a \in S$, therefore it follows from above that S is intra-regular. ■

The converse is not true in general from Example above.

Corollary 133 *If S is an AG-groupoid such that $aS = S$ holds for all $a \in S$, then $Sa = S$ holds for all $a \in S$.*

Theorem 134 *If S is intra-regular AG-groupoid with left identity, then $(BS)B = B \cap S$, where B is a bi-(generalized bi-) ideal of S.*

Proof. Let S be an intra-regular AG-groupoid with left identity, then clearly $(BS)B \subseteq B \cap S$. Now let $b \in B \cap S$, which implies that $b \in B$ and $b \in S$. Since S is intra-regular so there exist $x, y \in S$ such that $b = (xb^2)y$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned} b &= (x(bb))y = (b(xb))y = (y(xb))b = (y(x((xb^2)y)))b \\ &= (y((xb^2)(xy)))b = ((xb^2)(y(xy)))b = (((xy)y)(b^2x))b \\ &= ((bb)(((xy)y)x))b = ((bb)((xy)(xy)))b = ((bb)(x^2y^2))b \\ &= ((y^2x^2)(bb))b = (b((y^2x^2)b))b \in (BS)B. \end{aligned}$$

This shows that $(BS)B = B \cap S$. ■

The converse is not true in general. For this, let us consider an AG-groupoid S with left identity e in Example 128. It is easy to see that $\{a, b, f\}$ is a bi-(generalized bi-) ideal of S such that $(BS)B = B \cap S$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Corollary 135 *If S is intra-regular AG-groupoid with left identity, then $(BS)B = B$, where B is a bi-(generalized bi-) ideal of S.*

Theorem 136 *If S is intra-regular AG-groupoid with left identity, then $(SB)S = S \cap B$, where B is an interior ideal of S.*

Proof. Let S be an intra-regular AG-groupoid with left identity, then clearly $(SB)S \subseteq S \cap B$. Now let $b \in S \cap B$, which implies that $b \in S$ and $b \in B$. Since S is an intra-regular so there exist $x, y \in S$ such that $b = (xb^2)y$. Now by using paramedial law and left invertive law, we have

$$b = ((ex)(bb))y = ((bb)(xe))y = (((xe)b)b)y \in (SB)S.$$

Which shows that $(SB)S = S \cap B$. ■

The converse is not true in general. It is easy to see that form Example 128 that $\{a, b, f\}$ is an interior ideal of an AG-groupoid S with left identity e such that $(SB)S = B \cap S$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Corollary 137 *If S is intra-regular AG-groupoid with left identity, then $(SB)S = B$, where B is an interior ideal of S.*

Let S be an AG-groupoid, then $\emptyset \neq A \subseteq S$ is called semiprime if $a^2 \in A$ implies $a \in A$.

Theorem 138 *An AG-groupoid S with left identity is intra-regular if $L \cup R = LR$, where L and R are the left and right ideals of S respectively such that R is semiprime.*

Proof. Let S be an AG-groupoid with left identity, then clearly Sa and a^2S are the left and right ideals of S such that $a \in Sa$ and $a^2 \in a^2S$, because by using paramedial law, we have

$$a^2S = (aa)(SS) = (SS)(aa) = Sa^2.$$

Therefore by given assumption, $a \in a^2S$. Now by using left invertive law, medial law, paramedial law and medial law with left identity, we have

$$\begin{aligned} a &\in Sa \cup a^2S = (Sa)(a^2S) = (Sa)((aa)S) = (Sa)((Sa)(ea)) \\ &\subseteq (Sa)((Sa)(Sa)) = (Sa)((SS)(aa)) \subseteq (Sa)((SS)(Sa)) \\ &= (Sa)((aS)(SS)) = (Sa)((aS)S) = (aS)((Sa)S) \\ &= (a(Sa))(SS) = (a(Sa))S = (S(aa))S = (Sa^2)S. \end{aligned}$$

Which shows that S is intra-regular. ■

The converse is not true in general. In Example 128, the only left and right ideal of S is $\{a, b\}$, where $\{a, b\}$ is semiprime such that $\{a, b\} \cup \{a, b\} = \{a, b\}\{a, b\}$ but S is not an intra-regular because $d \in S$ is not an intra-regular.

Lemma 139 *[38] If S is intra-regular regular AG-groupoid, then $S = S^2$.*

Theorem 140 *For a left invertible AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $R \cap L = RL$, where R and L are any left and right ideals of S respectively.

Proof. $(i) \Longrightarrow (ii)$: Assume that S is intra-regular AG-groupoid with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a = (xa^2)y$. Let R and L be any left and right ideals of S respectively, then obviously $RL \subseteq R \cap L$. Now let $a \in R \cap L$ implies that $a \in R$ and $a \in L$. Now by using medial law with left identity, medial law and left invertive law, we have

$$\begin{aligned} a &= (xa^2)y \in (Sa^2)S = (S(aa))S = (a(Sa))S = (a(Sa))(SS) \\ &= (aS)((Sa)S) = (Sa)((aS)S) = (Sa)((SS)a) = (Sa)(Sa) \\ &\subseteq (SR)(SL) = ((SS)R)(SL) = ((RS)S)(SL) \subseteq RL. \end{aligned}$$

This shows that $R \cap L = RL$.

$(ii) \implies (i)$: Let S be a left invertible AG-groupoid with left identity, then for $a \in S$ there exists $a' \in S$ such that $a'a = e$. Since a^2S is a right ideal and also a left ideal of S such that $a^2 \in a^2S$, therefore by using given assumption, medial law with left identity and left invertive law, we have

$$\begin{aligned} a^2 &\in a^2S \cap a^2S = (a^2S)(a^2S) = a^2((a^2S)S) = a^2((SS)a^2) \\ &= (aa)(Sa^2) = ((Sa^2)a)a. \end{aligned}$$

Thus we get, $a^2 = ((xa^2)a)a$ for some $x \in S$.

Now by using left invertive law, we have

$$\begin{aligned} (aa)a' &= (((xa^2)a)a)a' \\ (a'a)a &= (a'a)(((xa^2)a) \\ a &= (xa^2)a. \end{aligned}$$

This shows that S is intra-regular. ■

Lemma 141 *[38] Every two-sided ideal of an intra-regular AG-groupoid S with left identity is idempotent.*

Theorem 142 *In an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $A = (SA)^2$, where A is any left ideal of S.

Proof. $(i) \implies (ii)$: Let A be a left ideal of an intra-regular AG-groupoid S with left identity, then $SA \subseteq A$ and $(SA)^2 = SA \subseteq A$. Now $A = AA \subseteq SA = (SA)^2$, which implies that $A = (SA)^2$.

$(ii) \implies (i)$: Let A be a left ideal of S, then $A = (SA)^2 \subseteq A^2$, which implies that A is idempotent and by using Lemma 149, S is intra-regular. ■

Theorem 143 *In an intra-regular AG-groupoid S with left identity, the following conditions are equivalent.*

(i) A is a bi-(generalized bi-) ideal of S.

(ii) $(AS)A = A$ and $A^2 = A$.

Proof. $(i) \implies (ii)$: Let A be a bi-ideal of an intra-regular AG-groupoid S with left identity, then $(AS)A \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist $x,\, y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law, medial law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\ &= ((a(xa))(y(xy)))a = ((ay)((xa)(xy)))a \\ &= ((xa)((ay)(xy)))a = ((xa)((ax)y^2))a \\ &= ((y^2(ax))(ax))a = (a((y^2(ax))x))a \in (AS)A. \end{aligned}$$

Thus $(AS)A = A$ holds. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\ &= ((a(xa))(y(xy)))a = (((y(xy))(xa))a)a = (((ax)((xy)y))a)a \\ &= (((ax)(y^2x))a)a = (((ay^2)(xx))a)a = (((ay^2)x^2)a)a \\ &= (((x^2y^2)a)a)a = (((x^2y^2)((x(aa))y))a)a \\ &= (((x^2y^2)((a(xa))y))a)a = (((x^2(a(xa)))(y^2y))a)a \\ &= (((a(x^2(xa)))y^3)a)a = (((a((xx)(xa)))y^3)a)a \\ &= (((a((ax)(xx)))y^3)a)a = ((((ax)(ax^2))y^3)a)a \\ &= ((((aa)(xx^2))y^3)a)a = (((y^3x^3)(aa))a)a \\ &= ((a((y^3x^3)a))a)a \subseteq ((AS)A)A \subseteq AA = A^2. \end{aligned}$$

Hence $A = A^2$ holds.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 144 *In an intra-regular AG-groupoid S with left identity, the following conditions are equivalent.*

(i) A is a quasi ideal of S.
(ii) $SQ \cap QS = Q$.
Proof. $(i) \Longrightarrow (ii)$: Let Q be a quasi ideal of an intra-regular AG-groupoid S with left identity, then $SQ \cap QS \subseteq Q$. Let $q \in Q$, then since S is intra-regular so there exist $x,\, y \in S$ such that $q = (xq^2)y$. Let $pq \in SQ$, then by using medial law with left identity, medial law and paramedial law, we have

$$\begin{aligned} pq &= p((xq^2)y) = (xq^2)(py) = (x(qq))(py) = (q(xq))(py) \\ &= (qp)((xq)y) = (xq)((qp)y) = (y(qp))(qx) \\ &= q((y(qp))x) \in QS. \end{aligned}$$

Now let $qy \in QS$, then by using left invertive law, medial law with left identity and paramedial law, we have

$$\begin{aligned} qp &= ((xq^2)y)p = (py)(xq^2) = (py)(x(qq)) = x((py)(qq)) \\ &= x((qq)(yp)) = (qq)(x(yp)) = ((x(yp))q)q \in SQ. \end{aligned}$$

Hence $QS = SQ$. As by using medial law with left identity and left invertive law, we have

$$q = (xq^2)y = (x(qq))y = (q(xq))y = (y(xq))q \in SQ.$$

Thus $q \in SQ \cap QS$ implies that $SQ \cap QS = Q$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 145 *In an intra-regular AG-groupoid S with left identity, the following conditions are equivalent.*

(i) A is an interior ideal of S.
(ii) $(SA)S = A$.
Proof. $(i) \Longrightarrow (ii)$: Let A be an interior ideal of an intra-regular AG-groupoid S with left identity, then $(SA)S \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist x, $y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) \\ &= (((xa^2)y)(xa))y = ((ax)(y(xa^2)))y = (((y(xa^2))x)a)y \in (SA)S. \end{aligned}$$

Thus $(SA)S = A$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Theorem 146 *In an intra-regular AG-groupoid S with left identity, the following conditions are equivalent.*

(i) A is a $(1, 2)$-ideal of S.
(ii) $(AS)A^2 = A$ and $A^2 = A$.
Proof. $(i) \Longrightarrow (ii)$: Let A be a $(1, 2)$-ideal of an intra-regular AG-groupoid S with left identity, then $(AS)A^2 \subseteq A$ and $A^2 \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist x, $y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(x((xa^2)y)))a = (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a \\ &= (((xy)y)(a^2x))a = ((y^2x)(a^2x))a = (a^2((y^2x)x))a \\ &= (a^2(x^2y^2))a = (a(x^2y^2))a^2 = (a(x^2y^2))(aa) \in (AS)A^2. \end{aligned}$$

Thus $(AS)A^2 = A$. Now by using medial law with left identity, left invertive law, paramedial law and medial law, we have

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(xa))((xa^2)y) = (xa^2)((y(xa))y) = (x(aa))((y(xa))y) \\ &= (a(xa))((y(xa))y) = (((y(xa))y)(xa))a = ((ax)(y(y(xa))))a \\ &= ((((xa^2)y)x)(y(y(xa))))a = (((xy)(xa^2))(y(y(xa))))a \\ &= (((xy)y)((xa^2)(y(xa))))a = ((y^2x)((x(aa))(y(xa))))a \\ &= ((y^2x)((xy)((aa)(xa))))a = ((y^2x)((aa)((xy)(xa))))a \\ &= ((aa)((y^2x)((xy)(xa))))a = ((aa)((y^2x)((xx)(ya))))a \\ &= ((((xx)(ya))(y^2x))(aa))a = ((((ay)(xx))(y^2x))(aa))a \\ &= ((((x^2y)a)(y^2x))(aa))a = (((xy^2)(a(x^2y)))(aa))a \\ &= ((a((xy^2)(x^2y)))(aa))a = ((a(x^3y^3))(aa))a \\ &\in ((AS)A^2)A \subseteq AA = A^2. \end{aligned}$$

Hence $A^2 = A$.
$(ii) \Longrightarrow (i)$ is obvious. ■

Lemma 147 *[38]Every non empty subset A of an intra-regular AG-groupoid S with left identity is a left ideal of S if and only if it is a right ideal of S.*

Theorem 148 *In an intra-regular AG-groupoid S with left identity, the following conditions are equivalent.*

(i) A is a $(1,2)$-ideal of S.
(ii) A is a two-sided ideal of S.

Proof. $(i) \Longrightarrow (ii)$: Assume that S is intra-regular AG-groupoid with left identity and let A be a $(1,2)$-ideal of S then, $(AS)A^2 \subseteq A$. Let $a \in A$, then since S is intra-regular so there exist x, $y \in S$ such that $a = (xa^2)y$. Now by using medial law with left identity, left invertive law and paramedial law, we have

$$\begin{aligned} sa &= s((xa^2)y) = (xa^2)(sy) = (x(aa))(sy) = (a(xa))(sy) \\ &= ((sy)(xa))a = ((sy)(xa))((xa^2)y) = (xa^2)(((sy)(xa))y) \\ &= (y((sy)(xa)))(a^2x) = a^2((y((sy)(xa)))x) \\ &= (aa)((y((sy)(xa)))x) = (x(y((sy)(xa))))(aa) \\ &= (x(y((ax)(ys))))(aa) = (x((ax)(y(ys))))(aa) \\ &= ((ax)(x(y(ys))))(aa) = ((((xa^2)y)x)(x(y(ys))))(aa) \\ &= (((xy)(xa^2))(x(y(ys))))(aa) = (((a^2x)(yx))(x(y(ys))))(aa) \\ &= ((((yx)x)a^2)(x(y(ys))))(aa) = (((y(ys))x)(a^2((yx)x)))(aa) \\ &= (((y(ys))x)(a^2(x^2y)))(aa) = (a^2(((y(ys))x)(x^2y)))(aa) \\ &= ((aa)(((y(ys))x)(x^2y)))(aa) = (((x^2y)((y(ys))x))(aa))(aa) \\ &= (a((x^2y)(((y(ys))x)a)))(aa) \in (AS)A^2 \subseteq A. \end{aligned}$$

Hence A is a left ideal of S and A is a two-sided ideal of S.

$(ii) \implies (i)$: Let A be a two-sided ideal of S. Let $y \in (AS)A^2$, then $y = (as)b^2$ for some $a, b \in A$ and $s \in S$. Now by using medial law with left identity, we have

$$y = (as)b^2 = (as)(bb) = b((as)b) \in AS \subseteq A.$$

Hence $(AS)A^2 \subseteq A$, therefore A is a $(1,2)$-ideal of S. ■

Lemma 149 *[38] Let S be an AG-groupoid, then S is intra-regular if and only if every left ideal of S is idempotent.*

Lemma 150 *[38]Every non empty subset A of an intra-regular AG-groupoid S with left identity is a two-sided ideal of S if and only if it is a quasi ideal of S.*

Theorem 151 *A two-sided ideal of an intra-regular AG-groupoid S with left identity is minimal if and only if it is the intersection of two minimal two-sided ideals of S.*

Proof. Let S be intra-regular AG-groupoid and Q be a minimal two-sided ideal of S, let $a \in Q$. As $S(Sa) \subseteq Sa$ and $S(aS) \subseteq a(SS) = aS$, which shows that Sa and aS are left ideals of S, so Sa and aS are two-sided ideals of S.

Now

$$\begin{aligned} S(Sa \cap aS) \cap (Sa \cap aS)S &= S(Sa) \cap S(aS) \cap (Sa)S \cap (aS)S \\ &\subseteq (Sa \cap aS) \cap (Sa)S \cap Sa \subseteq Sa \cap aS. \end{aligned}$$

This implies that $Sa \cap aS$ is a quasi ideal of S, so, $Sa \cap aS$ is a two-sided ideal of S. Also since $a \in Q$, we have

$$Sa \cap aS \subseteq SQ \cap QS \subseteq Q \cap Q \subseteq Q.$$

Now since Q is minimal, so $Sa \cap aS = Q$, where Sa and aS are minimal two-sided ideals of S, because let I be an two-sided ideal of S such that $I \subseteq Sa$, then $I \cap aS \subseteq Sa \cap aS \subseteq Q$, which implies that $I \cap aS = Q$. Thus $Q \subseteq I$. Therefore, we have

$$Sa \subseteq SQ \subseteq SI \subseteq I, \text{ gives } Sa = I.$$

Thus Sa is a minimal two-sided ideal of S. Similarly aS is a minimal two-sided ideal of S.

Conversely, let $Q = I \cap J$ be a two-sided ideal of S, where I and J are minimal two-sided ideals of S, then, Q is a quasi ideal of S, that is $SQ \cap QS \subseteq Q$. Let Q' be a two-sided ideal of S such that $Q' \subseteq Q$, then

$$SQ' \cap Q'S \subseteq SQ \cap QS \subseteq Q, \text{ also } SQ' \subseteq SI \subseteq I \text{ and } Q'S \subseteq JS \subseteq J.$$

Now

$$S(SQ^{'}) = (SS)\,(SQ^{'}) = (Q^{'}S)\,(SS) = (Q^{'}S)S = (SS)\,Q^{'} = SQ^{'},$$

which implies that $SQ^{'}$ is a left ideal and hence a two-sided ideal. Similarly $Q^{'}S$ is a two-sided ideal of S. Since I and J are minimal two-sided ideals of S, therefore $SQ^{'} = I$ and $Q^{'}S = J$. But $Q = I \cap J$, which implies that, $Q = SQ^{'} \cap Q^{'}S \subseteq Q^{'}$. This give us $Q = Q^{'}$ and hence Q is minimal. ■

4.4 Characterizations of Intra-regular AG-groupoids

Example 152 *Let $S = \{a, b, c, d, e\}$ be an AG-groupoid with left identity b in the following multiplication table.*

$.$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Clearly S is intra-regular because, $a = (aa^2)a$, $b = (cb^2)e$, $c = (dc^2)e$, $d = (cd^2)c$, $e = (be^2)e$.

Example 153 *Let $S = \{a, b, c, d, e\}$, and the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

Then clearly $(S, \cdot)$ is an AG-groupoid. Also $a = (aa^2)a$, $b = (bb^2)b$, $c = (ec^2)c$, $d = (ed^2)d$ and $e = (ee^2)e$. Therefore $(S, \cdot)$ is an intra-regular AG-groupoid. It is easy to see that $\{a\}$ and $\{a, b\}$ are ideals of S.

Theorem 154 *An AG-groupoid S is intra-regular if $Sa = S$ or $aS = S$ holds for all $a \in S$.*

Proof. Let S be an AG-groupoid such that $Sa = S$ holds for all $a \in S$, then $S = S^2$. Let $a \in S$, therefore by using medial law, we have

$$S = (SS)S = ((Sa)(Sa))S = ((SS)(aa))S \subseteq (Sa^2)S.$$

Which shows that S is intra-regular.

Let $a \in S$ and assume that $aS = S$ holds for all $a \in S$, then by using left invertive law, we have

$$S = SS = (aS)S = (SS)a = Sa.$$

Thus $Sa = S$ holds for all $a \in S$, therefore it follows from above that S is intra-regular. ■

Lemma 155 *Intersection of two ideals of an AG-groupoid with left identity is either empty or an ideal.*

Proof.

$$(A \cap B)S = AS \cap BS \subseteq A \cap B.$$

■

Lemma 156 *Product of two bi-ideals of an AG-groupoid with left identity is a bi-ideal.*

Lemma 157 *If I is an ideal of an intra-regular AG-groupoid S with left identity, then $I = I^2$.*

Proof. Clearly $I^2 \subseteq I$. Now let $i \in I$, then since S is intra-regular therefore there exists x and y in S such that $i = (xi^2)y$. Then $i = (xi^2)y \in (SI^2)S \subseteq I^2$. ■

Theorem 158 *The intersection of two quasi ideals of an AG-groupoid S is either empty or a quasi ideal of S.*

Proof. Let Q_1 and Q_2 be quasi-ideals of S. Suppose that $Q_1 \cap Q_2$ is non-empty, then

$$\begin{aligned} S(Q_1 \cap Q_2) \cap (Q_1 \cap Q_2)S &\subseteq (SQ_1 \cap SQ_2) \cap (Q_1S \cap Q_2S) \\ &\subseteq (SQ_1 \cap Q_1S) \cap (SQ_2 \cap Q_2S) \\ & Q_1 \cap Q_2. \end{aligned}$$

Hence $Q_1 \cap Q_2$ is a quasi-ideal of S. ■

Theorem 159 *[38]For an intra-regular AG-groupoid S with left identity the following statements are equivalent.*

(i) A is a left ideal of S.

(ii) A is a right ideal of S.

(iii) A is an ideal of S.

(iv) A is a bi-ideal of S.

(v) A is a generalized bi-ideal of S.

(vi) A is an interior ideal of S.

(vii) A is a quasi-ideal of S.

$(viii)$ $AS = A$ and $SA = A$.

Theorem 160 *Let S be an AG-groupoid with left identity e then the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *Every left ideal is idempotent.*

Proof. $(i) \Longrightarrow (ii)$
Let L be a left ideal of an intra-regular AG-groupoid S with left identity. Obviously $L^2 \subseteq L$. Now let $l \in L$. Since S is intra-regular therefore for l there exists x and y in S such that $l = (xl^2)y$. Then using left invertive law, we get

$$l = (xl^2)y = (l(xl))y = (y(xl))l \in (S(SL))L \subseteq L^2.$$

Therefore $L \subseteq L^2$. Hence $L = L^2$.
$(i) \Longrightarrow (ii)$
Since Sa is a left ideal contains a. Therefore using (ii) we get, $a \in Sa = (Sa)^2 = Sa^2 = (Sa^2)S$. ∎

Theorem 161 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.
(ii) Every quasi-ideal of S is idempotent.
Proof. $(i) \Longrightarrow (ii)$
Let Q be a quasi-ideal of S. Let $a \in Q$ which implies that $a^2 \in Q$ then since S is intra-regular so there exist $x, y \in S$ such that $a = (xa^2)y$. Now by theorem 159, Q is an ideal and Q^2 becomes an ideal. Therefore

$$a = (xa^2)y \in (SQ^2)S \subseteq Q^2.$$

Hence $Q = Q^2$.
$(ii) \Longrightarrow (i)$
Clearly Sa is a quasi-ideal. Now by (ii) Sa is idempotent. Therefore $a \in Sa = (Sa)^2$ but $(Sa)^2 = (Sa^2)S$. Hence $a \in Sa = (Sa^2)S$. ∎

Theorem 162 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.
(ii) $Q = (SQ)^2 \cap (QS)^2$, for every left ideal Q of S.
(iii) $Q = (SQ)^2 \cap (QS)^2$, for every quasi-ideal Q of S.
Proof. $(i) \Longrightarrow (vi)$
Let Q be a quasi-ideal of an intra-regular AG-groupoid S with left identity so by theorem 159, Q is an ideal and by theorem 161, Q is idempotent, then medial law we get

$$\begin{aligned}(SQ)^2 \cap (QS)^2 &= (SQ)(SQ) \cap (QS)(QS) = (SS)(QQ) \cap (QQ)(SS) \\ &= (SQ) \cap (QS) \subseteq Q.\end{aligned}$$

Now let $a \in Q$ and since S is intra-regular so there exists $x, y \in S$ such that $a = (xa^2)y$. Then using, left invertive law, paramedial law and medial law, we have

$$\begin{aligned}
a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) = (xa^2)((y(xa))y) \\
&= (y(y(xa)))((aa)x) = (aa)((y(y(xa)))x) = (x(y(y(xa))))(aa) \\
&\in S(QQ) = (SS)(QQ) = (SQ)(SQ) = (SQ)^2.
\end{aligned}$$

Thus $a \in (SQ)^2$. It is easy to see that $(SQ)^2 = (QS)^2$. Therefore $a \in (SQ)^2 \cap (QS)^2$. Thus $Q \subseteq (SQ)^2 \cap (QS)^2$. Hence $(SQ)^2 \cap (QS)^2 = Q$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Let Q be a left ideal of an AG-groupoid S with left identity then by (ii), $Q = (SQ)^2 \cap (QS)^2 \subseteq (SQ)^2 \subseteq Q^2$. Thus $Q = Q^2$. Hence by theorem 160, S is intra-regular. ■

Theorem 163 *Let S be an AG-groupoid with left identity e then the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $A \subseteq (AS)A$, for every quasi-ideal A and $A = A^2$.

Proof. $(i) \Rightarrow (ii)$

Let $a \in A$, and since S is intra-regular so there exists elements x, y in S such that $a = (xa^2)y$. Now using (1), left invertive law and medial law, we have

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a \\
&= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((x(aa))(y(xy)))a \\
&= ((a(xa))(y(xy)))a = ((ay)((xa)(xy)))a = ((xa)((ay)(xy)))a \\
&= ((xa)((ax)y^2))a = ((y^2(ax))(ax))a = (a((y^2(ax))x))a \in (AS)A.
\end{aligned}$$

Hence $A \subseteq (AS)A$. By theorem 159, A becomes an ideal and let $c^2 \in A$. Now since S in intra-regular so for c there exists u and v in S such that $(uc^2)v$. Then

$$c = (uc^2)v \in (SA)S \subseteq A.$$

Hence A is semiprime.

$(ii) \Rightarrow (i)$

It is same as the converse of theorem 161. ■

Theorem 164 *Let S be an AG-groupoid with left identity e then the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $R \cap L = RL$, for every right ideal R and every left ideal L and R is semiprime.

Proof. $(i) \Longrightarrow (ii)$

Let R, L be right and left ideals of an intra-regular AG-groupoid S with left identity then by theorem 159, R and L become ideals of S and so $RL \subseteq R \cap L$. Now $R \cap L$ is an ideal and by $R \cap L = (R \cap L)^2$. Thus $R \cap L = (R \cap L)^2 \subseteq RL$. Therefore $R \cap L = RL$. Next let $r^2 \in R$. Then since S is intra-regular therefore for r there exists x and y such that $r = (xr^2)y$. Thus

$$r = (xr^2)y \in (SR)S \subseteq R.$$

Hence R is semiprime.

$(ii) \Longrightarrow (i)$

Clearly Sa^2 is a right ideal contains a^2. Therefore by (ii) $a \in Sa^2$. Since Sa is left ideal and so we get

$$a \in Sa^2 \cap Sa = (Sa^2)(Sa) \subseteq (Sa^2)S.$$

■

Theorem 165 *Let S be an AG-groupoid with left identity e, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *$B = (BS)B$, for every bi-ideal B and $B = B^2$.*

Proof. $(i) \Longrightarrow (ii)$

Let B is bi-ideal of S then B is an ideal and $B = B^2$. Let $b \in B$, now since S is intra-regular therefore for b there exists x, y in S such that $b = (xb^2)y$. Also since $S = S^2$, therefore for y in S there exists u, v in S such that $y = uv$. Now using medial law and left invertive law, we get

$$\begin{aligned} b &= (xb^2)y = (xb^2)(uv) = (xu)(b^2v) = b^2((xu)v) \\ &= ((xu)v)b)b = [(xu)v][(xb^2)y])b = (xb^2)[[(xu)v]y])b \\ &= \{[x(bb)][(xu)v]y]\}b = \{[b(xb)][(xu)v]y]\}b \\ &= \{[y[(xu)v]][(xb)b]\}b = \{(xb)[y[(xu)v]]b]\}b \\ &= \{b[y[(xu)v]](bx)\}b = \{b[b[y[(xu)v]]x]\}b \subseteq (BS)B. \end{aligned}$$

Therefore $B = (BS)B$.

$(ii) \Longrightarrow (i)$

Since Sa is a bi-ideal contains a. Therefore using (ii) we get

$$a \in Sa = (Sa)^2 = Sa^2 = (Sa^2)S.$$

■

Theorem 166 *Let S be an AG-groupoid with left identity e, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *Every bi-ideal is idempotent.*

Proof. It is the part of theorem 165. ■

Theorem 167 *Let S be an AG-groupoid with left identity e, then the following conditions are equivalent.*
(i) S is intra-regular.
(ii) $L \cap R = LR$, for every right ideal R and every left ideal L and R is semiprime.

Proof. $(i) \Longrightarrow (ii)$
Let R is a right and L is a left ideal of an intra-regular AG-groupoid S with left identity. Then by theorem 159, R and L become ideals of S. Then clearly $LR \subseteq L \cap R$. Now let $a \in L \cap R$ which implies that $a \in L$ and $a \in R$. Then since S is intra-regular so for a there exists x, y in S such that $(xa^2)y$. Then using and left invertive law we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in LR.$$

Therefore $L \cap R \subseteq LR$. Hence $L \cap R = LR$.
Let $r^2 \in R$. Now since S in intra-regular therefore for r there exists u and v in S such that $r = (ur^2)v$. Thus

$$r = (ur^2)v \in (SR)S \subseteq R.$$

Hence R is semiprime.
$(ii) \Longrightarrow (i)$
Clearly Sa^2 is a right ideal contains a^2, therefore by (ii) it is semiprime. Thus $a \in Sa^2$. Also we know that Sa is a left ideal of S. Therefore using paramedial and medial law we get

$$a \in Sa \cap Sa^2 = (Sa)(Sa^2) = (a^2S)(aS) = (Sa^2)(aS) \subseteq (Sa^2)S.$$

■

Theorem 168 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.
(ii) $A \cap B = (AB)A$, for every bi-ideal A and every quasi-ideal B of S.
(iii) $A \cap B = (AB)A$, for every generalized bi-ideal A and every quasi-ideal B of S.
Proof. $(i) \Rightarrow (iii)$
Let A and B be a generalized bi-ideal and quasi-ideal of an intra-regular AG-groupoid with left identity. Now by theorem 159, A and B are ideals of S. Then $(AB)A \subseteq (AS)A \subseteq A$ and $(AB)A \subseteq (SB)S \subseteq B$, which implies that $(AB)A \subseteq A \cap B$. Next let $a \in A \cap B$, which implies that $a \in A$ and $a \in B$. Since S is intra-regular so for a there exist $x, y \in S$, such that

$a = (xa^2)y$, then using (1) and left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a == ((a(xa))(y(xy)))a \\ &= (((y(xy))(xa))a)a \subseteq (((S(SA)B)A \subseteq (AB)A. \end{aligned}$$

Thus $A \cap B = (AB)A$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Since Sa is both bi and quasi-ideal. Therefore by medial law, we get

$$\begin{aligned} a &\in Sa \cap Sa = ((Sa)(Sa))(Sa) = ((SS)(aa))(Sa) \\ &= (Sa^2)(Sa) \subseteq (Sa^2)S. \end{aligned}$$

■

Theorem 169 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.

(ii) $(A \cap B) \cap C = (AB)C$, for every left ideal A, every two-sided ideal B and every left ideal C of S and B is semiprime.

(iii) $(A \cap B) \cap C = (AB)C$, for every left ideal A, every right ideal B and every left ideal C of S and B is semiprime.

(iv) $(A \cap B) \cap C = (AB)C$, for every left ideal A, every interior ideal B and every left ideal C of S and B is semiprime.

Proof. $(i) \Rightarrow (iv)$

Let S be a intra-regular AG-groupoid with left identity. Let A, B and C be left, interior and left ideal of S respectively. Now by theorem 159, A, B and C become ideals of S. Then

$$(AB)C \subseteq (AS)S \subseteq A, (AB)C \subseteq (SB)S \subseteq B \text{ and} (AB)C \subseteq (SS)C \subseteq C.$$

Thus $(AB)C \subseteq (A \cap B) \cap C$. Now let $a \in (A \cap B) \cap C$, which implies that $a \in A$, $a \in B$ and $a \in C$. Now for a there exists $x, y \in S$, such that $a = (xa^2)y$, then by using (1) and left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((a(xa))(y(xy)))a \\ &= (((y(xy))(xa))a)a \subseteq (((S(SA)B)C \subseteq (AB)C. \end{aligned}$$

Therefore $(A \cap B) \cap C \subseteq (AB)C$. Hence $(A \cap B) \cap C = (AB)C$.

Next let $b^2 \in B$. Now for b there exists u and v in S such that $b = (ub^2)v$. Thus

$$b = (ub^2)v \in (SB)S \subseteq B.$$

Hence B is semiprime

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$

Sa is left ideal and Sa^2(contains a^2) is an ideal. By (ii), Sa^2 is semiprime, therefore $a \in Sa^2$. Now using paramedial, medial and left invertive law, we get

$$\begin{aligned} a &\in Sa \cap Sa^2 \cap Sa = ((Sa)(Sa^2))(Sa) \subseteq ((SS)(Sa^2))S \\ &= (((a^2S)(SS))S = (((a^2S)S)S = ((SS)a^2)S = (Sa^2)S. \end{aligned}$$

■

Theorem 170 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.

(ii) $A \cap B = (AB) \cap (BA)$, for every bi-ideal A and B of S.

(iii) $A \cap B = (AB) \cap (BA)$, for every bi-ideal A and every generalized bi-ideal B of S.

(iv) $A \cap B = (AB) \cap (BA)$, for every generalized bi-ideals A and B of S.

Proof. $(i) \Longrightarrow (iv)$

Let A and B be any generalized bi-ideal of an intra-regular AG-groupoid S with left identity, then by theorem 159, A and B are ideals of S. Clearly $AB \subseteq A \cap B$, now $A \cap B$ is an ideal and $A \cap B = (A \cap B)^2$. Now $A \cap B = (A \cap B)^2 \subseteq AB$. Thus $A \cap B = AB$ and then $A \cap B = B \cap A = BA$. Hence $A \cap B = (AB) \cap (BA)$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$

Let B be a ideal of an AG-groupoid S with left identity. Then by (ii) $B \cap B = (BB) \cap (BB) = B^2$, so by theorem 166, S is intra-regular. ■

Theorem 171 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.

(ii) $B \cap G = (BG)B$, for every bi-ideal B and every quasi-ideal G.

Proof. $(i) \Longrightarrow (ii)$

Let $a \in B \cap G$. Now by theorem 159, B and G become ideals of S. Then using (1) and left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y(xa^2(xy)))a = (xa^2)(y(xy)))a = (a(xa))(y(xy)))a \\ &= (y(xy)(xa))a)a \in ((S(Sa))a)a \subseteq ((S(SB))G)B = ((SB)G)B \subseteq (BG)B. \end{aligned}$$

Therefore $B \cap G \subseteq (BG)B$.

Next $(BG)B \subseteq (BS)B \subseteq B$ and $(BG)B \subseteq (SG)S \subseteq G$. Therefore $(BG)B \subseteq B \cap G$. Hence $B \cap G = (BG)B$.

$(ii) \Longrightarrow (i)$

Sa is both bi and quasi-ideal of an AG-groupoid S with left identity. Therefore by medial law we get

$$\begin{aligned} a &\in Sa \cap Sa = ((Sa)(Sa))(Sa) = ((SS)(aa))(Sa) \\ &= (Sa^2)(Sa) \subseteq (Sa^2)S. \end{aligned}$$

■

Theorem 172 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) *S is intra-regular.*

(ii) $B \cap I = BI(B \cap I \subseteq BI)$, *for every bi-ideal B and every quasi-ideal I.*

Proof. $(i) \Longrightarrow (ii)$

Let B and I be bi and quasi ideals of an AG-groupoid S with left identity. Then by theorem 159, B and I become ideals of S. Now clearly $BI \subseteq B \cap I$. Next let $a \in B \cap I$. Now since S is intra-regular so for a there exists x, y in S such that $a = (xa^2)y$. Now using left invertive law we get

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &\in (S(SB))I \subseteq (SB)I \subseteq BI. \end{aligned}$$

Therefore $B \cap I \subseteq BI$. Hence $B \cap I = BI$.

$(ii) \Longrightarrow (i)$

Sa is both bi and quasi-ideal. Therefore by medial law we get

$$a \in Sa \cap Sa = (Sa)(Sa) = (SS)a^2 = Sa^2 = (Sa^2)S.$$

■

Theorem 173 *For an AG-groupoid S with left identity the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *Every left ideal of S is idempotent.*

(iii) $A \cap B = AB$, *for every ideals A, B of S and A, B are semiprime.*

(iv) $A \cap B = AB$, *for every ideal A, every bi-ideal B of S and A, B are semiprime.*

(v) $A \cap B = AB$, *for every bi-ideals A, B of S and A, B are semiprime.*

(vi) *The set of left ideals forms a semilattice structure.*

Proof. $(i) \iff (ii)$

It is same as theorem 160.

$(i) \Longrightarrow (v)$

Let A, B are bi-ideals of an intra-regular AG-groupoid S with left identity. Then by theorem 159, A and B are ideals of S. Now clearly $AB \subseteq$

$A \cap B$. Since $A \cap B$ is an ideal and $(A \cap B)^2 = A \cap B$. Thus $A \cap B = (A \cap B)^2 \subseteq AB$. Therefore $(A \cap B) = AB$. Next let $a^2 \in A$. Now for a there exists x, y in S such that $a = (xa^2)y$. Thus $a = (xa^2)y \in (SA)S \subseteq A$. Hence A is semiprime. Similarly we can show that B is semiprime.

$(v) \implies (i)$

Assume that A is a bi-ideal of an AG-groupoid S with left identity then by (v) $A \cap A = AA$, that is, $A = A^2$ and by theorem 166, S is intra-regular.

