Functions of multivectors in 3D Euclidean geometric algebra via spectral decomposition (for physicists and engineers)

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Geometric algebra is a powerful mathematical tool for description of physical phenomena. The article [3] gives a thorough analysis of functions of multivectors in Cl_3 relaying on involutions, especially Clifford conjugation and complex structure of algebra. Here is discussed another elegant way to do that, relaying on complex structure and idempotents of algebra. Implementation of Cl_3 using ordinary complex algebra is briefly discussed.

Keywords: *function of multivector, idempotent, nilpotent, spectral decomposition, unipodal numbers, geometric algebra, Clifford conjugation, multivector amplitude, bilinear transformations*

1. Numbers

Geometric algebra is a promising platform for mathematical analysis of physical phenomena. The simplicity and naturalness of the initial assumptions and the possibility of formulation of (all?) main physical theories with the same mathematical language imposes the need for a serious study of this beautiful mathematical structure. Many authors have made significant contributions and there is some surprising conclusions. Important one is certainly the possibility of natural defining Minkowski metrics within Euclidean 3D space without introduction of negative signature, that is, without defining time as the fourth dimension ([1, 6]).

In Euclidean 3D space we define orthogonal unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ with the property

$$\boldsymbol{e}_i^2 = 1, \ \boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 0,$$

so one could recognize the rule for multiplication of Pauli matrices. Non-commutative product of two vectors is $\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}$, sum of symmetric (inner product) and anti-symmetric part (wedge product). Each element of the algebra (Cl_3) can be expressed as linear combination of elements of 2^3 – dimensional basis (*Clifford basis*)

$$\{1, \ \boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \boldsymbol{e}_3, \ \boldsymbol{e}_1\boldsymbol{e}_2, \ \boldsymbol{e}_3\boldsymbol{e}_1, \ \boldsymbol{e}_2\boldsymbol{e}_3, \ \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\},$$

where we have a scalar, three vectors, three bivectors and pseudoscalar. According to the number of unit vectors in the product we are talking about odd or even elements. If we define $j = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ it is easy to show that pseudoscalar j has two interesting properties in Cl_3: 1) $j^2 = -1$, 2) $jX = Xj$, for any element X of algebra, and behaves like an *ordinary imaginary unit*, which enables as to study a rich complex structure of Cl_3. This property we have for n = 3, 7, … [3]. Bivectors can be expressed as product of unit pseudoscalar and vectors, $j\vec{v}$.

We define a general element of the algebra (*multivector*)

$$M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb = z + \boldsymbol{F}, \quad z = t + jb, \quad \boldsymbol{F} = \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}}$$

where z is a complex scalar and an element of *center of algebra*, while $\boldsymbol{F}$, by analogy, is a *complex vector*. *Complex conjugation* ($j \in \mathbb{C}$) is $z^* = z^\dagger = t - jb$, $\boldsymbol{F}^* = \boldsymbol{F}^\dagger = \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}}$, where dagger means *reversion*, to be defined later in the text. The complex structure allows a different ways of expressing multivectors, one is

$$M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb = t + j\vec{\boldsymbol{n}} + j\left(b - j\vec{\boldsymbol{x}}\right),$$

where multivector of the form $a + vj\hat{\boldsymbol{v}}$ belongs to the even part of the algebra and can be associated with rotations, spinors or quaternions. Also we could treat multivector as ([12]) $M = \alpha_0 + \sum_{i=1}^{3} \alpha_i \boldsymbol{e}_i, \quad \alpha_k \in \mathbb{C}$ and implement it relying on ordinary complex numbers.

Main involutions in Clifford algebra are:

1) grade involution: $\hat{M} = t - \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} - jb$
2) reverse (*adjoint*): $M^\dagger = t + \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}} - jb = z^* + \boldsymbol{F}^*$
3) Clifford conjugation: $\bar{M} = t - \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}} + jb = \bar{z} + \bar{\boldsymbol{F}} = z - \boldsymbol{F}$,

where an asterisk stands for a complex conjugate. Grade involution is the transformation $\hat{\vec{\boldsymbol{x}}} = -\vec{\boldsymbol{x}}$ (*space inversion*), while reverse in *Cl*3 is like a complex conjugation, $\vec{\boldsymbol{x}}^\dagger = \vec{\boldsymbol{x}}, \ j^\dagger = -j$. Clifford conjugation is combination of two involutions $\bar{M} = \hat{M}^\dagger, \ \bar{\vec{\boldsymbol{x}}} = -\vec{\boldsymbol{x}}, \ \bar{j} = j$. Bivectors given as a wedge product could be expressed as $\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{y}} = j\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$, where $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$ is a *cross product*. An application of involutions is easy now.

Defining *paravector* $p = t + \vec{\boldsymbol{x}}$ we have $p\bar{p} = \left|t + \vec{\boldsymbol{x}}\right|^2 = \left(t + \vec{\boldsymbol{x}}\right)\left(t - \vec{\boldsymbol{x}}\right) = t^2 - x^2$ and we have a usual metric of special relativity.

From $M = \hat{M} \Rightarrow M = t + j\vec{\boldsymbol{n}}$, the even part of the algebra (*spinors*).

From $M = M^\dagger \Rightarrow M = t + \vec{\boldsymbol{x}}$, paravector; reverse is an anti-automorphism $\left(MM^\dagger\right)^\dagger = MM^\dagger$, so $MM^\dagger$ (square of *multivector magnitude*, [2]) is a paravector.

From $M = \bar{M} \Rightarrow M = t + jb = z$, a complex scalar. Clifford conjugation is anti-automorphism, $\overline{M\bar{M}} = M\bar{M}$, so $M\bar{M}$ (square of *multivector amplitude*, [2]) is a complex scalar and there is no other "amplitude" with such a property ([1]).

We define a *multivector amplitude* $\left|M\right|$ (hereinafter MA)

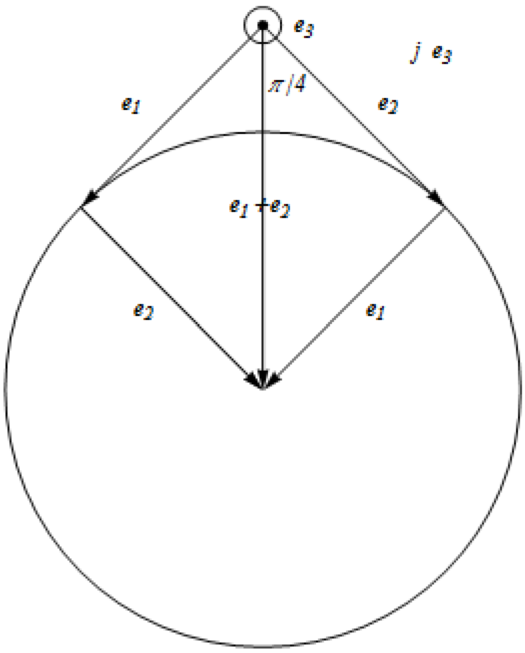
$$M\bar{M} = \left|M\right|^2 = t^2 - x^2 + n^2 - b^2 + 2j\left(tb - \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}}\right), \quad \left|M\right| \in \mathbb{C} \quad (3)$$

which we could express as

$$\sqrt{M\bar{M}} = \left|M\right| = \sqrt{\left(z + \boldsymbol{F}\right)\left(z - \boldsymbol{F}\right)} = \sqrt{z^2 - \boldsymbol{F}^2}, \quad \boldsymbol{F}^2 = x^2 - n^2 + 2j\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} \in \mathbb{C}.$$

For $\boldsymbol{F} = 0$, or $\boldsymbol{F}^2 = \boldsymbol{N}^2 = 0$ fallows $\left|M\right| = z$ ($\boldsymbol{N}^2 = x^2 - n^2, \ \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} = 0$ is a *nilpotent* in the algebra). For $\boldsymbol{F}^2 = c \in \mathbb{R}$ ($\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} = 0$, *whirl* [1]) here is used a designation $\boldsymbol{F}_{(c)}$. From $\boldsymbol{F}^2 \in \mathbb{C}$

we have $\boldsymbol{F}_{(1)} = \boldsymbol{F}/\sqrt{\boldsymbol{F}^2}$, $\boldsymbol{F}_{(1)}^2 = 1$, and $\hat{\boldsymbol{F}} = \boldsymbol{F}/\sqrt{-\boldsymbol{F}^2} = \boldsymbol{F}/\left|\boldsymbol{F}\right| = -j\boldsymbol{F}_{(1)}$, $\hat{\boldsymbol{F}}^2 = -1$ (*complex unit vector*). With $\boldsymbol{f} = \boldsymbol{F}_{(1)}$ we also define $u_\pm = \left(1 \pm \boldsymbol{f}\right)/2$, $u_\pm{}^2 = u_\pm$, $u_+ u_- = 0$, idempotents of the algebra ([1, 9]). Every idempotent in *Cl*3 can be expressed as $u_\pm = p_\pm + \boldsymbol{N}_\pm, \boldsymbol{N}_\pm^2 = 0$, where $p_\pm$ are simple idempotents, like $p_\pm = \left(1 \pm \boldsymbol{e}_1\right)/2$. For example, $u_+ = \left(1 + \boldsymbol{e}_1 + \boldsymbol{e}_2 + j\boldsymbol{e}_3\right)/2$, $\boldsymbol{N} = \boldsymbol{e}_2 + j\boldsymbol{e}_3$ (figure below).