$(i) \implies (vi)$

Let $\mathfrak{L}_S$ denote the set of all left ideas of an intra-regular AG-groupoid S with left identity and let I and $J \in \mathfrak{L}_S$. Now by theorem 159, I and J become ideals of S. Thus $IJ \subseteq I \cap J$. Now $I \cap J$ is an ideal and so $I \cap J = (I \cap J)^2$. Therefore $I \cap J \subseteq IJ$. Thus $I \cap J = IJ$ which clearly implies that $I \cap J = JI$. Now clearly all elements (ideals) of $\mathfrak{L}_S$ satisfy left invertive law. Therefore $\mathfrak{L}_S$ form an AG-groupoid. Also $IJ = JI$ and $I = I^2$, for all I and J in $\mathfrak{L}_S$. But we know that a commutative AG-groupoid becomes a commutative semigroup. Hence the set of all left ideals that is $\mathfrak{L}_S$ form a semilattice structure.

$(vi) \implies (i)$

If I is a left ideal of an AG-groupoid S with left identity, then by (vi), $I = I^2$. The rest is same as $(ii) \implies (i)$.

$(v) \implies (iv) \implies (iii)$ are obvious.

$(iii) \implies (i)$

Since Sa^2 is an ideal of an AG-groupoid S with left identity. Then by (iii) it becomes semiprime and since S itself is an ideal, therefore by (iii) we get

$$a \in Sa^2 = Sa^2 \cap S = (Sa^2)S.$$

■

4.5 Characterizations of Intra-regular AG**-groupoids

It is easy to see that every AG-groupoid with left identity becomes an AG**-groupoid but the converse is not true (see the example below)

Example 174 *Let $S = \{1, 2, 3, 4, 5\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4	5
1	1	2	4	4	5
2	5	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	2	4	4	4	4

$(S,.)$ is neither commutative nor associative because $5 = 1.5 \neq 5.1 = 2$ and $2 = (2.1).1 \neq 2.(2.1) = 5$. Also by AG-test in [48], it is easy to check that S is an AG^{**}-groupoid.

Here we begin with examples of intra-regular AG-groupoids.

Example 175 *Let $S = \{1, 2, 3, 4, 5, 6\}$, then by AG-test in [48], $(S, \cdot)$ is an AG-groupoid with left identity 5 as given in the following multiplication table:*

$\cdot$	1	2	3	4	5	6
1	5	6	1	2	3	4
2	4	5	6	1	2	3
3	3	4	5	6	1	2
4	2	3	4	5	6	1
5	1	2	3	4	5	6
6	6	1	2	3	4	5

Clearly $(S, \cdot)$ is intra-regular because, $1 = \left(4 \cdot 1^2\right) \cdot 2, 2 = \left(3 \cdot 2^2\right) \cdot 4, 3 = \left(2 \cdot 3^2\right) \cdot 6, 4 = \left(1 \cdot 4^2\right) \cdot 2, 5 = \left(5 \cdot 5^2\right) \cdot 5, 6 = \left(3 \cdot 6^2\right) \cdot 2$.

Example 176 *Let $S = \{a, b, c, d, e\}$, and the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

Then clearly $(S, \cdot)$ is an AG-groupoid. Also $a = (aa^2)a$, $b = (bb^2)b$, $c = (ec^2)c$, $d = (ed^2)d$ and $e = (ee^2)e$. Therefore $(S, \cdot)$ is an intra-regular AG-groupoid. It is easy to see that $\{a\}$ and $\{a, b\}$ are ideals of S.

It is easy to note that if S is intra-regular AG-groupoid then $S = S^2$.

Lemma 177 *Intersection of two ideals of an AG-groupoid is an ideal.*

Lemma 178 *Product of two bi-ideals of an AG^{**}-groupoid is a bi-ideal.*

Lemma 179 *Let S be an AG^{**}-groupoid such that $S = S^2$, then every right ideal is a left ideal.*

Proof. Let R be a right ideal of S, then using left invertive law, we get

$$SR = (SS)R = (RS)S \subseteq RS \subseteq R.$$

■

Lemma 180 *If I is an ideal of an intra-regular AG^{**}-groupoid S, then $I = I^2$.*

Proof. It is same as in [38]. ■

Lemma 181 *Let S be an AG^{**}-groupoid S such that $S = S^2$, then a subset I of S is a right ideal of S if and only if it is an interior ideal of S.*

Proof. It is same as in [38]. ■

Corollary 182 *Every interior ideal of S becomes a left ideal of S.*

Theorem 183 *Let S be an intra-regular AG^{**}-groupoid, then the following statements are equivalent.*
(i) *A is a left ideal of S.*
(ii) *A is a right ideal of S.*
(iii) *A is an ideal of S.*
(iv) *A is a bi-ideal of S.*
(v) *A is a generalized bi-ideal of S.*
(vi) *A is an interior ideal of S.*
(vii) *A is a quasi-ideal of S.*
$(viii)$ *$AS = A$ and $SA = A$.*

Proof. $(i) \Rightarrow (viii)$

Let A be a left ideal of S. Then clearly $SA \subseteq A$. Now let $a \in A$ and since S is intra-regular for a there exists x, y in S such that $a = (xa^2)y$. Using left invertive law we get

$$a = (xa^2)y = [\{x(aa)\}]y = [\{a(xa)\}]y = [\{y(xa)\}]a \in SA.$$

Thus $A \subseteq SA$. Therefore $SA = A$.

Now let $a \in A$ and $s \in S$, since S is an intra-regular, so there exist x, $y \in S$ such that $a = \left(xa^2\right) y$, therefore by left invertive law, we have

$$\begin{aligned} as &= \left(\left(xa^2\right) y\right) s = \left(\left(x\left(aa\right)\right) y\right) s \in \left(\left(S\left(AA\right)\right) S\right) S \subseteq \left(\left(S\left(SA\right)\right) S\right) S \subseteq \left(\left(SA\right) S\right) S \\ &= \left(SS\right)\left(SA\right) = S\left(SA\right) = A. \end{aligned}$$

Thus $AS \subseteq A$. Next let $a \in A$, then since $S = S^2$ so for y in S there exists y_1, y_2 in S such that $y = y_1 y_2$. Then using medial law, paramedial law we get

$$a = (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] \in AS.$$

Therefore $AS = S$.

$(viii) \Rightarrow (vii) \Rightarrow (vi) \Rightarrow (v)$ are same as in [38].

$(v) \Rightarrow (iv)$

Let A be a generalized bi-ideal of S. Let $a, b \in A$, and since S is intra-regular so there exist x, y in S such that $a = \left(xa^2\right) y$, then we have

$$\begin{aligned} ab &= \left(\left(xa^2\right) y\right) b = [a^2\{(y_2y_1)x\}]b = [\{(y_2y_1)x\}a^2]b \\ &= [a(\{(y_2y_1)x\}a)]b \in \left(AS\right) A \subseteq A. \end{aligned}$$

Hence A is a bi-ideal of S.
$(iv) \Rightarrow (iii)$ is same as in [38]
$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. ■

Lemma 184 *In an intra-regular AG^{**}-groupoid S, $IJ = I \cap J$, for all ideals I and J in S.*

Proof. Let I and J be ideals of S, then obviously $IJ \subseteq I \cap J$. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$, then $(I \cap J)^2 \subseteq IJ$, also $I \cap J$ is an ideal of S, so we have $I \cap J = (I \cap J)^2 \subseteq IJ$. Hence $IJ = I \cap J$. ■

An AG-groupoid S is called totally ordered under inclusion if P and Q are any ideals of S such that either $P \subseteq Q$ or $Q \subseteq P$.

An ideal P of an AG-groupoid S is called strongly irreducible if $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, for all ideals A, B and P of S.

Lemma 185 *Every ideal of an intra-regular S is prime if and only if it is strongly irreducible.*

Proof. It is an easy. ■

Theorem 186 *Every ideal of an intra-regular AG-groupoid S is prime if and only if S is totally ordered under inclusion.*

Proof. Assume that every ideal of S is prime. Let P and Q be any ideals of S, so, $PQ = P \cap Q$, where $P \cap Q$ is ideal of S, so is prime, therefore $PQ \subseteq P \cap Q$, which implies that $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$, which implies that $P \subseteq Q$ or $Q \subseteq P$. Hence S is totally ordered under inclusion.

Conversely, assume that S is totally ordered under inclusion. Let I, J and P be any ideals of S such that $IJ \subseteq P$. Now without loss of generality assume that $I \subseteq J$ then

$$I = I^2 = II \subseteq IJ \subseteq P.$$

Therefore either $I \subseteq P$ or $J \subseteq P$, which implies that P is prime. ■

Theorem 187 *Let S be an intra-regular AG^{**}-groupoid such that $S = S^2$, then the set of all ideals I_S of S, forms a semilattice structure.*

Proof. Let A, $B \in I_S$, since A and B are ideals of S, therefore using medial law, we have

$$\begin{aligned} (AB)S &= (AB)(SS) = (AS)(BS) \subseteq AB. \\ \text{Also } S(AB) &= (SS)(AB) = (SA)(SB) \subseteq AB. \end{aligned}$$

Thus AB is an ideal of S. Hence I_s is closed. Also we have, $AB = A \cap B = B \cap A = BA$, which implies that I_S is commutative, so is associative. Now $A^2 = A$, for all $A \in I_S$. Hence I_S is semilattice. ■

Theorem 188 *Let S be an AG^{**}-groupoid such that $S = S^2$, then the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For every generalized bi-ideal B, $B = B^2$.

Proof. Assume that S is an intra-regular AG**-groupoid and B is a generalized bi-ideal of S. Let $b \in B$, and since S is intra-regular so there exist c, d in S such that $b = \left(cb^2\right)d$, then we have

$$\begin{aligned} b &= \left(cb^2\right)d = \{c(bb)\}d = \{b(cb)\}d = \{d(cb)\}b \\ &= [d\{c(\left(cb^2\right)d)\}]b = [d\{\left(cb^2\right)(cd)\}]b = [\left(cb^2\right)\{d(cd)\}]b \\ &= [\{(cd)d\}\left(b^2c\right)]b = [b^2\left(\{(cd)d\}c\right)]b = [(c\{(cd)d\})\,b^2]b \\ &= [b((c\{(cd)d\})\,b)]b \in ((BS)\,B)B \subseteq BB. \end{aligned}$$

Thus $B \subseteq B^2$. Let $a, b \in B$, then $ab = [a(\{(y_2y_1)x\}a)]b \in (BS)B \subseteq B$, therefore $B^2 \subseteq B$. Hence $B^2 = B$.

Conversely, consider the subset Sa of S, then using paramedial law, medial law and left invertive law, we get

$$((Sa)S)(Sa) \subseteq S(Sa) = (SS)(Sa) = (aS)S = (SS)a = Sa.$$

Therefore Sa is a generalized bi-ideal. Now by assumption Sa is idempotent, so by using medial law, we have

$$a \in (Sa)(Sa) = ((Sa)(Sa))(Sa) = ((SS)(aa))(Sa) \subseteq \left(Sa^2\right)(SS) = \left(Sa^2\right)S.$$

Hence S is intra-regular. ■

Corollary 189 *Let S be an AG^{**}-groupoid such that $S = S^2$, then the following conditions are equivalent.*
(i) S is intra-regular.
(ii) For every bi-ideal B, $B = B^2$.

Theorem 190 *For an AG^{**}-groupoid S, then S is intra-regular if and only if every ideal I is semiprime.*

Proof. $(i) \Longrightarrow (ii)$

Let S be an intra-regular AG**-groupoid. Now let $a \in S$ such that $a^2 \in I$. For $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Therefore $a = (xa^2)y \in (SI)S \subseteq I$. Hence I is semiprime.

$(ii) \Longrightarrow (i)$

Obviously Sa^2 is an ideal contains a^2. And by (ii) it is semiprime so $a \in Sa^2$. Therefore $a \in Sa^2 = (Sa^2)S$. Hence S is intra-regular. ■

Corollary 191 *For an AG^{**}-groupoid S, then S is intra-regular if and only if every right ideal is semiprime.*

Theorem 192 *For an AG^{**}-groupoid S, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *For generalized bi-ideals B_1 and B_2, $B_1 \cap B_2 = (B_1B_2)B_1$.*

Proof. $(i) \Longrightarrow (ii)$
Let B_1 and B_2 be generalized bi-ideals of an intra-regular AG**-groupoid S. Now B_1 and B_2 become ideals of S. Therefore $(B_1B_2)B_1 \subseteq (B_1S)B_1 \subseteq B_1$ and $(B_1B_2)B_1 \subseteq (SB_2)S \subseteq B_2$. This implies that $(B_1B_2)B_1 \subseteq B_1 \cap B_2$. Now $B_1 \cap B_2$ becomes an ideal and we get,

$$\begin{aligned} B_1 \cap B_2 &= (B_1 \cap B_2)^2 = (B_1 \cap B_2)^2 (B_1 \cap B_2) \\ &= ((B_1 \cap B_2)(B_1 \cap B_2))(B_1 \cap B_2) \subseteq (B_1B_2)B_1. \end{aligned}$$

Thus $B_1 \cap B_2 \subseteq (B_1B_2)B_1$. Hence $B_1 \cap B_2 = (B_1B_2)B_1$.
$(i) \Longrightarrow (ii)$
Let B be a bi-ideal of an AG**-groupoid S, then using (ii), we get
$B = B \cap B = (BB)B \subseteq B^2B \subseteq BB = B^2$. Hence by theorem 188, S is intra-regular. ■

Corollary 193 *For an AG^{**}-groupoid S, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *For bi-ideals B_1 and B_2, $B_1 \cap B_2 = (B_1B_2)B_1$.*

Theorem 194 *If A is an interior ideal of an intra-regular AG^{**}-groupoid S such that $S = S^2$, then A^2 is also interior ideal.*

Proof. Using medial law we obtained,

$$\begin{aligned} \left(SA^2\right)S &= ((SS)(AA))(SS) = ((SA)(SA))(SS) \\ &= ((SA)S)((SA)S) \subseteq AA = A^2. \end{aligned}$$

■

Theorem 195 *For an AG^{**}-groupoid S, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *Every two sided ideal is semiprime.*
(iii) *Every right ideal is semiprime.*
(iv) *Every interior ideal is semiprime.*
(v) *Every generalized interior ideal is semiprime.*

Proof. $(i) \Longrightarrow (v)$
Let I be a generalized interior ideal of an intra-regular AG**-groupoid S. Let $a^2 \in I$. Then since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that, $a = (xa^2)y$. Then $a = (xa^2)y \in (SI)S \subseteq I$.
$(v) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.
$(ii) \Longrightarrow (i)$
It is same as the converse of theorem 190. ■

Theorem 196 *For an AG^{**}-groupoid S, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *Every two sided ideal is semiprime.*
(iii) *Every bi-ideal is semiprime.*
(iv) *Every generalized bi-ideal is semiprime.*

Proof. $(i) \Longrightarrow (iv)$

Let B be any generalized bi-ideal of an intra-regular AG**-groupoid S. Let $a^2 \in B$, since S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that, $a = (xa^2)y$. No B becomes an ideal of S. Therefore $a = (xa^2)y \in (SB)S \subseteq B$.

$(iv) \Longrightarrow (iii) \Longrightarrow (ii)$ are obvious.

$(ii) \Longrightarrow (i)$

It is same as $(ii) \Longrightarrow (i)$ of theorem 195. ∎

Theorem 197 *For an AG^{**}-groupoid S such that $S = S^2$, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *Every left ideal is idempotent.*
(iii) *For every left ideal L of $S, L = (SL)^2 \cap (LS)^2$.*

Proof. $(i) \Longrightarrow (ii)$

Let L be any left ideal of an intra-regular AG**-groupoid S so using medial law and paramedial law we get

$$\begin{aligned}(SL)^2 \cap (LS)^2 &= (SL)(SL) \cap (LS)(LS) = (SS)(LL) \cap (LL)(SS) \\ &= (SS)(LL) \cap (SS)(LL) = (SS)(LL) = (SL)(SL) \subseteq LL \subseteq L.\end{aligned}$$

Now let $a \in L$ and since S is intra-regular so there exists $x, y \in S$ such that $a = (xa^2)y$. Then using left invertive law, medial law and paramedial law, we get

$$\begin{aligned}a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))((xa^2)y) = (xa^2)((y(xa))y) \\ &= (y(y(xa)))((aa)x) = (aa)((y(y(xa)))x) = (x(y(y(xa))))(aa) \\ &\in S(LL) = (SS)(LL) = (SL)(SL) = (SL)^2.\end{aligned}$$

Thus $a \in (SL)^2$. It is easy to see that $(SL)^2 = (LS)^2$. Therefore $a \in (SL)^2 \cap (LS)^2$.

Thus $L \subseteq (SL)^2 \cap (LS)^2$. Hence $(SL)^2 \cap (LS)^2 = L$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Clearly Sa is a left ideal contains a, therefore by (ii) it is idempotent. Therefore using medial law, we get

$$a \in Sa = (Sa)(Sa) = (Sa^2) = (Sa^2)S.$$

Hence S is intra-regular. ∎

Theorem 198 *For an AG^{**}-groupoid S such that $S = S^2$, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *For every bi-ideal of S, $B = (SB)^2B \cap (BS)^2B$.*

Proof. $(i) \Longrightarrow (ii)$
Let B be a bi-ideal of an intra-regular AG**-groupoid S so by using medial law and paramedial law we get,

$$\begin{aligned} (SB)^2B \cap (BS)^2B &= ((SB)(SB))B \cap ((BS)(BS))B \\ &= ((BB)(SS))B \cap ((BB)(SS))B \\ &= (B^2S^2)B \cap (B^2S^2)B = (B^2S^2)B \\ &\subseteq (BS)B \subseteq B. \end{aligned}$$

Now let $a \in B$ and since S is intra-regular so there exists $x, y \in S$ such that $a = (xa^2)y$. Then using left invertive law, paramedial law and medial law, we have,

$$\begin{aligned} a &= (xa^2)y = (a(xa))y = (y(xa))a = (y(xa))a = (y(x((xa^2)y)))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((xy)(a^2(xy)))a \\ &= (a^2((xy)(xy)))a = (a(xy))^2a. \end{aligned}$$

Therefore $a \in ((B(SS))^2 B = (BS)^2 B$. This implies that $a \in (BS)^2 B$. Hence $B \subseteq (BS)^2 B$. Now since $(BS)^2 = (SB)^2$, thus $B \subseteq (SB)^2B$. Therefore $B \subseteq (SB)^2B \cap (BS)^2B$. Hence $B = (SB)^2B \cap (BS)^2B$.
$(ii) \Rightarrow (i)$
Let B be a bi-ideal of an AG-groupoid S, then by (ii), medial law, para medial law, left invertive law and (1), we get

$$\begin{aligned} B &= (SB)^2B \cap (BS)^2B = (SB)^2B = (S^2B^2)B = (B^2S)B \\ &= (BS)(BB) = B[(BS)B] \subseteq B^2. \end{aligned}$$

Thus $B \subseteq B^2$ but $B^2 \subseteq B$. Therefore $B = B^2$ and hence by corollary 189, S is intra-regular. ■

Theorem 199 *Let S be an AG^{**}-groupoid such that $S = S^2$, then the following are equivalent*
(i) *S is intra-regular,*
(ii) *Every ideal of S is semiprime.*
(ii) *Every quasi-ideal of S is semiprime.*

Proof. Let Q be a quasi-ideal of an intra-regular AG**-groupoid S and let $a^2 \in Q$. Then using paramedial and medial laws we get

$$a = a^2((y_2y_1)x) = (x(y_2y_1))a^2 \in QS \cap SQ \subseteq Q.$$

Therefore $a \in Q$. Hence Q is semiprime.
Converse is same as $(ii) \Longrightarrow (i)$ of theorem 195. ■

5

Some Characterizations of Strongly Regular AG-groupoids

In this chapter, we introduce a new class of AG-groupoids namely strongly regular and characterize it using its ideals.

5.1 Regularities in AG-groupoids

An AG-groupoid S is said to be regular if for every a in S there exists some x in S such that $a = (ax)a$.

An AG-groupoid S is said to be intra-regular if for every a in S there exists some x, y in S such that $a = (xa^2)y$.

An AG-groupoid S is said to be strongly regular if for every a in S there exists some x in S such that $a = (ax)a$ and $ax = xa$.

Here we begin with examples of AG-groupoids.

Example 200 *Let $S = \{1, 2, 3\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	1

Clearly $(S, \cdot)$ is an AG-groupoid without left identity.

Example 201 *Let $S = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	4	3	3	3
3	3	3	3	3
4	2	3	3	3

Clearly $(S, \cdot)$ is an AG-groupoid with left identity 1.

Example 202 *Let $S = \{1, 2, 3\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Clearly $(S, \cdot)$ is a strongly regular AG-groupoid with left identity 1.

Note that every strongly regular AG-groupoid is regular, but converse is not true, for converse consider the following example.

Example 203 *Let $S = \{1, 2, 3\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Clearly $(S, \cdot)$ is regular AG-groupoid, but not strongly regular.

Theorem 204 *Every strongly regular AG-groupoid is intra-regular.*
Proof. *Let S be strongly regular AG-groupoid,then for every $a \in S$ there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$, then using left invertive law we get*

$$\begin{aligned} a &= (ax)a = (ax)[(ax)a] = (ax)[(xa)a] = (ax)(a^2x) \\ &= [(a^2x)a]x = [(ax)a^2]x = (ua^2)x, \text{ where } u = ax. \end{aligned}$$

Hence S is intra-regular.

Converse of above theorem is not true, for converse consider the following example. ■

Example 205 *Let $S = \{1, 2, 3, 4, 5, 6, 7\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4	5	6	7
1	1	3	5	7	2	4	6
2	4	6	1	3	5	7	2
3	7	2	4	6	1	3	5
4	3	5	7	2	4	6	1
5	6	1	3	5	7	2	4
6	2	4	6	1	3	5	7
7	5	7	2	4	6	1	3

Clearly $(S, \cdot)$ is intra-regular AG-groupoid, but not strongly regular.

5.2 Some Characterizations of Strongly Regular AG-groupoids

Theorem 206 *For an AG-groupoid S with left identity the following are equivalent,*

(i) S is strongly regular,

(ii) $L \cap A \subseteq LA$ and L is strongly regular AG-subgroupoid, where L is any left ideal and A is any subset of S.

Proof. $(i) \Longrightarrow (ii)$

Let S be a strongly regular AG-groupoid with left identity. Let $a \in L \cap A$, now since S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Then

$$a = (ax)a = (xa)a \in (SL)A \subseteq LA.$$

Thus $L \cap A \subseteq LA$. Let $a \in L$, thus $a \in S$ and since S is strongly regular so there exists an x in S such that $a = (ax)a$ and $ax = xa$. Let $y = (xa)x$, then using left invertive law, we get

$$y = (xa)x = (ax)x = x^2 a \in SL \subseteq L.$$

Now using left invertive law and (1), we get

$$ya = [(xa)x]a = (ax)(xa) = (xa)(ax) = a[(xa)x] = ay.$$

Now using left invertive law we get

$$\begin{aligned} a &= (ax)a = (ax)[(ax)a] = (ax)[(xa)a] = (ax)(a^2 x) \\ &= a^2[(ax)x] = (aa)[(xa)x] = (aa)y = (ya)a = (ay)a. \end{aligned}$$

Therefore L is strongly regular.

$(ii) \Longrightarrow (i)$

Since S itself is a left ideal, therefore by assumption S is strongly regular. ■

Theorem 207 *For an AG-groupoid S with left identity the following are equivalent,*

(i) S is strongly regular,

(ii) $B \cap A \subseteq BA$ and B is strongly regular AG-subgroupoid, where B is any bi ideal and A is any subset of S.

Proof. $(i) \Longrightarrow (ii)$

Let S be a strongly regular AG-groupoid with left identity. Let $a \in B \cap A$, now since S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Then using left invertive law, we get

$$\begin{aligned} a &= (ax)a = [\{(ax)a\}x]a = [(xa)(ax)]a \\ &= [(ax)(xa)]a = [\{(xa)x\}a]a = [\{(x\{(ax)a\})x\}a]a \\ &= [\{\{(ax)(xa)\}x\}a]a = [\{\{x(xa)\}(ax)\}a]a = [(a\{\{x(xa)\}x\})a]a \\ &= [(at)a]a \in [(BS)B]A \subseteq BA, \text{where } t = x(xa). \end{aligned}$$

Thus $B \cap A \subseteq BA$. Let $a \in B$, thus $a \in S$ and since S is strongly regular so there exists an x in S such that $a = (ax)a$ and $ax = xa$. Let $y = (xa)x$, then using left invertive law, paramedial and medial law, we get

$$\begin{aligned} y &= (xa)x = (ax)x = x^2a = x^2[(ax)a] = x^2[(xa)a] = x^2(a^2x) \\ &= a^2(x^2x) = a^2t = (aa)t = (ta)a = [t\{(ax)a\}]a = [t\{(xa)a\}]a \\ &= [t(a^2x)]a = [a^2(tx)]a = [(aa)(tx)]a = [(xt)(aa)]a = [a\{(xt)a\}]a \\ &= (av)a \in (BS)B \subseteq B, \text{where } t = (x^2x) \text{ and } v = (xt)a. \end{aligned}$$

Now using left invertive law, we get

$$ya = [(xa)x]a = (ax)(xa) = (xa)(ax) = a[(xa)x] = ay.$$

Now using left invertive law we get

$$\begin{aligned} a &= (ax)a = (ax)[(ax)a] = (ax)[(xa)a] = (ax)(a^2x) \\ &= a^2[(ax)x] = (aa)[(xa)x] = (aa)y = (ya)a = (ay)a. \end{aligned}$$

Therefore B is strongly regular.

$(ii) \Longrightarrow (i)$

Since S itself is a bi ideal, therefore by assumption S is strongly regular.

■

Theorem 208 *For an AG-groupoid S with left identity the following are equivalent,*

(i) *S is strongly regular,*

(ii) *$Q \cap A \subseteq QA$ and Q is strongly regular AG-subgroupoid, where Q is any quasi ideal and A is any subset of S.*

Proof. $(i) \Longrightarrow (ii)$

Let S be a strongly regular AG-groupoid with left identity. Let $a \in Q \cap A$, now since S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Now using left invertive law, we get

$$\begin{aligned} ax &= [(ax)a]x = [(xa)a]x = (a^2x)x = x^2a^2 = x^2(aa) \\ &= a(x^2a) \in QS. \\ ax &= [(ax)a]x = (xa)(ax) = (ax)(xa) = [(xa)x]a \in SQ. \end{aligned}$$

Thus $ax \in QS \cap SQ \subseteq Q$.

Also $a = (ax)a \in QA$. Let $a \in Q$, thus $a \in S$ and since S is strongly regular so there exists an x in S such that $a = (ax)a$ and $ax = xa$. Let $y = (xa)x$, then using left invertive law, paramedial, medial law, we get

$$y = (xa)x = (ax)x = x^2a \in SQ,$$

and

$$y = (xa)x = (xa)(ex) = (xe)(ax) = a[(xe)x] \in QS.$$

Thus $y \in QS \cap SQ \subseteq Q$. Now using left invertive law and (1), we get

$$ya = [(xa)x]a = (ax)(xa) = (xa)(ax) = a[(xa)x] = ay.$$

Now using left invertive law, we get

$$\begin{aligned} a &= (ax)a = (ax)[(ax)a] = (ax)[(xa)a] = (ax)(a^2x) \\ &= a^2[(ax)x] = (aa)[(xa)x] = (aa)y = (ya)a = (ay)a. \end{aligned}$$

Therefore Q is strongly regular.

$(ii) \Longrightarrow (i)$

Since S itself is a quasi ideal, therefore by assumption S is strongly regular. ■

Theorem 209 *Let S be a strongly regular AG-groupoid with left identity. Then, for every $a \in S$, there exists $y \in S$ such that $a = (ay)a$, $y = (ya)y$ and $ay = ya$.*

Proof. Let $a \in S$, since S is strongly regular, there exists $x \in S$ such that $a = (ax)a$ and $ax = xa$. Now using paramedial law and medial law, we get

$$\begin{aligned} a &= (ax)a = (xa)a = [x\{(ax)a\}]a = [x\{(ax)(ea)\}]a \\ &= [x\{(ae)(xa)\}]a = [(ae)\{x(xa)\}]a = [(ae)\{(ex)(ax)\}]a \\ &= [(ae)\{(xa)(xe)\}]a = [(xa)\{(ae)(xe)\}]a = [(xa)\{(ex)(ea)\}]a \\ &= [(xa)(xa)]a = [(ax)(ax)]a = [a\{(ax)x\}]a = [a\{(xa)x\}]a \\ &= (ay)a, \text{ where } y = (xa)x. \end{aligned}$$

Now using and left invertive law, we get

$$\begin{aligned} y &= (xa)x = [x\{(ax)a\}]x = [(ax)(xa)]x \\ &= [\{(xa)x\}a]x = (ya)x = [y\{(ax)a\}]x \\ &= [x\{(ax)a\}]y = [x\{(ax)a\}]y = [(ax)(xa)]y \\ &= [\{(xa)x\}a]y = (ya)y. \end{aligned}$$

Now using left invertive law, we get

$$ay = a[(xa)x] = (xa)(ax) = (ax)(xa) = [(xa)x]a = ya.$$

■

Theorem 210 *For an AG-groupoid S with left identity the following are equivalent,*

(i) S is strongly regular,

(ii) S is left regular, right regular and $(Sa)S$ is a strongly regular AG-subgroupoid, of S for every $a \in S$.

(iii) For every $a \in S$, we have $a \in aS$ and $(Sa)S$ is a strongly regular AG-subgroupoid, of S.

Proof. $(i) \Longrightarrow (ii)$

Let $a \in S$, and S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Now left invertive law ,we get

$$a = (ax)a = (xa)a = a^2x.$$

This implies that S is right regular. Now using medial law and paramedial law, we get

$$\begin{aligned} a &= (ax)a = (ax)[(ax)a] = [a(ax)](xa) \\ &= [a(xa)](xa) = [x(aa)](xa) = (xa^2)(xa) \\ &= [x(aa)](xa) = [(ex)(aa)](xa) = [(aa)(xe)](xa) \\ &= [a^2(xe)](xa) = [(xa)(xe)]a^2 = ua^2, \text{ where } u = [(xa)(xe)]. \end{aligned}$$

Let $b \in (Sa)S \subseteq S$, thus $b \in S$, and since S is strongly regular, so there exist $x_1 \in S$, such that $b = (bx_1)b$ and $x_1 = (x_1b)x_1$ and $bx_1 = x_1b$, since $b \in (Sa)S \Rightarrow b = (za)t$, for some $z, t \in S$. Using paramedial, medial law, left invertive law, we get

$$\begin{aligned} x_1 &= (x_1b)x_1 = (x_1b)(ex_1) = (x_1e)(bx_1) = b[(x_1e)x_1] = bu \\ &= [(za)t]u = (ut)(za) = (az)(tu) = [(tu)z]a = va = v(a^2x) \\ &= a^2(vx) = (aa)(vx) \in (Sa)S, \text{ where } u = (x_1e)x_1 \text{ and } v = (tu)z. \end{aligned}$$

This shows that $(Sa)S$ is strongly regular.

$(ii) \Longrightarrow (iii)$

Let $a \in S$, and S is left regular so there exists some $y \in S$ such that $a = ya^2$.

Now using (1), we get

$$a = ya^2 = y(aa) = a(ya) \in a(SS) = aS.$$

$(iii) \Longrightarrow (i)$

Let $a \in aS$ so there exists some $t \in S$ such that $a = at$, also $a \in Sa$ so there exists some $z \in S$ such that $a = za$.

Now

$$a = at = (za)t \in (Sa)S,$$

and as $(Sa)S$ strongly regular so there exists some x in S such that $a = (ax)a$ and $ax = xa$. So S is strongly regular. ■

Theorem 211 *For an AG-groupoid S with left identity the following are equivalent,*

(i) *S is strongly regular,*

(ii) *$(Sa)S$ is strongly regular and S is left duo.*

Proof. $(i) \Longrightarrow (ii)$

Let $a \in (Sa)S$, so $a \in S$ and since S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Let $y = (xa)x$ for any $y \in S$. Now using (1) and left invertive law ,we get

$$\begin{aligned} y &= (xa)x = [x\{(ax)a\}]x = [(ax)(xa)]x \\ &= [\{(xa)x\}a]x = (ya)x \in (Sa)S. \end{aligned}$$

Now using paramedial law,medial law, we get

$$\begin{aligned} a &= (ax)a = (xa)a = [x\{(ax)a\}]a = [x\{(ax)(ea)\}]a \\ &= [x\{(ae)(xa)\}]a = [(ae)\{x(xa)\}]a = [(ae)\{(ex)(ax)\}]a \\ &= [(ae)\{(xa)(xe)\}]a = [(xa)\{(ae)(xe)\}]a = [(xa)\{(ex)(ea)\}]a \\ &= [(xa)(xa)]a = [(ax)(ax)]a = [a\{(ax)x\}]a = [a\{(xa)x\}]a \\ &= (ay)a, \end{aligned}$$

and using (1) and left invertive law, we get

$$ay = a[(xa)x] = (xa)(ax) = (ax)(xa) = [(xa)x]a = ya.$$

This shows that $(Sa)S$ is strongly regular.

Let L be any left ideal in $S \Rightarrow SL \subseteq L$. Let $a \in L, s \in S$. Since S is strongly regular, so there exists some $x \in S$, such that, $a = (ax)a$ and $ax = xa$. Now $as \in LS$

$$as = [(ax)a]s = [(xa)a]s = (a^2x)s = (sx)a^2 = (sx)(aa) \in S(SL) \subseteq SL \subseteq L.$$

This shows that L is also right ideal and S is left duo.

$(ii) \Longrightarrow (i)$

Using medial and paramedial laws we get $(Sa)(SS) = (SS)(aS) = (Sa)S$. Now since S is left duo, so $aS \subseteq Sa$. Also we can show that $Sa \subseteq aS$. Thus $Sa = aS$. Now let $a \in S$, also $a \in Sa = aS \Rightarrow a = ta$ and $a = av$ for some $t, v \in S$. Now

$$a = av = (ta)v \in (Sa)S.$$

As $(Sa)S$ is strongly regular, so there exists some $u \in (Sa)S$, such that $a = (au)a$ and $au = ua$. Hence S is regular. ■

Theorem 212 *For an AG-groupoid S with left identity the following are equivalent,*

(i) *S is strongly regular,*

(ii) *Sa is strongly regular for all a in S.*

Proof. $(i) \Longrightarrow (ii)$

Let $a \in Sa$, so $a \in S$ and S is strongly regular so there exists some $x \in S$ such that $a = (ax)a$ and $ax = xa$. Let $y = (xa)x$ for some $y \in S$. Now using left invertive law we get

$$y = (xa)x = (ax)x = x^2a \in Sa.$$

Now using paramedial law,medial law, we get

$$\begin{aligned} a &= (ax)a = (xa)a = [x\{(ax)a\}]a = [x\{(ax)(ea)\}]a \\ &= [x\{(ae)(xa)\}]a = [(ae)\{x(xa)\}]a = [(ae)\{(ex)(ax)\}]a \\ &= [(ae)\{(xa)(xe)\}]a = [(xa)\{(ae)(xe)\}]a = [(xa)\{(ex)(ea)\}]a \\ &= [(xa)(xa)]a = [(ax)(ax)]a = [a\{(ax)x\}]a = [a\{(xa)x\}]a \\ &= (ay)a, \end{aligned}$$

and using left invertive law, we get

$$\begin{aligned} ay &= a[(xa)x] = (xa)(ax) = (ax)(xa) \\ &= [(xa)x]a = ya. \end{aligned}$$

Which implies that Sa is strongly regular.

$(ii) \Longrightarrow (i)$

Let $a \in S$, so $a \in Sa$ and Sa is strongly regular which implies S is strongly regular. ■

6

Fuzzy Ideals in Abel-Grassmann's Groupoids

In this chapter we introduce the fuzzy ideals in AG-groupoids and discuss their related properties.

A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called a fuzzy AG-subgroupoid of $\mathcal{S}$ if $f(xy) \geq f(x) \wedge f(y)$ for all x, $y \in \mathcal{S}$. A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called a fuzzy left (right) ideal of $\mathcal{S}$ if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all x, $y \in \mathcal{S}$. A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called a fuzzy two-sided ideal of $\mathcal{S}$ if it is both a fuzzy left and a fuzzy right ideal of $\mathcal{S}$. A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called a fuzzy quasi-ideal of S if $f \circ S \cap S \circ f \subseteq f$. A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called a fuzzy generalized bi-ideal of $\mathcal{S}$ if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in \mathcal{S}$. A fuzzy AG-subgroupoid f of an AG-groupoid $\mathcal{S}$ is called a fuzzy bi-ideal of $\mathcal{S}$ if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in \mathcal{S}$. A fuzzy AG-subgroupoid f of an AG-groupoid $\mathcal{S}$ is called a fuzzy interior ideal of $\mathcal{S}$ if $f((xa)y) \geq f(a)$, for all x, a and $y \in \mathcal{S}$.

Let f and g be any two fuzzy subsets of an AG-groupoid $\mathcal{S}$, then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in \mathcal{S}, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of $\mathcal{S}$

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x), \text{ for all } x \text{ in } \mathcal{S}$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x), \text{ for all } x \text{ in } \mathcal{S}.$$

The proof of the following three lemma's are same as in [37].

Lemma 213 *Let f be a fuzzy subset of an AG-groupoid S. Then the following properties hold.*

(i) f is a fuzzy AG-subgroupoid of S if and only if $f \circ f \subseteq f$.
(ii) f is a fuzzy left(right) ideal of S if and only if $S \circ f \subseteq f (f \circ S \subseteq f)$.
(iii) f is a fuzzy two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Lemma 214 *Let f be a fuzzy AG-subgroupoid of an AG-groupoid S. Then f is a fuzzy bi-ideal of S if and only if $(f \circ S) \circ f \subseteq f$.*

Lemma 215 *Let f be a fuzzy AG-subgroupoid of an AG-groupoid S. Then f is a fuzzy interior ideal of S if and only if $(S \circ f) \circ S \subseteq f$.*

The principal left, right and two-sided ideals of an AG-groupoid S is denoted by $L[a^2]$, $R[a^2]$ and $J[a^2]$. Note that the principal left, right and two-sided ideals generated by a^2 are equals, that is,

$$L[a^2] = R[a^2] = J[a^2] = a^2 S = Sa^2 S = Sa^2 = \{sa^2 : s \in S\}.$$

The characteristic function C_A for a subset A of an AG-groupoid S is defined by

$$C_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

The proof of the following three lemma's are same as in [29].