2. Implementation

From $M = \alpha_0 + \sum_{i=1}^{3} \alpha_i \boldsymbol{e}_i = \alpha_0 + A$, $\alpha_k \in \mathbb{C}$, it is easy to implement the algebra on computer using ordinary complex numbers only. In [15] are defined products:

$A \circ B = \sum_{i=1}^{3} \alpha_i \beta_i$ (*generalized inner product*),

$A \otimes B = \det\left[\boldsymbol{e}_i,\ \alpha_i,\ \beta_i\right]$, (*generalized outer product*), and

$AB = A \circ B + A \otimes B$ (*generalized geometric product*).

Now we have $\left(\alpha_A + A\right)\left(\alpha_B + B\right) = \alpha_A \alpha_B + \alpha_B A + \alpha_A B + AB$. We can find α_i for multivector $M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb$ using linear independency.

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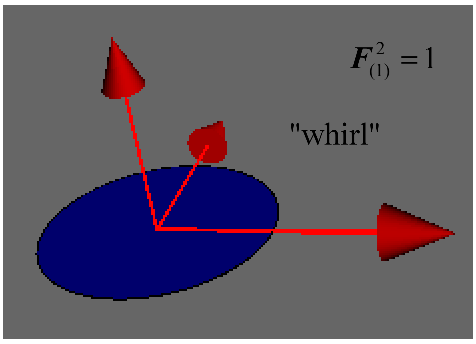
3. Spectral decomposition



Starting from multivector

$$M = z + \boldsymbol{F} = z + \sqrt{\boldsymbol{F}^2}\,\boldsymbol{f} = x + y\boldsymbol{f},\quad \boldsymbol{F}^2 \neq 0$$

we see the form of unipodal-like numbers. Defining $M_\pm = x \pm y$ and recalling a relation (easy to proof) $\boldsymbol{f} u_\pm = \pm u_\pm$ follows

$$M u_\pm = \left(x \pm y\boldsymbol{f}\right) u_\pm = \left(x \pm y\right) u_\pm = M_\pm u_\pm,$$

so we have a projection. Spectral basis $u_\pm$ is very useful because the binomial expansion of a multivector is very simple

$$M^2 = \left(M_+ u_+ + M_- u_-\right)^2 = M_+^2 u_+ + M_-^2 u_- \Rightarrow M^n = M_+^n u_+ + M_-^n u_-,\quad n \in \mathbb{Z},$$

where $n <$ 0 is possible for $\left|M\right| \neq 0$ ($M^{-1} = \bar{M}/\left(M\bar{M}\right)$).

Defining conjugation $(a+b\boldsymbol{f})^{-}=a-b\boldsymbol{f}$ (obviously the Clifford conjugation) we have $(a+b\boldsymbol{f})^{-}(a+b\boldsymbol{f})=a^2-b^2$, where $\sqrt{a^2-b^2}=\sqrt{MM^{-}}=\sqrt{M\bar{M}}$ is a *multivector amplitude*. In spectral basis using $u_{\pm}^{-}=u_{\mp}$ we have $MM^{-}=(M_+u_+ + M_-u_-)(M_+u_- + M_-u_+)=M_+M_-$.

Starting from $M=z+\boldsymbol{F}$ we have

$$M=z+\sqrt{\boldsymbol{F}^2}\boldsymbol{f}=x+y\boldsymbol{f}=\rho\left(\frac{x}{\rho}+\frac{y}{\rho}\boldsymbol{f}\right)=\rho(\cosh\varphi+\boldsymbol{f}\sinh\varphi),\quad \rho=\sqrt{x^2-y^2},$$

obtaining the polar form of a multivector. If a multivector amplitude is zero we have light-like multivector and there is no a polar form. Now defining $\tanh\varphi=\vartheta$ ("velocity") we have

$$M=\rho(\cosh\varphi+\boldsymbol{f}\sinh\varphi)=\rho\gamma(1+\vartheta\boldsymbol{f}),\quad \gamma^{-1}=\sqrt{1-\vartheta^2}$$

and in the spectral basis

$$M=\rho\gamma(1+\vartheta\boldsymbol{f})=k_+u_+ + k_-u_- \Rightarrow k_{\pm}=\rho\gamma(1\pm\vartheta)=\rho K^{\pm 1},$$

where $K=\sqrt{(1+\vartheta)/(1-\vartheta)}$ is a generalized *Bondi factor* ($\varphi=\log K$). Now we have

$$(K_1u_+ + K_1^{-1}u_-)(K_1u_+ + K_1^{-1}u_-)=K_1K_2u_+ + K_1^{-1}K_2^{-1}u_- = Ku_+ + K^{-1}u_- \Rightarrow K=K_1K_2,$$

or

$$\begin{aligned}&\gamma_1\gamma_2(1+\vartheta_1\boldsymbol{f})(1+\vartheta_2\boldsymbol{f})=\gamma_1\gamma_2(1+(\vartheta_1+\vartheta_2)\boldsymbol{f}+\vartheta_1\vartheta_2)=\\&\gamma_1\gamma_2(1+\vartheta_1\vartheta_2)(1+(\vartheta_1+\vartheta_2)/(1+\vartheta_1\vartheta_2)\boldsymbol{f})\Rightarrow\\&\quad\gamma=\gamma_1\gamma_2(1+\vartheta_1\vartheta_2),\quad \vartheta=(\vartheta_1+\vartheta_2)/(1+\vartheta_1\vartheta_2),\end{aligned}$$

and we have "velocity addition rule". So, every multivector could be mathematically treated like an ordinary boost in special relativity. For $\rho=1$ we have a "boost" $\gamma(1+\vartheta\boldsymbol{f})=Ku_+ + K^{-1}u_-$ as transformation that preserves multivector amplitude and should be considered as a part of the Lorentz group ([1, 2]). As a simple example of a unit complex vector we already mentioned $\boldsymbol{f}=\boldsymbol{e}_1+\boldsymbol{e}_2+j\boldsymbol{e}_3=\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_1\boldsymbol{e}_2$, completely in Cl_2, suggesting that one could analyze problem in basis $(1,\ \boldsymbol{F}_{(1)})$ or related spectral basis in $\boldsymbol{e}_1\boldsymbol{e}_2$ plane and rotate all elements to obtain relations for an arbitrary orientation of a plane, using powerful apparatus of geometric algebra for rotations.

Mapping basis $(1, \boldsymbol{f})$ to $(e^{\phi f},\ \boldsymbol{f}e^{\phi f})$ we obtain new orthogonal basis and new components of multivector

$$\begin{aligned}&a+b\boldsymbol{f}\to a'e^{\phi f}+b'\boldsymbol{f}e^{\phi f}=(a'+b'\boldsymbol{f})e^{\phi f},\\&a'=ae^{-\phi f},\quad b'=be^{-\phi f},\quad |a+b\boldsymbol{f}|=|a'+b'\boldsymbol{f}|.\end{aligned}$$

4. Functions of multivectors

Using series expansion it is straight forward to find a closed formulae for (analytic at least) functions. If $\boldsymbol{F}^2=0$ we have $f(M)=f(z)$ and it is easy to find closed form using theory of functions on the complex field. Otherwise, from the series expansion

$$f(x) = f(0) + \sum_n \frac{f^{(n)}(0)x^n}{n!}$$

using $M^2 = M_+^n u_+ + M_-^n u_-$ we have

$$f(M) = f(M_+)u_+ + f(M_-)u_-$$

and, again, it is "easy" to find a closed form because of $M_\pm \in \mathbb{C}$. For $M = \boldsymbol{F} = \sqrt{\boldsymbol{F}^2}\boldsymbol{f}$ we have $M_\pm = \pm\sqrt{\boldsymbol{F}^2} \Rightarrow f(M) = f\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + f\left(-\sqrt{\boldsymbol{F}^2}\right)u_-$. If function is even we have $f(\boldsymbol{F}) = f\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ + u_-) = f\left(\sqrt{\boldsymbol{F}^2}\right)$ and similarly for odd functions $f(\boldsymbol{F}) = f\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ - u_-) = f\left(\sqrt{\boldsymbol{F}^2}\right)\boldsymbol{f}$. For $M = z + \boldsymbol{F},\ \boldsymbol{F}^2 = \boldsymbol{N}^2 = 0$ there is no spectral decomposition ($\boldsymbol{f}$ is not defined), but we have $M^n = (z + \boldsymbol{N})^n = z^n + nz^{n-1}\boldsymbol{N}$, giving $f(z + \boldsymbol{N}) = f(z) + f'(z)\boldsymbol{N}$. We also have a special cases

$$\begin{aligned}
&f(u_\pm) = f(\pm 1)u_\pm,\\
&f(\boldsymbol{f}) = f(u_+ - u_-) = f(1)u_+ + f(-1)u_-,\\
&f(\hat{\boldsymbol{F}}) = f(-ju_+ + ju_-) = f(-j)u_+ + f(j)u_-.
\end{aligned}$$