Lemma 216 *Let A be a non-empty subset of an AG-groupoid S. Then the following properties hold.*

(i) A is an AG-subgroupoid if and only if C_A is a fuzzy AG-subgroupoid of S.
(ii) A is a left(right, two-sided) ideal of S if and only if C_A is a fuzzy left(right, two-sided) of S.
(iii) A is a bi-ideal of S if and only if C_A is a fuzzy bi-ideal of S.

Lemma 217 *Let A be a non-empty subset of an AG-groupoid S. Then A is a bi-ideal of S if and only if C_A is a fuzzy bi-ideal of S.*

Lemma 218 *Let A be a non-empty subset of an AG-groupoid S. Then A is an interior ideal of S if and only if C_A is a fuzzy interior ideal of S.*

Example 219 *Let $S = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" be defined on S as follows:*

$\cdot$	1	2	3	4
1	4	4	2	2
2	4	4	4	4
3	4	4	2	4
4	4	4	4	4

Then $(S, \cdot)$ is an AG-groupoid.

Let us denote the set of all fuzzy subsets of an AG-groupoid S by $F(S)$. Note that $S(x) = 1$, for all $x \in S$.

Lemma 220 *If S is an AG-groupoid with left identity e, then $F(S)$ is an AG-groupoid with left identity S.*

Proof. Let f be any subset of $F(S)$, and for any $a \in S$, since e is left identity of S. So, $a = ea$, then we have

$$(S \circ f)(a) = \bigvee_{a=ea} \{S(e) \wedge f(a)\} = \bigvee_{a=ea} \{1 \wedge f(a)\} = f(a).$$

Now for uniqueness, suppose S and S' be the two left identities of $F(S)$, then $S \circ S' = S'$, and $S' \circ S = S$. Now by using (1), we have $S = S' \circ S = (S' \circ S') \circ S = (S \circ S') \circ S' = S' \circ S' = S'$. ■

Lemma 221 *In an AG-groupoid $F(S)$, every right identity S is a unique left identity.*

Proof. Let f be any subset of $F(S)$, since S is a right identity of $F(S)$, then $f \circ S = f$. Now we have

$$S \circ f = (S \circ S) \circ f = (f \circ S) \circ S = f \circ S = f.$$

■

Lemma 222 *An AG-groupoid $F(S)$ with right identity is a commutative semigroup.*

Proof. Since $F(S)$ is an AG-groupoid with right identity S. So by lemma 221, S is left identity of $F(S)$. Let f, g and $h \in F(S)$, then, we have

$$f \circ g = (S \circ f) \circ g = (g \circ f) \circ S = g \circ f, \text{ and}$$

$$(f \circ g) \circ h = (h \circ g) \circ f = (g \circ h) \circ f = f \circ (g \circ h).$$

■

6.1 Inverses in AG-groupoids

Let f be any fuzzy subset of an AG-groupoid S with left identity. A fuzzy subset f' of S is called left(right) inverse of f, if $f' \circ f = S(f \circ f' = S)$. f' is said to be inverse of f if it is both left inverse and right inverse.

Lemma 223 *Every right inverse in an AG-groupoid $F(S)$, is left inverse.*

Proof. Let f' and f be any fuzzy subsets of S and f' is the right inverse of f. Then by using (1), we have

$$f' \circ f = (S \circ f') \circ f = (f \circ f') \circ S = S \circ S = S,$$

which implies that f' is left inverse of f. Now for uniqueness, let f' and f'' be the two left inverses of f. So $f' \circ f = S$ and $f'' \circ f = S$. Now by using (1), we have

$$f'' = S \circ f'' = (f' \circ f) \circ f'' = (f'' \circ f) \circ f' = S \circ f' = f'.$$

■

Let f, g and h be any fuzzy subsets of an AG-groupoid S, then $F(S)$ is called left (right) cancellative AG-groupoid if $f \circ g = f \circ h$ $(g \circ f = h \circ f)$ implies that $g = h$, and $F(S)$ is called a cancellative AG-groupoid if it is both right and left cancellative.

Lemma 224 *A left cancellative AG-groupoid $F(S)$ is a cancellative AG-groupoid.*

Proof. Let $F(S)$ be left cancellative and f, g and h be any fuzzy subsets of an AG-groupoid S. Now let $g \circ f = h \circ f$, which implies that $(g \circ f) \circ k = (h \circ f) \circ k$, where $k \in F(S)$, now we have $(k \circ f) \circ g = (k \circ f) \circ h$, which implies that $g = h$. ■

Lemma 225 *A right cancellative AG-groupoid $F(S)$ with left identity S is a cancellative AG-groupoid.*

Proof. Let f, g , h and k be any fuzzy subsets of an AG-groupoid S. Let $F(S)$ is right cancellative then $g \circ f = h \circ f$ implies that $g = f$. Let $f \circ g = f \circ h$ which implies that $g \circ f = h \circ f$ in [29] which implies that $g = f$. ■

Lemma 226 *An AG-groupoid $F(S)$ is a semigroup if and only if $f \circ (g \circ h) = (h \circ g) \circ f$. where f, g, h and k are fuzzy subsets of S.*

Proof. Let $f \circ (g \circ h) = (h \circ g) \circ f$, holds for all fuzzy subsets f, g, h and k of S. Then by using (1), we have $f \circ (g \circ h) = (h \circ g) \circ f = (f \circ g) \circ h$.

Conversely, suppose that $F(S)$ is a semigroup, then it is easy to see that $f \circ (g \circ h) = (f \circ g) \circ h$. ■

Lemma 227 *If f and g be any fuzzy bi-ideals of an AG-groupoid S with left identity, then $f \circ g$ and $g \circ f$ are fuzzy bi-ideals of S.*

Proof. Let f and g be any fuzzy bi-ideals of an AG-groupoid S with left identity e, then

$$\begin{aligned}
(f \circ g) \circ (f \circ g) &= (f \circ f) \circ (g \circ g) \subseteq f \circ g, \text{ and} \\
((f \circ g) \circ S) \circ (f \circ g) &= ((f \circ g) \circ (S \circ S)) \circ (f \circ g) \\
&= ((f \circ S) \circ (g \circ S)) \circ (f \circ g) \\
&= ((f \circ S) \circ f) \circ ((g \circ S) \circ g) \subseteq f \circ g.
\end{aligned}$$

Hence $f \circ g$ is a fuzzy bi-ideal of S. Similarly $g \circ f$ is a fuzzy bi-ideal of S. ■

Lemma 228 *Every fuzzy ideal of an AG-groupoid S, is a fuzzy bi-ideal and a fuzzy interior ideal of S.*

Proof. Let S be an AG-groupoid and f be any fuzzy ideal of S, then for a, $b \in S$, we have $f(ab) \geq f(a)$ and $f(ab) \geq f(b)$, therefore $f(ab) \geq f(a) \wedge f(b)$, which implies that f is a fuzzy AG-subgroupoid. Now for any x, y, $z \in S$, we have $f((xy)z) \geq f(xy) \geq f(x)$, and $f((xy)z) \geq f(z)$, which implies that $f((xy)z) \geq f(x) \wedge f(z)$. Hence f is a fuzzy bi-ideal. Similarly it is easy to see that $f((xa)y) \geq f(a)$. ■

Lemma 229 *Let f be a fuzzy subset of a completely regular AG-groupoid S with left identity, then the following are equivalent.*

(i) f is a fuzzy ideal of S.
(ii) f is a fuzzy interior ideal of S.
Proof. $(i) \Rightarrow (ii)$, it is obvious.
$(ii) \Rightarrow (i)$
Since S is a completely regular AG-groupoid so for all a, $b \in S$ there exist x, $y \in S$ such that $a = (ax)a$ and $ax = xa$, $b = (by)b$ and $by = yb$, now by using we have

$$f(ab) = f(((ax)a)b) = f((ba)(ax)) \geq f(a).$$

Now we get

$$f(ab) = f(a((by)b)) = f(a((yb)b)) = f((yb)(ab)) \geq f(b).$$

■

Theorem 230 *Every fuzzy generalized bi-ideal of a completely regular AG-groupoid S with left identity, is a fuzzy bi-ideal of S.*

Proof. Let f be any fuzzy generalized bi-ideal of an AG-groupoid S. Then, since S is completely regular, so for each $a \in S$ there exist $x \in S$ such that $a = (ax)a$ and $ax = xa$. Thus

$$\begin{aligned} f(ab) &= f(((ax)a)b) = f(((ax)(ea))b) = f(((xa)(ea))b) \\ &= f(((xe)(aa))b) = f((a((xe)a))b) \geq f(a) \wedge f(b). \end{aligned}$$

■

Theorem 231 *Let f, g and h be any fuzzy subset of an AG-groupoid S, then the following are equivalent.*

(i) $f \circ (g \cup h) = (f \circ g) \cup (f \circ h)$; $(g \cup h) \circ f = (g \circ f) \cup (h \circ f)$.
(ii) $f \circ (g \cap h) = (f \circ g) \cap (f \circ h)$; $(g \cap h) \circ f = (g \circ f) \cap (h \circ f)$.
Proof. It is same as in [37]. ■

Lemma 232 *Let f, g and h be any fuzzy subsets of an AG-groupoid S, if $f \subseteq g$, then $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$.*

Proof. It is same as in [37]. ∎

A subset P of an AG-groupoid S is called semiprime, if for all $a \in S$, $a^2 \in P$ implies $a \in P$.

A fuzzy subset f of an AG-groupoid S is called fuzzy semiprime, if $f(a) \geq f(a^2)$, for all $a \in S$.

6.2 Fuzzy Semiprime Ideals

Lemma 233 *In an intra-regular AG-groupoid S, every fuzzy interior ideal is fuzzy semiprime.*

Proof. Since S intra-regular so for $a \in S$ there exist x, $y \in S$ such that $a = (xa^2)y$, so we have

$$f(a) = f((xa^2)y) \geq f(a^2).$$

∎

Theorem 234 *A non-empty subset A of an AG-groupoid S, is semiprime if and only if the characteristic function C_A of A is fuzzy semiprime.*

Proof. Let $a^2 \in A$, since A is semiprime so $a \in A$, hence $C_A(a) = 1 = C_A(a^2)$. Also if $a^2 \notin A$, then $C_A(a) \geq 0 = C_A(a^2)$. In both cases, we have $C_A(a) \geq C_A(a^2)$ for all $a \in S$, which implies that C_A is fuzzy semiprime.

Conversely, assume that $a^2 \in A$, since C_A is a fuzzy semiprime, so we have $C_A(a) \geq C_A(a^2) = 1$, and so $C_A(a) = 1$, which implies that $a \in A$. ∎

Theorem 235 *For any fuzzy AG-subgroupoid f of an AG-groupoid S, the following are equivalent.*

(i) f is a fuzzy semiprime.
(ii) $f(a) = f(a^2)$, for all $a \in S$.

Proof. $(i) \Rightarrow (ii)$

Let $a \in S$, then since f is a fuzzy AG-subgroupoid of S, so we have

$$f(a) \geq f(a^2) = f(aa) \geq f(a) \wedge f(a) = f(a).$$

$(ii) \Rightarrow (i)$, it is obvious. ∎

An element a of an AG-groupoid S is called intra-regular if there exists elements x, $y \in S$ such that $a = (xa^2)y$. An AG-groupoid S is called intra-regular if every element of S is intra-regular.

Theorem 236 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.
(ii) $f(a) = f(a^2)$, for all fuzzy two sided ideal f of S, for all $a \in S$.

(iii) $f(a) = f(a^2)$, for all fuzzy interior ideal f of S, for all $a \in S$.
Proof. $(i) \Rightarrow (iii)$

Let f be any fuzzy interior ideal of an intra-regular AG-groupoid S. Now for any $a \in S$, there exist x, $y \in S$, such that $a = (xa^2)y$. Then we have

$$\begin{aligned} f(a) &= f\left(\left(xa^2\right)y\right) \geq f(a^2) = f(aa) = f(a\left(\left(xa^2\right)y\right)) = f(a((ya^2)x)) \\ &= f(a((y(aa))x)) = f(a((a(ya))x)) = f(a((x(ya))a)) \\ &= f((x(ya))(aa)) = f((xa)((ya)a)) \geq f(a). \end{aligned}$$

Clearly $(iii) \Rightarrow (ii)$.
$(ii) \Rightarrow (i)$

Let $J\left[a^2\right]$ be the principal two sided ideal generated by a^2. Then, $C_{J[a^2]}$ is a fuzzy two sided ideal of S. Since $a^2 \in J\left[a^2\right]$, so by (ii) we have $C_{J[a^2]}(a) = C_{J[a^2]}\left(a^2\right) = 1$, hence $a \in J\left[a^2\right] = (Sa^2)S$, which implies that there exist x, $y \in S$ such that $a = (xa^2)y$. ■

Theorem 237 *Let f be a fuzzy interior ideal of an intra-regular AG-groupoid S with left identity, then $f(ab) = f(ba)$, for all a, b in S.*

Proof. Let S be an intra-regular AG-groupoid and a, $b \in S$, then

$$\begin{aligned} f(ab) &= f\left((ab)^2\right) = f((ab)(ab)) = f((ba)(ba)) \\ &= f((e(ba))(ba)) = f((ab)((ba)e)) \\ &= f((a(ba))(be)) \geq f(ba) = f\left((ba)^2\right) \\ &= f((ba)(ba)) = f((ab)(ab)) \\ &= f((e(ab))(ab)) = f((ba)((ab)e)) \\ &= f((b(ab))(ae)) \geq f(ab). \end{aligned}$$

■

The following three propositions are well-known.

Proposition 238 *Every locally associative AG-groupoid has associative powers.*

Proposition 239 *In a locally associative AG-groupoid S, $a^m a^n = a^{m+n}$, $\forall$ $a \in S$ and positive integers m, n.*

Proposition 240 *In a locally associative AG-groupoid S, $(a^m)^n = a^{mn}$, for all $a \in S$ and positive integers m, n.*

Theorem 241 *Let f be a fuzzy semiprime interior ideal of a locally associative AG-groupoid S with left identity, then $f(a^n) = f(a^{n+1})$, for all positive integer n.*

Proof. Let n be any positive integer and f be any fuzzy interior ideal of S, then we have

$$\begin{aligned} f(a^n) &\geq f\left((a^n)^2\right) = f(a^{2n}) \geq f(a^{4n}) = f(a^{n+2}a^{3n-2}) \\ &= f((aa^{n+1})a^{3n-2}) \geq f(a^{n+1}). \end{aligned}$$

■

An AG-groupoid S is called archimedean if for all $a, b \in S$, there exist a positive integer n such that $a^n \in (Sb)S$.

Theorem 242 *Let S be an archimedean locally associative AG-groupoid with left identity, then every fuzzy semiprime fuzzy interior ideal of S is a constant function.*

Proof. Let f be any fuzzy semiprime fuzzy interior ideal of S and $a, b \in S$. Thus we have $f(a) \geq f(a^2) \geq f(a^4) \geq ... \geq f(a^{2n}) = f(a^m)$, where $2n = m$. Now since S is archimedean, so there exist a positive integer m and $x, y \in S$ such that $a^m = (xb)y$. Therefore $f(a) \geq f((xb)y) \geq f(b)$. Similarly we can prove that $f(b) \geq f(a)$. Hence f is a constant function. ■

An AG-groupoid S is called left (right) simple, if it contains no proper left (right) ideal and is called simple if it contain no proper two sided ideal.

An AG-groupoid S is called fuzzy simple, if every fuzzy subset of S is a constant function.

Theorem 243 *An AG-groupoid S is simple if and only if $S = a^2S = Sa^2 = (Sa^2)S$, for all a in S.*

Proof. *It is easy.* ■

An AG-groupoid S is called semisimple if every two-sided ideal of S is idempotent. It is easy to prove that S is semisimple if and only if $a \in ((Sa)S)((Sa)S)$, that is, for every $a \in S$, there exist $x, y, u, v \in S$ such that $a = ((xa)y)((ua)v)$.

Theorem 244 *Every fuzzy two-sided ideal of a semisimple AG-groupoid S is an idempotent.*

Proof. Let f be fuzzy two-sided ideal of S. Obviously

$$(f \circ f)(a) = \bigvee_{a=((xa)y)((ua)v)} \{f((xa)y) \wedge f((ua)v)\} \geq f(a).$$ ■

Theorem 245 *Let f and g be any fuzzy ideal of a semisimple AG-groupoid S, then $f \circ g$ is a fuzzy ideal in S.*

Proof. Clearly $f \circ g \subseteq f \cap g$. Now

$$\begin{aligned} (f \circ g)(a) &= \bigvee_{a=((xa)y)((ua)v)} \{f((xa)y) \wedge g((ua)v)\} \\ &\geq f((xa)y) \wedge g((ua)v) \geq f(xa) \wedge g(ua) \geq f(a) \wedge g(a) \\ &= (f \cap g)(a). \end{aligned}$$

Hence $f \circ g$ is a fuzzy ideal of S. ■

Theorem 246 *Every fuzzy interior ideal of a semisimple AG-groupoid S with left identity, is a fuzzy two-sided ideal of S.*

Proof. Let f be a fuzzy interior ideal of S, and a, $b \in S$, then since S is semisimple, so there exist x, y, u, $v \in S$ and p, q, r, $s \in S$ such that $a = ((xa)y)((ua)v)$ and $b = ((pb)q)((rb)s)$. Thus we have

$$\begin{aligned} f(ab) &= f(((xa)y)((ua)v)b) = f((((xa)(ua))(yv))b) \\ &= f((((xu)(aa))(yv))b) = f((((aa)(ux))(yv))b) \\ &= f((((yv)(ux))(aa))b) = f(((aa)((ux)(yv)))b) \\ &= f((b((ux)(yv)))(aa)) = f((ba)(((ux)(yv))a)) \\ &= f((((((ux)(yv))a)a)b) \geq f(a). \end{aligned}$$

Now we have

$$\begin{aligned} f(ab) &= f(a(((pb)q)((rb)s))) = f(a(((pb)(rb))(qs))) \\ &= f(a(((pr)(bb))(qs))) = f(a(((bb)(rp))(qs))) \\ &= f(a((((rp)b)b)(qs))) \\ &= f((((rp)b)b)(a(qs))) \geq f(b). \end{aligned}$$

■

Theorem 247 *The set of fuzzy ideals of a semisimple AG-groupoid S forms a semilattice structure.*

Proof. Let Θ_I be the set of fuzzy ideals of a semisimple AG-groupoid S and f, g and $h \in \Theta_I$, then clearly Θ_I is closed and we have $f = f^2$ and $f \circ g = f \cap g$, where f and g are ideals of S. Clearly $f \circ g = g \circ f$, and then, we get $(f \circ g) \circ h = (h \circ g) \circ f = f \circ (g \circ h)$. ■

A fuzzy ideal f of an AG-groupoid S is said to be strongly irreducible if and only if for fuzzy ideals g and h of S, $g \cap h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$.

The set of fuzzy ideals of an AG-groupoid S is called totally ordered under inclusion if for any fuzzy ideals f and g of S either $f \subseteq g$ or $g \subseteq f$.

A fuzzy ideal h of an AG-groupoid S is called fuzzy prime ideal of S, if for any fuzzy ideals f and g of S, $f \circ g \subseteq h$, implies that $f \subseteq h$ or $g \subseteq h$.

Theorem 248 *In a semisimple AG-groupoid S, a fuzzy ideal is strongly irreducible if and only if it is fuzzy prime.*

Proof. It follows from theorem 245. ■

Theorem 249 *Every fuzzy ideal of a semisimple AG-groupoid S is fuzzy prime if and only if the set of fuzzy ideals of S is totally ordered under inclusion.*

Proof. It is easy. ■

7

$(\in, \in \vee q)$ and $(\in, \in \vee q_k)$-fuzzy Bi-ideals of AG-groupoids

In this chapter we characterize intra-regular AG-groupoids by the properties of the lower part of $(\in, \in \vee q)$-fuzzy bi-ideals. Moreover we characterize AG-groupoids using $(\in, \in \vee q_k)$-fuzzy.

Let f be a fuzzy subset of an AG-groupoid S and $t \in (0,1]$. Then $x_t \in f$ means $f(x) \geq t$, $x_t q f$ means $f(x) + t > 1$, $x_t \alpha \vee \beta f$ means $x_t \alpha f$ or $x_t \beta f$, where α, β denotes any one of $\in$, q, $\in \vee q$, $\in \wedge q$. $x_t \alpha \wedge \beta f$ means $x_t \alpha f$ and $x_t \beta f$, $x_t \overline{\alpha} f$ means $x_t \alpha f$ does not holds.

Let f and g be any two fuzzy subsets of an AG-groupoid $\mathcal{S}$, then for $k \in [0,1)$, the product $f \circ_{0.5} g$ is defined by,

$$(f \circ_{0.5} g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c) \wedge 0.5\}, & \text{if there exist } b, c \in \mathcal{S}, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

The following definitions for AG-groupoids are same as for semigroups in [56].

Definition 250 *A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies,*
$x_t \in \delta$, $y_r \in \delta$ implies that $(xy)_{\min\{t,r\}} \in \vee q \delta$.

Definition 251 *A fuzzy subset δ of S is called an $(\in, \in \vee q)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0,1]$, it satisfies,*
$x_t \in \delta$ implies $(yx)_t \in \vee q \delta$ ($x_t \in \delta$ implies $(xy)_t \in \vee q \delta$).

Definition 252 *A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (0,1]$ the following condition holds.*
$y_t \in f$ implies $((xy)z)_t \in \vee q f$.

Definition 253 *A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called an $(\in, \in \vee q)$-fuzzy quasi-ideal of S if it satisfies, $f(x) \geq \min(f \circ C_S(x), C_S \circ f(x), 0.5)$, where C_S is the fuzzy subset of S mapping every element of S on 1.*

Definition 254 *A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if $x_t \in f$ and $z_r \in f$ implies $((xy)z)_{\min\{t,r\}} \in \vee q f$, for all $x, y, z \in S$ and $t, r \in (0,1]$.*

Definition 255 *A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following conditions hold*

(i) If $x_t \in f$ and $y_r \in S$ implies $(xy)_{\min\{t,r\}} \in \vee qf$,

(ii) If $x_t \in f$ and $z_r \in f$ implies $((xy)\,z)_{\min\{t,r\}} \in \vee qf$.

The proofs of the following four theorems are same as in [56].

Theorem 256 *Let δ be a fuzzy subset of S. Then δ is an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S if and only if $\delta(xy) \geq \min\{\delta\,(x)\,, \delta(y), 0.5\}$.*

Theorem 257 *A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy left (right) ideal of S if and only if*

$\delta(xy) \geq \min\{\delta(y), 0.5\}$ $(\delta(xy) \geq \min\{\delta(x), 0.5\})$.

Theorem 258 *A fuzzy subset f of an AG-groupoid S is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if it satisfies the following conditions.*

(i) $f\,(xy) \geq \min\{f\,(x)\,, f\,(y)\,, 0.5\}$ for all $x, y \in S$ and $k \in [0, 1)$.

(ii) $f\,((xy)z) \geq \min\{f\,(y)\,, 0.5\}$for all $x, y, z \in S$ and $k \in [0, 1)$.

Theorem 259 *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if*

(i) $f(xy) \geq \min\{f\,(x)\,, f(y), 0.5\}$ for all $x, y \in S$ and $k \in [0, 1)$,

(ii) $f((xy)z) \geq \min\{f(x), f\,(z)\,, 0.5\}$ for all $x, y, z \in S$ and $k \in [0, 1)$.

Example 260 *Let $S = \{a, b, c\}$ be an AG-groupoid with the following Cayley table:*

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	c	c

One can easily check that $\{a\}, \{b\}, \{c\}, \{a, c\}$ and $\{a, b, c\}$ are all bi-ideals of S. Let f be fuzzy subsets of S such that $f(a) = 0.9, f(b) = 0.2$ and $f(c) = 0.6$. Then f is an $(\in, \in \vee q)$-fuzzy ideal of S.

Definition 261 *An element a of an AG-groupoid S is called intra-regular if there exist x, y in S such that $a = (xa^2)y$. An AG-groupoid S is called intra-regular if every element of S is intra-regular.*

Theorem 262 *Let S be an AG-groupoid with left identity. Then S is intra-regular if and only if $R \cap L = RL$ and R is semiprime, for every left ideal L and every right ideal R of S.*

Proof. Let R, L be right and left ideals of an intra-regular AG-groupoid S with left identity. Then R and L become ideals of S and so $RL \subseteq R \cap L$. Now we have $R \cap L$ is an ideal of S. We can also deduce that $R \cap L = (R \cap L)^2 \subseteq RL$. Hence we obtain $R \cap L = RL$.

Next, we show that R is semiprime. So let $r^2 \in R$. Since S is intra-regular, there exist $x, y \in S$ such that $r = (xr^2)y$. Thus we have

$$r = (xr^2)y \in (SR)S \subseteq R.$$

Therefore, R is semiprime.

Conversely, assume that $R \cap L = RL$ and R is semiprime, for any left ideal L and right ideal R of S. We need to show that S is intra-regular. To see this, note that for any $a \in S$, Sa^2 is a right ideal and Sa is a left ideal of S. Clearly, $a \in Sa$. Since Sa^2 is semiprime and $a^2 \in Sa^2$, we also have $a \in Sa^2$. Hence it follows that

$$a \in Sa^2 \cap Sa = (Sa^2)(Sa) \subseteq (Sa^2)S,$$

which shows that a is intra-regular. Therefore, S is intra-regular as required. ■

Example 263 *Let $S = \{1, 2, 3, 4, 5\}$ be an AG-groupoid with the following Cayley table:*

$*$	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

It is clear that S is intra-regular since $1 = (3 * 1^2) * 2,\ 2 = (1 * 2^2) * 5,\ 3 = (5 * 3^2) * 2,\ 4 = (2 * 4^2) * 1,\ 5 = (3 * 5^2) * 1$. Let us define a fuzzy subset f on S such that $f(1) = 0.8,\ f(2) = 0.7,\ f(3) = 0.5,\ f(4) = 0.9$ and $f(5) = 0.6$. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Theorem 264 *For an AG-groupoid S with left identity, the following conditions are equivalent:*
(i) S is intra-regular.
(ii) $B = B^2$ for every bi-ideal B of S.

Proof. Let B be a bi-ideal of an intra-regular AG-groupoid S with left identity. Thus B is an ideal of S. Then it follows that $B = B^2$.

Conversely, assume that $B = B^2$ for every bi-ideal B of S. For any $a \in S$, Sa is a bi-ideal contains a. Thus we have

$$a \in Sa = (Sa)^2 = Sa^2 = (Sa^2)S,$$

which shows that a is intra-regular. Therefore, S is intra-regular as required. ■

The following results can be proved by similar techniques as used in [56].

Lemma 265 *Let L be a non-empty subset of an AG-groupoid S and C_L be the characteristic function of L. Then L is a left ideal of S if and only if the lower part C_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Lemma 266 *Let R be a non-empty subset of an AG-groupoid S and C_R be the characteristic function of R. Then R is a right ideal of S if and only if the lower part C_R^- is an $(\in, \in \vee q)$-fuzzy right ideal of S.*

Lemma 267 *Let A and B be non-empty subsets of an AG-groupoid S. Then we have the following:*

(1) $(C_A \wedge C_B)^- = C_{(A\cap B)}^-$.

(2) $(C_A \vee C_B)^- = C_{(A\cup B)}^-$.

(3) $(C_A \circ C_B)^- = C_{(AB)}^-$.

Theorem 268 *A fuzzy subset f of an AG-groupoid S is $(\in, \in \vee q)$-fuzzy semiprime if and only if $f(x) \geq f(x^2) \wedge 0.5$ for all $x \in S$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in, \in \vee q)$-fuzzy semiprime. If there exists some $x_0 \in S$ such that $f(x_0) < t_0 = f\left(x_0^2\right) \wedge 0.5$. Then $(x_0^2)_{t_0} \in f$, but $(x_0)_{t_0} \overline{\in} f$. In addition, we have $(x_0)_{t_0} \in \vee q f$ since f is $(\in, \in \vee q)$-fuzzy semiprime. On the other hand, we have $f(x_0) + t_0 \leq t_0 + t_0 \leq 1$. Thus $(x_0)_{t_0} \overline{q} f$, and so $(x_0)_{t_0} \overline{\in \vee q} f$. This is a contradiction. Hence $f(x) \geq f(x^2) \wedge 0.5$ for all $x \in S$.

Conversely, assume that f is a fuzzy subset of an AG-groupoid S such that $f(x) \geq f(x^2) \wedge 0.5$ for all $x \in S$. Let $x_t^2 \in f$. Then $f\left(x^2\right) \geq t$, and so $f(x) \geq f\left(x^2\right) \wedge 0.5 \geq t \wedge 0.5$. Now, we consider the following two cases:

(i) If $t \leq 0.5$, then $f(x) \geq t$. That is, $x_t \in f$. Thus we have $x_t \in \vee q f$.

(ii) If $t > 0.5$, then $f(x) \geq 0.5$. It follows that $f(x) + t \geq 0.5 + t > 1$. That is, $x_t q f$, and so $x_t \in \vee q f$ also holds. Therefore, we conclude that f is $(\in, \in \vee q)$-fuzzy semiprime as required. ∎

Theorem 269 *Let A be a non-empty subset of an AG-groupoid S with left identity. Then A is semiprime if and only if C_A^- is fuzzy semiprime.*

Proof. Suppose that A is a non-empty subset of an AG-groupoid S with left identity and A is semiprime. For any $a \in S$, if $a^2 \in A$, then we have $a \in A$ since A is semiprime. It follows that $C_A^-(a) = C_A^-(a^2) = 0.5$. If $a^2 \overline{\in} A$, then we have $C_A^-(a) \geq 0 = C_A^-(a^2)$. This shows that C_A^- is fuzzy semiprime.

Conversely, assume that C_A^- is fuzzy semiprime. Thus we have $C_A^-(x) \geq C_A^-(x^2)$ for all x in S. If $x^2 \in A$, then $C_A^-(x^2) = 0.5$. Hence $C_A^-(x) \geq C_A^-(x^2) \geq 0.5$, which implies $x \in A$. Therefore, A is semiprime as required. ∎

Definition 270 *An $(\in, \in \vee q)$-fuzzy AG-subgroupoid of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy interior ideal of S if*

$$a_t \in f \Rightarrow ((xa)y)_t \in \vee q f,$$

for all $x, a, y \in S$ and $t \in (0, 1]$.

Theorem 271 *Let f be a fuzzy subset of an AG-groupoid S. Then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if it satisfies:*
(i) $f(xy) \geq \min\{f(x),\ f(y), 0.5\}$, for all $x, y \in S$.
(ii) $f((xa)y) \geq \min\{f(a), 0.5\}$, for all $x, a, y \in S$.

Proof. It is easy. ■

7.1 Characterizations of Intra-regular AG-groupoids

In this section, we give some characterizations of intra-regular AG-groupoids based on the properties of their $(\in, \in \vee q)$-fuzzy ideals.

Theorem 272 *For an AG-groupoid S with left identity e, the following conditions are equivalent:*
(i) S is intra-regular.
(ii) $(f \wedge g)^- = (f \circ g)^-$ and f is fuzzy semiprime, where f is an $(\in, \in \vee q)$-fuzzy right ideal and g is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Proof. Assume that S is an intra-regular AG-groupoid with left identity e. Let f be an $(\in, \in \vee q)$-fuzzy right ideal of S and g be $(\in, \in \vee q)$-left ideal of S. For $a \in S$, we have

$$\begin{aligned}
(f \circ g)^-(a) = (f \wedge g)(a) \wedge 0.5 &= (\bigvee_{a=yz} \{f(y) \wedge g(z)\}) \wedge 0.5 \\
&= (\bigvee_{a=yz} (\{f(y) \wedge g(z)\} \wedge 0.5) \\
&= (\bigvee_{a=yz} (\{f(y) \wedge 0.5\} \wedge \{g(z) \wedge 0.5\} \wedge 0.5) \\
&\leq \bigvee_{a=yz} \{f(yz) \wedge g(yz) \wedge 0.5\} \\
&= f(a) \wedge g(a) \wedge 0.5 = (f \wedge g)^-(a).
\end{aligned}$$

Thus $(f \circ g)^{-} \leq (f \wedge g)^{-}$. Since S is intra-regular, for $a \in S$, there exist $x, y \in S$ such that $a = (xa^2)y$. Now we get

$$\begin{aligned} a &= (xa^2)y = (x(a.a)y = (a(xa))y = (y(xa))a \\ &= ((ey)(xa))a = ((ax)(ye)) = ((ay)(xe))a \\ &= ((ax)(ye))a = ((ax)(ye))((xa^2)y) = ((ax)(ye))((x(aa))y) \\ &= ((ax)(ye))((a(xa))y) = ((ax)(ye))((y(xa))a). \end{aligned}$$

Hence we deduce that

$$\begin{aligned} (f \circ g)^{-}(a) &= (f \circ g)(a) \wedge 0.5 \\ &= \bigvee_{a=pq} \{f(p) \wedge g(q)\} \wedge 0.5 \\ &= [\bigvee_{a=pq} \{f(p) \wedge g(q)\}] \wedge 0.5 \\ &= \bigvee_{pq=((ax)(ye))((y(xa))a)} \{f(p) \wedge g(q)\} \wedge 0.5 \\ &\geq f((ax)(ye)) \wedge g((y(xa))a) \wedge 0.5 \\ &\geq f(ax) \wedge g(a) \wedge 0.5 \geq f(a) \wedge g(a) \wedge 0.5 \\ &= (f \wedge g)(a) \wedge 0.5 = (f \wedge \bar{g})(a). \end{aligned}$$

This shows that $(f \circ g)^{-} \geq (f \wedge g)^{-}$. Thus we obtain $(f \circ g)^{-} = (f \wedge g)^{-}$.

Next we shall show that f is fuzzy semiprime. Since $S = S^2$, thus for $x \in S$ there exist u, v in S such that $x = uv$. Then we get

$$f(a) = f\left(\left(xa^2\right)y\right) \geq f\left(xa^2\right) = f\left((uv)(aa)\right) = f\left((aa)(vu)\right) \geq f\left(a^2\right).$$

Therefore, f is fuzzy semiprime as required.

Conversely, suppose that S is an AG-groupoid with left identity e, such that $(f \wedge g)^{-} = (f \circ g)^{-}$ and f is fuzzy semiprime for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S. Let R and L be right and left ideals of S respectively. Then, C_L^{-} and C_R^{-} are $(\in, \in \vee q)$-fuzzy left ideal and $(\in, \in \vee q)$-fuzzy right ideal of S, respectively. By assumption, C_R^{-} is also fuzzy semiprime. Then we deduce that R is semiprime. Then we have

$$C_{(RL)}^{-} = (C_R \circ C_L)^{-} = (C_R \wedge C_L)^{-} = C_{(R \cap L)}^{-}.$$

Thus $RL = R \cap L$. Hence S is intra-regular as required. ■

Note that $RL \subseteq R \cap L$ for every right ideal R and left ideal L of an AG-groupoid S. We immediately obtain the following.

Theorem 273 *For an AG-groupoid S with left identity e, the following conditions are equivalent:*

(i) S is intra-regular.