Obviously, for an odd function we have $f(\boldsymbol{f}) = f(1)\boldsymbol{f},\ f(\hat{\boldsymbol{F}}) = -f(j)\boldsymbol{f}$ and for even functions $f(\boldsymbol{f}) = f(1),\ f(\hat{\boldsymbol{F}}) = f(j)\boldsymbol{f}$.

For an inverse function we have

$$f^{-1}(y) = x \Rightarrow f(x) = y \Rightarrow f(x_\pm) = y_\pm \Rightarrow x_\pm = f^{-1}(y_\pm).$$

For a *light-like* multivectors ($M\bar{M} = 0$) we have

$$M = z + \sqrt{\boldsymbol{F}}\boldsymbol{f} = \mathrm{z} + \mathrm{z}_{\boldsymbol{F}}\boldsymbol{f},\ \mathrm{z}^2 - \boldsymbol{F}^2 = 0 = (\mathrm{z} - \mathrm{z}_{\boldsymbol{F}})(\mathrm{z} + \mathrm{z}_{\boldsymbol{F}}),$$

and two possibilities:

1) $z = z_{\boldsymbol{F}} \Rightarrow M_+ = 2z_{\boldsymbol{F}},\ M_- = 0 \Rightarrow f(M) = f(2z_{\boldsymbol{F}})u_+$
2) $z = -z_{\boldsymbol{F}} \Rightarrow M_+ = 0,\ M_- = -2z_{\boldsymbol{F}} \Rightarrow f(M) = f(-2z_{\boldsymbol{F}})u_-$

Once a spectral decomposition of a function is analyzed there remains just to use the well known properties of functions of a complex variables.

5. Examples

Here we (again) define $M = z + \boldsymbol{F} = z + \sqrt{\boldsymbol{F}^2}\boldsymbol{f} = z + z_{\boldsymbol{F}}\boldsymbol{f}$ and $M_+ = z + z_{\boldsymbol{F}},\ M_- = z - z_{\boldsymbol{F}}$.

For the inverse of M we have ($|M| \neq 0$)

$$M^{-1}=\frac{1}{M_+u_++M_-u_-}=\frac{M_+u_-+M_-u_+}{\left(M_+u_++M_-u_-\right)\left(M_+u_-+M_-u_+\right)}=\frac{M_+u_-+M_-u_+}{M_+M_-}=\frac{u_+}{M_+}+\frac{u_-}{M_-},$$

as expected. Now it is obvious that

$$M^{-n}=\frac{1}{\left(M_+u_++M_-u_-\right)^n}=\frac{u_+}{\left(M_+\right)^n}+\frac{u_-}{\left(M_-\right)^n}.$$

We can find square root using

$$\sqrt{M}=S=S_+u_++S_-u_-\Rightarrow M=M_+u_++M_-u_-=\left(S_+\right)^2u_++\left(S_-\right)^2u_-\Rightarrow S_\pm=\pm\sqrt{M_\pm}.$$

So, generally we have $M^{\pm 1/n}=S\Rightarrow S_\pm=\left(M_\pm\right)^{\pm 1/n}$, $n\in\mathbb{N}$. As a simple example (an interested reader could compare with [3]) $\sqrt{\boldsymbol{e}_i}=\pm\left(j+\boldsymbol{e}_i\right)/\sqrt{2j}$.

The exponential function is easy one, $e^M=e^{M_+}u_++e^{M_-}u_-$ and now we have exponentials of complex numbers (just use ordinary $i=\sqrt{-1}$ and replace $i\to j$ at the end). Logarithm is the inverse function to exponential, so we have

$$\log M=X\Rightarrow e^X=M=M_+u_++M_-u_-=\exp\left(X_+\right)u_++\exp\left(X_-\right)u_-\Rightarrow X_\pm=\log M_\pm.$$

In [3] is derived formula $\log M=\log\left|M\right|+\varphi\hat{\boldsymbol{F}}$, $\varphi=\arctan\left(\left|\boldsymbol{F}\right|/z\right)$, but those are equivalent:

$$z_{\boldsymbol{F}}=\sqrt{\boldsymbol{F}^2}=-j\left|\boldsymbol{F}\right|,\ \hat{\boldsymbol{F}}=-j\boldsymbol{f}\Rightarrow$$

$$\log\left(M_+\right)u_++\log\left(M_-\right)u_-=\frac{\log\left(M_+\right)+\log\left(M_-\right)}{2}+\frac{\log\left(M_+\right)-\log\left(M_-\right)}{2}\boldsymbol{f}=$$

$$\log\left|M\right|-j\hat{\boldsymbol{F}}\log\left(\sqrt{\left[1-j\left(\left|\boldsymbol{F}\right|/z\right)\right]/\left[1+j\left(\left|\boldsymbol{F}\right|/z\right)\right]}\right)=\log\left|M\right|+\hat{\boldsymbol{F}}\arctan\left(\left|\boldsymbol{F}\right|/z\right)=\log\left|M\right|+\varphi\hat{\boldsymbol{F}}.$$

Now we can find, for $a\in\mathbb{R}$, $M^a=X\Rightarrow\log X=a\log M\Rightarrow X=e^{a\log M}$,

but the same appears to be correct for $a=z+\boldsymbol{F}$ and one can find, for example,

$$M^{\vec{\boldsymbol{v}}}=X\Rightarrow\log X=\vec{\boldsymbol{v}}\log M\Rightarrow X=e^{\vec{\boldsymbol{v}}\log M},$$

although here some caution is needed because of possibility $\log X=\left(\log M\right)\vec{\boldsymbol{v}}\Rightarrow X=e^{\left(\log M\right)\vec{\boldsymbol{v}}}$. Also, relation $M=e^{\log M}$ is generally not valid and needs some care due to the multivalued nature of the logarithm operation. Nevertheless, expressions like $j^{\boldsymbol{e}_1}=\exp\left(\boldsymbol{e}_1\log j\right)=\exp\left(j\pi\boldsymbol{e}_1/2\right)=j\boldsymbol{e}_1$, or $\left(\boldsymbol{e}_1\boldsymbol{e}_2\right)^{\boldsymbol{e}_3}=j$ are quite possible in Cl_3. Simple examples ($u_\pm=\left(1\pm\boldsymbol{e}_1\right)/2$):

1. $\boldsymbol{e}_1^{\boldsymbol{e}_1}=X\Rightarrow\boldsymbol{e}_1\log\boldsymbol{e}_1=\log X$,
   $\boldsymbol{e}_1=u_+-u_-\Rightarrow\log\boldsymbol{e}_1=u_+\log 1+u_-\log\left(-1\right)=j\pi u_-\Rightarrow$
   $\boldsymbol{e}_1\log\boldsymbol{e}_1=-j\pi u_-\Rightarrow X=\exp\left(-j\pi u_-\right)=\exp\left(-j\pi\right)u_-=-u_-$
   (solution $\boldsymbol{e}_1$ is not valid because of $\boldsymbol{e}_1\log\boldsymbol{e}_1=-j\pi u_-=-\log\boldsymbol{e}_1$).
2. $\boldsymbol{e}_1^{\boldsymbol{e}_2}=X\Rightarrow\log X=\boldsymbol{e}_2\log\boldsymbol{e}_1=j\pi\boldsymbol{e}_2u_-$,
   but $\boldsymbol{e}_2u_-\boldsymbol{e}_2u_-=\boldsymbol{e}_2\boldsymbol{e}_2u_+u_-=0$, so $X=\exp\left(j\pi\boldsymbol{e}_2u_-\right)=1+j\pi\boldsymbol{e}_2u_-$,

or $\log X = (\log \boldsymbol{e}_1)\boldsymbol{e}_2 = j\pi u_- \boldsymbol{e}_2 \Rightarrow X = 1 + j\pi \boldsymbol{e}_2 u_+$,

and finally $X = 1 + j\pi \boldsymbol{e}_2 u_\pm$ and $X^n = 1 + jn\pi \boldsymbol{e}_2 u_\pm$, $n \in \mathbb{Z}$

(solution 1 is not valid because of $\boldsymbol{e}_2 \log \boldsymbol{e}_1 \neq 0$, it is a nilpotent).