(ii) $(f \wedge g)^{-} \leq (f \circ g)^{-}$ and f is fuzzy semiprime, where f is an $(\in, \in \vee q)$-fuzzy right ideal and g is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Theorem 274 *For an AG-groupoid S with left identity e, the following conditions are equivalent:*
(i) S is intra-regular.
(ii) $((h \wedge f) \wedge g)^{-} \leq ((h \circ f) \circ g)^{-}$ and h is fuzzy semiprime, for every $(\in, \in \vee q)$-fuzzy right ideal h, $(\in, \in \vee q)$-fuzzy bi-ideal f and $(\in, \in \vee q)$-fuzzy left ideal g of S.
(iii) $((h \wedge f) \wedge g)^{-} \leq ((h \circ f) \circ g)^{-}$and h is fuzzy semiprime, for every $(\in, \in \vee q)$-fuzzy right ideal h, $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and $(\in, \in \vee q)$-fuzzy left ideal g of S.
Proof. $(i) \Rightarrow (iii)$: *Let S be an intra-regular AG-groupoid with left identity e. For any $a \in S$, there exist x and y in S such that $a = (xa^2)y$. Now we get*

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= ((ey)(xa))a = ((ye)(xa))a = \left((ye)\left(x\left(\left(xa^2\right)y\right)\right)\right)a \\ &= ((ye)(x((xa^2)(ey)))a = ((ye)(x((ye)(a^2x))a \\ &= ((ye)(x(a^2((ye)x))a = (((ye)\left(a^2(x((ye)x))\right))a \\ &= ((ye)(a^2((ye)x^2)))a = (a^2((ye)((ye)x^2)))a \\ &= ((ba)a)a, \text{ where } b = (ye)\left((ye)x^2\right). \end{aligned}$$

Now, we have

$$\begin{aligned} ba &= b((xa^2)y) = (xa^2)(by) = (yb)(xa^2) \\ &= (yx)(a^2b) = a^2(yx)b = (((yx)b)a)a \\ &= (((yx)b)((xa^2)y))a = (t((xa^2)y))a, \text{ where } t = (yx)b. \\ (t((xa^2)y))a &= (((xa^2)(ty)))a = (((yt)(a^2x)))a \\ &= (a^2((yt)x))a = (a^2u)a, \text{ where } u = (yt)x. \end{aligned}$$

Thus $a = (((a^2u)a)a)a$. Furthermore, we can deduce that

$$\begin{aligned} ((h \circ f) \circ g)^{-}(a) &= [\bigvee_{a=pq} ((h \circ f)(p) \wedge g(q) \wedge 0.5] \\ &= \bigvee_{pq=((ba)a)a} [(h \circ f)(p) \wedge g(q)] \wedge 0.5 \\ &= \bigvee_{pq=((ba)a)a} [(h \circ f)((ba)a) \wedge g(a)] \wedge 0.5 \\ &\geq (h \circ f)((ba)a) \wedge g(a) \wedge 0.5 \\ &= \bigvee_{(ba)a=((a^2u)a)a} \{h((a^2u)a) \wedge f(a) \wedge 0.5\} \wedge g(a) \wedge 0.5 \\ &\geq h(a^2) \wedge f(a) \wedge g(a) \wedge 0.5 \\ &\geq h(a) \wedge f(a) \wedge g(a) \wedge 0.5 = ((h \wedge f) \wedge g)^{-}(a). \end{aligned}$$

This shows that $((h \wedge f) \wedge g)^{-} \leq ((h \circ f) \circ g)^{-}$. *In addition, for any* $a \in S$, *there exist* $x, y \in S$ *such that* $a = \left(xa^{2}\right) y$ *since* S *is intra-regular. Thus we get*

$$\begin{aligned} h(a) &= h\left(xa^{2}\right) y = h(xa^{2}) = h(x(aa)) = h(a(xa)) = h((ea)(xa)) \\ &= h(ax)(ae) = h(aa)(xe) = h\left(a^{2}(xe)\right) \geq h\left(a^{2}\right). \end{aligned}$$

Therefore, h *is fuzzy semiprime as required.*

$(iii) \Rightarrow (ii)$: *Straightforward.*

$(ii) \Rightarrow (i)$: *Let* h *be an* $(\in, \in \vee q)$*-fuzzy semiprime right ideal and let* g *be an* $(\in, \in \vee q)$*-fuzzy left ideal of* S. *Then*

$$\begin{aligned} (h \circ C_s)(a) &= \bigvee_{a=bc} h(b) \wedge C_s(c) = \bigvee_{a=bc} h(b) \wedge 1 \\ &\leq \bigvee_{a=bc} h(bc) = h(a). \end{aligned}$$

Since C_s *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S, *for any* $a \in S$ *we have*

$$(h \wedge g)^{-}(a) = ((h \wedge C_s) \wedge g)^{-}(a) \leq ((h \circ C_s) \circ g)^{-}(a) \leq (h \circ g)^{-}(a).$$

Therefore, $(h \wedge g)^{-} \leq (h \circ g)^{-}$. *Then by Theorem 273, we deduce that* S *is intra-regular as required.* ■

Lemma 275 *A non-empty subset* B *of an AG-groupoid* S *is a bi-ideal if and only if* C_B^{-} *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S.

Proof. It is similar to the proof of Lemma 9 in [55]. ■

Theorem 276 *For an AG-groupoid* S *with left identity* e, *the following conditions are equivalent:*

(i) S *is intra-regular.*

(ii) $f^{-} = ((f \circ C_s) \circ f)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy generalized bi-ideal* f *of* S *and* $f \circ f = f$.

(iii) $f^{-} = ((f \circ C_s) \circ f)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* f *of* S *and* $f \circ f = f$.

Proof. $(i) \Rightarrow (ii)$: Let S be an intra-regular AG-groupoid with left identity e. For any $a \in S$, there exist x and y in S such that $a = (xa^{2})y$. We already obtained $a = (((a^2u)a)a)a$. Moreover, we have

$$\begin{aligned}((f \circ C_S) \circ f)^-(a) &= ((f \circ C_S) \circ f)(a) \wedge 0.5 \\ &= \bigvee_{a=pq} \{(f \circ C_S)(p) \wedge f(q)\} \ \wedge 0.5 \\ &= \bigvee_{pq=(((a^2u)a)a)a} \{(f \circ C_S)(p) \wedge f(q)\} \ \wedge 0.5 \\ &\geq (f \circ C_S)(((a^2u)a)a) \wedge f(a) \wedge 0.5 \\ &= \bigvee_{bc=((a^2u)a)a} \{f((a^2u)a) \wedge C_S(a)\} \wedge f(a) \wedge 0.5 \\ &\geq f((a^2u)a) \wedge 1 \wedge f(a) \wedge 0.5 \\ &\geq f(a^2) \wedge f(a) \wedge 0.5 \wedge f(a) \wedge 0.5 \\ &\geq f(a) \wedge f(a) \wedge 0.5 \geq f(a) \wedge 0.5 = f^-(a),\end{aligned}$$

which shows that $((f \circ C_S) \circ f)^- \geq f^-$.

On the other hand, since f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S, we can deduce that

$$\begin{aligned}((f \circ C_S) \circ f)^-(a) &= ((f \circ C_S) \circ f)(a) \wedge 0.5 \\ &= \bigvee_{a=bc} \{(f \circ C_S)(b) \wedge f(c)\} \ \wedge 0.5 \\ &= \bigvee_{a=bc} \{ \bigvee_{b=pq} \{f(p) \wedge C_S(q)\} \wedge f(c)\} \wedge 0.5 \\ &= \bigvee_{a=yz} \{ \bigvee_{b=pq} \{f(p) \wedge 1\} \wedge f(c)\} \wedge 0.5 \\ &= \bigvee_{a=bc} \{ \bigvee_{b=pq} \{f(p)\} \wedge f(c)\} \wedge 0.5 \\ &\leq \bigvee_{a=(pq)c} \{f((pq)c) \wedge 0.5\} \leq f(a) \wedge 0.5 = f^-(a).\end{aligned}$$

Thus $((f \circ C_S) \circ f)^- \leq f^-$, and so $((f \circ C_S) \circ f)^- = f^-$ as required.

Now $(C_S \circ f)(a) = \bigvee_{a=ea} \{C_S(e) \wedge f(a)\} = f(a)$. Therefore, $f \circ f \leq C_S \circ f = f$. Since S is intra-regular, thus we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

In addition, we also have

$$\begin{aligned}y(xa) &= y(x(xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) \\ &= (xa^2)(xy^2) = (a(xa))(xy^2) = ((xy^2)(xa))a.\end{aligned}$$

Let us denote $xy^2 = t$. Then we deduce that

$$\begin{aligned}((xy^2)(xa) &= t(xa) = t(x((xa^2)y)) = t((xa^2)(xy)) \\ &= (xa^2)(t(xy)) = ((xy)t)(a^2x) = a^2(((xy)t)x) \\ &= (x(xy)t))(aa) = a((x((xy)t))a)\end{aligned}$$

Thus we obtain $a = (y(xa))a = (a((x((xy)t))a)a)a$. Now, it follows that

$$\begin{aligned}f \circ f(a) &= \bigvee_{a=(y(xa))a} \{f(y(xa)) \wedge f(a)\} \\ &= \bigvee_{y(xa)=a((x((xy)t))a)a} \{f(a((x((xy)t))a)) \wedge f(a)\} \\ &\geq f(a) \wedge f(a) \geq f(a),\end{aligned}$$

which shows that $f \circ f \geq f$. Therefore, $f \circ f = f$.

$(ii) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (i)$: Let B be any bi-ideal of an AG-groupoid S with left identity e. Then C_B is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Thus we have $C_B \circ C_B = C_B$. Also it is clear that $C_B \circ C_B = C_{B^2}$. Hence $C_B = C_{B^2}$ and so $B = B^2$. Hence deduce that S is intra-regular as required. ■

Theorem 277 *For an AG-groupoid S with left identity e, the following conditions are equivalent:*

(i) S is intra-regular.

(ii) $(f \wedge g)^- = ((f \circ g) \circ f)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and $(\in, \in \vee q)$-fuzzy interior ideal g of S.

Proof. $(i) \Rightarrow (ii)$*: Let S be an intra-regular AG-groupoid with left identity e. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal and g be an $(\in, \in \vee q)$-fuzzy interior ideal of S. Since C_s itself is an $(\in, \in \vee q)$-fuzzy ideal of S, for any $a \in S$, we have*

$$\begin{aligned}((f \circ g) \circ f)^-(a) &\leq ((f \circ C_s) \circ f)(a) \wedge 0.5 \\ &= \bigvee_{a=pq} \{(f \circ C_s)(p) \wedge f(q)\} \wedge 0.5 \\ &= \bigvee_{a=pq} \{ \bigvee_{p=bc} \{f(b) \wedge C_s(c)\} \wedge f(q)\} \wedge 0.5 \\ &= \bigvee_{a=pq} \{ \bigvee_{p=bc} \{f(b) \wedge 1\} \wedge f(q)\} \wedge 0.5 \\ &= \bigvee_{a=pq} \{ \bigvee_{p=bc} f(b) \wedge f(q)\} \wedge 0.5 \\ &\leq \bigvee_{a=(bc)q} \{f((bc)q\} \wedge 0.5 \quad = f(a) \wedge 0.5 = f^-(a).\end{aligned}$$

Note also that

$$\begin{aligned}((f\circ g)\circ f)^{-}(a) &\leq ((C_s\circ g)\circ C_s)(a)\wedge 0.5\\ &= \bigvee_{a=bc}\{(C_s\circ g)(b)\wedge C_s(c)\}\wedge 0.5\\ &= \bigvee_{a=bc}\{\bigvee_{b=pq}\{C_s(p)\wedge g(q)\}\wedge 1\}\wedge 0.5\\ &= \bigvee_{a=(pq)c}\{1\wedge g(q)\}\wedge 0.5=\bigvee_{a=(pq)c}\{g(q)\}\wedge 0.5\\ &\leq \bigvee_{a=(pq)c}\{g((pq)c)\}\wedge 0.5=g(a)\wedge 0.5=g^{-}(a).\end{aligned}$$

Hence $((f\circ g)\circ f)\leq (f^{-}\wedge g^{-})=(f\wedge g)^{-}$*. Now, since* S *is intra-regular, for* $a\in S$ *there exist elements* $x,y\in S$ *such that* $a=(xa^2)y$*. We already obtained* $a=(((a^2u)a)a)a$*. Thus we have*

$$\begin{aligned}((f\circ g)\circ f)^{-}(a) &= \bigvee_{a=(((a^2u)a)a)a}\{(f\circ g)(y(xa))\wedge f(a)\}\wedge 0.5\\ &\geq (f\circ g)(((a^2u)a)a)\wedge f(a)\wedge 0.5\\ &= \bigvee_{((a^2u)a)a=bc}\{f(((a^2u)a)\wedge g(a)\wedge 0.5\}\wedge f(a)\wedge 0.5\\ &= \bigvee_{((a^2u)a)a=bc} f(((a^2u)a)\wedge g(a)\wedge f(a)\wedge 0.5\\ &\geq f(a^2)\wedge f(a)\wedge 0.5\wedge g(a)\wedge f(a)\wedge 0.5\\ &\geq f(a)\wedge f(a)\wedge g(a)\wedge f(a)\wedge 0.5\\ &\geq f(a)\wedge g(a)\wedge f(a)\wedge 0.5\\ &\geq f(a)\wedge g(a)\wedge 0.5=(f\wedge g)^{-}(a),\end{aligned}$$

which gives $((f\circ g)\circ f)^{-}\geq (f\wedge g)^{-}$*. Therefore,* $((f\circ g)\circ f)^{-}=(f\wedge g)^{-}$ *as required.*

$(ii)\Rightarrow(i)$*: Assume that* S *is an AG-groupoid with left identity such that* $(f\wedge g)^{-}=((f\circ g)\circ f)^{-}$ *for every* $(\in,\in\vee q)$*-fuzzy bi-ideal* f *and* $(\in,\in\vee q)$*-fuzzy interior ideal* g *of* S*. Let* f *be any* $(\in,\in\vee q)$*-fuzzy bi-ideal of* S*. Since* C_S *itself is an* $(\in,\in\vee q)$*-fuzzy interior ideal of* S*, we have*

$$f^{-}(a)=f(a)\wedge 0.5=(f\wedge C_S)(a)\wedge 0.5=(f\wedge C_S)^{-}(a)=((f\circ C_S)\circ f)^{-}(a),$$

for all $a\in S$*. That is,* $((f\circ C_S)\circ f)^{-}=f^{-}$*. Hence* S *is intra-regular as required.* ■

Theorem 278 *For AG-groupoid* S *with left identity* e*, the following conditions are equivalent:*

(i) S is intra-regular.

(ii) $A \cap B \subseteq AB$, for every bi-ideal B and left ideal A of S.

(iii) $(f \wedge g)^- \leq (f \circ g)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and $(\in, \in \vee q)$-fuzzy left ideal g of S.

(iv) $(f \wedge g)^- \leq (f \circ g)^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.

Proof. $(i) \Rightarrow (iv)$: Let S be an intra-regular AG-groupoid with left identity e. Let f and g be any $(\in, \in \vee q)$-fuzzy generalized bi-ideal and any $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. For any $a \in S$, there exist x and y in S such that $a = (xa^2)y$. Thus

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ((ey)(xa))a \\ &= ((ax)(ye))a = (((ye)x)a)a = (ta)a, \text{ where } t = (ye)x. \end{aligned}$$

Thus

$$a = (ta)a = (t(xa^2)y)a = ((xa^2)(ty))a = ((x(aa))(ty))a = ((a(xa))(ty))a.$$

Furthermore, we have

$$\begin{aligned} (f \circ g)^-(a) &= (f \circ g)(a) \wedge 0.5 \\ &= \bigvee_{a=pq} \{f(p) \wedge g(q)\} \wedge 0.5 \\ &= \bigvee_{a=((a(xa))(ty))a} \{f(p) \wedge g(q)\} \wedge 0.5 \\ &\geq f(a(xa)) \wedge g((ty)a) \wedge 0.5 \\ &\geq f(a) \wedge f(a) \wedge g(a) \wedge 0.5 \\ &= f(a) \wedge g(a) \wedge 0.5 \\ &= (f \wedge g) \wedge 0.5 = (f \wedge g)^-(a). \end{aligned}$$

That is, $(f \wedge g)^- \leq (f \circ g)^-$.

$(iv) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (ii)$: Assume that S is an AG-groupoid with left identity such that $(f \wedge g)^- \leq (f \circ g)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and $(\in, \in \vee q)$-fuzzy left ideal g of S. Let A and B be bi-ideal and left ideal of S, respectively. Then C_A^- and C_B^- are $(\in, \in \vee q)$-fuzzy bi-ideal and $(\in, \in \vee q)$-fuzzy left ideal of S. Thus by hypothesis we get

$$C_{A \cap B}^- = (C_A \wedge C_B)^- \leq (C_A \circ C_B)^- = C_{AB}^-.$$

It follows that $A \cap B \subseteq AB$.

$(ii) \Rightarrow (i)$: Since Sa is both a bi-ideal and left ideal of an AG-groupoid S with left identity. Using the medial law, the left invertive law and the paramedial law, we have

$$\begin{aligned}a &\in Sa\cap Sa=(Sa)(Sa)=(SS)(aa)=(a^2S)S\\ &=((aa)(SS))S=((SS)(aa))S=(Sa^2)S,\end{aligned}$$

for all $a\in S$. Hence S is intra-regular as required. ∎

Theorem 279 *For an AG-groupoid S with left identity e, the following conditions are equivalent:*
(i) S is intra-regular.
(ii) $(f\circ f)^-\geq f^-$ for every $(\in,\in\vee q)$-fuzzy bi-ideal f of S.
(iii) $(f\circ g)^-\geq f^-\wedge g^-$ for every $(\in,\in\vee q)$-fuzzy bi-ideals f and g of S.

Proof. $(i)\Rightarrow(iii)$: Let S be an intra-regular AG-groupoid with left identity e. Let f and g be $(\in,\in\vee q)$-fuzzy bi-ideals of S. For any $a\in S$, there exist x,y in S such that $a=\left(xa^2\right)y$. Thus we get

$$a=\left(xa^2\right)y=\left(x\left(aa\right)\right)y=\left(a\left(xa\right)\right)y=(y(xa))a,$$

and

$$\begin{aligned}y(xa)&=y(x\left((xa^2)y\right))=y\left(\left(xa^2\right)\left(xy\right)\right)\\ &=\left(xa^2\right)\left(xy^2\right)=\left(a\left(xa\right)\right)\left(xy^2\right)=((xy^2)(xa))a\\ &=(\left(xy^2\right)\left(x((xa^2)y)\right)a=((xy^2)((xa^2)(xy)))a\\ &=((xa^2)((xy^2))(a^2x))a=(\left((xy)\left(xy^2\right)\right)\left(a^2x\right))a\\ &=(a^2(((xy)\left(xy^2\right))x))a=((x((xy)(xy^2)))(aa))a\\ &=\left(a\left(x((xy)\left(xy^2\right))\right)a\right)a.\end{aligned}$$

Thus $a=(y(xa))a=\left(a\left(x((xy)\left(xy^2\right))\right)a\right)a$. Now, we have

$$\begin{aligned}(f\circ g)^-(a)&=(f\circ g)(a)\wedge 0.5\\ &=\left(\bigvee_{a=(a(x((xy)(xy^2)))a)a} f\left(a\left(x((xy)\left(xy^2\right))\right)a\right)\wedge g\left(a\right)\right)\wedge 0.5\\ &=\left(\bigvee_{a=(a(x((xy)(xy^2)))a)a} f\left(a\left(x((xy)\left(xy^2\right))\right)a\right)\wedge g\left(a\right)\right)\wedge 0.5\\ &\geq\left(f\left(a\left(x((xy)\left(xy^2\right))\right)a\right)\wedge g\left(a\right)\right)\wedge 0.5\\ &\geq\left(f\left(a\right)\wedge f(a)\wedge 0.5\right)\wedge\left(g\left(a\right)\wedge 0.5\right)\\ &\geq\left[f\left(a\right)\wedge g\left(a\right)\right]\wedge 0.5=(f\wedge g)(a)\wedge 0.5=(f\wedge g)^-(a).\end{aligned}$$

This shows that $(f\circ g)^-\geq f^-\wedge g^-$ as required.
$(iii)\Rightarrow(ii)$: Straightforward.

$(ii) \Rightarrow (i)$: Assume that S is an AG-groupoid with left identity e such that $(f \circ f)^- \geq f^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S. Let B be a bi-ideal of S. Then C_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. By hypothesis, we have $(C_B \circ C_B)^- = C_{B^2}^- \geq C_B^-$, and so $B \subseteq B^2$. Clearly, we have $B^2 \subseteq B$ since B is a bi-ideal of S. Therefore, $B^2 = B$. Hence S is intra-regular. ■

7.2 $(\in, \in \vee q_k)$-fuzzy Ideals of Abel-Grassmann's

7.3 Main results

We begin with the following definition.

Definition 280 *An element a of an AG-groupoid S is called **intra-regular** if there exists $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.*

Let S be an intra-regular AG-groupoid with left identity. Then, for x in S there exist u and v in S such that $x = uv$. Now, using paramedial, medial, left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = [(uv)(aa)]y = [(aa)(vu)]y = [y(vu)]a^2 = a[\{y(vu)\}a] \quad (2) \\ &= [y(vu)]a^2 = (ya)[(vu)a] = [a(vu)](ay) = [(ay)(vu)]a. \end{aligned}$$

Note. It is obvious from (2) that the results for intra-regular AG-groupoid with left identity is significantly different from those of semigroups and monoids.

The characteristic function C_A for a subset A of an AG-groupoid S is defined by

$$C_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy subgroupoid of S if for all $x, y \in S$ and $t, r \in (0, 1]$ the following condition holds.

$x_t \in f$ and $y_r \in f$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k f$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy left(right) ideal of S if for all $x, y \in S$ and $t, r \in (0, 1]$ the following condition holds.

$y_t \in f$ implies that $(xy)_t \in \vee q_k f$ $\left(y_t \in f \text{ implies that } (yx)_t \in \vee q_k f\right)$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S if it is both $(\in, \in \vee q_k)$-fuzzy left and $(\in, \in \vee q_k)$-fuzzy right ideal of S.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following condition holds.

$x_t \in f$ and $z_r \in f$ implies that $((xy)\, z)_{\min\{t,r\}} \in \vee q_k f$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following condition holds.

$x_t \in f$ and $z_r \in f$ implies that $((xy)\, z)_{\min\{t,r\}} \in \vee q_k f$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy interior of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following condition holds.

(a) $x_t \in f$ and $y_r \in f$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k f$.

(b) $a_t \in f$ implies that $((xa)\, y)_t \in \vee q_k f$.

A fuzzy subset f of S is called an $(\in, \in \vee q_k)$-fuzzy generalized interior of S if for all $x, y, z \in S$ and $t \in (0, 1]$ the following condition holds.

$a_t \in f$ implies that $((xa)\, y)_t \in \vee q_k f$.

A fuzzy subset f of an AG-groupoid S is called $(\in, \in \vee q_k)$-fuzzy semiprime if $f(a) \geq f(a^2) \wedge \frac{1-k}{2}$, for all a in S.

Definition 281 *Let A be any subset of S. Then, the characteristic function $(C_A)_k$ is defined as,*

$$(C_A)_k\,(x) = \begin{cases} \geq \frac{1-k}{2}, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

The proof of the following two lemma's are same as in [55].

Lemma 282 *For an AG-groupoid S, the following holds.*

(i) *A non empty subset J of AG-groupoid S is an ideal if and only if $(C_J)_k$ is an $(\in, \in \vee q_k)$-fuzzy ideal.*

(ii) *A non empty subset L of AG-groupoid S is left ideal if and only if $(C_L)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal.*

(iii) *A non empty subset R of AG-groupoid S is right ideal if and only if $(C_R)_k$ is an $(\in, \in \vee q_k)$-fuzzy right ideal.*

(iv) *A non empty subset B of AG-groupoid S is an bi-ideal if and only if $(C_B)_k$ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal.*

(v) *A non empty subset I of AG-groupoid S is an interior ideal if and only if $(C_I)_k$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal.*

(vi) *A non empty subset I of AG-groupoid S is semiprime if and only if $(C_I)_k$ is an $(\in, \in \vee q_k)$-fuzzy semiprime.*

(vii) *A right ideal R of an AG-groupoid S is semiprime if and only if $(C_R)_k$ is $(\in, \in \vee q_k)$-fuzzy semiprime.*

Let f and g be any two fuzzy subsets of an AG-groupoid S. Then, for $k \in [0, 1)$, the product $f \circ_k g$ is defined by,

$$(f \circ_k g)\,(a) = \begin{cases} \bigvee\limits_{a=bc} \left\{ f(b) \wedge g(c) \wedge \dfrac{1-k}{2} \right\}, & \text{if there exist } b, c \in S, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

Definition 283 *Let f and g be fuzzy subsets of an AG-groupoid S. We define the fuzzy subsets f_k, $f \wedge_k g$, $f \vee_k g$ and $f \circ_k g$ of S as follows,*

(i) $f_k(a) = f(a) \wedge \frac{1-k}{2}$.
(ii) $(f \wedge_k g)(a) = (f \wedge g)(a) \wedge \frac{1-k}{2}$.
(iii) $(f \vee_k g)(a) = (f \vee g)(a) \wedge \frac{1-k}{2}$.
(iv) $(f \circ_k g)(a) = (f \circ g)(a) \wedge \frac{1-k}{2}$, *for all* $a \in S$.

Lemma 284 *Let* A, B *be non empty subsets of an AG-groupoid* S*. Then, the following holds.*
(i) $(C_{A\cap B})_k = (C_A \wedge_k C_B)$.
(ii) $(C_{A\cup B})_k = (C_A \vee_k C_B)$.
(iii) $(C_{AB})_k = (C_A \circ_k C_B)$.

Example 285 *Let* $S = \{1,2,3,4,5,6\}$*, and the binary operation "·" be defined on* S *as follows.*

·	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	1	1	1
3	1	1	5	6	3	4
4	1	1	4	5	6	3
5	1	1	3	4	5	6
6	1	1	6	3	4	5

Clearly $1 = (1 \cdot 1^2) \cdot 1, 2 = (2 \cdot 2^2) \cdot 2, = 3(3 \cdot 3^2) \cdot 5, 4 = (6 \cdot 4^2) \cdot 3, 5 = (5{\cdot}5^2){\cdot}5, 6 = (4{\cdot}6^2){\cdot}3$. *Clearly* $\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,4,5,\}$ *and* $\{1,2,3,4,5,6\}$ *are ideal of* S*. Define a fuzzy subset* $f : S \longrightarrow [0,1]$ *as follows:*

$$f(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.8 \text{ for } x = 2 \\ 0.5 \text{ for } x = 3 \\ 0.5 \text{ for } x = 4 \\ 0.5 \text{ for } x = 5 \\ 0.5 \text{ for } x = 6 \end{cases}$$

Then, clearly f *is an* $(\in, \in \vee q_k)$*-fuzzy ideal of* S.

Theorem 286 *For an AG-groupoid* S *with left identity, the following are equivalent.*
(i) S *is intra-regular.* (ii) *For bi-ideals* B_1 *and* B_2 *of* S, $B_1 \cap B_2 = (B_1B_2)B_1$. (iii) *For* $(\in, \in \vee q_k)$*-fuzzy bi-ideals* f *and* g *of* S, $f \wedge_k g \leq (f \circ_k g) \circ_k f$. (iv) *For* $(\in, \in \vee q_k)$*-fuzzy generalized bi-ideals* f *and* g *of* S, $f \wedge_k g \leq (f \circ_k g) \circ_k f$.

Proof. $(i) \Longrightarrow (iv)$: Let f and g be $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals of an intra-regular AG-groupoid S. Since S is intra-regular therefore for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)\,y$. Now, by using left invertive law, medial law and paramedial law we get,

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = \left(y(x((xa^2)y))\right)a \\
&= \left(y((xa^2)(xy))\right)a = \left((xa^2)(y(xy))\right)a = \left((xa^2)(x(yy))\right)a \\
&= \left((xa^2)(xy^2)\right)a = \left((xx)(a^2y^2)\right)a = \left(x^2(a^2y^2)\right)a \\
&= \left(a^2(x^2y^2)\right)a = \left((aa)(x^2y^2)\right)a = \left((y^2a)(x^2a)\right)a \\
&= \left((y^2x^2)(aa)\right)a = \left(a((y^2x^2)a)\right)a = \left(a((y^2x^2)((xa^2)y))\right)a \\
&= \left(a((xa^2)((y^2x^2)y))\right)a = \left(a((x(y^2x^2))(a^2y))\right)a \\
&= \left(a(a^2((x(y^2x^2))y))\right)a = \left(a^2(a((x(y^2x^2))y))\right)a \\
&= \left((aa)(a((x(y^2x^2))y))\right)a = \left(((a((x(y^2x^2))y))a)a\right)a.
\end{aligned}$$

Thus,

$$\begin{aligned}
& ((f \circ_k g) \circ_k f)(a) \\
&= \bigvee_{a=pq} (f \circ_k g)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(((a((x(y^2x^2))y))a)a)a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&\geq \left\{ f(a((x(y^2x^2))y)))a) \wedge g(a) \right\} \wedge f(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ \left(f(a) \wedge \frac{1-k}{2} \right) \wedge g(a) \right\} \wedge f(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a)\} \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} \\
&= (f \wedge g)(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).
\end{aligned}$$

So, $f \wedge_k g \leq (f \circ_k g) \circ_k f$.

$(iv) \Longrightarrow (iii)$: is obvious.

$(iii) \Longrightarrow (ii)$: Assume that B_1 and B_2 are bi-ideals of S. Then $(C_{B_1})_k$ and $(C_{B_2})_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideals. Thus we have, $(C_{B_1 \cap B_2})_k = (C_{B_1} \wedge_k C_{B_2}) \leq (C_{B_1} \circ_k C_{B_2}) \circ_k C_{B_1} = (C_{(B_1B_2)B_1})_k$. Hence, $B_1 \cap B_2 \subseteq (B_1B_2)B_1$.

$(ii) \Longrightarrow (i)$: Since Sa is a bi-ideal of S contains a. Thus, using (ii), and medial law we have,

$$a \in Sa \cap Sa \subseteq ((Sa)(Sa))Sa = ((SS)(aa))(SS) = (Sa^2)S.$$

Hence, S is intra-regular. ■

Theorem 287 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular. (ii) For left ideals, L_1, L_2 of S, $L_1 \cap L_2 \subseteq L_1L_2 \cap L_2L_1$. (iii) For $(\in, \in \vee q_k)$-fuzzy left ideals, f, g of S, $f \wedge_k g \leq f \circ_k g \wedge g \circ_k f$.

Proof. $(i) \Longrightarrow (iii)$: Let f and g be $(\in, \in \vee q_k)$-fuzzy left ideals of an intra-regular AG-groupoid S respectively. Since S is intra-regular therefore for $a \in S$ there exists $x, y \in S$ such that $a = \left(xa^2\right) y$. Now, using left invertive law we get,
$a = \left(xa^2\right) y = (x\,(aa))\, y = (a\,(xa))\, y = (y\,(xa))\, a$. Therefore,

$$\begin{aligned}
(f \circ_k g)(a) &= \bigvee_{a=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=(y(xa))a=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} \\
&\geq f(y\,(xa)) \wedge g(a) \wedge \frac{1-k}{2} \\
&\geq \left(f(a) \wedge \frac{1-k}{2}\right) \wedge g(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).
\end{aligned}$$

Thus, $f \wedge_k g \leq f \circ_k g$. Similarly, we can show that $f \wedge_k g \leq g \circ_k f$. Thus, we have $f \wedge_k g \leq f \circ_k g \cap g \circ_k f$.

$(iii) \Longrightarrow (ii)$: Assume that L_1 and L_2 be any left ideals of S. Then, $(C_{L_1})_k$ and $(C_{L_2})_k$ are $(\in, \in \vee q_k)$-fuzzy left ideals of S therefore, we have,

$$(C_{L_1 \cap L_2})_k = (C_{L_1} \wedge_k C_{L_2}) \leq (C_{L_1} \circ_k C_{L_2}) = (C_{L_1L_2})_k.$$

This implies that $L_1 \cap L_2 \subseteq L_1L_2$. Similarly, we can show that $L_1 \cap L_2 \subseteq L_2L_1$. Thus, $L_1 \cap L_2 \subseteq L_1L_2 \cap L_2L_1$.

$(ii) \Longrightarrow (i)$: Since Sa is a left ideal of S contains a. Thus, using (ii), paramedial law, medial law, we get,

$$\begin{aligned}
a &\in Sa \cap Sa \subseteq (Sa)(Sa) \cap (Sa)(Sa) \subseteq (Sa)(Sa) = (aa)(SS) \\
&= S((aa)S) = (SS)((aa)S) = (Sa^2)SS = (Sa^2)S.
\end{aligned}$$

Hence, S is intra-regular. ■

Theorem 288 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular. (ii) For bi-ideal B, right ideal R ,and left ideal L of S, $B \cap R \cap L \subseteq (BR)L$ and R is semiprime. (iii) For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy right ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$ and g is $(\in, \in \vee q_k)$-fuzzy semiprime. (iv) For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy interior ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$ and g is $(\in, \in \vee q_k)$-fuzzy semiprime. (v) For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy generalized interior ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$ and g is $(\in, \in \vee q_k)$-fuzzy semiprime.

Proof. $(i) \Longrightarrow (v)$: Let f,g, h are $(\in, \in \vee q_k)$-fuzzy generalized bi, fuzzy generalized interior and fuzzy left ideals of an intra-regular AG-groupoid S respectively. Now, as S is an intra-regular AG-groupoid so for $a \in S$ there exists $x, y \in S$ such that using left invertive law we get,

$$\begin{aligned} a &= \left(xa^2\right) y = (x\,(aa))y = (a(xa))y = \left(a\left(x(\left(xa^2\right) y)\right)\right) y \\ &= \left(a\left(\left(xa^2\right)(xy)\right)\right) y = \left(y\left(\left(xa^2\right)(xy)\right)\right) a = (y\,((x(aa))\,(xy)))\,a \\ &= (y\,((a(xa))\,(xy)))\,a = ((a\,(xa))\,(y\,(xy)))\,a = (((y\,(xy))\,(xa))a)a \\ &= (aa)\,((y(xy))\,(xa))\,. \end{aligned}$$

Therefore,

$$\begin{aligned} & ((f \circ_k g) \circ_k h)\,(a) \\ = & \bigvee_{a=pq} (f \circ_k g)\,(p) \wedge h\,(q) \wedge \frac{1-k}{2} \\ = & \bigvee_{a=pq} \left\{ \left(\bigvee_{p=uv} (f\,(u) \wedge g\,(v)) \wedge \frac{1-k}{2} \right) \wedge h\,(q) \wedge \frac{1-k}{2} \right\} \\ = & \bigvee_{a=(uv)q} (f\,(u) \wedge g\,(v)) \wedge h\,(q) \wedge \frac{1-k}{2} \\ = & \bigvee_{a=(uv)q=(aa)((y(xy))(xa))} (f\,(u) \wedge g\,(v)) \wedge h\,(q) \wedge \frac{1-k}{2} \\ \geq & (f\,(a) \wedge g\,(a)) \wedge h((y(xy))\,(xa)) \wedge \frac{1-k}{2} \\ = & (f\,(a) \wedge g\,(a)) \wedge \left(h(a) \wedge \frac{1-k}{2} \right) \wedge \frac{1-k}{2} \\ \geq & (f\,(a) \wedge g\,(a)) \wedge h(a) \wedge \frac{1-k}{2} \\ = & ((f \wedge_k g) \wedge_k h)\,(a)\,. \end{aligned}$$

Thus, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$. As given that S is intra-regular so for $a \in S$ there exists $x, y \in S$ such that $a = \left(xa^2\right) y$. This implies that

$g(a) = g\left(\left(xa^2\right)y\right) \geq g\left(a^2\right)$.

$(v) \Longrightarrow (iv) \Longrightarrow (iii)$: are obvious.

$(iii) \Longrightarrow (ii)$: Assume that B be any bi, R be right and L be left ideal of S respectively. Now $(C_B)_k$, $(C_R)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$-fuzzy bi, right and left ideals of S respectively. Now, by (iii) $(C_R)_k$ is fuzzy semiprime. Therefore R is semiprime. Thus we have

$$(C_{(B\cap R)\cap L})_k = (C_B \wedge_k C_R) \wedge_k C_L \leq (C_B \circ_k C_R) \circ_k C_L = (C_{(BR)L})_k.$$

Hence, $(B \cap R) \cap L \subseteq (BR)L$.

$(ii) \Longrightarrow (i)$: We know that Sa is both bi and left ideal and Sa^2 right ideal of S containing a and a^2, respectively. And by (ii) Sa^2 is semiprime. So,, $a \in Sa^2$. Thus, using (ii) , medial law we have,

$$\begin{aligned} a &\in \left(Sa \cap Sa^2\right) \cap Sa \subseteq (\left(Sa\right)(Sa^2))Sa = (\left(Sa\right)S)((Sa^2)S\big) \\ &= (\left(SS\right)S)((Sa^2)S\big) = \left(SS\right)((Sa^2)S\big) = S((Sa^2)S \\ &= (Sa^2)(SS) = (Sa^2)S. \end{aligned}$$

Hence, S is intra-regular. ■

7.4 Regular AG-groupoids

In this section we have characterized regular Abel-Grassmann's groupoid in terms of its $(\in, \in \vee q_k)$-fuzzy ideals.

Definition 289 *An element a of an AG-groupoid S is called **regular** if there exist x in S such that $a = (ax)a$ and S is called **regular**, if every element of S is regular.*

Lemma 290 *Let S be an AG-groupoid. If $a = a(ax)$, for some x in S. Then $a = a^2y$, for some y in S.*

Proof. Using medial law, we get

$$a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y, \text{ where } y = (ax)x.$$

■

Lemma 291 *Let S be an AG-groupoid with left identity. If $a = a^2x$, for some x in S. Then $a = (ay)a$, for some y in S.*

Proof. Using medial law, left invertive law, paramedial law and medial law, we get

$$\begin{aligned} a &= a^2x = (aa)x = ((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2) \\ &= a^2((xx^2)a^2)) = ((xx^2)a^2)a)a = ((aa^2)(xx^2))a = ((x^2x)(a^2a))a \\ &= [a^2\{(x^2x)a\}]a = [\{a(x^2x)\}(aa)]a = [a(\{a(x^2x)\}a)]a \\ &= (ay)a, \text{ where } y = \{a(x^2x)\}a. \end{aligned}$$

■

Lemma 292 *In AG-groupoid S, with left identity, the following holds.*

(i) $(aS)\,a^2 = (aS)\,a.$
(ii) $(aS)\,((aS)\,a) = (aS)\,a.$
(iii) $S\,((aS)\,a) = (aS)\,a.$
(iv) $(Sa)\,(aS) = a\,(aS)\,.$
(v) $(aS)\,(Sa) = (aS)\,a.$
(vi) $[a(aS)]S = (aS)a.$
(vi) $[(Sa)S](Sa) = (aS)(Sa).$
(vii) $(Sa)S = (aS).$
$(viii)$ $S(Sa) = Sa.$
(ix) $Sa^2 = a^2S.$

Proof. It is easy. ■

Example 293 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

It is easy to check that $\{1, 2\}$ is the quasi-ideal of S. Clearly S is regular because $1 = 1 \circ 1$, $2 = (2 \circ 3) \circ 2$ and $3 = (3 \circ 2) \circ 3$. Let us define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.8 \text{ for } x = 2 \\ 0.6 \text{ for } x = 3 \end{cases}$$

Then clearly f is an $(\in, \in \vee q_k)$-fuzzy ideal of S.