Trigonometric and hyperbolic trigonometric functions are straightforward and ctg one could obtain as inverse of tan. For example

$$\sin \boldsymbol{F} = \sin\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + \sin\left(-\sqrt{\boldsymbol{F}^2}\right)u_- = \sin\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ - u_-) = \boldsymbol{F}_1 \sin\left(\sqrt{\boldsymbol{F}^2}\right),$$

$$\cos \boldsymbol{F} = \cos\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + \cos\left(-\sqrt{\boldsymbol{F}^2}\right)u_- = \cos\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ + u_-) = \cos\left(\sqrt{\boldsymbol{F}^2}\right).$$

For $\boldsymbol{F} = \boldsymbol{N}$ there is no spectral decomposition, but using $\exp(z\boldsymbol{N}) = 1 + z\boldsymbol{N}$ we have

$\sin \boldsymbol{N} = \boldsymbol{N}$, $\cos \boldsymbol{N} = 1$, $\sinh \boldsymbol{N} = \boldsymbol{N}$, $\cosh \boldsymbol{N} = 1$. There is no $\boldsymbol{N}^{-1}$, so ctg$\boldsymbol{N}$ is not defined.

Series in powers of argument are crucial for all analytic functions (in geometric algebra too) and we can use presented spectral decomposition to obtain components of such functions in spectral basis.

Conclusion

Geometric algebra of Euclidean 3D space (Cl_3) is really rich in structure and gives the possibility to analyze functions defined on multivectors, extending thus theory of functions of real and complex variables, providing intuitive geometrical interpretation also. From simple fact that for a complex vector ($\boldsymbol{F}^2 \neq 0$) we can write $\boldsymbol{F}/\sqrt{\boldsymbol{F}^2} = \boldsymbol{f}$, $\boldsymbol{f}^2 = 1$, $\boldsymbol{F}^2 \in \mathbb{C}$ follows nice possibility to explore idempotent structure $u_\pm = (1 \pm \boldsymbol{f})/2$ and spectral decomposition of multivectors. Using the orthogonality of the spectral basis vectors (idempotents) $u_\pm$ it is shown that all multivectors ([1]) can be treated as the unipodal numbers (i.e. hypercomplex numbers over a complex field). A definition of functions is then quite simple and natural and strongly counts on the theory of functions of complex variable. Complex numbers and vectors (bivectors, trivectors) are thus united in one promising system.

Appendix

A1. *Bilinear transformations*

Regarding that bilinear transformations of multivectors do not change some property of multivectors one could ask yourself: what property? In [2] was shown that the multivector amplitude, defined using the Clifford conjugation, is the unique involution that is commutative ($M\bar{M} = \bar{M}M$) and belongs to the center of the algebra. From text we see that multivector amplitude is defined as $M_+ M_- \in \mathbb{C}$ using just natural conjugation $a + b\boldsymbol{f} \to a - b\boldsymbol{f}$, where $\boldsymbol{f}$ is our "hypercomplex unit". But this is just Clifford conjugation and we see now new meaning for it: it is just a "hypercomplex conjugation". This is a strong argument for regarding Lorentz transformations to be a group of bilinear transformations that preserve multivector amplitude. It is verified on paravectors giving known special relativity, but now we should extend it on a whole multivector including in this way all multivector symmetries of Euclidean 3D space. For multivector X (transformation) now we have $X_+ X_- = 1$ and

$$X = e^{M} \Rightarrow M = \log X = \log|X| + \varphi\hat{\boldsymbol{F}} = \log 1 + \varphi\hat{\boldsymbol{F}} = \varphi\hat{\boldsymbol{F}} = \boldsymbol{F},$$

giving thus a general bilinear (twelve parameters) transformation $M' = e^{\vec{p}+j\vec{q}} M e^{\vec{r}+j\vec{s}}$.

We started from a new definition of a vector multiplication (Clifford or geometric product) and we see that 3D Euclidean space has a rich complex and hyper-complex structure. The Clifford conjugation is the natural choice to define a multivector amplitude. A bilinear transformations that preserve multivector amplitude form a group containing the ordinary Lorentz transformations (restricted), but it is really natural to assume that a physical reality should be richer. Some consequences of that assumption are discussed in [1] and [2].

A2. *Hyperbolic inner and outer products*

Given two multivectors $M_1 = z_1 + z_{1F}\boldsymbol{f}$ and $M_2 = z_2 + z_{2F}\boldsymbol{f}$ we define a square of *multivector distance* (*conjugate products*, [13]) as

$$M_1^{-} M_2 = \left(z_1 - z_{1F}\boldsymbol{f}\right)\left(z_2 + z_{2F}\boldsymbol{f}\right) = z_1 z_2 - z_{1F} z_{2F} + \left(z_1 z_{2F} - z_2 z_{1F}\right)\boldsymbol{f} = hi + ho\boldsymbol{f},$$

where *hi* and *ho* stands for hyperbolic inner and hyperbolic outer products. If $M_1 = M_2 = M$ we have $M^{-}M = z^2 - z_F^2 + \left(zz_F - zz_F\right)\boldsymbol{f} = z^2 - z_F^2$, just square of multivector amplitude. This suggests that *hi* and *ho* have to do something about being “parallel” or “orthogonal” besides being “near” and “close”. For complex and hypercomplex plane (with real coordinates) meaning is obvious (fig A2).

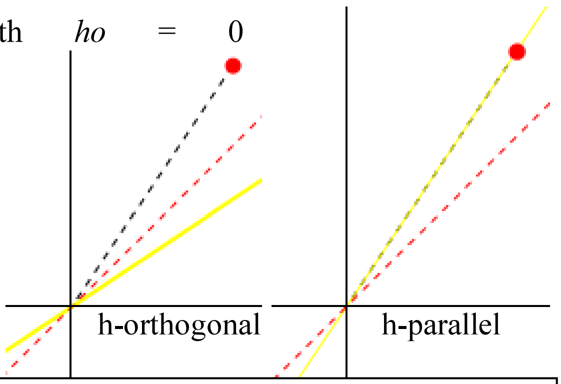


fig A2: *Yellow line represents h-numbers parallel or orthogonal to given h-number* (red dot).

With *ho* = 0 multivectors are said to be “h-parallel”, while for *hi* = 0 they are “h-orthogonal”. For *hi* = *ho* = 0 multivector distance is null and we said it to be “h-light-like”, where h- stands for hyperbolic.

In “boost” formalism we have

$$hi = \rho_1\rho_2\gamma_1\gamma_2\left(1 - \vartheta_1\vartheta_2\right),$$

$$ho = \rho_1\rho_2\gamma_1\gamma_2\left(\vartheta_2 - \vartheta_1\right)/\left(1 - \vartheta_1\vartheta_2\right).$$

So, “h-parallel” multivectors have equal “velocities” and $hi = \rho_1\rho_2$, while for “h-orthogonal” multivectors “velocities” are reciprocal and *ho* becomes infinite (orthogonal multivectors belong to different hyper-quadrants delimited by light-like hyper-planes).

Lema: *Let* $M_1^{-}M_2 = 0$ *for* $M_1 \neq 0$ *and* $M_2 \neq 0$. Then $M_1 M_2 \neq 0$, *and vice versa.*

$$M_1 M_2 = \left(M_{1+}u_+ + M_{1-}u_-\right)\left(M_{2+}u_+ + M_{2-}u_-\right) = M_{1+}M_{2+}u_+ + M_{1-}M_{2-}u_-,$$

$$M_1^{-}M_2 = \left(M_{1+}u_- + M_{1-}u_+\right)\left(M_{2+}u_+ + M_{2-}u_-\right) = M_{1-}M_{2+}u_+ + M_{1+}M_{2-}u_-,$$

and we have $M_{1-}M_{2+} = 0$ or $M_{1+}M_{2-} = 0$, which means $M_{1-} = 0$ and $M_{2-} = 0$ or $M_{1+} = 0$ and $M_{2+} = 0$, but either case gives $M_1 M_2 \neq 0$. Converse proof is similar.

A3. *Polynomials*

Suppose we have simple equation $M^2+1=0$, then objects that squares to -1 are solutions. Using spectral decomposition we could explore it further, so,

$$M^2+1=\left(M_+u_++M_-u_-\right)^2+u_++u_-=\left(M_+^2+1\right)u_++\left(M_-^2+1\right)u_-=0\Rightarrow$$

$M_+^2+1=0,\ M_-^2+1=0$, so we have two equations over complex numbers. Obvious solutions are $\sqrt{-1},\ j,\ je_i,\ \hat{\boldsymbol{F}}$, but it is possible to investigate further. There is an infinite number of solutions, obviously, due to algebraically expanded paradigm of number.

Another simple equation is $M^2=M_+^2u_++M_-^2u_-=0$. One obvious solution is $M=\boldsymbol{N}$ (nilpotent) which we cannot obtain using spectral decomposition (because there is no one for a multivector $\boldsymbol{N}$), so, some caution is necessary.

In physics we are using a lot of special polynomials and their roots and we probably should reconsider those in geometric algebra.

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