Theorem 294 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) *S is regular.*
(ii) *For bi-ideal B, ideal I and left ideal L of S, $B \cap I \cap L \subseteq (BI)\,L$.*
(iii) *$B\,[a] \cap I\,[a] \cap L\,[a] \subseteq (B\,[a]\,I\,[a])\,L\,[a]\,,$ for some a in S.*

Proof. $(i) \Rightarrow (ii)$

Assume that B, I and L are bi-ideal, ideal and left ideal of a regular AG-groupoid S respectively. Let $a \in B \cap I \cap L$. This implies that $a \in I$, $a \in B$ and $a \in L$. Since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a = (B\,((SI)\,S))\,L = (BI)\,L.$

Thus $B \cap I \cap L \subseteq (BI)\,L.$

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

$B[a] = a \cup a^2 \cup (aS)a$, $I[a] = a \cup Sa \cup aS$ and $L[a] = a \cup Sa$ are principle bi-ideal, principle ideal and principle left ideal of S generated by a respectively. Thus by left invertive law and paramedial law we have,

$$\begin{aligned}
& \left(a \cup a^2 \cup (aS)a\right) \cap (a \cup Sa \cup aS) \cap (a \cup Sa) \\
\subseteq\ & \left(\left(a \cup a^2 \cup (aS)a\right)(a \cup Sa \cup aS)\right) \\
& (a \cup Sa) \\
\subseteq\ & \{S(a \cup Sa \cup aS)\}(a \cup Sa) \\
\subseteq\ & \{Sa \cup S(Sa) \cup S(aS)\}(a \cup Sa) \\
=\ & (Sa \cup aS)(a \cup Sa) \\
=\ & (Sa)a \cup (Sa)(Sa) \cup (aS)a \cup (aS)(Sa) \\
=\ & a^2S \cup a^2S \cup (aS)a \cup (aS)a \\
=\ & a^2S \cup (aS)a.
\end{aligned}$$

Hence S is regular. ■

Theorem 295 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) S is regular.
(ii) For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.
(iii) For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (iii)$
Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law we have, $a = (ax)a = (((ax)a)x)a =$

$((xa)(ax))a = (a((xa)x))a$. Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Assume that B, I and L are bi-ideal, ideal and left ideal of S respectively. Then $(C_B)_k$, $(C_I)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S respectively. Therefore we have, $(C_{B\cap I\cup L})_k = (C_B \wedge_k C_I) \wedge_k C_L \leq (C_B \circ_k C_I) \circ_k C_L = (C_{(BI)L})_k = (C_{(BI)L})_k$. Therefore $B \cap I \cap L \subseteq (BI)L$. Hence S is regular. ■

Theorem 296 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For left ideal L, ideal I and quasi-ideal Q of S, $L \cap I \cap Q \subseteq (LI)Q$.*

(ii) *$L[a] \cap I[a] \cap Q[a] \subseteq (L[a]I[a])Q[a]$, for some a in S.*

Proof. $(i) \Rightarrow (ii)$

Assume that L, I and Q are left ideal, ideal and quasi-ideal of regular AG-groupoid S. Let $a \in L \cap I \cap Q$. This implies that $a \in L$, $a \in I$ and $a \in Q$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (L((SI)S))Q \subseteq (LI)Q$. Thus $L \cap I \cap Q \subseteq (LI)Q$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

$L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are left ideal, ideal and quasi-ideal of S generated a respectively. Thus by medial law we have,

$$\begin{aligned}
& (a \cup Sa) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
\subseteq\; & ((a \cup Sa)(a \cup Sa \cup aS)) \\
& (a \cup (Sa \cap aS)) \\
\subseteq\; & \{(a \cup Sa) S\} (a \cup aS) \\
=\; & \{aS \cup (Sa) S\} (a \cup aS) \\
=\; & (aS)(a \cup aS) \\
=\; & (aS) a \cup (aS)(aS) \\
=\; & (aS) a \cup a^2 S.
\end{aligned}$$

Hence S is regular. ■

Theorem 297 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) For $(\in, \in \vee q_k)$-fuzzy left ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy quasi-ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (ii)$

Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law, we have, $a = (ax) a = (((ax) a) x) a = ((xa)(ax)) a = (a((xa) x)) a$. Thus,

$$\begin{aligned}
& ((f \circ_k g) \circ_k h)(a) \\
=\; & \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
=\; & \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
=\; & \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
=\; & \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
\geq\; & \{f(a) \wedge g((xa) x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
\geq\; & \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
=\; & \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
=\; & ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(ii) \Longrightarrow (i)$

Assume that L, I and Q are left ideal, ideal and quasi-ideal of S respectively. Thus $(C_L)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore we have, $(C_{L\cap I\cup Q})_k = (C_L \wedge_k C_I) \wedge_k C_Q \leq (C_L \circ_k C_I) \circ_k C_Q = (C_{(LI)Q})_k = (C_{(LI)Q})_k$. Therefore $L \cap I \cap Q \subseteq (LI)\, Q$. Hence S is regular. ■

Theorem 298 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, ideal I and quasi-ideal Q of S, $B \cap I \cap Q \subseteq (BI)\, Q$.*

(iii) *$B[a] \cap I[a] \cap Q[a] \subseteq (B[a]\, I[a])\, Q[a]$, for some a in S.*

Proof. $(i) \Rightarrow (ii)$

Assume that B, I and Q are bi-ideal, ideal and quasi-ideal of regular AG-groupoid S. Let $a \in B \cap I \cap Q$. This implies that $a \in B$, $a \in I$ and $a \in Q$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\, a = (((ax)\, a)\, x)\, a = ((xa)\,(ax))\, a = (a\,((xa)\, x))\, a \in (B\,((SI)\, S))\, Q \subseteq (BI)\, Q$. Thus $B \cap I \cap Q \subseteq (BI)\, Q$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $B[a] = a \cup a^2 \cup (aS)\, a$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are principle bi-ideal, principle ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii) and medial law and left invertive law we have,

$$\begin{aligned}
& \left(a \cup a^2 \cup (aS)\, a\right) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
\subseteq\ & \left(\left(a \cup a^2 \cup (aS)\, a\right)(a \cup Sa \right. \\
& \left. \cup aS)\right)(a \cup (Sa \cap aS)) \\
\subseteq\ & (S(a \cup Sa \cup aS))\,(a \cup aS) \\
=\ & (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
=\ & (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
=\ & (aS \cup Sa)\,(a \cup aS) \\
=\ & (aS)\, a \cup (aS)\,(aS) \cup (Sa)\, a \cup (Sa)\,(aS) \\
=\ & (aS)\, a \cup a^2 S \cup a\,(aS)\,.
\end{aligned}$$

Hence S is regular. ■

Theorem 299 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy quasi ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

(iii) For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy quasi ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (iii)$

Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law, we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a$. Thus,

$$\begin{aligned}
& ((f \circ_k g) \circ_k h)(a) \\
= & \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
= & \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
\geq & \{f(a) \wedge g\,((xa)\,x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
\geq & \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
= & \{f\,(a) \wedge g\,(a) \wedge \frac{1-k}{2}\} \wedge h\,(a) \wedge \frac{1-k}{2} \\
= & ((f \wedge_k g) \wedge_k h)\,(a)\,.
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Assume that B, I and Q be bi-ideal, ideal and quasi-ideal of S respectively. Then $(C_B)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore we have, $(C_{B \cap I \cup Q})_k = (C_B \wedge_k C_I) \wedge_k C_Q \leq (C_B \circ_k C_I) \circ_k C_Q = (C_{(BI)Q})_k = (C_{(BI)Q})_k$. Therefore $B \cap I \cap Q \subseteq (BI)\,Q$. Hence S is regular. ■

Theorem 300 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) For an ideals I_1, I_2 and I_3 of S, $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2)\,I_3$.

(iii) $I\,[a] \cap I\,[a] \cap I\,[a] \subseteq (I\,[a]\,I\,[a])\,I\,[a]\,,$ for some a in S.

Proof. $(i) \Rightarrow (ii)$

Assume that I_1, I_2, and I_3 are ideals of a regular AG-groupoid S. Let $a \in I_1 \cap I_2 \cap I_3$. This implies that $a \in I_1$, $a \in I_2$ and $a \in I_3$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a \in (I_1\,((SI_2)\,S))\,I_3 \subseteq (I_1 I_2)\,I_3$. Thus $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2)\,I_3$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $I\,[a] = a \cup Sa \cup aS$ is a principle ideal of S generated by a. Thus by (iii), left invertive law, medial law and paramedial law we have,

$$\begin{aligned}
& (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \\
\subseteq\ & ((a \cup Sa \cup aS)\,(a \cup Sa \cup aS))\,(a \cup Sa \cup aS) \\
\subseteq\ & \{(a \cup Sa \cup aS)\,S\}\,(a \cup Sa \cup aS) \\
=\ & \{aS \cup (Sa)\,S \cup (aS)\,S\}\,(a \cup Sa \cup aS) \\
=\ & \{aS \cup Sa\}\,(a \cup Sa \cup aS) \\
=\ & (aS)\,a \cup (aS)\,(Sa) \cup (aS)\,(aS) \cup (Sa)\,a \\
& \cup (Sa)\,(Sa) \cup (Sa)\,(aS) \\
=\ & (aS)\,a \cup a^2 S.
\end{aligned}$$

Hence S is regular. ■

Theorem 301 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For quasi-ideals Q_1, Q_2 and ideal I of S, $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)\,Q_2$.*

(iii) $Q\,[a] \cap I\,[a] \cap Q\,[a] \subseteq (Q\,[a]\,I\,[a])\,Q\,[a]$, *for some a in S.*

Proof. $(i) \Rightarrow (ii)$

Assume that Q_1 and Q are quasi-ideal and I is an ideal of a regular AG-groupoid S. Let $a \in Q_1 \cap I \cap Q_2$. This implies that $a \in Q_1$, $a \in I$ and $a \in Q_2$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a \in (Q_1\,((SI)\,S))\,Q_2 \subseteq (Q_1 I)\,Q_2$. Thus $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)\,Q_2$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

$Q\,[a] = a \cup (Sa \cap aS)$ and $I\,[a] = a \cup Sa \cup aS$ are principle quasi-ideal and principle ideal of S generated by a respectively. Thus by (iii), left invertive law, medial law, we have,

$$\begin{aligned}
& (a \cup (Sa \cap aS)) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
\subseteq\ & ((a \cup (Sa \cap aS))\,(a \cup Sa \cup aS)) \\
& (a \cup (Sa \cap aS)) \\
\subseteq\ & \{(a \cup aS)\,S\}\,(a \cap aS) \\
=\ & \{aS \cup (aS)\,S\}\,(a \cap aS) \\
=\ & (aS \cup Sa)\,(a \cap aS) \\
=\ & \{(aS)\,a \cup (aS)\,(aS) \cup (Sa)\,a \cup (Sa)\,aSa \\
=\ & (aS)\,a \cup a^2 S \cup a\,(aS)\,.
\end{aligned}$$

Hence S is regular. ■

Theorem 302 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) S is regular.
(ii) For $(\in, \in \vee q_k)$-fuzzy quasi-ideals f, h, and $(\in, \in \vee q_k)$-fuzzy ideal g, of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

Proof. $(i) \Rightarrow (ii)$
Assume that f, h are $(\in, \in \vee q_k)$-fuzzy quasi-ideal and g is $(\in, \in \vee q_k)$-fuzzy ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law, we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a$. Thus,

$$\begin{aligned}
& ((f \circ_k g) \circ_k h)(a) \\
=\ & \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
=\ & \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
=\ & \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
=\ & \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
\geq\ & \{f(a) \wedge g\,((xa)\,x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
\geq\ & \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
=\ & \{f\,(a) \wedge g\,(a) \wedge \frac{1-k}{2}\} \wedge h\,(a) \wedge \frac{1-k}{2} \\
=\ & ((f \wedge_k g) \wedge_k h)\,(a)\,.
\end{aligned}$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

$(ii) \Longrightarrow (i)$

Assume that Q_1 and Q_2 are quasi-ideals and I is an ideal of S respectively. Thus $(C_{Q_1})_k$, $(C_I)_k$ and $(C_{Q_2})_k$ are $(\in, \in \vee q_k)$-fuzzy quasi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore we have,

$$\begin{aligned} (C_{Q_1 \cap I \cup Q_2})_k &= (C_{Q_1} \wedge_k C_I) \wedge_k C_{Q_2} \leq (C_{Q_1} \circ_k C_L) \circ_k C_{Q_2} \\ &= (C_{(Q_1 I) Q_2})_k = (C_{(Q_1 I) Q_2})_k. \end{aligned}$$

Therefore $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$. Hence S is regular. ■

Theorem 303 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, $B = (BS) B$.*

(iii) *For generalized bi-ideal B, $B = (BS) B$.*

Proof. $(i) \Rightarrow (iii)$

Assume that B is generalized bi-ideal of a regular AG-groupoid S. Clearly $(BS) B \subseteq B$. Let $b \in B$. Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx) b \in (BS) B$. Thus $B = (BS) B$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Since $I[a] = a \cup a^2 \cup (aS) a$ is a principle bi-ideal of S generated by a respectively. Thus by (ii), we have,

$$\begin{aligned} & a \cup a^2 \cup (aS) a \\ = & \left(\left(a \cup a^2 \cup (aS) a \right) S \right) \left(a \cup a^2 \cup (aS) a \right) \\ = & \left\{ \left(aS \cup a^2 S \cup ((aS) a) S \right) \left(a \cup a^2 \cup (aS) a \right) \right. \\ = & \left(aS \cup a^2 S \cup a (aS) \right) \left(a \cup a^2 \cup (aS) a \right) \\ = & (aS) a \cup (aS) a^2 \cup (aS) ((aS) a) \\ & \cup \left(a^2 S \right) a \cup \left(a^2 S \right) a^2 \cup \left(a^2 S \right) ((aS) a) \\ & \cup (a (aS)) a \cup (a (aS)) a^2 \cup (a (aS)) ((aS) a) \\ = & (aS) a \cup a^2 S \cup (aS) a \cup a^2 S \cup a^2 S \cup a^2 S \\ & \cup (aS) a \cup (aS) a \cup (aS) a \\ = & a^2 S \cup (aS) a. \end{aligned}$$

Hence S is regular. ■

Theorem 304 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, of S, $f_k = (f \circ_k S) \circ_k f$.*

(iii) *For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, of S, $f_k = (f \circ_k S) \circ_k f$.*

Proof. $(i) \Rightarrow (iii)$

Assume that f is $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx)\, b$. Therefore we have,

$$\begin{aligned}
& ((f \circ_k S) \circ_k f)(b) \\
= & \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
= & \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{b=(uv)q} \left(\{f(u) \wedge S(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{b=(bx)b=(uv)q} \left(\{f(u) \wedge S(v)\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
\geq & \{f(b) \wedge S\,(x)\} \wedge f(b) \wedge \frac{1-k}{2} \\
\geq & f(b) \wedge 1 \wedge f(b) \wedge \frac{1-k}{2} \\
= & f\,(b) \wedge \frac{1-k}{2} = f_k\,(b)\,.
\end{aligned}$$

Thus $(f \circ_k S) \circ_k f \geq f_k$. Since f is $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. So we have,

$$\begin{aligned}
& ((f \circ_k S) \circ_k f)(b) \\
= & \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
= & \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge 1 \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{b=pq} \left(\bigvee_{p=uv} f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \\
= & \bigvee_{b==pq} \left\{ \bigvee_{p=uv} \left(f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \right\} \\
\leq & \bigvee_{b==(uv)q} \left(f\left((uv)q\right) \wedge \frac{1-k}{2} \right) \\
= & f(b) \wedge \frac{1-k}{2} = f_k(b).
\end{aligned}$$

This implies that $(f \circ_k S) \circ_k f \leq f_k$. Thus $(f \circ_k S) \circ_k f = f_k$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Assume that B is a bi-ideal of S. Then $(C_B)_k$, is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Therefore by by (ii) and, we have, $(C_B)_k = (C_B \circ_k C_S) \circ_k C_B = (C_{(BS)B})_k$. Therefore $B = (BS)B$. Hence S is regular. ■

8

Interval Valued Fuzzy Ideals of AG-groupoids

In this chapter we discuss interval valued fuzzy ideals of AG-groupoids.

8.1 Basics

Definition 305 *An interval value fuzzy subset $\widetilde{f}$ on AG-groupoid is called an interval value $(\in, \in, \vee q_{k})$ fuzzy AG-subgroupiod of S if $x_{\tilde{t}} \in \widetilde{f}$ and $y_{\tilde{s}} \in \widetilde{f}$ this implies that $(xy)_{\min\{\tilde{t},\tilde{s}\}} \in \vee q_k \widetilde{f}$ for all $x, y \in S$ and $\tilde{t}, \tilde{s} \in D[0,1]$.*

Definition 306 *An interval valued fuzzy subset $\widetilde{f}$ on an AG-groupiod is called an interval $(\in, \in \vee_{qk})$ fuzzy left(respt right) ideal of an AG -groupiod of S If $y_{\tilde{t}} \in \widetilde{f}$ This implies that $(xy)_{\tilde{t}} \in \vee q_k \widetilde{f}$(respt $x_{\tilde{t}} \in \widetilde{f}$ implies that $(xy)_{\tilde{t}} \in \vee q_k\ \widetilde{f}$).*

Definition 307 *A fuzzy subset $\widetilde{f}$ of an AG-groupiod S is called an interval valued $(\in, \in \vee q_k)$-fuzzy semi prime if $x^2_{\tilde{t}} \in \widetilde{f}$ implies that $x_{\tilde{t}} \in \widetilde{f}$ for all $x \in S$.*

Theorem 308 *An interval value fuzzy subset $\widetilde{f}$ of an AG-groupoid S is an interval valued $(\in, \in, \vee q_k)$-fuzzy AG-sub groupoid if and only if $\widetilde{f}(xy) \geq \min\{\widetilde{f}(x), \widetilde{f}(y), \frac{1-k}{2}\}$.*

Proof. *Let $x, y \in S$ and $\tilde{t}, \tilde{s} \in D[0,1]$. We assume that $x, y \in S$ such that $\widetilde{f}(xy) < \min\{\widetilde{f}(x), \widetilde{f}(y)\}$. we choose $\tilde{t} \in D[0,1]$such that $\widetilde{f}(xy) < \tilde{t} \leq \min\{\widetilde{f}(x), \widetilde{f}(y)\}$ this implies that $(xy)_{\tilde{t}} \overline{\in \vee q_k} \widetilde{f}$ and $\min\{\widetilde{f}(x), \widetilde{f}(y), \frac{1-k}{2}\} \geq \tilde{t}$ This implies that $\widetilde{f}(x) \geq \tilde{t}$ and $\widetilde{f}(y) \geq \tilde{t}$ further $x_{\tilde{t}} \in \widetilde{f}$ and $y_{\tilde{t}} \in \widetilde{f}$ but $(xy)_{t^\sim} \overline{\epsilon \vee q_k} \widetilde{f}$·which is contradiction due to our wrong supposition so $\widetilde{f}(xy) \geq \min\{\widetilde{f}(x), \widetilde{f}(y), \frac{1-k}{2}\}$*

Conversely, suppose that $\widetilde{f}(xy) \geq \min\{\widetilde{f}(x), \widetilde{f}(y), \frac{1-k}{2}\}$. $x_{\tilde{t}} \in \widetilde{f}$ and $y_{\tilde{s}} \in \widetilde{f}$ for $\tilde{t}, \tilde{s} \in D[0,1]$ then by definition we write it as $\widetilde{f}(x) \geq \tilde{t}$ and $\widetilde{f}(y) \geq \tilde{t}$ so $\widetilde{f}(xy) \geq \{\widetilde{f}(x), \widetilde{f}(y)\} \geq \min\{\tilde{t}, \frac{1-k}{2}\}$. Here arises two cases:

Case(i): If $\tilde{t} \leq \frac{1-k}{2}$. Then $\widetilde{f}(xy) \geq \tilde{t}$ it mean that $(xy)_{t^\sim} \in \widetilde{f}$.

Case(ii) If $\tilde{t} > \frac{1-k}{2}$. Then $\widetilde{f}(xy) + \tilde{t} + \frac{1-k}{2} > [1,1]$ that as $(xy)_{\tilde{t}} \in q\widetilde{f}$

From both cases we get $(xy)_{\tilde{t}} \in \vee q_k \widetilde{f}$. Therefore $\widetilde{f}$is an$(\in, \in, \vee q_k)$ fuzzy AG-groupiod of S. ■

Lemma 309 *If L is a left ideal if and only if $(C_l)_k$ is $(\in, \in \vee_{q_k})$ fuzzy left ideal of S.*

Proof. (i) Let $x \in L$ and $s \in S$ this implies that $xs \in L$ now by definition we have $(C_L)_k(x) \geq [1,1]$ and $(C_L)(xs) \geq [1,1]$ therefore

$$(C_L)_k(xs) \geq \min\{(C_L)_k(x), \frac{1-k}{2}\}$$

(ii)If $x \notin L$ and $s \in S$ This implies that $xs \notin L$. Then by definition we have $(C_L)_k(x) \geq [0,0]$ and $(C_L)_k(xs) \geq [0,0]$

$$(C_L)_k(xs) \geq \min\{(C_L)_k(x), \frac{1-k}{2}\}.$$

Conversely let $x \in L$, $y \in S$ Now we have to prove that $xy \in L$ Then by definition we get $(C_L)_k(x) \geq [1,1]$ and now we get

$$(C_L)_k(xy) \geq \{(C_L)_k(x), \frac{1-k}{2}\} \geq \{[1,1], \frac{1-k}{2}\} \geq \frac{1-k}{2},$$

so we have

$$(C_L)_k(xy) \geq \frac{1-k}{2}.$$

This implies that $xy \in L$. ■

Theorem 310 *An interval valued fuzzy subset $\widetilde{f}$ of an AG-groupiod S is an interval value $(\in, \in \vee q_k)$ fuzzy left ideal if and if*

$$\widetilde{f}(xy) \geq \min\{\widetilde{f}(y), \frac{1-k}{2}\}.$$

Proof. Let $x, y \in S$ and $\tilde{t}, \tilde{s} \in D[0,1]$. Let $\widetilde{f}$ be an $(\in, \in \vee q_k)$ fuzzy AG-groupiod of S on contrary we assume that $x, y \in S$ Such that $\widetilde{f}(xy) \leq \widetilde{f}(y)$ we choose $\tilde{t} \in D[0,1]$ such that

$$\widetilde{f}(xy) < \tilde{t} \leq \min\{\widetilde{f}(y), \frac{1-k}{2}\}.$$

Then we have $\widetilde{f}(y) \geq \tilde{t}$ This implies that $y_{\tilde{t}} \in \widetilde{f}$ and $\widetilde{f}(xy) < \tilde{t}$. This implies that $(xy)_{\tilde{t}} \in \vee q_k\ \widetilde{f}$ but this is contradiction due to our wrong supposition Hence

$$\widetilde{f}(xy) \geq \min\{\widetilde{f}(y), \frac{1-k}{2}\}.$$

Conversely let $x, y \in S$ and $\tilde{t}, \tilde{s} \in D[0,1]$ and $y_{\tilde{t}} \in \widetilde{f}$. Now by definition we have

$$\widetilde{f}(xy) \geq \min\{\tilde{t}, \frac{1-k}{2}\}.$$

Here we consider two cases:

(i) if $\tilde{t} \leq \frac{1-k}{2}$. Then we have $\widetilde{f}(xy) \geq \tilde{t}$ This implies that $(xy)_{\tilde{t}} \in \widetilde{f}$
(ii) If $\tilde{t} > \frac{1-k}{2}$. Then we have

$$\widetilde{f}(xy) + \tilde{t} + k > [1, 1].$$

This implies that $(xy)_{\tilde{t}} \in q_k\ \widetilde{f}$. From both cases we have to prove $(xy)_{\tilde{t}} \in \vee q_k\ \widetilde{f}$ ■

Theorem 311 *An interval valued fuzzy subset $\widetilde{f}$ of an AG-groupiod S is called an interval valued $(\in, \in \vee q_k)$ fuzzy left ideal of S If and only if $U(\widetilde{f}, \tilde{t})$ is left ideal of S for all $[0, 0] < \tilde{t} \leq \frac{1-k}{2}$.*

Proof. Assume that $\widetilde{f}$ is an $(\in, \in \vee q_k)$ fuzzy left ideal of S. Let us consider $y \in U(\widetilde{f}, \tilde{t})$ then $\widetilde{f}(y) \geq \tilde{t}$. Then we write $\widetilde{f}(xy) \geq \min\{\widetilde{f}(y), \frac{1-k}{2}\} \geq \min\{\tilde{t}, \frac{1-k}{2}\} \geq \tilde{t}$ this implies that $\widetilde{f}(xy) \geq \tilde{t}$, this implies that $xy \in U(\widetilde{f}, \tilde{t})$. Hence $U(\widetilde{f}, \tilde{t})$ is left ideal of S.

Conversely Let $x, y \in L$ and $\tilde{t} \in D[0, 1]$. Assume that $\widetilde{f}(xy) < \tilde{t} \leq \{\widetilde{f}(y), \frac{1-k}{2}\}$. Then $\widetilde{f}(xy) < \tilde{t}$. This implies that $\widetilde{f}(xy) + \tilde{t} + k < [1, 1]$ further implies that $(xy)\overline{\in \vee q}_k U(\widetilde{f}, \tilde{t})$ and $\{\widetilde{f}(y), \frac{1-k}{2}\} \geq \tilde{t}$, $\widetilde{f}(y) \geq \tilde{t}$ this implies that $y \in U(\widetilde{f}, \tilde{t})$ but $xy\overline{\in \vee q}_k U(\widetilde{f}, \tilde{t})$. This is contradiction due to our wrong supposition. Thus $\widetilde{f}(xy) \geq \min\{\widetilde{f}(y), \frac{1-k}{2}\}$. ■

8.2 Main Results using Interval-valued Generalized Fuzzy Ideals

Theorem 312 *Let S be an AG-groupiod with left identity then the following condition are equivalent.*
(i) S is intra regular.
(ii) For every left ideal L and for any subset I, $L \cap I \subseteq LI$.
(iii) For every interval-valued $(\in, \in \vee q_k)$ fuzzy left ideal $\widetilde{f}$ and for every interval-valued $(\in, \in \vee q_k)$ be any fuzzy subset $\tilde{g}$ then $\widetilde{f} \wedge_k \tilde{g} \leq \widetilde{f} \circ_k \tilde{g}$.

Proof. $(i) \Longrightarrow (iii)$ Assume that S is intra regular AG-groupiod and $\widetilde{f}$ and $\tilde{g}$ are interval-valued $(\in, \in \vee q_k)$ fuzzy left and interval-valued $(\in, \in \vee q_k)$ be any fuzzy subset of S. Since S is intra regular therefore for any a in S Then their exist $x, y \in S$ such that

$$a = (xa^2)y = (x(aa)y) = (a(xa))y = y(xa)a.$$

For any a in S, their exist u and v in S Such that $a = uv$ then we have

$$
\begin{aligned}
(\widetilde{f} \circ_k \tilde{g})(a) &= \vee_{a=uv}(\widetilde{f}(u) \wedge \tilde{g}(v) \wedge \frac{1-k}{2}) \\
&\geq \{\widetilde{f}(y(xa) \wedge \tilde{g}(a) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\
&\geq \{\widetilde{f}(xa) \wedge \tilde{g}(a) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\
&\geq \widetilde{f}(a) \wedge \tilde{g}(a) \wedge \frac{1-k}{2} \\
&= (\widetilde{f} \wedge \tilde{g})(a) \wedge \frac{1-k}{2} \\
&= (\widetilde{f} \wedge_k \tilde{g})(a).
\end{aligned}
$$

This implies that $\widetilde{f} \wedge_k \tilde{g} \leq \widetilde{f} \circ_k \tilde{g}$.

$(iii) \Longrightarrow (ii)$ Now let us assume that L be any left ideal and I be any subset of S. Now $(C_L)_k$ and $(C_I)_k$ are the interval-valued $(\in, \in \vee q_k)$ fuzzy left and interval-valued $(\in, \in \vee q_k)$ be fuzzy subset of S. Therefore

$$(C_{L\cap I}) = (C_L \wedge_k C_I) \subseteq C_L \circ_k C_I = (C_{LI})_k \subseteq (C_{LI})_k.$$

this implies that $L \cap I \subseteq LI$.

Now $(ii) \Longrightarrow (i)$ $L \cap I \subseteq LI$.

$a \in Sa \cap Sa \subseteq (Sa)(Sa) = (SS)(aa) = Sa^2 = (Sa^2)S$. Hence S is intra-regular. ■

Theorem 313 *Let S be an AG-groupiod with left identity then following condition are equivalent.*

(i) S is intra regular.

(ii) For any subset I and for any left ideal L Then $I \cap L \subseteq IL$.

(iii) For every interval valued $(\in, \in \vee q_k)$ fuzzy subset $\widetilde{f}$ and for every interval valued $(\in, \in \vee q_k)$ fuzzy left ideal $\tilde{g}$ then $\widetilde{f} \wedge_k \tilde{g} \leq \widetilde{f} \circ_k \tilde{g}$.

Proof. $(i) \Longrightarrow (iii)$ Let us assume that S is intra regular AG-groupiod and $\widetilde{f}$ are interval valued $(\in, \in \vee q_k)$ fuzzy subset and $\tilde{g}$ are interval valued $(\in, \in \vee q_k)$ fuzzy left ideal of S. Since S is intra regular then for any $a \in S$ then their exist $x, y \in S$ such that

$$
\begin{aligned}
a &= (xa^2)y = (x(aa)y = ((a(xa))y = y((xa))a = y(xa) = y(x(xa^2)y) \\
&= y((xa^2))(xy) = (xa^2)(y(xy) = (xa^2)(xy^2) = (y^2x)(a^2x) \\
&= (y^2a^2)(x^2) = (x^2a^2)(y^2) = a^2((x^2y^2)a).
\end{aligned}
$$

For any a in S their exist u and v in S Such that $a = uv$ then

$$\begin{aligned}
(\widetilde{f} \circ_k \tilde{g})(a) &= \vee_{a=uv}(\widetilde{f}(u) \wedge \tilde{g}(v)) \wedge \frac{1-k}{2} \\
&\geq \{\widetilde{f}(a) \wedge \tilde{g}((x^2y^2)a) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\
&\geq \{\widetilde{f}(a) \wedge \tilde{g}(a) \wedge \frac{1-k}{2}\} \wedge \frac{1-k}{2} \\
&= (\widetilde{f} \wedge \tilde{g})(a) \wedge \frac{1-k}{2} \\
&= \widetilde{f} \wedge_k \tilde{g}(a) \\
\widetilde{f} \wedge_k \tilde{g} &\leq \widetilde{f} \circ_k \tilde{g}.
\end{aligned}$$

$(iii) \Longrightarrow (ii)$ Now let us assume that I be any subset of S and L be any left ideal of S. Now $(C_I)_k$ and $(C_L)_k$ are the $(\in, \in \vee q_k)$ fuzzy subset and $(\in, \in \vee q_k)$ fuzzy left ideal of S. Therefore

$$(C_{I\cap L}) = (C_I \wedge_k C_L) \leq C_I \circ_k C_L = (C_{IL})_k.$$

This implies that $I \cap L \subseteq IL$.
$(ii) \implies (i)$

$$a \in Sa \cap Sa \subseteq (Sa)(Sa) \subseteq (aa)(SS) = Sa^2 = (Sa^2)S.$$

Hence S is intra regular . ■

Theorem 314 *A fuzzy subset $\widetilde{f}$ of an AG-groupoid S is an interval valued $(\in, \in \vee q_k)$-fuzzy semi prime if and only if $\widetilde{f}(x) \geq \min\{\ \widetilde{f}(x^2), \frac{1-k}{2}\}$,for all $x \in S$.*

Proof. Assume that $\widetilde{f}$ is an interval valued $(\in, \in \vee q_k)$-fuzzy semi prime so let $x^2_{\tilde{t}} \in \widetilde{f}$. This implies that $\widetilde{f}(x^2) \geq \tilde{t}$ Therefore we have $\widetilde{f}(x) \geq \{\ \widetilde{f}(x^2), \frac{1-k}{2}\} = \tilde{t}$ so $\widetilde{f}(x) \geq \tilde{t}$ This implies that $x_{\tilde{t}} \in \widetilde{f}$.

Conversely let us assume that $\widetilde{f}(x) < \min\{\ \widetilde{f}(x^2), \frac{1-k}{2}\}$, for all $x \in S$. Then we choose $\tilde{t} \in (0,1]$. Now let us assume that $\widetilde{f}(x) < \tilde{t} \leq \min\{\ \widetilde{f}(x), \frac{1-k}{2}\}$ then we have $\widetilde{f}(x) < \tilde{t}$. This implies that $\widetilde{f}(x) + \tilde{t} + k < [1,1]$. Further implies that $x_{\tilde{t}} \overline{\in \vee q_k}\ \widetilde{f}$ and then $\min\{\ \widetilde{f}(x^2), \frac{1-k}{2}\} \geq \tilde{t}$ here we consider $\widetilde{f}(x^2) \geq \tilde{t}$. This implies that $x^2_{\tilde{t}} \in \widetilde{f}$ or $x^2_{\tilde{t}} \in \vee q_k\ \widetilde{f}$. But this implies that $x_{\tilde{t}} \overline{\in \vee q_k}\ \widetilde{f}$. This contradiction arises due to our wrong supposition thus we have final result $\widetilde{f}(x) \geq \min\{\ \widetilde{f}(x^2), \frac{1-k}{2}\}$. ■

Example 315 *Let $S = \{1, 2, 3\}$, and the binary operation "$\circ$"be define on S as follows:*

$\circ$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Then (S,$\circ$) is an AG-groupoid. Define a fuzzy subset $f : S \to [0,1]$ as follows.

$$f(x) = \begin{cases} 0.77 \ \textit{if } x = 1 \\ 0.66 \ \textit{if } x = 2 \\ 0.55 \ \textit{if } x = 3 \end{cases}$$

Then clearly f is $(\in, \in \vee q_k)$-fuzzy left ideal.

Example 316 *Let $S = \{1,2,3\}$ and binary operation "$\circ$" be defined on S as fallows:*

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset $f : S \longrightarrow [0,1]$ as fallows

Theorem 317 *Let S be an AG -groupiod with left identity then the following conditions are equivalent.*
(i) S is intra regular.
(ii) For any left ideal L and for any subset A of S so $A \cap L \subseteq (AL)A$.
(iii) For every interval valued $(\in, \in \vee q_k)$-fuzzy subset $\widetilde{f}$ and every interval valued $(\in, \in \vee q_k)$ -fuzzy left ideal g of S then $\widetilde{f} \wedge_k \widetilde{g} \subseteq (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{f}$.

Proof. $(i) \implies (iii)$ Let $\widetilde{f}$ be the interval valued $(\in, \in \vee q_k)$-fuzzy subset and $\widetilde{g}$ be the interval valued $(\in, \in \vee q_k)$-fuzzy left ideal of an intra regular AG-groupoid S with left identity then for any a in S their exist x,$y \in S$ such that so we use medial law and paramedial law and $(ab)c = b(ac)$

$$\begin{aligned} a &= (xa^2)y = (x(aa)y) = (a(xa)y) = (y(xa))a \\ y(xa) &= y(x(xa^2)y) = y((xa^2)(xy)) = (xa^2)(y(xy)) \\ &= (xa^2)(xy^2) = (xx)(a^2y^2) = (x^2)(a^2y^2) = a^2(x^2y^2) \\ &= ((y^2x^2)a^2) = ((y^2x^2)aa)a = a((y^2x^2)a)a. \end{aligned}$$

For any a in S their exist u and v in S such that $a = uv$

$$
\begin{aligned}
& (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{f}(a) \\
= & \bigvee_{a=uv} ((\widetilde{f} \circ \widetilde{g})(u)) \wedge \widetilde{f}(v) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
\geq & ((\widetilde{f} \circ \widetilde{g})(a(y^2x^2)a) \wedge \widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
= & (\vee_{a((y^2x^2)a)=pq}(\widetilde{f}(p) \wedge \widetilde{g}(q)) \wedge \widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
\geq & ((\widetilde{f}(a) \wedge \widetilde{g}(y^2x^2)a) \wedge \widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
\geq & (\widetilde{f}(a) \wedge \widetilde{g}(a)) \wedge \widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
= & (\widetilde{f} \wedge_k \widetilde{g})(a) \text{ so we have} \\
(\widetilde{f} \wedge_k \widetilde{g}) \leq & (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{f}.
\end{aligned}
$$

$(iii) \implies (ii)$ Let A be any subset and L be the left ideal of S then we get $(C_A)_k$ are interval valued $(\in, \in \vee q_k)$-fuzzy subset and $(C_L)_k$ are interval valued $(\in, \in \vee q_k)$-fuzzy left ideal of S then we get

$$
\begin{aligned}
(C_{(A \cap L) \cap A})_k &= (C_{A \cap L})_k \cap (C_A)_k \subseteq (C_A \circ_k C_L) \circ_k C_A \\
&= C_{(AL)} \circ_k C_{(A)} = C_{(AL)A}.
\end{aligned}
$$

Hence$(A \cap L) \subseteq (AL)A$.

$(ii) \implies (i)$ Since a is any subset and Sa be the left ideal containing a so we get the result

$$a \in Sa \cap Sa \subseteq (Sa)(Sa) = Sa^2 = (Sa^2)S.$$

Hence S is intra regular. ■

Theorem 318 *Let S be an AG-groupiod with left identity then the following condition are equivalent.*

(i) S is intra regular.

(ii) $A \cap B \subseteq AB$, for every two sided A and for every bi-ideal B of S.

(iii) For every interval valued $(\in, \in \vee q_k)$-fuzzy two sided $\widetilde{f}$ and for every interval value $(\in, \in \vee q_k)$ -fuzzy bi-ideal $\widetilde{g}$ then $\widetilde{f} \wedge_k \widetilde{g} \leq \widetilde{f} \circ_k \widetilde{g}$.

Proof. $(i) \implies (iii)$ Let us assume that S is intra regular AG-groupoid and $\widetilde{f}$ be interval valued $(\in, \in \vee q_k)$-fuzzy two sided and $\widetilde{g}$ be interval valued $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Since S is intra regular AG-groupoid therefore for any a in S. Then their exist $x, y \in S$ such that

$$\begin{aligned} a &= (xa^2)y = (x(aa)y) = (a(xa))y = (y(xa))a \\ y(xa)a &= y(x((xa^2)y))a = y((xa^2)(xy) = (xa^2)(y(xy) = a^2(x^2y^2) \\ &= ((x^2y^2)a)a = (y^2x^2)(aa) = a((y^2x^2)a) = a(ta) \\ &= a(t(xa^2)y)) = a((xa^2)(ty)) = a((yt)(a^2x)) \\ &= a(a^2((yt)x))) = a((yt)x)a)a) = a((ia)a) \\ &= a((i(xa^2)y))a = a((xa^2)(iy))a) = a((yi)(a^2x))a) \\ &= a(a^2((yi)x))a) = a((x(yi))(aa))a) \\ &= a(a((x(yi)i)a)a) = a((at))a), \end{aligned}$$

so for any a in S their exist u and v in S so $a = uv$ then we get

$$\begin{aligned} (\widetilde{f} \circ_k \widetilde{g})(a) &= \bigvee_{a=uv} (\widetilde{f}(u) \wedge \widetilde{g}(v)) \wedge \frac{1-k}{2} \\ &\geq (\widetilde{f}(a(a\acute{t})) \wedge \widetilde{g}(a)) \wedge \frac{1-k}{2} \\ &\geq (\widetilde{f}(a) \wedge \widetilde{g}(a)) \wedge \frac{1-k}{2} \\ &\geq (\widetilde{f} \wedge_k \widetilde{g})(a) \\ (\widetilde{f} \wedge_k \widetilde{g}) &\leq (\widetilde{f} \circ_k \widetilde{g}). \end{aligned}$$

$(iii) \implies (ii)$ Let A be any two sided and B be any bi-ideal of S so we get $(C_A)_k$ is interval valued $(\in, \in \vee q_k)$-fuzzy two sided ideal and $(C_B)_k$ is interval valued $(\in, \in \vee q_k)$-fuzzy bi-ideal of S then we get

$$(C_{A\cap B})_k = C_A \circ_k C_B \leq C_A \circ_k C_B \leq (C_{AB})_k.$$

Hence $A \cap B \subseteq AB$.

$(ii) \implies (i)$ Sa is bi-ideal of an AG-groupoid S containing a and $\{a\} \cup \{a^2\} \cup (aS)a$ is an ideal of S then we get

$$\begin{aligned} a &\in (Sa) \cap (a \cup a^2 \cup (aS)a) \subseteq (Sa)(a \cup a^2 \cup (aS)a) \\ &= (Sa)a \cup (Sa)a^2 \cup (Sa)(aS)a \subseteq Sa^2. \end{aligned}$$

Hence S is intra regular. ■

Theorem 319 *Let S be an AG-groupoid with left identity then the following condition are equivalent*

(i) S is intra regular.

(ii) $(Q_1 \cap Q_2) \cap L \subseteq (Q_1Q_2)L$, for all quasi ideal Q_1 and Q_2 and left ideal L of S.

(iii) $(\widetilde{f} \wedge_k \widetilde{g}) \wedge_k \widetilde{h} \leq (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h}$, for all interval valued $(\in, \in \vee q_k)$-fuzzy quasi ideals $\widetilde{f}$ and $\widetilde{g}$ and left ideal $\widetilde{h}$ of S.

Proof. $(i) \implies (iii)$ Let us assume that S is intra regular AG-groupoid with left identity $\widetilde{f}$ and $\widetilde{g}$ are the interval valued $(\in, \in \vee q_k)$ fuzzy quasi ideals and $\widetilde{h}$ be the interval valued $(\in, \in \vee q_k)$ fuzzy left ideal of S. For each a in S then their exist $x, y \in S$ such that

$$\begin{aligned} a &= (xa^2)y = (x(aa)y) = (a(xa)y) = (y(xa))a = (y(x(xa^2)y))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((xa^2)(xy^2))a \\ &= ((y^2x)(a^2x))a = (a^2(y^2x)x))a = ((aa)(y^2x)x))a \\ &= ((a(y^2x))(ax))a. \end{aligned}$$

Now for any a in S their exist u and v in S such that $a = uv$ then

$$\begin{aligned} & ((\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h})(a) \\ = & \bigvee_{a=uv} (\widetilde{f} \circ_k \widetilde{g})(u) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\ = & \bigvee_{a=uv} (\bigvee_{u=pq} \widetilde{f}(p) \wedge \widetilde{g}(p) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\ = & \bigvee_{a=(pq)v} (\widetilde{f}(p) \wedge \widetilde{g}(q) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\ = & \bigvee_{((a(y^2x))(ax)a} (\widetilde{f}(p) \wedge \widetilde{g}(q) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\ \geq & (\widetilde{f}(a) \wedge \widetilde{f}(a) \wedge \widetilde{g}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\ \geq & ((\widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{g}(a)) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\ = & (\widetilde{f}(a) \wedge \widetilde{g}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\ = & (\widetilde{f} \wedge_k \widetilde{g})(a) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\ = & ((\widetilde{f} \wedge_k \widetilde{g})(a) \wedge_k \widetilde{h}(a) \wedge \frac{1-k}{2} \\ = & ((\widetilde{f} \wedge_k \widetilde{g})(a) \wedge_k \widetilde{h}(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Hence $(\widetilde{f} \wedge_k \widetilde{g}) \wedge_k \widetilde{h} \leq (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h}$

$(iii) \implies (ii)$ Let Q_1, Q_2 and L are the fuzzy quasi ideals and fuzzy left ideal of S. Then $C_{Q_1}$and C_{Q_2} and C_L are interval valued $(\in, \in \vee q_k)$ fuzzy quasi ideals and interval valued fuzzy left ideal of S

$$\begin{aligned} (C_{(Q_1Q_2)_L})_k(a) &= (C_{Q_1})_k(a) \circ (C_{Q_2})_k(a) \circ (C_L)_k(a) \\ &\geq ((C_{Q_1} \wedge_k C_{Q_2}) \wedge_k C_L)(a) \\ &= (C_{(Q_1 \cap Q_2) \cap L)_k}(a) \\ \text{Hence } (Q_1 \cap Q_2) \cap L &\subseteq (Q_1Q_2)L. \end{aligned}$$

$(ii) \implies (i)$ Let Q and L are the quasi and left ideal of S. Now

$$a \in (Sa \cap Sa) \cap Sa \subseteq [(Sa)(Sa)](Sa) \subseteq Sa^2 = (Sa^2)S.$$

Hence S is intra regular. ■

Theorem 320 *Let S be an AG-groupoid with left identity then the following condition are equivalent.*
(i) S is intra regular.
(ii) $(L_1 \cap L_2) \cap Q \subseteq (L_1L_2)Q$,$L$ are the fuzzy left ideal and Q are the fuzzy quasi ideal of S.
(iii) $(\widetilde{f} \wedge_k \widetilde{g}) \wedge_k \widetilde{h} \leq (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h}$, for all interval valued $(\in, \in \vee q_k)$-fuzzy left ideals $\widetilde{f}$ and $\widetilde{g}$ and quasi ideal $\widetilde{h}$ of S.

Proof. $(i) \implies (iii)$ Let us assume that S is intra regular AG-groupoid with left identity $\widetilde{f}$ and $\widetilde{g}$ are the interval valued $(\in, \in \vee q_k)$-fuzzy left ideals and $\widetilde{h}$ be the interval valued $(\in, \in \vee q_k)$-fuzzy quasi ideal of S For each a in S then their exist $x, y \in S$ such that

$$\begin{aligned} a &= (xa^2)y = (x(aa)y) = (a(xa)y) = (y(xa))a = (y(x(xa^2)y))a \\ &= (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a = ((xa^2)(xy^2))a \\ &= ((y^2x)(a^2x))a = (a^2((y^2x)x))a = ((aa)(y^2x))a \\ &= ((a(y^2x))(ax))a. \end{aligned}$$

Now for any a in S their exist u and v in S such that $a = uv$ then

$$
\begin{aligned}
& ((\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h})(a) \\
= & \bigvee_{a=uv} (\widetilde{f} \circ_k \widetilde{g})(u) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\
= & \bigvee_{a=uv} (\bigvee_{u=pq} \widetilde{f}(p) \wedge \widetilde{g}(q) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\
= & \bigvee_{a=(pq)v} (\widetilde{f}(p) \wedge \widetilde{g}(q) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\
= & \bigvee_{((a(y^2x))(ax)a} (\widetilde{f}(p) \wedge \widetilde{g}(q) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(v) \wedge \frac{1-k}{2} \\
\geq & (\widetilde{f}(a) \wedge \widetilde{f}(a) \wedge \widetilde{g}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\
\geq & ((\widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{g}(a)) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\
= & (\widetilde{f}(a) \wedge \widetilde{f}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\
= & (\widetilde{f}(a) \wedge \widetilde{g}(a) \wedge \frac{1-k}{2}) \wedge \widetilde{h}(a) \wedge \frac{1-k}{2} \\
= & ((\widetilde{f} \wedge_k \widetilde{g})(a) \wedge_k \widetilde{h}(a) \wedge \frac{1-k}{2} \\
= & ((\widetilde{f} \wedge_k \widetilde{g})(a) \wedge_k \widetilde{h}(a) \wedge \frac{1-k}{2} \\
= & ((\widetilde{f} \wedge_k \widetilde{g})(a) \wedge_k \widetilde{h}(a) \wedge \frac{1-k}{2}.
\end{aligned}
$$

Hence $(\widetilde{f} \wedge_k \widetilde{g}) \wedge_k \widetilde{h} \leq (\widetilde{f} \circ_k \widetilde{g}) \circ_k \widetilde{h}$.

$(iii) \implies (ii)$ Let $L_1, L_2$and Q are the fuzzy left ideals and fuzzy quasi ideal of S. Then C_{L_1} and C_{L_2} and C_Q are interval valued $(\in, \in \vee q_k)$ fuzzy left ideals and interval valued fuzzy quasi ideal of S

$$
\begin{aligned}
(C_{(L_1L_2)_Q})_k(a) &= (C_{L_1})_k(a) \circ (C_{L_2})_k(a) \circ (C_Q)_k(a) \\
&\geq ((C_{L_1} \wedge_k C_{L_2}) \wedge_k C_Q)(a) = (C_{(L_1 \cap L_2) \cap Q)_k}(a).
\end{aligned}
$$

Hence $= (L_1 \cap L_2) \cap Q \subseteq (L_1L_2)Q$.

$(ii) \implies (i)$ Let L and Q are the left and quasi ideal of S. Now

$$a \in (Sa \cap Sa) \cap Sa \subseteq [(Sa)(Sa)](Sa) = (Sa^2)S.$$

Hence S is intra regular. ■

9

Generalized Fuzzy Ideals of Abel-Grassmann's Groupoids

In this chapter we characterize a Abel-Grassmann's groupoid in terms of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

9.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of AG-groupoids

For the following definitions see [65].

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X^\delta_{\gamma B}$ be the fuzzy subset of X by $X^\delta_{\gamma B}(x) \geq \delta$ for all $x \in B$ and $X^\delta_{\gamma B}(x) \leq \gamma$ otherwise. Clearly, $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.

(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ,δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

Lemma 321 *Let f and g are fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$ *for all $x \in X$.*

Proof. It is same as in [65]. ■

Lemma 322 *Let f, g and $h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

Proof. It is same as in [65]. ■

It is shown in [65] that "$=_{(\gamma,\delta)}$" is equivalence relation on $\mathcal{F}(X)$. It is also notified that $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 323 *For an AG-groupoid S, the following holds.*

(i) A non empty subset I of AG-groupoid S is an ideal if and only if $X^\delta_{\gamma I}$ is $((\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal.

(ii) A non empty subset L of AG-groupoid S is left ideal if and only if $X^\delta_{\gamma L}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal.

(iii) A non empty subset R of AG-groupoid S is right ideal if and only if $X_{\gamma R}^{\delta}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal.

(iv) A non empty subset B of AG-groupoid S is bi-ideal if and only if $X_{\gamma B}^{\delta}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal.

(v) A non empty subset Q of AG-groupoid S is quasi-ideal if and only if $X_{\gamma Q}^{\delta}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal.

Lemma 324 *Let A, B be any non empty subsets of an AG -groupoid S with left identity. Then we have*

(1) *$A \subseteq B$ if and only if $X_{\gamma A}^{\delta} \subseteq \vee q_{(\gamma,\delta)} X_{\gamma B}^{\delta}$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0,1]$.*

(2) $X_{\gamma A}^{\delta} \cap X_{\gamma B}^{\delta} =_{(\gamma,\delta)} X_{\gamma(A\cap B)}^{\delta}$.

(3) $X_{\gamma A}^{\delta} \circ X_{\gamma B}^{\delta} =_{(\gamma,\delta)} X_{\gamma(AB)}^{\delta}$.

Proof. It is same in [65]. ■

Lemma 325 *If S is an AG-groupoid with left identity then $(ab)^2 = a^2b^2 = b^2a^2$ for all a and b in S.*

Proof. It follows by medial and paramedial laws. ■

Definition 326 *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, it satisfies $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*

Theorem 327 *Let f be a fuzzy subset of an AG groupoid S with left identity. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG subgroupoid of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ where } \gamma, \delta \in [0,1].$$

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroupoid of S. Assume that there exists $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t$. This implies that $f(xy) < t$, which further implies that $(xy)_{\min t}\overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t$. Therefore $\min\{f(x), f(y)\} \geq t$ which implies that $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, implies that $x_t \in_\gamma f$, $y_s \in_\gamma f$. But $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}}\overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, $f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$, this is a contradiction. Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Definition 328 *A fuzzy subset f of an AG-groupoid S with left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and t, s $(\gamma, 1]$ it satisfies $y_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_t \in_\gamma \vee q_\delta f$).*

Theorem 329 *A fuzzy subset f of an AG-groupoid S with left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt right) ideal of S. if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\} \text{ (respt } \max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}).$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ which further implies that $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ which implies that $f(y) \geq t > \gamma$, this implies that $y_t \in_\gamma f$. But $(xy)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\}.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_s \in_\gamma f$ but $(xy)_t \overline{\in_\gamma \vee q_\delta} f$, then $f(y) \geq t > \gamma$, $f(xy) < \min\{f(y), \delta\}$ and $f(xy) + t \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(y), \delta\}$ which is a contradiction. Hence $y_t \in_\gamma f$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$) for all x, y in S. ■

Definition 330 *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if for all x, y and $z \in S$ and $t, s \in (\gamma, 1]$, the following conditions hold.*

(1) if $x_t \in_\gamma f$ and $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

(2) if $x_t \in_\gamma f$ and $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

Theorem 331 *A fuzzy subset f of an AG-groupoid S with left identity is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if*

(I) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

(II) $\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof. $(1) \Leftrightarrow (I)$ is the same as theorem 327.

$(2) \Rightarrow (II)$ Assume that $x, y \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t$ which implies that $f((xy)z) < t$ this implies that $((xy)z)_t \bar{\in}_\gamma f$ which further implies that $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$. Also

$\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$. But $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

$(II) \Rightarrow (2)$ Assume that x, y in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ but $((xy)z)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, $f((xy)z) < \min\{f(x), f(y), \delta\}$ and $f((xy)z) + \min\{t, s\} \leq 2\delta$. It follows that $f((xy)z) < \delta$ and so $\max\{f((xy)z), \gamma\} < \min\{f(x), f(y), \delta\}$ a contradiction. Hence $x_t \in_\gamma f$, $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Example 332 *Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 \text{ if } x = 1, \\ 0.44 \text{ if } x = 2, \\ 0.42 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$-fuzzy left ideal,*
- *f is not an $(\in, \in \vee q_{0.11})$-fuzzy left ideal,*
- *f is not a fuzzy left ideal.*

Example 333 *Let $S = \{1, 2, 3\}$ and the binary operation $\circ$ be defined on S as follows:*

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then clearly $(S, \circ)$ is an AG-groupoid. Defined a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.44 \text{ if } x = 1, \\ 0.6 \text{ if } x = 2, \\ 0.7 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy left ideal of S.*
- *f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy right ideal of S.*

Theorem 334 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *$B[a] \cap I[a] \cap L[a] \subseteq (B[a]\,I[a])\,L[a]$, for some a in S.*

(iii) *For bi-ideal B, ideal I and left ideal L of S, $B \cap I \cap L \subseteq (BI)\,L$.*

(iv) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.*

(v) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.*

(vi) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.*

Proof. $(i) \Rightarrow (vi)$

Assume that f, g and h are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$\begin{aligned} a &= (ax)\,a = [\{(ax)\,a\}x]a = (ax)\{(ax)\,a\} = [\{(ax)a\}x]\{(ax)a\} \\ &= \{(xa)(ax)\}\{(ax)a\} = [\{(ax)a\}(ax)](xa) \end{aligned}$$

Thus,

$$\begin{aligned} & \max\{((f \circ g) \circ h)(a), \gamma\} \\ =\ & \max\left\{ \bigvee_{a=xy} \{(f \circ g)(x) \wedge h(y)\}, \gamma \right\} \\ \geq\ & \max\{(f \circ g)[\{(ax)a\}(ax)] \wedge h(xa), \gamma\} \\ =\ & \max\{ \bigvee_{\{(ax)a\}(xa)=uv} (f(u) \wedge g(v)) \wedge h(xa), \gamma\} \\ \geq\ & \max\{f((ax)a) \wedge g(ax) \wedge h(xa), \gamma\} \\ =\ & \min\{\max\{f((ax)a), \gamma\}, \max\{g(ax), \gamma\}, \max\{h(xa), \gamma\}\} \\ \geq\ & \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\} \\ =\ & \min\{\min\{f(a) \wedge g(a) \wedge h(a), \delta\} \\ =\ & \min\{[f \cap g \cap h](a), \delta\} \end{aligned}$$

Thus $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(vi) \implies (v)$ is obvious.

$(v) \implies (iv)$ is obvious.

$(iv) \implies (iii)$

Assume that B, I and L are bi-ideal, ideal and left ideal of S respectively. Then $\chi_{\gamma B}^{\delta}$, $\chi_{\gamma I}^{\delta}$ and $\chi_{\gamma L}^{\delta}$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy ideal and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal of S respectively. Therefore we have,

$$\begin{aligned} \chi_{\gamma(B\cap I\cap L)}^{\delta} &= {}_{(\gamma,\delta)}\chi_{\gamma B}^{\delta} \cap \chi_{\gamma I}^{\delta} \cap \chi_{\gamma L}^{\delta} \subseteq \vee q_{(\gamma,\delta)}(\chi_{\gamma B}^{\delta} \odot \chi_{\gamma I}^{\delta}) \odot \chi_{\gamma L}^{\delta} \\ &= {}_{(\gamma,\delta)}(\chi_{\gamma BI}^{\delta}) \odot \chi_{\gamma L}^{\delta} ={}_{(\gamma,\delta)} \chi_{\gamma(BI)L}^{\delta}. \end{aligned}$$

Therefore $B \cap I \cap L \subseteq (BI)\,L$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

$B[a] = a \cup a^2 \cup (aS)\,a$, $I[a] = a \cup Sa \cup aS$ and $L[a] = a \cup Sa$ are principle bi-ideal, principle ideal and principle left ideal of S generated by a respectively. Thus by (ii), left invertive law, paramedial law and using law $a(bc) = b(ac)$. we have,

$$\begin{aligned} & \left(a \cup a^2 \cup (aS)\,a\right) \cap (a \cup Sa \cup aS) \cap (a \cup Sa) \\ \subseteq\ & \left(\left(a \cup a^2 \cup (aS)\,a\right)(a \cup Sa \cup aS)\right)(a \cup Sa) \\ \subseteq\ & \{S\,(a \cup Sa \cup aS)\}\,(a \cup Sa) \\ \subseteq\ & \{Sa \cup S\,(Sa) \cup S\,(aS)\}\,(a \cup Sa) \\ =\ & (Sa \cup aS)\,(a \cup Sa) \\ =\ & (Sa)\,a \cup (Sa)\,(Sa) \cup (aS)\,a \cup (aS)\,(Sa) \\ =\ & a^2S \cup a^2S \cup (aS)\,a \cup (aS)\,a \\ =\ & a^2S \cup (aS)\,a. \end{aligned}$$

Hence S is regular. ■

Theorem 335 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *$L[a] \cap I[a] \cap Q[a] \subseteq (L[a]\,I[a])\,Q[a]$, for some a in S..*

(iii) *For left ideal L, ideal I and quasi-ideal Q of S, $L \cap I \cap Q \subseteq (LI)\,Q$*

(iv) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal f, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy ideal g, and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy quasi- ideal h of S.*

(v) *$f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal f, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy right ideal g, and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy quasi- ideal h of S.*

Proof. $(i) \Rightarrow (ii)$

Assume that f, g and h are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy right ideal and $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)\,a = [\{(ax)\,a\}x]a = \{(xa)(ax)\}a.$$

Thus

$$
\begin{aligned}
& \max\{((f\circ g)\circ h)(a),\gamma\} \\
= & \max\left\{\bigvee_{a=xy}\{(f\circ g)(x)\wedge h(y)\},\gamma\right\} \\
\geq & \max\{(f\circ g)\{(xa)(ax)\}\wedge h(a),\gamma\} \\
= & \max\{\bigvee_{\{(xa)(ax)\}=pq}(f(p)\wedge g(q))\wedge h(a),\gamma\} \\
\geq & \max\{f(xa)\wedge g(ax)\wedge h(a),\gamma\} \\
= & \min\{\max\{f(xa),\gamma\},\max\{g(ax),\gamma\},\max\{h(a),\gamma\}\} \\
\geq & \min\{\min\{f(a),\delta\},\min\{g(a),\delta\},\min\{h(a),\delta\}\} \\
= & \min\{\min\{f(a)\wedge g(a)\wedge h(a),\delta\} \\
= & \min\{[f\cap g\cap h](a),\delta\}
\end{aligned}
$$

Hence $f\cap g\cap h\subseteq \vee q_{(\gamma,\delta)}(f\circ g)\circ h$.

$(v)\Rightarrow(iv)$ is obvious.

$(iv)\Rightarrow(iii)$

Assume that L, I and Q are left ideal, ideal and quasi-ideal of S respectively. Then $\chi^{\delta}_{\gamma B}$, $\chi^{\delta}_{\gamma I}$ and $\chi^{\delta}_{\gamma L}$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideal of S respectively. Therefore we have,

$$
\begin{aligned}
\chi^{\delta}_{\gamma(L\cap I\cap Q)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L}\cap\chi^{\delta}_{\gamma I}\cap\chi^{\delta}_{\gamma Q}\subseteq\vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma L}\odot\chi^{\delta}_{\gamma I})\odot\chi^{\delta}_{\gamma Q} \\
&= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma LI})\odot\chi^{\delta}_{\gamma Q} ={}_{(\gamma,\delta)}\ \chi^{\delta}_{\gamma(LI)Q}.
\end{aligned}
$$

Therefore $L\cap I\cap Q\subseteq(LI)\,Q$.

$(iii)\Rightarrow(ii)$ is obvious.

$(ii)\Rightarrow(i)$

$L\,[a]=a\cup Sa$, $I\,[a]=a\cup Sa\cup aS$ and $Q\,[a]=a\cup(Sa\cap aS)$ are left ideal, ideal and quasi-ideal of S generated a respectively. Thus by (iii) and medial law we have,

$$
\begin{aligned}
(a\cup Sa)\cap(a\cup Sa\cup aS)\cap(a\cup(Sa\cap aS)) &\subseteq ((a\cup Sa)\,(a\cup Sa\cup aS)) \\
& \quad (a\cup(Sa\cap aS)) \\
&\subseteq \{(a\cup Sa)\,S\}\,(a\cup aS) \\
&= \{aS\cup(Sa)\,S\}\,(a\cup aS) \\
&= (aS)\,(a\cup aS) \\
&= (aS)\,a\cup(aS)\,(aS) \\
&= (aS)\,a\cup a^2S.
\end{aligned}
$$

Hence S is regular. ■

Theorem 336 *For an AG-groupoid S, with left identity, the following are equivalent.*
(i) *S is regular.*
(ii) $B[a] \cap I[a] \cap Q[a] \subseteq (B[a]\, I[a])\, Q[a]$, *for some a in S.*
(iii) *For bi-ideal B, ideal I and quasi-ideal Q of S, $B \cap I \cap Q \subseteq (BI)\, Q$.*
(iv) $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. *for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal h of S.*
(v) $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. *for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal h of S.*

Proof. $(i) \Rightarrow (v)$
Assume that f, g and h are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)\, a = (((ax)\, a)\, x)\, a = ((xa)\, (ax))\, a = [a\{(xa)\, x\}]a.$$

Thus,

$$\begin{aligned}
& \max\{((f \circ g) \circ h)(a), \gamma\} \\
= & \max\left\{ \bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma \right\} \\
\geq & \max\{(f \circ g)[a\{(xa)\, x\}] \wedge h(a), \gamma\} \\
= & \max\{ \bigvee_{a\{(xa)x\}=pq} (f(p) \wedge g(q)) \wedge h(a), \gamma\} \\
\geq & \max\{f(a) \wedge g\{(xa)x\} \wedge h(a), \gamma\} \\
= & \min\{\max\{f(a), \gamma\}, \max\{g\{xa)x\}, \gamma\}, \max\{h(a), \gamma\}\} \\
\geq & \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\} \\
= & \min\{\min\{f(a) \wedge g(a) \wedge h(a), \delta\} \\
= & \min\{[f \cap g \cap h](a), \delta\}
\end{aligned}$$

Thus $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.
$(v) \Rightarrow (iv)$ is obvious.
$(iv) \Rightarrow (iii)$
Assume that B, I and Q are bi-ideal, ideal and quasi-ideal of regular AG-groupioud of S respectively. Then $\chi^\delta_{\gamma B}$, $\chi^\delta_{\gamma I}$ and $\chi^\delta_{\gamma Q}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S respectively. Therefore we have,

$$\begin{aligned}
\chi^\delta_{\gamma(B\cap I\cap Q)} &= {}_{(\gamma,\delta)}\chi^\delta_{\gamma L} \cap \chi^\delta_{\gamma I} \cap \chi^\delta_{\gamma Q} \subseteq \vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma B} \odot \chi^\delta_{\gamma I}) \odot \chi^\delta_{\gamma Q} \\
&= {}_{(\gamma,\delta)}(\chi^\delta_{\gamma BI}) \odot \chi^\delta_{\gamma Q} =_{(\gamma,\delta)} \chi^\delta_{\gamma(BI)Q}.
\end{aligned}$$

Therefore $B \cap I \cap Q \subseteq (BI)\,Q$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Since $B\,[a] = a \cup a^2 \cup (aS)\,a,\ I\,[a] = a \cup Sa \cup aS$ and $Q\,[a] = a \cup (Sa \cap aS)$ are principle bi-ideal, principle ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii) and using law $a(bc) = b(ac)$ medial law and left invertive law we have,

$$\begin{aligned}
& \left(a \cup a^2 \cup (aS)\,a\right) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
\subseteq\ & \left(\left(a \cup a^2 \cup (aS)\,a\right)(a \cup Sa \cup aS)\right)(a \cup (Sa \cap aS)) \\
\subseteq\ & (S(a \cup Sa \cup aS))\,(a \cup aS) \\
=\ & (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
=\ & (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
=\ & (aS \cup Sa)\,(a \cup aS) \\
=\ & (aS)\,a \cup (aS)\,(aS) \cup (Sa)\,a \cup (Sa)\,(aS) \\
=\ & (aS)\,a \cup a^2S \cup a\,(aS)\,.
\end{aligned}$$

Hence S is regular. ■

Theorem 337 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) $I\,[a] \cap I\,[a] \cap I\,[a] \subseteq (I\,[a]\,I\,[a])\,I\,[a]$, for some a in S.

(iii) For an ideals I_1, I_2 and I_3 of S, $I_1 \cap I_2 \cap I_3 \subseteq (I_1I_2)\,I_3$.

(iv) $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals f,g and h of S.

Proof. $(i) \Rightarrow (iv)$

Assume that f, g and h are any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have,

$$a = (ax)\,a = [\{(ax)\,a\}x]a = ((xa)\,(ax))\,a.$$

Thus,

$$\begin{aligned}
& \max\{((f\circ g)\circ h)(a),\gamma\} \\
= & \max\left\{\bigvee_{a=bc}\{(f\circ g)(b)\wedge h(c)\},\gamma\right\} \\
\geq & \max\{(f\circ g)\{(xa)(ax)\}\wedge h(a),\gamma\} \\
= & \max\{\bigvee_{(xa)(ax)=pq}\{f(p)\wedge g(q)\}\wedge h(a),\gamma\} \\
\geq & \max\{f(xa)\wedge g(ax)\wedge h(a),\gamma\} \\
= & \min\{\max\{f(xa),\gamma\},\max\{g(ax),\gamma\},\max\{h(a),\gamma\}\} \\
\geq & \min\{\min\{f(a),\delta\},\min\{g(a),\delta\},\min\{h(a),\delta\}\} \\
= & \min\{\min\{f(a)\wedge g(a)\wedge h(a),\delta\} \\
= & \min\{[f\cap g\cap h](a),\delta\}
\end{aligned}$$

Thus $f\cap g\cap h\subseteq \vee q_{(\gamma,\delta)}(f\circ g)\circ h$.
$(iv)\Rightarrow(iii)$
Assume that I_1, I_2 and I_3 are any ideals of regular AG-groupioud of S respectively. Then $\chi^{\delta}_{\gamma I_1}$, $\chi^{\delta}_{\gamma I_2}$ and $\chi^{\delta}_{\gamma I_3}$ are any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy ideals of S respectively. Therefore we have,

$$\begin{aligned}
\chi^{\delta}_{\gamma(I_1\cap I_2\cap I_3)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma I_1}\cap\chi^{\delta}_{\gamma I_2}\cap\chi^{\delta}_{\gamma I_3}\subseteq\vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I_1}\odot\chi^{\delta}_{\gamma I_2})\odot\chi^{\delta}_{\gamma I_3} \\
&= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma I_1 I_2})\odot\chi^{\delta}_{\gamma I_3} ={}_{(\gamma,\delta)}\ \chi^{\delta}_{\gamma(I_1 I_2)I_3}.
\end{aligned}$$

Therefore $I_1\cap I_2\cap I_3\subseteq(I_1I_2)I_3$.
$(ii)\Rightarrow(iii)$ is obvious.
$(ii)\Rightarrow(i)$
Since $I[a]=a\cup Sa\cup aS$ is a principle ideal of S generated by a. Thus by (ii), left invertive law, medial law and paramedial law we have,

$$\begin{aligned}
& (a\cup Sa\cup aS)\cap(a\cup Sa\cup aS)\cap(a\cup Sa\cup aS) \\
\subseteq & ((a\cup Sa\cup aS)(a\cup Sa\cup aS)) \\
& (a\cup Sa\cup aS) \\
\subseteq & \{(a\cup Sa\cup aS)S\}(a\cup Sa\cup aS) \\
= & \{aS\cup(Sa)S\cup(aS)S\}(a\cup Sa\cup aS) \\
= & \{aS\cup Sa\}(a\cup Sa\cup aS) \\
= & (aS)a\cup(aS)(Sa)\cup(aS)(aS)\cup(Sa)a \\
& \cup(Sa)(Sa)\cup(Sa)(aS) \\
= & (aS)a\cup a^2S.
\end{aligned}$$

Hence S is regular. ■

Theorem 338 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) S is regular.

(ii) $Q[a] \cap I[a] \cap Q[a] \subseteq (Q[a] I[a]) Q[a]$, for some a in S.

(iii) For quasi-ideals Q_1, Q_2 and ideal I of S, $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$.

(iv) $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals f and h, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

Proof. $(i) \Rightarrow (iv)$

Assume that f, h are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals and g is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and law is $a(bc) = b(ac)$, we have,

$$a = (ax)\,a = [\{(ax)\,a\}x]a = ((xa)\,(ax))\,a = a\{(xa)\,x\}a.$$

Thus,

$$\begin{aligned}
& \max\{((f \circ g) \circ h)(a), \gamma\} \\
= & \max\left\{\bigvee_{a=pq} \{(f \circ g)(p) \wedge h(q)\}, \gamma\right\} \\
\geq & \max\{(f \circ g)[a\{(xa)\,x\}] \wedge h(a), \gamma\} \\
= & \max\{\bigvee_{a\{(xa)x\}=uv} (f(u) \wedge g(v)) \wedge h(a), \gamma\} \\
\geq & \max\{f(a) \wedge g\{(xa)x\} \wedge h(a), \gamma\} \\
= & \min\{\max\{f(a), \gamma\}, \max\{g\{xa)x\}, \gamma\}, \max\{h(a), \gamma\}\} \\
\geq & \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\} \\
= & \min\{\min\{f(a) \wedge g(a) \wedge h(a), \delta\} \\
= & \min\{[f \cap g \cap h](a), \delta\}
\end{aligned}$$

Thus $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(iv) \Rightarrow (iii)$

Assume that Q_1 and Q_2 are quasi-ideals and I is an ideal of a regular AG-groupoid S. Then $\chi^\delta_{\gamma Q_1}$ and $\chi^\delta_{\gamma Q_2}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal, $\chi^\delta_{\gamma I}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Therefore we have,

$$\begin{aligned}
\chi^\delta_{\gamma(Q_1 \cap I \cap Q_2)} &= {}_{(\gamma,\delta)}\chi^\delta_{\gamma Q_1} \cap \chi^\delta_{\gamma I} \cap \chi^\delta_{\gamma Q_2} \subseteq \vee q_{(\gamma,\delta)}(\chi^\delta_{\gamma Q_1} \odot \chi^\delta_{\gamma I}) \odot \chi^\delta_{\gamma Q_2} \\
&= {}_{(\gamma,\delta)}(\chi^\delta_{\gamma Q_1 I}) \odot \chi^\delta_{\gamma Q_2} =_{(\gamma,\delta)} \chi^\delta_{\gamma(Q_1 I)Q_2}.
\end{aligned}$$

Thus $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)\,Q_2$.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

$Q[a] = a \cup (Sa \cap aS)$ and $I[a] = a \cup Sa \cup aS$ are principle quasi-ideal and principle ideal of S generated by a respectively. Thus by (iii), left invertive law, medial law and we have,

$$\begin{aligned} & (a \cup (Sa \cap aS)) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\ \subseteq\ & ((a \cup (Sa \cap aS))(a \cup Sa \cup aS)) \\ & (a \cup (Sa \cap aS)) \\ \subseteq\ & \{(a \cup aS) S\} (a \cap aS) \\ =\ & \{aS \cup (aS) S\} (a \cap aS) \\ =\ & (aS \cup Sa)(a \cap aS) \\ =\ & \{(aS) a \cup (aS)(aS) \cup (Sa) a \cup (Sa) aSa \\ =\ & (aS) a \cup a^2 S \cup a (aS). \end{aligned}$$

Hence S is regular. ■

Theorem 339 *For an AG-groupoid S with left identity, the following are equivalent.*
(i) *S is regular.*
(ii)*For principle bi-ideal $B[a]$, $B[a] = (B[a]S) B[a]$.*
(iii)*For bi-ideal B, $B = (BS) B$.*
(iv) *For generalized bi-ideal B, $B = (BS) B$.*
(v) *For $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f, of S $f =_{(\gamma,\delta)} (f \circ S) \circ f$.*
(vi) *For $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, of S, $f =_{(\gamma,\delta)} (f \circ S) \circ f$.*

Proof. $(i) \Rightarrow (vi)$
Assume that f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx) b$. Therefore we have,

$$\begin{aligned} & \max\{((f \circ S) \circ f)(b), \gamma\} \\ =\ & \max\left\{ \bigvee_{b=xy} \{(f \circ S)(x) \wedge f(y)\}, \gamma \right\} \\ \geq\ & \max\{(f \circ s)(bx) \wedge f(b), \gamma\} \\ =\ & \max\{ \bigvee_{bx=uv} (f(u) \wedge S(v)) \wedge f(b), \gamma\} \\ \geq\ & \max\{f(b) \wedge S(x) \wedge f(b), \gamma\} \\ =\ & \min\{\max\{f(b), \gamma\}, 1, \max\{f(b), \gamma\}\} \\ \geq\ & \min\{\min\{f(b), \delta\}, 1, \min\{f(b), \delta\}\} \\ =\ & \min\{\min\{f(b) \wedge 1 \wedge f(b), \delta\} \\ =\ & \min\{f(b), \delta\} \end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)} (f \circ S) \circ f$. Since f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. So we have $(f \circ S) \circ f \subseteq \vee q_{(\gamma,\delta)} f$.

Hence $f =_{(\gamma,\delta)} (f \circ S) \circ f$.

$(vi) \Rightarrow (v)$ is obvious.

$(v) \Rightarrow (iv)$

Assume that B is a bi-ideal of S. Then $\chi^{\delta}_{\gamma B}$, is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Therefore we have,

$$\begin{aligned} \chi^{\delta}_{\gamma B} &= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma B} \odot \chi^{\delta}_{\gamma S}) \odot \chi^{\delta}_{\gamma B} \\ &= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma BS}) \odot \chi^{\delta}_{\gamma B} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma (BS)B}. \end{aligned}$$

Therefore $B = (BS)\, B$.

$(iv) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$

Since $B\,[a] = a \cup a^2 \cup (aS)\, a$ is a principle bi-ideal of S generated by a respectively. Thus by (ii), we have,

$$\begin{aligned} & a \cup a^2 \cup (aS)\, a \\ =\ & [\{a \cup a^2 \cup (aS)\, a\} S] \left(a \cup a^2 \cup (aS)\, a\right) \\ =\ & [aS \cup a^2 S \cup \{(aS)\, a\} S] \left(a \cup a^2 \cup (aS)\, a\right) \\ =\ & \left(aS \cup a^2 S \cup a\,(aS)\right) \left(a \cup a^2 \cup (aS)\, a\right) \\ =\ & (aS)\, a \cup (aS)\, a^2 \cup (aS)\, ((aS)\, a) \\ & \cup \left(a^2 S\right) a \cup \left(a^2 S\right) a^2 \cup \left(a^2 S\right) ((aS)\, a) \\ & \cup (a\,(aS))\, a \cup (a\,(aS))\, a^2 \cup (a\,(aS))\, ((aS)\, a) \\ =\ & (aS)\, a \cup a^2 S \cup (aS)\, a \cup a^2 S \cup a^2 S \cup a^2 S \\ & \cup (aS)\, a \cup (aS)\, a \cup (aS)\, a \\ =\ & a^2 S \cup (aS)\, a. \end{aligned}$$

Hence S is regular ■

Theorem 340 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *$B[a] \cap Q\,[a] \subseteq (B\,[a]\, S)\, Q\,[a]$, for some a in S.*

(iii) *For bi-ideal B and quasi-ideal Q of S, $B \cap Q \subseteq (BS)\, Q$.*

(iv) *$f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ S) \circ g$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.*

(v) *$f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ S) \circ g$. for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.*

Proof. $(i) \Rightarrow (v)$

Assume that f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and also using law $a(bc) = b(ac)$, we have, $a = (ax)\, a = [\{(ax)\, a\} x] a$.

Thus,

$$\begin{aligned}
& \max\{((f\circ S)\circ g)(a),\gamma\} \\
= & \max\left\{\bigvee_{a=bc}\{(f\circ S)(b)\wedge g(c)\},\gamma\right\} \\
\geq & \max\{(f\circ S)[\{(ax)a\}x]\wedge g(a),\gamma\} \\
= & \max\{\bigvee_{\{(ax)a\}x=pq}(f(p)\wedge S(q))\wedge g(a),\gamma\} \\
\geq & \max\{f((ax)a)\wedge S(x)\wedge g(a),\gamma\} \\
= & \min\{\max\{f((ax)a),\gamma\},1,\max\{g(a),\gamma\}\} \\
\geq & \min\{\min\{f(a),\delta\},1,\min\{g(a),\delta\}\} \\
= & \min\{\min\{f(a)\wedge 1\wedge g(a),\delta\} \\
= & \min\{[f\cap g](a),\delta\}
\end{aligned}$$

Thus $f\cap g\subseteq\vee q_{(\gamma,\delta)}(f\circ S)\circ g$

$(v)\Rightarrow(iv)$ is obvious.

$(iv)\Rightarrow(iii)$

Assume that B and Q are bi-ideal and quasi-ideal of regular AG-groupioud of S respectively. Then $\chi^{\delta}_{\gamma B}$ and $\chi^{\delta}_{\gamma Q}$ are $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy bi-ideal and $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy quasi-ideal of S respectively. Therefore we have,

$$\begin{aligned}
\chi^{\delta}_{\gamma(B\cap Q)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma B}\cap\chi^{\delta}_{\gamma S}\cap\chi^{\delta}_{\gamma Q}\subseteq\vee q_{(\gamma,\delta)}(\chi^{\delta}_{\gamma B}\odot\chi^{\delta}_{\gamma S})\odot\chi^{\delta}_{\gamma Q} \\
&= {}_{(\gamma,\delta)}(\chi^{\delta}_{\gamma BS})\odot\chi^{\delta}_{\gamma Q}={}_{(\gamma,\delta)}\ \chi^{\delta}_{\gamma(BS)Q}.
\end{aligned}$$

Therefore $B\cap Q\subseteq(BS)Q$.

$(iii)\Rightarrow(ii)$ is obvious.

$(ii)\Rightarrow(i)$

Since $B[a]=a\cup a^2\cup(aS)a$ and $Q[a]=a\cup(Sa\cap aS)$ are principle bi-ideal and principle quasi-ideal of S generated by a respectively. Thus by (ii), law $a(bc)=b(ac)$, medial law and left invertive law we have,

$$\begin{aligned}
& \{a\cup a^2\cup(aS)a\}\cap\{a\cup(Sa\cap aS)\} \\
\subseteq & \{\big(a\cup a^2\cup(aS)a\big)S\}(a\cup(Sa\cap aS)) \\
\subseteq & \{aS\cup a^2S\cup((aS)a)S\}(a\cup Sa) \\
= & \{aS\cup a^2S\cup(Sa)(aS)\}(a\cup Sa) \\
= & \{(aS)a\cup(a^2S)a\cup\{(Sa)(aS)\}a\cup(aS)(Sa) \\
& \cup(a^2S)(Sa)\cup(Sa)(aS)(Sa)\} \\
\subseteq & (aS)a\cup Sa^2\cup(aS)a\cup(aS)a\cup Sa^2\cup(aS)a \\
= & (aS)a\cup a^2S
\end{aligned}$$

Hence S is regular. ■

9.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Quasi-ideals of AG-groupoids

The following is an example of generalized fuzzy quasi-ideal in an AG-groupoid.

Example 341 *Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	3	3	2
2	2	2	2
3	2	2	2

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.21 & \text{if } x = 1 \\ 0.23 & \text{if } x = 2 \\ 0.24 & \text{if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.23})$-fuzzy left ideal,*
- *f is not an $(\in, \in \vee q_{0.23})$-fuzzy left ideal.*

Definition 342 *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if it satisfies $y_t \in_\gamma f, (xy)_t \in_\gamma \vee q_\delta f$ ($x_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$), for all $t, s \in (0, 1]$, and $\gamma, \delta \in [0, 1]$.*

Theorem 343 *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt. right) ideal if and only if $\max\{f(ab), \gamma\} \geq \min\{f(b), \delta\}$, (respt. $\max\{f(ab), \gamma\} \geq \min\{f(a), \delta\}$) for all $a, b \in S$.*

Lemma 344 *Every intra regular AG-groupoid S is regular.*

Proof. It is easy. ■

Lemma 345 *In an AG-groupoid with left identity S the following holds*
(i) $(aS)(Sa) = (aS)a$, for all a in S,
(ii) $\{(Sa)(aS)\}(Sa) \subseteq (aS)a$, for all a in S.

Proof. (i) Using left invertive law, paramedial law, medial law and 1 we get

$$(aS)(Sa) = \{(Sa)S\}a = \{(Sa)(SS)\}a = \{(SS)(aS)\}a = \{S(aS)\}a = (aS)a.$$

(ii) now using paramedial and medial laws, and using (i) of this lemma we get:

$$\begin{aligned}\{(Sa)(aS)\}(Sa) &= (aS)\{(aS)(Sa)\} = (aS)[\{(Sa)S\}a] \\ &\subseteq (aS)(Sa) \subseteq (aS)a.\end{aligned}$$

■

Lemma 346 *In an AG-groupoid with left identity S the following holds*
(i) $a^2S = (Sa^2)S$, for all a in S,
(ii) $Sa^2 = (Sa^2)S$, for all a in S.

Proof. (i) Using (1) we get

$$a^2S = a^2(SS) = S(a^2S).$$

(ii)

$$Sa^2 = (SS)a^2 = (a^2S)S = \{(a^2)(SS)\}S = \{(SS)(a^2)\}S = (Sa^2)S.$$

■

Lemma 347 *A subset I of an AG-groupoid is left (bi, quasi, two sided) ideal if and only if $\mathcal{X}^{\delta}_{\gamma I}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy left(bi, quasi, two sided) ideal.*

Proof. It is easy. ■

Theorem 348 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $B[a] \cap L[a] \subseteq (B[a]S)L[a]$, for all a in S,
(iii) $B \cap L \subseteq (BS)L$, where B and L are bi and left ideals of S,
(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g$, where f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi and left ideals of S.

Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$. Then using paramedial and medial laws, we get

$$a = (ax)a = (ax)[(ax)a] = [a(ax)](xa).$$

$$
\begin{aligned}
& \max\{(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g(a), \gamma\} \\
= & \max\left[\bigvee_{a=bc}\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})(b) \wedge g(c)\right\}, \gamma\right] \\
\geq & \max\left[(f \circ \mathcal{X}^{\delta}_{\gamma S})(a(ax)) \wedge g(xa), \gamma\right] \\
= & \max\left[\min\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})(a(ax)), g(xa)\right\}, \gamma\right] \\
= & \max\left[\min\left\{\bigvee_{a(ax)=pq}\{(f(p) \wedge \mathcal{X}^{\delta}_{\gamma S}(q)\}, g(xa)\right\}, \gamma\right] \\
\geq & \max\left[\min\left\{(f(a), \mathcal{X}^{\delta}_{\gamma S}(ax), g(xa)\right\}, \gamma\right] \\
= & \max\left[\min\left\{(f(a), 1, g(xa)\right\}, \gamma\right] \\
= & \max\left[\min\left\{(f(a), g(xa)\right\}, \gamma\right] \\
= & \min\left[\max\{f(a), \gamma\}, \max\{g(xa), \gamma\}\right] \\
\geq & \min\left[\min\{f(a), \delta\}, \min\{g(a), \delta\}\right] \\
= & \min\{(f \cap g)(a), \delta\}.
\end{aligned}
$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g$.

$(iv) \Longrightarrow (iii)$ Let B and L are bi and left ideals of S. Then $\mathcal{X}^{\delta}_{\gamma B}$ and $\mathcal{X}^{\delta}_{\gamma L}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi and left ideals of S. Now by (iv)

$$
\begin{aligned}
\mathcal{X}^{\delta}_{\gamma B \cap L} &= \mathcal{X}^{\delta}_{\gamma B} \cap \mathcal{X}^{\delta}_{\gamma L} \subseteq \vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma B} \circ \mathcal{X}^{\delta}_{\gamma S}) \circ \mathcal{X}^{\delta}_{\gamma L} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma BS}) \circ \mathcal{X}^{\delta}_{\gamma L} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma (BS)L}.
\end{aligned}
$$

Thus $B \cap L \subseteq (BS)L$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$ Using left invertive law, paramedial law, medial law, we get

$$
\begin{aligned}
a &\in [a \cup a^2 \cup (aS)a] \cap (a \cup Sa) \subseteq [\{a \cup a^2 \cup (aS)a\}S](a \cup Sa) \\
&= [aS \cup a^2 S \cup \{(aS)a\}S](a \cup Sa) \\
&= (aS)a \cup (aS)(Sa) \cup (a^2 S)a \cup (a^2 S)(Sa) \\
& \quad \cup [\{(aS)a\}S]a \cup [\{(aS)a\}S](Sa) \\
&\subseteq (aS)a \cup (aS)(Sa) \cup Sa^2 \cup (aS)\{(aS)a\} \\
& \quad \cup \{(Sa)(aS)\}(Sa) \\
&\subseteq (aS)a \cup (Sa^2)S.
\end{aligned}
$$

Hence S is regular. ■

Theorem 349 *If S is an AG-groupoid with left identity then the following are equivalent*

(i) *S is regular,*

(ii) $I[a] \cap L[a] \subseteq (I[a]S)L[a]$ *for all* a *in* S,
(iii) $I \cap L \subseteq (IS)L$ *for ideal* I *and left ideal* L,
(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}_{\gamma S}^{\delta}) \circ g$, *where* f *and* g *are* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy ideal and left ideals of* S.

Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$. Then using paramedial and medial laws, we get

$$a = (ax)a = (ax)(ea) = (ae)(xa).$$

Then

$$\begin{aligned}
& \max\{(f \circ \mathcal{X}_{\gamma S}^{\delta}) \circ g(a), \gamma\} \\
= \;& \max\left[\bigvee_{a=bc} \left\{(f \circ \mathcal{X}_{\gamma S}^{\delta})(b) \wedge g(c)\right\}, \gamma\right] \\
= \;& \max\left[\bigvee_{a=bc} \left\{(f \circ \mathcal{X}_{\gamma S}^{\delta})(ae) \wedge g(xa)\right\}, \gamma\right] \\
\geq \;& \max\left[\left\{(f \circ \mathcal{X}_{\gamma S}^{\delta})(ae) \wedge g(xa)\right\}, \gamma\right] \\
= \;& \max\left[\min\left\{(f \circ \mathcal{X}_{\gamma S}^{\delta})(ae), g(xa)\right\}, \gamma\right] \\
= \;& \max\left[\min\left\{(\bigvee_{ae=pq} f(p) \wedge \mathcal{X}_{\gamma S}^{\delta}(q), g(xa)\right\}, \gamma\right] \\
\geq \;& \max\left[\min\left\{(f(a), \mathcal{X}_{\gamma S}^{\delta}(e), g(xa)\right\}, \gamma\right] \\
= \;& \max\left[\min\left\{(f(a), 1, g(xa)\right\}, \gamma\right] \\
= \;& \max\left[\min\left\{(f(a), g(xa)\right\}, \gamma\right] \\
= \;& \min\left[\max\{f(a), \gamma\}, \max\{g(xa), \gamma\}\right] \\
\geq \;& \min\left[\min\{f(a), \delta\}, \min\{g(a), \delta\}\right] \\
= \;& \min\{(f \cap g)(a), \delta\}.
\end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}_{\gamma S}^{\delta}) \circ g$.
$(iv) \Longrightarrow (iii)$ Let I and L are ideal and left ideal of S respectively. Then $\mathcal{X}_{\gamma I}^{\delta}$ and $\mathcal{X}_{\gamma L}^{\delta}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and left ideal of S respectively. Now by (iv)

$$\begin{aligned}
\mathcal{X}_{\gamma I \cap L}^{\delta} &= \mathcal{X}_{\gamma I}^{\delta} \cap \mathcal{X}_{\gamma L}^{\delta} \subseteq \vee q_{(\gamma,\delta)}(\mathcal{X}_{\gamma I}^{\delta} \circ \mathcal{X}_{\gamma S}^{\delta}) \circ \mathcal{X}_{\gamma L}^{\delta} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)}(\mathcal{X}_{\gamma IS}^{\delta}) \circ \mathcal{X}_{\gamma L}^{\delta} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} \mathcal{X}_{\gamma (IS)L}^{\delta}.
\end{aligned}$$

Thus $I \cap L \subseteq (IS)L$.
$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$ Using left invertive law, paramedial law, medial law, we get

$$\begin{aligned} a &\in (a \cup aS \cup Sa) \cap (a \cup Sa) \subseteq \{(a \cup aS \cup Sa)S\}(a \cup Sa) \\ &= \{aS \cup (aS)S \cup (Sa)S\}(a \cup Sa) \\ &\subseteq \{aS \cup Sa \cup (Sa)S\}(a \cup Sa) \\ &= (aS)a \cup (Sa)a \cup \{(Sa)S\}a \cup (aS)(Sa) \\ &\quad \cup (Sa)(Sa) \cup \{(Sa)S\}(Sa) \\ &\subseteq (aS)a \cup (Sa^2)S. \end{aligned}$$

Hence S is regular. ■

Theorem 350 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $B[a] \subseteq (B[a]S)(SB[a])$ for all a in S,
(iii) $B \subseteq (BS)(SB)$, where B is bi-ideal,
(iv) $f \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ (\mathcal{X}^{\delta}_{\gamma S} \circ f)$, where f is fuzzy bi-ideal.
Proof. *$(i) \Rightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$. then using medial law we get*

$$\begin{aligned} a &= (ax)a = [\{(ax)a\}x]a = [\{(ax)a\}x](ea) \\ &= [\{(ax)a\}e](xa) = [\{(ax)a\}e][x\{(ax)a\}]. \\ &\quad \max[\{(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ (\mathcal{X}^{\delta}_{\gamma S} \circ f)\}(a), \gamma] \\ &= \max\left[\bigvee_{a=bc} \left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})(b) \wedge (\mathcal{X}^{\delta}_{\gamma S} \circ f)(c)\right\}, \gamma\right] \\ &\geq \max\left[\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})[\{(ax)a\}e] \wedge (\mathcal{X}^{\delta}_{\gamma S} \circ f)[x\{(ax)a\}]\right\}, \gamma\right] \\ &= \max\left[\min\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})[\{(ax)a\}e], (\mathcal{X}^{\delta}_{\gamma S} \circ f)[x\{(ax)a\}]\right\}, \gamma\right] \\ &\geq \max\left[\min\left\{\min\{f\{(ax)a\}, \mathcal{X}^{\delta}_{\gamma S}(e)\}, \min\{\mathcal{X}^{\delta}_{\gamma S}(x), f((ax)a)\}\right\}, \gamma\right] \\ &= \max\left[\min\left\{\min\{f\{(ax)a\}, 1\}, \min\{1, f\{(ax)a\}\}\right\}, \gamma\right] \\ &= \max[\min\{f((ax)a), f((ax)a)\}, \gamma] \\ &= \min[\max\{f((ax)a), \gamma\}, \max\{f((ax)a, \gamma)\}] \\ &\geq \min[\min\{f(a), \delta\}, \min\{f((a), \delta\} \\ &= \min\{f(a), \delta\}. \end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)}\{(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ (\mathcal{X}^{\delta}_{\gamma S} \circ f)\}$.
$(iv) \Longrightarrow (iii)$ Let B be bi-ideal of S. Then $\mathcal{X}^{\delta}_{\gamma B}$ $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of S. Now by (iv)

$$\mathcal{X}^{\delta}_{\gamma B} \subseteq \vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma B} \circ \mathcal{X}^{\delta}_{\gamma S})(\mathcal{X}^{\delta}_{\gamma S} \circ \mathcal{X}^{\delta}_{\gamma B}) = \vee q_{(\gamma,\delta)}\mathcal{X}^{\delta}_{\gamma (BS)(SB)}.$$

Thus $B \subseteq (BS)(SB)$.

$(iii) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$ Using left invertive law, we get

$$\begin{aligned}
[a \cup a^2 \cup (aS)a] &\subseteq [\{a \cup a^2 \cup (aS)a\}S][S\{a \cup a^2 \cup (aS)a\}] \\
&= [aS \cup a^2S \cup \{(aS)a\}S][Sa \cup Sa^2 \cup \{S((aS)a)\}] \\
&= [aS \cup a^2S \cup (Sa)(aS)][Sa \cup Sa^2 \cup (aS)(Sa)] \\
&\subseteq (aS)a.
\end{aligned}$$

Hence S is regular. ■

Theorem 351 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $L[a] \cap B[a] \subseteq (L[a]S)B[a]$ for all a in S,
(iii) $L \cap B \subseteq (LS)B$ for left ideal L and bi-ideal B,
(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g$, where f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left and bi-ideals of S respectively.
Proof. *$(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$.*

$$\begin{aligned}
a &= (ax)a = (ax)\{(ax)a\}. \\
& \max\{(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g(a), \gamma\} \\
&= \max\left[\bigvee_{a=bc}\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})(b) \wedge g(c)\right\}, \gamma\right] \\
& \max\{(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g(a), \gamma\} \\
&\geq \max\left[(f \circ \mathcal{X}^{\delta}_{\gamma S})(ax) \wedge g((ax)a), \gamma\right] \\
&= \max\left[\min\left\{(f \circ \mathcal{X}^{\delta}_{\gamma S})(ax), g((ax)a)\right\}, \gamma\right] \\
&= \max\left[\min\left\{\bigvee_{a(ax)=pq}\{(f(p) \wedge \mathcal{X}^{\delta}_{\gamma S}(q)\}, g((ax)a)\right\}, \gamma\right] \\
&\geq \max\left[\min\left\{\min\{(f(a), \mathcal{X}^{\delta}_{\gamma S}(x)\}, g((ax)a)\right\}, \gamma\right] \\
&= \max\left[\min\left\{\min\{(f(a), 1\}, g((ax)\,a)\right\}, \gamma\right] \\
&= \max\left[\min\left\{(f(a), g((ax)a)\right\}, \gamma\right] \\
&= \min\left[\max\{f(a), \gamma\}, \max\{g((ax)\,a), \gamma\}\right] \\
&\geq \min\left[\min\{f(a), \delta\}, \min\{g(a), \delta\}\right] \\
&= \min\{(f \cap g)(a), \delta\}.
\end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ \mathcal{X}^{\delta}_{\gamma S}) \circ g$.

$(iv) \Longrightarrow (iii)$ Let L and B are ideal and left ideal of S respectively. Then $\mathcal{X}^{\delta}_{\gamma L}$ and $\mathcal{X}^{\delta}_{\gamma B}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal and bi-ideal of S respectively.

Now by (iv)

$$\begin{aligned}\mathcal{X}^{\delta}_{\gamma L\cap B} &= \mathcal{X}^{\delta}_{\gamma L}\cap\mathcal{X}^{\delta}_{\gamma B}\subseteq\vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma L}\circ\mathcal{X}^{\delta}_{\gamma S})\circ\mathcal{X}^{\delta}_{\gamma B}\\ &= {}_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}(\mathcal{X}^{\delta}_{\gamma LS})\circ\mathcal{X}^{\delta}_{\gamma B} ={}_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}\mathcal{X}^{\delta}_{\gamma(LS)B}.\end{aligned}$$

Thus $L\cap B\subseteq(LS)B$.
$(iii)\Longrightarrow(ii)$ *is obvious.*
$(ii)\Longrightarrow(i)$ *Using left invertive law, paramedial law, medial law, we get*

$$\begin{aligned}a &\in (a\cup Sa)\cap\{a\cup a^2\cup(aS)a\}\subseteq\{(a\cup Sa)S\}\{a\cup a^2\cup(aS)a\}\\ &= (aS)a\cup(aS)a^2\cup(aS)\{(aS)a\}\cup\{(Sa)S\}a\\ &\quad\cup\{(Sa)S\}a^2\cup\{(Sa)S\}\{(aS)a\}\\ &\subseteq (aS)a\cup(aS)a^2\cup(aS)\{(aS)S\}\cup(aS)(Sa)\\ &\quad\cup\{(Sa)(SS)\}(aa)\cup[\{(aS)a\}S](Sa)\\ &\subseteq (aS)a\cup(aS)a^2\cup(aS)\{(aS)S\}\cup(aS)(Sa)\\ &\quad\cup\{(Sa)(SS)\}(aa)\cup\{(Sa)(aS)\}(Sa)\\ &\subseteq a(Sa)\cup(Sa^2)S.\end{aligned}$$

Hence S *is regular.* ■

Theorem 352 *If* S *is an* AG*-groupoid with left identity then the following are equivalent*
(i) S *is regular,*
(ii) $L[a]\cap Q[a]\cap I[a]\subseteq(L[a]Q[a])I[a]$ *for all* a *in* S,
(iii) $L\cap Q\cap I\subseteq(LQ)I$ *for left ideal* L, *quasi-ideal* Q *and ideal* I *of* S,
(iv) $f\cap g\cap h\subseteq\vee q_{(\gamma,\delta)}(f\circ g)\circ h$, *where* f, g *and* h *are* $(\in_\gamma,\in_\gamma\vee q_\delta)$*-fuzzy left ideal, right ideal and ideal of* S.

Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$. Now using left invertive law we get

$$\begin{aligned}
a &= (ax)a = [\{(ax)a\}x]a = \{(xa)(ax)\}a. \\
& \max\{(f \circ g) \circ h(a), \gamma\} \\
&= \max\left[\bigvee_{a=bc} \{(f \circ g)(b) \wedge h(a)\}, \gamma\right] \\
&= \max\left[\bigvee_{a=bc} \{(f \circ g)((xa)(ax)) \wedge h(a)\}, \gamma\right] \\
&\geq \max\left[\min\{(f \circ g)((xa)(ax)), h(a)\}, \gamma\right] \\
&= \max\left[\min\left\{\bigvee_{\{(xa)(ax)\}=pq} f(p) \wedge g(q)), h(a)\right\}, \gamma\right] \\
&\geq \max\left[\min\{\min\{f(xa), g(ax)\}, h(a)\}, \gamma\right] \\
&= \max[\min\{f(xa), g(ax), h(a)\}, \gamma] \\
&= \min\left[\max\{f(xa), \gamma\}, \max\{g(ax), \gamma\}, \max\{h(a), \gamma\}\right] \\
&\geq \min\left[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\right] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

Thus $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(iv) \Longrightarrow (iii)$ Let L , J and I are left ideal and right ideal and ideal of S respectively. Then $\mathcal{X}_{\gamma L}^{\delta}$, $\mathcal{X}_{\gamma J}^{\delta}$ and $\mathcal{X}_{\gamma I}^{\delta}$are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal , right ideal and ideal of S respectively. Now by (iv)

$$\begin{aligned}
\mathcal{X}_{\gamma L\cap J\cap I}^{\delta} &= \mathcal{X}_{\gamma L}^{\delta} \cap \mathcal{X}_{\gamma J}^{\delta} \cap \mathcal{X}_{\gamma I}^{\delta} \subseteq \vee q_{(\gamma,\delta)}(\mathcal{X}_{\gamma L}^{\delta} \circ \mathcal{X}_{\gamma J}^{\delta}) \circ \mathcal{X}_{\gamma I}^{\delta} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)}(\mathcal{X}_{\gamma LJ}^{\delta}) \circ \mathcal{X}_{\gamma I}^{\delta} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)}\mathcal{X}_{\gamma(LJ)I}^{\delta}.
\end{aligned}$$

Thus $L \cap J \cap I \subseteq (LJ)I$. Hence $L \cap Q \cap I \subseteq (LQ)I$, where Q is a quasi-ideal.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$ Using left invertive law, paramedial law, medial law, we get

$$\begin{aligned}
a &\in (a \cup Sa) \cap [a \cup \{(Sa) \cap (aS)\}] \cap (a \cup Sa \cup aS) \\
&\subseteq [(a \cup Sa)\{a \cup \{(Sa) \cap (aS)\}\}](a \cup Sa \cup aS) \\
&= \{(a \cup Sa)(a \cup Sa)\}(a \cup Sa \cup aS) \\
&= \{a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa)\}(a \cup Sa \cup aS) \\
&= (a^2)(a) \cup (a^2)(Sa) \cup (a^2)(aS) \cup \{a(Sa)\}a \cup \{a(Sa)\}(Sa) \\
&\quad \cup\{a(Sa)\}(aS) \cup \{(Sa)a\}a \cup \{(Sa)a\}(Sa) \cup \{Sa)a\}(aS) \\
&\quad \cup\{(Sa)(Sa)\}a \cup \{(Sa)(Sa)\}(Sa) \cup \{(Sa)(Sa)\}(aS) \\
&\subseteq (Sa^2)S \cup (Sa)S.
\end{aligned}$$

Hence S is regular. ■

Theorem 353 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $I[a] \cap B[a] \subseteq I[a](I[a]B[a])$ for all a in S,
(iii) $I \cap B \subseteq I(IB)$ for ideal I and bi-ideal B,

(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ (f \circ g)$, where f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and bi-ideal of S.
Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$.

$$a = (ax)a = (ax)\{(ax)a\} = (ax)[(ax)\{(ax)a\}].$$

$$\begin{aligned}
& \max\{f \circ (f \circ g)(a), \gamma\} \\
= \; & \max\left[\left\{\bigvee_{a=bc} f(b) \wedge f \circ g(c)\right\}, \gamma\right] \\
\geq \; & \max\left[\{f(ax) \wedge f \circ g(a)\}, \gamma\right] \\
= \; & \max\left[\min\{f(ax), f \circ g(a)\}, \gamma\right] \\
= \; & \max\left[\min\left\{f(ax), \left\{\bigvee_{a=pq} f(p) \wedge g(q)\right\}\right\}, \gamma\right] \\
\geq \; & \max\left[\min\{f(ax), \{f(p) \wedge g(q)\}\}, \gamma\right] \\
= \; & \max\left[\min\{f(ax), \min\{f(ax), g((ax)a)\}\}, \gamma\right] \\
= \; & \min\left[\max\{f(ax), \gamma\}, \max\{f(ax), \gamma\}, \max\{g((ax)a), \gamma\}\right] \\
\geq \; & \min\left[\min\{f(a), \delta\}, \min\{f(a), \delta\}, \min\{g(a), \delta\}\right] \\
= \; & \min\{f \cap g, \delta\}.
\end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ (f \circ g)$.
$(iv) \Longrightarrow (iii)$ Let I and B are ideal and bi-ideal of S respectively. Then $\mathcal{X}^{\delta}_{\gamma I}$ and $\mathcal{X}^{\delta}_{\gamma B}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal and bi-ideal of S respectively. Now by (iv)

$$\begin{aligned}
\mathcal{X}^{\delta}_{\gamma I \cap B} &= \mathcal{X}^{\delta}_{\gamma I} \cap \mathcal{X}^{\delta}_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} (\mathcal{X}^{\delta}_{\gamma I}) \circ (\mathcal{X}^{\delta}_{\gamma I} \circ \mathcal{X}^{\delta}_{\gamma B}) \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} (\mathcal{X}^{\delta}_{\gamma I}) \circ \mathcal{X}^{\delta}_{\gamma IB} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma I(IB)}.
\end{aligned}$$

Thus $I \cap B \subseteq I(IB)$.
$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$Using left invertive law, paramedial law, medial law, we get

$$\begin{aligned}
& (a \cup Sa \cup aS) \cap \{a \cup a^2 \cup (aS)a\} \\
\subseteq \;& (a \cup Sa \cup aS)[(a \cup Sa \cup aS)\{a \cup a^2 \cup (aS)a\}] \\
= \;& (a \cup Sa \cup aS)[S\{a \cup a^2 \cup (aS)a\}] \\
= \;& (a \cup Sa \cup aS)\{Sa \cup Sa^2 \cup S((aS)a)\} \\
= \;& a(Sa) \cup a(Sa^2) \cup a[S\{(aS)a\}] \cup (Sa)(Sa) \\
& \cup (Sa)(Sa^2) \cup (Sa)[S\{(aS)a\}] \cup (aS)(Sa) \\
& \cup (aS)(Sa^2)(aS)[S\{(aS)a\}] \\
\subseteq \;& (aS)a \cup (Sa^2)S.
\end{aligned}$$

Hence S is regular. ■

Theorem 354 *If S is an AG-groupoid with left identity then the following are equivalent*

(i) *S is regular,*

(ii) *$L[a] \subseteq \{L[a](L[a]S)\}L[a]$ for all a in S,*

(iii) *$L \subseteq \{L(LS)\}L$ for left ideal L of S,*

(iv) *$f \subseteq \vee q_{(\gamma,\delta)}\{f \circ (f \circ \mathcal{X}^{\delta}_{\gamma S})\} \circ f$ where f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so there exists x in S such that $a = (ax)a$. now using left invertive law, Paramedial law, medial law and putting $xe = x', x'x = x"$ we get:

$$\begin{aligned}
a &= (ax)a = [\{(ax)a\}x]a = \{(xa)(ax)\}a \\
&= \{a(ax)\}(xa) = \{(ea)(ax)\}(xa) \\
&= \{(xa)(ae)\}(xa) = [\{(ae)a\}x](xa) \\
&= x[\{((ae)a)x\}a] = (ex)[\{((ae)a)x\}a] \\
&= [a\{((ae)a)x\}](xe) = [a\{((ae)a)x\}]x' \\
&= [\{(ae)a\}(ax)]x' = \{x'(ax)\}\{(ae)a\} \\
&= \{a(x'x)\}\{(ae)a\} = \{a(ae)\}\{(x'x)a\} \\
&= (x'x)[\{a(ae)\}a] = (x'x)[\{a(ae)\}(ea)] \\
&= (x'x)[(ae)\{(ae)a\}] = (ae)[(x'x)\{(ae)a\} \\
&= [\{(ae)a\}(x'x)](ea) = [\{(ae)a\}x"]a \\
&= [\{(x"a)(ae)\}a] = [\{(ea)(ax")\}a] \\
&= [\{a(ax")\}a]. \\
& \max\{\mathcal{X}_{\gamma S}^{\delta} \circ (f \circ f), \gamma\} \\
&= \max\left[\bigvee_{a=bc} \{f \circ (f \circ \mathcal{X}_{\gamma S}^{\delta})\}(b) \wedge f(c), \gamma\right] \\
&\geq \max\left[\{f \circ (f \circ \mathcal{X}_{\gamma S}^{\delta})\}\{a(ax")\} \wedge f(a), \gamma\right] \\
&= \max\left[\min\left\{\{f \circ (f \circ \mathcal{X}_{\gamma S}^{\delta})\}\{a(ax")\}\}, f(a)\right\}, \gamma\right] \\
&= \max[\min\left[[\{\bigvee_{\{a(ax")\}\}=pq} f(p) \wedge (f \circ \mathcal{X}_{\gamma S}^{\delta})(q)\}, f(a)], \gamma\right] \\
&\geq \max\left[\min[\min\{\{f(a), (f \circ \mathcal{X}_{\gamma S}^{\delta})(ax")\}, f(a)\}], \gamma\right] \\
&= \max\left[\min[\min\{\{f(a), (f \circ \mathcal{X}_{\gamma S}^{\delta})(ax")\}, f(a)\}], \gamma\right] \\
&= \max\left[\min[\min\{\{f(a), \{\bigvee_{ax"=st} \{f(s) \wedge \mathcal{X}_{\gamma S}^{\delta}(t)\}\}, f(a)\}], \gamma\right] \\
&\geq \max\left[\min[\min\{f(a), \{f(a) \wedge \mathcal{X}_{\gamma S}^{\delta}(x")\}, f(a)\}], \gamma\right] \\
&= \max\left[\min[\min\{f(a), \min\{f(a), \mathcal{X}_{\gamma S}^{\delta}(x")\}, f(a)\}], \gamma\right] \\
&= \max[\min[\min\{f(a), \min\{f(a), 1\}, f(a)\}], \gamma] \\
&= \max[\min[\min\{\{f(a), f(a), f(a)\}], \gamma] \\
&= \max[\min\{f(a), \gamma\}] \\
&= \min[\max\{f(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}].
\end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)}\{f \circ (f \circ \mathcal{X}_{\gamma S}^{\delta})\} \circ f$.

$(iv) \Longrightarrow (iii)$ Let L be left ideal of S. Then $\mathcal{X}^{\delta}_{\gamma L}$ $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left ideal of S. Now by (iv)

$$\mathcal{X}^{\delta}_{\gamma L} \subseteq \vee q_{(\gamma,\delta)}[\{\mathcal{X}^{\delta}_{\gamma L} \circ (\mathcal{X}^{\delta}_{\gamma L} \circ \mathcal{X}^{\delta}_{\gamma S})\} \circ \mathcal{X}^{\delta}_{\gamma L}] = \vee q_{(\gamma,\delta)} \mathcal{X}^{\delta}_{\gamma[\{L(LS)\}L]}.$$

Thus $L \subseteq [\{L(LS)\}L]$.
$(ii) \Longrightarrow (i)$ is obvious.

$$\begin{aligned} a &\in Sa \subseteq [(Sa)\{(Sa)S\}](Sa) \\ &= [\{S(Sa)\}(aS)](Sa) \\ &= a[\{S(Sa)\}S](Sa) \\ &= (aS)(Sa) \\ &\subseteq (aS)a. \end{aligned}$$

Hence S is regular. ■

Theorem 355 *If S is an AG-groupoid with left identity then the following are equivalent*
(i) S is regular,
(ii) $I[a] \cap B[a] \subseteq I[a](SB[a])$ for all a in S,
(iii) $I \cap B \subseteq I(SB)$ for left ideal I and bi-ideal B,
(iv) $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ (\mathcal{X}^{\delta}_{\gamma S} \circ g)$, where f and g are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy interior ideal and bi-ideal of S.

Proof. $(i) \Longrightarrow (iv)$ Let $a \in S$, then since S is regular so using left invertive law we get

$$\begin{aligned} a &= (ax)a = \{(ea)x\}a = \{(xa)e\}\{(ax)a\} \\ &= (ax)[\{(xa)e\}a] = (ax)\{(ae)(xa)\} \\ &= (ax)[x\{(ae)a\}]. \end{aligned}$$

$$
\begin{aligned}
& \max\{f \circ (\mathcal{X}_{\gamma S}^{\delta} \circ g), \gamma\} \\
= & \max\left[\left\{\bigvee_{a=pq} f(p) \wedge (\mathcal{X}_{\gamma S}^{\delta} \circ g)(q)\right\}, \gamma\right] \\
\geq & \max\left[f(ax) \wedge (\mathcal{X}_{\gamma S}^{\delta} \circ g)[x\{(ae)a\}], \gamma\right] \\
= & \max\left[\min\{f(ax), (\mathcal{X}_{\gamma S}^{\delta} \circ g)\{x((ae)a)\}\}, \gamma\right] \\
= & \max\left[\min\{f(ax), \{\bigvee_{x((ae)a)\}=st} (\mathcal{X}_{\gamma S}^{\delta}(s) \wedge g(t), \gamma\right] \\
\geq & \max\left[\min\{f(ax), \{(\mathcal{X}_{\gamma S}^{\delta}(x) \wedge g((ae)a)\}, \gamma\right] \\
= & \max\left[\{\min\{f(ax), \min\{(\mathcal{X}_{\gamma S}^{\delta}(x), g((ae)a)\}\}, \gamma\right] \\
= & \max\left[\{\min\{f(ax), \min\{1, g((ae)a)\}\}, \gamma\right] \\
= & \max\left[\{\min\{f(ax), g((ae)a)\}, \gamma\right] \\
= & \min\left[\max\{f(ax), \gamma\}, \max\{g((ae)a), \gamma\}\right] \\
\geq & \min[\min\{f(a), \delta\}, \min\{g(a), \delta\} \\
= & \min\{f \cap g(a), \delta\}.
\end{aligned}
$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ (\mathcal{X}_{\gamma S}^{\delta} \circ g)$.

$(iv) \Longrightarrow (iii)$ Let I and B are ideal and bi-ideal of S respectively. Then $\mathcal{X}_{\gamma I}^{\delta}$ and $\mathcal{X}_{\gamma B}^{\delta}$ are $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy ideal and bi-ideal of S respectively. Now by (iv)

$$
\begin{aligned}
\mathcal{X}_{\gamma I\cap B}^{\delta} &= \mathcal{X}_{\gamma I}^{\delta} \cap \mathcal{X}_{\gamma B}^{\delta} \subseteq \vee q_{(\gamma,\delta)} \mathcal{X}_{\gamma I}^{\delta} \circ (\mathcal{X}_{\gamma S} \circ \mathcal{X}_{\gamma B}^{\delta}) \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} \mathcal{X}_{\gamma\{I(SB)\}}^{\delta}.
\end{aligned}
$$

Thus $I \cap B \subseteq I(SB)$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$Using $\{S(Sa)\} \subseteq (Sa)$ and we get

$$
\begin{aligned}
a &\in (aS \cup Sa) \cap \{a \cup a^2 \cup (aS)a\} \\
&\subseteq (aS \cup Sa)[S\{a \cup a^2 \cup (aS)a\}] \\
&= (aS \cup Sa)[Sa \cup Sa^2 \cup S\{(aS)a\}] \\
&= (aS \cup Sa)\{Sa \cup Sa^2 \cup (aS)(Sa)\} \\
&= (aS)(Sa) \cup (aS)(Sa^2) \cup (aS)\{(aS)(Sa)\} \\
&\quad \cup (Sa)(Sa) \cup (Sa)(Sa^2) \cup (Sa)\{(aS)(Sa)\} \\
&\subseteq (aS)a.
\end{aligned}
$$

Hence S is regular. ■

10

On Fuzzy Soft Intra-regular Abel-Grassmann's Groupoids

In this chapter we characterize intra-regular AG-groupoids in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals.

Definition 356 *Let S be an AG-groupoid and U be an initial universe and let E be a set of parameters. A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of U.*

Generally, the soft set, i.e, a pair (F, A) with $A \subseteq B$ and $F : A \to P(S)$.

Definition 357 *Let (F, A) and (G, B) be soft sets over S, then (G, B) is called a soft subset of (F, A) if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in B$.*

Generally we write it as $(G, B)\tilde{\subseteq}(F, A)$. (F, A) is the soft supperset of (G, B), if (G, B) is a soft subset of (F, A).

Definition 358 *A soft set (F, A) over an AG-groupoid S is called a soft AG-groupoid over S if $(F, A) \odot (F, A) \subseteq (F, A)$.*

Definition 359 *A soft set (F, A) over an AG-groupoid S is called a soft left (right)ideal over S, $\Sigma(S, E) \odot (F, A) \subseteq (F, A)((F, A) \odot \Sigma(S, E) \subseteq (F, A))$.*

A soft set over S is a soft ideal if it is both a soft left and a soft right ideal over S.

Definition 360 *Let $V \subseteq U$. A fuzzy soft set $\langle F, A\rangle$ over V is said to be a relative whole (γ, δ)- fuzzy soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if $F(\varepsilon) = f^{\delta}_{\gamma V}$ for all $\varepsilon \in A$.*

Definition 361 *A new ordering relation is defined on $\mathcal{F}(X)$ denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.*

For any $\mu, \nu \in \mathcal{F}(X)$, by $\mu \subseteq \vee q_{(\gamma,\delta)}\nu$, we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta \nu$ for all $x \in X$ and $r \in (\gamma, 1]$.

Definition 362 *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that $\langle F, A\rangle$ is an (γ, δ)-fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle \subseteq_{(\gamma,\delta)} \langle G, B\rangle$ if*

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A, F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.

Definition 363 *For any fuzzy soft set $\langle F,A\rangle$ over an AG-groupiod S, $\epsilon \in A$ and $r \in (\gamma, 1]$, denote $F(\epsilon)_r = \{x \in S \,|x_r \in_\gamma F(\epsilon)\}$, $\langle F(\epsilon)\rangle_r = \{x \in S \,|x_r \in_\gamma q_\delta F(\epsilon)\}$, $[F(\epsilon)]_r = \{x \in S \,|x_r \in_\gamma \vee q_\delta F(\epsilon)\}$.*

Definition 364 *Suppose f be a fuzzy subset of an AG-groupoid S, $A \in [0,1]$. Define the map $F : A \longrightarrow P(S)$ as*

$F(\alpha) = \{x \in S : \ f(x) \geq \alpha\}$ for all $\alpha \in A$.

Indeed $F(\alpha)$ is parameterized family of α-level subsets, corresponding to f. Therefore (F, A) is a soft set over S.

We also define another map, $F_q : A \longrightarrow P(S)$ as follows

$F_q(\alpha) = \{x \in S : \ f(x) + \alpha > 1\}$ for all $\alpha \in A$. Then (F_q, A) is a soft set over S.

Define a map $F^* : A \longrightarrow P(S)$ as follows

$F^*(\alpha) = \{x \in S : \ f(x) > \alpha\}$ for all $\alpha \in A$. Therefore (F^*, A) is a soft set over S.

Example 365 *Let $S = \{a, b, c, d\}$ and the binary operation "$\cdot$" defines on S as follows:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	d	d	c
c	a	d	d	d
d	a	d	d	d

Then $(S, \cdot)$ is an AG-groupoid. Let $E = \{0.3, 0.4\}$ and define a fuzzy soft set $\langle G, A\rangle$ over S as follows:

$$G(\epsilon)(x) = \begin{cases} 2\epsilon \text{ if } x \in \{a,b\} \\ \frac{1}{2} \text{ otherwise} \end{cases}$$

Then $\langle G, A\rangle$ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.4})$- fuzzy soft left ideal of S.

Let $E = \{0.6, 0.7\}$ and define a fuzzy soft set $\langle F, A\rangle$ over S as follows:

$$F(\epsilon)(x) = \begin{cases} \epsilon \text{ if } x \in \{a,b\} \\ \frac{1}{2} \text{ otherwise} \end{cases}$$

Then $\langle F, A\rangle$ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$- fuzzy soft bi-ideal of S.

Theorem 366 *A fuzzy subset f of an AG-groupoid S is fuzzy interior ideal if and only if (F, A) is a soft interior ideal of S where $A = [0,1]$.*

Proof. Let f be a fuzzy interior ideal of S then for all $x, a, y \in S$, $f((xa)y) \geq f(a)$. Now let $a \in F(\alpha)$ this implies that $\{a \in S : f(a) \geq \alpha\}$ for all $\alpha \in A$. This implies that $f(a) \geq \alpha$ implies that $f((xa)y) \geq f(a) \geq \alpha$ implies that $((xa)y) \in F(\alpha)$ implies that $F(\alpha)$ is an interior ideal implies that (F, A) is soft interior ideal.

Conversely, let (F, A) is soft interior ideal of S we show that f is fuzzy interior ideal of S. Let $f((xa)y) < f(a)$ for some $x, y, a \in S$ and choose

$\alpha \in A$ such that $f((xa)y) < \alpha \leq f(a)$ this implies that $a \in F(\alpha)$ but $(xa)y \notin F(\alpha)$ which is a contradiction. Hence f is fuzzy interior ideal of S. ■

Theorem 367 *A fuzzy subset f of a AG-groupoid S is fuzzy bi-ideal if and only if (F, A) is a soft bi-ideal of S where $A = [0, 1]$.*

Proof. Let f be a fuzzy bi-ideal of S then for all $x, y, z \in S$, $f((xy)z) \geq f(x) \wedge f(z)$. Now let $x, z \in F(\alpha)$ this implies that

$$\{x, y \in S : \ f(x) \geq \alpha, \ f(z) \geq \alpha\} \text{ for all } \alpha \in A.$$

This implies that $f(x) \geq \alpha$, $f(z) \geq \alpha$ implies that $f(x) \wedge f(z) \geq \alpha$ implies that $f((xy)z) \geq \alpha$ implies that $((xy)z) \in F(\alpha)$ implies that $F(\alpha)$ is an bi-ideal implies that (F, A) is soft bi-ideal over S.

Conversely, let (F, A) is soft bi-ideal of S. we show that f is fuzzy bi-ideal of S. Let $f((xy)z) < f(x) \wedge f(z)$ for some $x, y, z \in S$ and choose $\alpha \in A$ such that $f((xy)z) < \alpha \leq f(x) \wedge f(z)$ this implies that $x, z \in F(\alpha)$ but $(xy)z \notin F(\alpha)$ which is a contradiction. Hence f is fuzzy bi-ideal of S. ■

Theorem 368 *A fuzzy subset f of an AG-groupoid S is fuzzy interior ideal if and only if (F_q, A) is a soft interior ideal of S where $A = [0, 1]$.*

Proof. Let f be a fuzzy interior ideal of S then for all $x, a, y \in S$, $f((xa)y) \geq f(a)$. Now let $a \in F(\alpha)$ this implies that $\{a \in S : f(a) + \alpha > 1\}$ for all $\alpha \in A$. This implies that $f(a) + \alpha > 1$ implies that $f((xa)y) + \alpha > 1$ implies that $((xa)y) \in F_q(\alpha)$ implies that $F_q(\alpha)$ is an interior ideal implies that (F_q, A) is soft interior ideal.

Conversely, let (F_q, A) is soft interior ideal of S we show that f is fuzzy interior ideal of S. Let $f((xa)y) < f(a)$ for some $x, a, y \in S$ and choose $\alpha \in A$ such that $f((xa)y) < \alpha \leq f(a)$ this implies that $a \in F(\alpha)$ but $(xa)y \notin F(\alpha)$ which is a contradiction. Hence f is fuzzy interior ideal of S. ■

Theorem 369 *A fuzzy subset f of an AG-groupoid S is fuzzy bi-ideal if and only if (F_q, A) is a soft bi-ideal of S where $A = [0, 1]$.*

Proof. Let f be a fuzzy bi-ideal of S then for all $x, y, z \in S$, $f((xy)z) \geq f(x) \wedge f(z)$. Now let $x, z \in F_q(\alpha)$ this implies that

$$\{x, z \in S : f(x) + \alpha \geq 1, \ f(z) + \alpha \geq 1\} \text{ for all } \alpha \in A.$$

This implies that $f(x) + \alpha \geq 1$, $f(z) + \alpha \geq 1$ implies that $f(x) \wedge f(z) + \alpha \geq 1$ implies that $f((xy)z) + \alpha \geq 1$ implies that $((xy)z) \in F_q(\alpha)$ implies that $F_q(\alpha)$ is an bi-ideal implies that (F_q, A) is soft bi-ideal over S.

Conversely, let (F_q, A) is soft bi-ideal of S we show that f is fuzzy bi-ideal of S. Let $f((xy)z) < f(x) \wedge f(z)$ for some $x, y, z \in S$ and choose

$\alpha \in A$ such that $f((xy)z) + \alpha < 1 \leq f(x) \wedge f(z) + \alpha$ this implies that $x, z \in F(\alpha)$ but $(xy)z \notin F(\alpha)$ which is a contradiction. Hence f is fuzzy bi-ideal of S. ■

Theorem 370 *Let f be a fuzzy subset of an AG-groupoid S, then $(F, (0.5, 1])$ is a soft interior ideal if and only if* $\max\{f((xa)y), 0.5\} \geq f(a)$.

Proof. Let $(F, (0.5, 1])$ be a soft interior ideal over S, then $F(\alpha)$ is an interior ideal of S for each $\alpha \in (0.5, 1]$ such that $\max\{f((xa)y), 0.5\} < f(a)$. Choose an $\alpha \in (0.5, 1]$ such that $\max\{f((xa)y), 0.5\} < \alpha < f(a)$. Then $a \in F(\alpha)$ but $((xa)y) \notin F(\alpha)$ which is a contradiction,therefore $\max\{f((xa)y), 0.5\} \geq f(a)$.

Conversely, let $\max\{f((xa)y), 0.5\} \geq f(a)$ and $(F, (0.5, 1])$ be a soft set over S. Let $a \in F(\alpha)$,where $\alpha \in (0.5, 1]$. Then $\max\{f((xa)y), 0.5\} \geq f(a) \geq \alpha > 0.5$. So $((xa)y) \in F(\alpha)$. Therefore $F(\alpha)$ is an interior ideal of S. Hence $(F, (0.5, 1])$ is a soft interior ideal over S. ■

Theorem 371 *Let f be a fuzzy subset of an AG-groupoid S, then $(F, (0.5, 1])$ is a soft bi-ideal if and only if* $\max\{f((xy)z), 0.5\} \geq f(x) \wedge f(z)$.

Proof. Let $(F, (0.5, 1])$ be a soft bi-ideal over S, then $F(\alpha)$ is an bi-ideal of S for each $\alpha \in (0.5, 1]$ such that $\max\{f((xy)z), 0.5\} < f(x) \wedge f(z)$. Choose an $\alpha \in (0.5, 1]$ such that $\max\{f((xy)z), 0.5\} < \alpha < f(x) \wedge f(z)$. Then $x, z \in F(\alpha)$ but $((xy)z) \notin F(\alpha)$ which is a contradiction,therefore $\max\{f((xy)z), 0.5\} \geq f(x) \wedge f(z)$.

Conversely, let $\max\{f((xy)z), 0.5\} \geq f(x) \wedge f(z)$ and $(F, (0.5, 1])$ be a soft set over S. Let $x, z \in F(\alpha)$,where $\alpha \in (0.5, 1]$. Then $\max\{f((xy)z), 0.5\} \geq f(x) \wedge f(z) \geq \alpha > 0.5$. So $((xy)z) \in F(\alpha)$. Therefore $F(\alpha)$ is an bi-ideal of S. Hence $(F, (0.5, 1])$ is a soft bi-ideal over S. ■

Theorem 372 *A fuzzy subset f of an AG-groupoid S is $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if $(F, (0, 0.5])$ is a soft interior ideal over S.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy interior ideal of S, for $x, a, y \in S$, $f((xa)y) \geq f(a) \wedge 0.5$.Now let $a \in F(\alpha)$, then $f(a) \geq \alpha$ so $a_\alpha \in f$ this implies that $((xa)y)_\alpha \in \vee q f$ that is $f((xa)y) \geq \alpha$ or $f((xa)y) + \alpha > 1$. If $f((xa)y) \geq \alpha$ then $((xa)y) \in F(\alpha)$. If $f((xa)y) + \alpha > 1$ then $f((xa)y) > 1 - \alpha \geq \alpha$ because $\alpha \in (0, 0.5]$. So $((xa)y) \in F(\alpha)$ .Thus $F(\alpha)$ is an interior ideal of S for all $\alpha \in (0, 0.5]$. Consequently $(F, (0, 0.5])$ is a soft interior ideal over S.

Conversely, Suppose that $(F, (0, 0.5])$ is a soft interior ideal over S. Then $F(\alpha)$ is an interior ideal of S for all $\alpha \in (0, 0.5]$. We have to show that f is an $(\in, \in \vee q)$-fuzzy interior ideal of S. If possible let there exists some $x, a, y \in S$ such that $f((xa)y) < f(a) \wedge 0.5$. Choose an $\alpha \in (0, 0.5]$ such that $f((xa)y) < \alpha < f(a) \wedge 0.5$, this shows that $a \in F(\alpha)$ but $((xa)y) \notin F$. Which is a contradiction, Thus $f((xa)y) \geq f(a) \wedge 0.5$. Thus f is $(\in, \in \vee q)$-fuzzy interior ideal of S. ■

Theorem 373 *A fuzzy subset f of an AG-groupoid S is $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if $(F, (0, 0.5])$ is a soft bi-ideal over S.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S, for $x, y, z \in S$, $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5$.Now let $x, z \in F(\alpha)$, then $f(x) \geq \alpha$ and $f(z) \geq \alpha$ so $x_\alpha, z_\alpha \in f$ this implies that $((xy)z)_\alpha \in \vee qf$ that is $f((xy)z) \geq \alpha$ or $f((xy)z) + \alpha > 1$. If $f((xy)z) \geq \alpha$ then $(xy)z \in F(\alpha)$. If $f((xy)z) + \alpha > 1$ then $f((xy)z) > 1 - \alpha \geq \alpha$ because $\alpha \in (0, 0.5]$. So $(xy)z \in F(\alpha)$. Thus $F(\alpha)$ is bi-ideal of S for all $\alpha \in (0, 0.5]$. Consequently $(F, (0, 0.5])$ is a soft bi-ideal over S.

Conversely, assume that $(F, (0, 0.5])$ is a soft bi-ideal over S. Then $F(\alpha)$ is an bi-ideal of S for all $\alpha \in (0, 0.5]$. We have to show that f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. If possible let there exists some $x, y, z \in S$ such that $f((xy)z) < f(x) \wedge f(z) \wedge 0.5$. Choose an $\alpha \in (0, 0.5]$ such that$f((xy)z) < \alpha < f(x) \wedge f(z) \wedge 0.5$, this shows that $x, z \in F(\alpha)$ but $((xy)z) \notin F$. Which is a contradiction, thus $f((xy)z) \geq f(x) \wedge f(z) \wedge 0.5$. Thus f is $(\in, \in \vee q)$-fuzzy bi-ideal of S. ■

Theorem 374 *Let f be a fuzzy subset of an AG-groupoid S. Then f is a (q, q)-fuzzy interior ideal if and only if $(F_q, (0.5, 1])$ is a soft interior ideal over S.*

Proof. Let f be an (q, q)-fuzzy interior ideal of S and suppose that $a \in F_q(\alpha)$ where $\alpha \in (0.5, 1]$, then $f(a) + \alpha \geq 1$, that is $a_\alpha qf$. Then for each $x, y \in S$, $((xa)y)_\alpha qf$ That is $f((xa)y) + \alpha \geq 1$. Hence $((xa)y) \in F_q(\alpha)$. Thus $F_q(\alpha)$ is an interior ideal of S. Consequently $(F_q, (0.5, 1])$ is a soft interior ideal over S.

Conversely suppose that $(F_q, (0.5, 1])$ is a soft interior ideal over S. Assume that there exists some $a \in S$ and $\alpha \in (0.5, 1]$, such that $a_\alpha qf$ but $((xa)y)_\alpha q^- f$ that is $f((xa)y) + \alpha < 1 \leq f(a) + \alpha$ for some $x, a, y \in S$. Then $a \in F_q(\alpha)$ but $((xa)y) \notin F_q(\alpha)$, which is a contradiction therefore $f((xa)y) + \alpha \geq 1$. Hence $((xa)y)_\alpha qf$. Which shows that f is a (q, q)-fuzzy interior ideal of S. ■

Theorem 375 *Let f be a fuzzy subset of an AG-groupoid S. Then f is a (q, q)-fuzzy bi-ideal if and only if $(F_q, (0.5, 1])$ is a soft bi-ideal over S.*

Proof. Let f be an (q, q)-fuzzy bi-ideal of S and suppose that $x, z \in F_q(\alpha)$ where $\alpha \in (0.5, 1]$, then $f(x) + \alpha \geq 1$ and $f(z) + \alpha \geq 1$, that is $x_\alpha qf$,and.$z_\alpha qf$. Then for each $x, z \in S$, $((xy)z)_\alpha qf$ That is $f((xy)z) + \alpha \geq 1$. Hence $((xy)z) \in F_q(\alpha)$. Thus $F_q(\alpha)$ is an bi-ideal of S. Consequently $(F_q, (0.5, 1])$ is a soft bi-ideal over S.

Conversely suppose $(F_q, (0.5, 1])$ is a soft bi-ideal over S. Assume that there exists some $x, z \in S$ and $\alpha \in (0.5, 1]$, such that $x_\alpha qf$ and.$z_\alpha qf$ but $((xy)z)_\alpha qf$ that is $f((xy)z) + \alpha < 1 \leq f(x) \wedge f(z) + \alpha$ for some $x, y, z \in S$. Then $x, z \in F_q(\alpha)$ but $((xy)z) \notin F_q(\alpha)$, which is a contradiction therefore

$f\left((xy)\,z\right)+\alpha \geq 1$. Hence $\left((xy)\,z\right)_{\alpha}qf$. Which shows that f is a (q,q)-fuzzy bi-ideal of S. $\blacksquare$

Definition 376 *The restricted product (H,C) of two soft sets (F,A) and (G,B) over a semigroup S is defined as the soft set $(H,C)=(F,A)\odot(G,B)$ where $C=A\cap B$ and H is a set valued function from C to $P(S)$ defined as $H(c)=F(c)\circ G(c)$ for all $c\in C$.*

Definition 377 *Let X be a non empty set. A fuzzy subset f of X is defined as a mapping from X into $[0,1]$, where $[0,1]$ is the usual interval of real numbers. The set of all fuzzy subsets of X is denoted by $\mathcal{F}(X)$.*

Definition 378 *A fuzzy subset f of X of the form*

$$f(y)=\begin{cases} r(\neq 0 & \text{if } y=x,\\ 0 & \text{otherwise}\end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r\in(0,1]$.

Let $\gamma,\delta\in[0,1]$ be such that $\gamma<\delta$. For any $Y\subseteq X$,we define $\chi_{\gamma Y}^{\delta}$ be the fuzzy subset of X by $\chi_{\gamma Y}^{\delta}(x)\geq\delta$ for all $x\in Y$ and $\chi_{\gamma Y}^{\delta}(x)\leq\gamma$ otherwise. Clearly,$\chi_{\gamma Y}^{\delta}$ is the characteristic function of Y if $\gamma=0$ and $\delta=1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that
(i) $x_r\in_{\gamma} f$ if $f(x)\geq r>\gamma$.
(ii) $x_r q_{\delta} f$ if $f(x)+r>2\delta$.
(iii) $x_r\in_{\gamma}\vee q_{\delta}f$ if $x_r\in_{\gamma}$or $x_r q_{\delta}f$.
$(iiii)$ $x_r\in_{\gamma}\wedge q_{\delta}f$ if $x_r\in_{\gamma}$and $x_r q_{\delta}f$.

Definition 379 *Let S be an AG-groupoid and $\mu,\nu\in\mathcal{F}(S)$ .Define the product of μ and ν, denoted by $\mu\circ\nu$, by*

$$(\mu\circ\nu)(x)=\begin{cases}\sup_{x=yz}\min\{\mu(y),\nu(z)\} & \textit{if there exist } y,z\in S \textit{ such that } x=yz,\\ & 0,\ \textit{otherwise.}\end{cases}$$

for all $x\in S$.

The following definitions are basics are available in [16].

Definition 380 *A pair $\langle F,A\rangle$ is called fuzzy soft set over U, where $A\subseteq E$ and F is a mapping given by $F:A\rightarrow\mathcal{F}(U)$.*

In general, for every $\varepsilon\in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called fuzzy value set of parameter ε. The set of all fuzzy soft sets over U with parameters from E is called a fuzzy soft class, and it is denoted by $\mathcal{F}\rho(U,E)$.

Definition 381 *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft sets over U. We say that $\langle F,A\rangle$ is a fuzzy soft subset of $\langle G,B\rangle$ and write $\langle F,A\rangle\Subset\langle G,B\rangle$ if*

(i) $A\subseteq B$;
(ii) For any $\varepsilon\in A, F(\varepsilon)\subseteq G(\varepsilon)$.

$\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be fuzzy soft equal and write $\langle F, A\rangle = \langle G, B\rangle$ if $\langle F, A\rangle \Subset \langle G, B\rangle$ and $\langle G, B\rangle \Subset \langle F, A\rangle$.

Definition 382 *The extended intersection of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is called fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and*

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \,\widetilde{\cap}\, \langle G, B\rangle$.

Definition 383 *The extended union of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is a fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and*

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \,\widetilde{\cup}\, \langle G, B\rangle$.

Definition 384 *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U such that $A \cap B \neq \phi$. The restricted intersection of $\langle F, A\rangle$ and $\langle G, B\rangle$ is defined to be fuzzy soft set $\langle H, C\rangle$, where $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ for all $\varepsilon \in C$.This is denoted by $\langle H, C\rangle = \langle F, A\rangle \cap \langle G, B\rangle$.*

Definition 385 *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U such that $A \cap B \neq \phi$. The restricted union of $\langle F, A\rangle$ and $\langle G, B\rangle$ is defined to be fuzzy soft set $\langle H, C\rangle$, where $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon) \cup G(\varepsilon)$ for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \cup \langle G, B\rangle$.*

Definition 386 *The product of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over an semigroup S is a fuzzy soft set over S, denoted by $\langle F \circ G, C\rangle$, where $C = A \cup B$ and*

$$(F \circ G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \circ G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F \circ G, C\rangle = \langle F, A\rangle \odot \langle G, B\rangle$.

Definition 387 *A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid is called an $(\in_\gamma, \in_\gamma \vee q_{\ \delta})$-fuzzy soft left (resp., right) ideal over S if it satisfies*

$$\Sigma(S, A) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \ (resp., \langle F, A\rangle \odot \Sigma(S, A) \Subset_{(\gamma,\delta)} \langle F, A\rangle)$$

A fuzzy soft set over S is called $(\in_\gamma, \in_\gamma \vee q_{\ \delta})$-fuzzy soft ideal over S if it is both an $(\in_\gamma, \in_\gamma \vee q_{\ \delta})$-fuzzy soft left ideal and an $(\in_\gamma, \in_\gamma \vee q_{\ \delta})$-fuzzy soft right ideal over S.

Definition 388 *A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q\ _\delta)$-fuzzy soft bi-ideal over S if it satisfies*

(i) $\langle F, A\rangle \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$;
(ii) $\langle F, A\rangle \odot \Sigma(S, A) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$.

Definition 389 *A fuzzy soft set $\langle F, A\rangle$ over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy soft quasi-ideal over S if it satisfies*

$$\langle F, A\rangle \odot \Sigma(S, A) \mathbin{\widetilde{\cap}} \Sigma(S, A) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$$

Theorem 390 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy left (resp. right) ideal over S if it satisfies*
$(for\ all\ x,\ y \in S)(\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\}\ (resp. \min\{f(x), \delta\}))$.

Proof. It is same as in .

Definition 391 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy left ideal over S if*
$(for\ all\ x,\ y \in S)(for\ all\ t, \delta \in (\gamma, 1))(y_t \in_\gamma f \implies (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f)$.

Definition 392 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy right ideal over S if*
$(for\ all\ x,\ y \in S)(for\ all\ t, \delta \in (\gamma, 1))(x_t \in_\gamma f \implies (xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f)$.

■

Definition 393 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy ideal over S if it is both $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy left ideal and $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy right ideal.*

Theorem 394 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy bi-ideal over S if it satisfies*
(i) $(for\ all\ x,\ y \in S)(\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\})$;
(ii) $(for\ all\ x,\ y,\ z \in S)(\max\{f(xyz), \gamma\} \geq \min\{f(x), f(z), \delta\})$.

Proof. It is easy. ■

Theorem 395 *A fuzzy set f over an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q\ _\delta)-$fuzzy interior ideal over S if it satisfies*
(i) $(for\ all\ x,\ y \in S)(\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\})$;
(ii) $(for\ all\ x,\ a,\ z \in S)(\max\{f(xaz), \gamma\} \geq \min\{f(a), \delta\})$.

Proof. It is easy. ■

Theorem 396 *Let S be an AG-groupoid and $\langle F, A\rangle$ a fuzzy soft set over S. Then*

(i) $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)-$fuzzy soft left ideal (resp., right, bi-ideal, quasi-ideal) over S if and only if non-empty subset $F(\varepsilon)_r$ is a left ideal (resp. right, bi-ideal, quasi-ideal) of S for all $\varepsilon \in A$ and $r \in (\gamma, \delta]$.

(ii) If $2\delta = 1 + \gamma$, then $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)-$fuzzy soft left ideal (resp., right, bi-ideal, quasi-ideal) over S if and only if non-empty subset $\langle F(\varepsilon)\rangle_r$ is a left ideal (resp. right, bi-ideal, quasi-ideal) of S for all $\varepsilon \in A$ and $r \in (\delta, 1]$.

(iii) $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)-$fuzzy soft left ideal (resp., right, bi-ideal, quasi-ideal) over S if and only if non-empty subset $[F(\varepsilon)]_r$ is a left ideal (resp. right, bi-ideal, quasi-ideal) of S for all $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta - \gamma, 1\}]$.

Proof. It is straightforward. ■

Corollary 397 *Let S be an AG-groupoid and $P \subseteq S$. Then P is a left ideal (resp. right ideal, bi-ideal, quasi-ideal) of S if and only if $\Sigma(P, A)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal (resp., right ideal, bi-ideal, quasi-ideal) over S for any $A \subseteq E$.*

10.1 Some Characterizations Using Generalized Fuzzy Soft Bi-ideals

Theorem 398 *For an AG-groupoid with left identity e, the following are equivalent.*

(i) *S is intra-regular.*

(ii) *$B = B^2$, for any bi-ideal B.*

Proof. It is easy. ■

Theorem 399 *Let S be an AG-groupoid with left identity e. Then S is intra-regular if and only if $\langle F, A\rangle \,\widetilde{\cap}\, \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi- ideal $\langle F, A\rangle$ and $\langle G, B\rangle$ over S.*

Proof. Let S be an intra-regular and $\langle F, A\rangle$ and $\langle G, B\rangle$ are any two $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal of S . Now let x be an element of S, $\varepsilon \in A \cup B$ and $\langle F, A\rangle \,\widetilde{\cap}\, \langle G, B\rangle = \langle H, A \cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A - B$. Then $H(\varepsilon) = F(\varepsilon) = (F \circ G)(\varepsilon)$.

Case 2: $\varepsilon \in B - A$. Then $H(\varepsilon) = G(\varepsilon) = (F \circ G)(\varepsilon)$.

Case 3: $\varepsilon \in A \cap B$.Then $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $(F \circ G)(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \Subset \vee q_{(\gamma,\delta)} F(\varepsilon) \circ G(\varepsilon)$. Since S is intra-regular, therefore for any a in S there exist x and y in S such that $a = (xa^2)y$.

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\
&= (y(x((xa^2)y))a = (y((xa^2)(xy)))a = ((xa^2)(y(xy)))a \\
&= ((a(xa))(xy^2))a = (((xy^2)(xa))a)a \\
&= (((xy^2)(x((xa^2)y)))a)a = (((xy^2)(xa^2)(xy)))a)a \\
&= ((xa^2)((xy^2)(xy)))a)a = (((xy)(xy^2)(a^2x))a)a \\
&= ((a^2(((xy)(xy^2))x))a)a \\
&= ((a^2u)a)a, \text{ where } u = ((xy)(xy^2))x \\
&= [\{a^2(eu)\}a]a = [\{(ue)a^2\}a]a = [\{a((ue)a)\}a]a \\
&= [(av)a]a, \text{ where } v = (ue)a).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\max\{(F(\varepsilon) \circ G(\varepsilon))(a), \gamma\} \\
=\ &\max\left\{\sup_{a=pq} \min\{F(\varepsilon)(p), G(\varepsilon)(q)\}, \gamma\right\} \\
\geq\ &\max\{\min\{F(\varepsilon)((av)a), G(\varepsilon)(a)\}, \gamma\} \\
=\ &\min\{\max\{F(\varepsilon)((av)a), \gamma\}, \max\{G(\varepsilon)(a), \gamma\}\} \\
\geq\ &\min\{\min\{F(\varepsilon)(a), F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(a), \delta\}\} \\
=\ &\min\{\min\{F(\varepsilon)(a), \delta\}, \min\{G(\varepsilon)(a), \delta\}\} \\
=\ &\min\{(F(\varepsilon) \cap G(\varepsilon))(a), \delta\}.
\end{aligned}$$

It follows that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon) \circ G(\varepsilon)$. That is $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} (F \circ G)(\varepsilon)$. Thus in any case, we have

$$H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} (F \circ G)(\varepsilon).$$

Therefore

$$\langle F, A \rangle \,\widetilde{\cap}\, \langle G, B \rangle \Subset_{(\gamma,\delta)} \langle F, A \rangle \odot \langle G, B \rangle.$$

Conversely assume that the given condition hold. Let B be any bi-ideal of S then $\Sigma(B, E)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)-$fuzzy soft bi-ideal of S. Now by the assumption, we have $\Sigma(B, E) \,\widetilde{\cap}\, \Sigma(B, E) \Subset_{(\gamma,\delta)} \Sigma(B, E) \odot \Sigma(B, E)$. Hence we have

$$\begin{aligned}
\chi^\delta_{\gamma B} &= {}_{(\gamma,\delta)}\chi^\delta_{\gamma(B \cap B)} =_{(\gamma,\delta)} \chi^\delta_{\gamma B} \cap \chi^\delta_{\gamma B} \subseteq q_{(\gamma,\delta)} \chi^\delta_{\gamma B} \odot \chi^\delta_{\gamma B} \\
&= {}_{(\gamma,\delta)}\chi^\delta_{\gamma BB} =_{(\gamma,\delta)} \chi^\delta_{\gamma B^2}.
\end{aligned}$$

It follows that $B \subseteq B^2$. Also $B^2 \subseteq B$. This implies that $B = B^2$. Therefore S is intra-regular. ■

Theorem 400 *In intra-regular AG-groupoid S with left identity the following are equivalent.*
(i) *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy right ideal.*
(ii) *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy left ideal.*
(iii) *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal.*
(iv) *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy interior ideal.*
(v) *A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal.*

Proof. It is easy. ■

Theorem 401 *Let S be an AG-groupoid with left identity then the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *For all left ideals A, B, $A \cap B \subseteq BA$.*
(iii) *For all $(\in, \in \vee q_k)$-fuzzy left ideals f and g, $f \wedge_k g \leq g \circ_k f$.*
(iv)*For all $(\in, \in \vee q_k)$-fuzzy bi-ideals f and g, $f \wedge_k g \leq g \circ_k f$.*
(v)*For all $(\in, \in \vee q_k)$- generalized fuzzy bi-ideals f and g, $f \wedge_k g \leq g \circ_k f$.*

Proof. It is easy. ■

Theorem 402 *Let S be an AG-groupoid with left identity e. Then S is intra-regular if and only if $\langle G, R\rangle \widetilde{\cap} \langle F, Q\rangle \Subset_{(\gamma,\delta)} \langle F, Q\rangle \odot \langle G, R\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F, Q\rangle$ and for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle G, R\rangle$ over S.*

Proof. Let S be an intra-regular and $\langle F, Q\rangle$ and $\langle G, R\rangle$ are any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal of S . Now let x be an element of S, $\varepsilon \in Q \cup R$ and $\langle F, Q\rangle \widetilde{\cap} \langle G, R\rangle = \langle H, Q \cup R\rangle$. We consider the following cases.

Case 1: $\varepsilon \in Q - R$. Then $H(\varepsilon) = F(\varepsilon) = (F \circ G)(\varepsilon)$.

Case 2: $\varepsilon \in R - Q$. Then $H(\varepsilon) = G(\varepsilon) = (F \circ G)(\varepsilon)$.

Case 3: $\varepsilon \in Q \cap R$.Then $H(\varepsilon) = F(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $(F \circ G)(\varepsilon) = F(\varepsilon) \circ G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \Subset_{(\gamma,\delta)} G(\varepsilon) \circ F(\varepsilon)$. Since S is intra-regular, therefore for any a in S there exist x and y in S such that $a = (xa^2)y$.

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = (y(xa))(ea) \\ &= (ye)((xa)a) = (xa)((ye)a) = (xa)(ta), \text{ where } t = (ye). \end{aligned}$$

Then we have

$$\begin{aligned}
& \max\left\{(G(\varepsilon)\circ F(\varepsilon))(a),\gamma\right\} \\
= & \max\left\{\sup_{a=uv}\min\left\{G(\varepsilon)(u),F(\varepsilon)(v)\right\},\gamma\right\} \\
\geq & \max\left\{\min\left\{G(\varepsilon)\left(xa\right),F(\varepsilon)(ta)\right\},\gamma\right\} \\
= & \min\left\{\max\{G(\varepsilon)(xa)),\gamma\},\max\{F(\varepsilon)(ta),\gamma\}\right\} \\
\geq & \min\left\{\min\left\{G(\varepsilon)(a),\delta\right\},\min\{F(\varepsilon)(a),\delta\}\right\} \\
= & \min\left\{\min\left\{G(\varepsilon)(a),F(\varepsilon)(a),\delta\right\}\right\} \\
= & \min\left\{(G(\varepsilon)\cap F(\varepsilon))(a),\delta\right\} \\
= & \min\left\{(F(\varepsilon)\cap G(\varepsilon))(a),\delta\right\}.
\end{aligned}$$

It follows that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}G(\varepsilon)\circ F(\varepsilon)$. That is $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(G\circ F)(\varepsilon)$. Thus in any case, we have

$$H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(G\circ F)(\varepsilon).$$

Therefore,

$$\langle F,Q\rangle\,\widetilde{\cap}\,\langle G,R\rangle\Subset_{(\gamma,\delta)}\langle G,R\rangle\odot\langle F,Q\rangle.$$

Conversely assume that the given condition hold. Let L_1 and L_2 are any two left ideal of S then, $\Sigma(L_1,E)$ and $\Sigma(L_2,E)$ are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left ideal of S. Now by the assumption, we have $\Sigma(L_1,E)\,\widetilde{\cap}\,\Sigma(L_2,E)\Subset_{(\gamma,\delta)}\Sigma(L_2,E)\odot\Sigma(L_1,E)$.Hence we have

$$\begin{aligned}
\chi^{\delta}_{\gamma(L_1\cap L_2)} &= {}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L_1}\cap\chi^{\delta}_{\gamma L_2} \\
&\subseteq q_{(\gamma,\delta)}\chi^{\delta}_{\gamma L_2}\odot\chi^{\delta}_{\gamma L_1}={}_{(\gamma,\delta)}\chi^{\delta}_{\gamma L_2L_1}
\end{aligned}$$

It follows that $L_1\cap L_2\subseteq L_2L_1$. Therefore by theorem S is intra-regular. ■

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An AG-groupoid is an algebraic structure that lies in between a groupoid and a commutative semigroup. It has many characteristics similar to that of a commutative semigroup. If we consider $x^2y^2 = y^2x^2$, which holds for all x, y in a commutative semigroup, on the other hand one can easily see that it holds in an AG-groupoid with left identity e and in AG**-groupoids. This simply gives that how an AG-groupoid has closed connections with commutative agebras.

We extend now for the first time the AG-groupoid to the Neutrosophic AG-groupoid. A neutrosophic AG-groupoid is a neutrosophic algebraic structure that lies between a neutrosophic groupoid and a neutrosophic commutative semigroup.

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