Incoherence of the relativistic dynamics: $E = mc^2$ contradicts Special Relativity!

Radwan M. Kassir

radwan.elkassir@dargroup.com

Abstract

The 'relativistic" mass concept is rooted in the problematic longitudinal and transverse mass equations emerging from the Lorentz transformation, as presented by Einstein in his 1905 paper on Special Relativity. These equations, although actual outcomes of the Special Relativity, and verified in this paper through both simplified dimensional analyses and conservation of energy principle, had later been implicitly dropped and replaced by an ad-hoc relativistic mass equation, needed to maintain the consistency of the Special Relativity with the conservation of momentum law—although it results in its violation of the law of conservation of energy. Maintaining the latter law, results in the same transverse mass equation as obtained in Einstein's said paper. The relativistic mass adopted in the literature is but an attempt to conceal contradictions in the Special Relativity, and a convenient means for arriving at the relativistic kinetic energy formula implying the desired mass-energy equivalence equation E = mc^2. In this paper, the incoherence of the Special Relativity emerging from its established mass formulae is revealed through simplified physical demonstrations. Depending on the force definition and the "moving" mass equation used, four different formulae for the relativistic kinetic energy are obtained, all validated from the Special Relativity perspective, creating a detrimental incoherence in the theory. All these formulae are reduced to the classical kinetic energy equation for v << c (v = velocity, c = speed of light). It is revealed that the energy equation E = mc^2 is not a valid consequence of the Special Relativity.

Keywords: *Special Relativity, longitudinal mass, transverse mass, relativistic mass, relativistic momentum, Newton's second law, relativistic kinetic energy,* $E = mc^2$

1. Introduction

In his 1905 paper[1] on the Special Relativity, Einstein predicted (from the Lorentz transformations for the space-time and electromagnetic field components) the longitudinal and transverse mass of moving electron as functions of its velocity, extended to ponderable material point, as measured in the "stationary" system. This was based on defining the force acting on the electron as being equal to $mass \times acceleration$ (Newton's second law of motion). The longitudinal "moving" mass obtained as such along with the mentioned force definition, resulted in the relativistic kinetic energy of the material point moving in the longitudinal direction with a velocity v as being $E_k = \left(\gamma(v) - 1\right) \times m_o c^2$, where m_o is the material point rest mass, c the speed of light, and $\gamma(v) = (1 - v^2/c^2)^{-1/2}$. However, in this context, $\gamma(v) \times m_o$ was not the predicted mass of the moving material point, which was rather $\gamma^3(v) \times m_o$. Thus, there was no such implication as to the mass-energy equivalence—which Einstein attempted to demonstrate in later works[2 3]—from the above kinetic energy equation. In addition, the transverse mass, as well as the longitudinal mass, doesn't satisfy the conservation of momentum within the Special Relativity framework. Thus, the Special Relativity derived "directional" relative mass equations were later implicitly dropped, and replaced by the relativistic mass $\gamma(v) \times m_o$, required for the conservation of momentum. If the relativistic mass was used in deriving the relativistic kinetic energy equation, the equation $E_k = \left(\gamma(v) - 1\right) \times m_o c^2$ would be obtained if the force was rather defined as the momentum change rate $\left(force = d(mv)/dt\right)$—equivalent to the former definition $(force = m \times dv/dt)$ if the mass was invariant. In such a case, the kinetic energy equation becomes $E_k = mc^2 - m_o c^2$, with the mass-energy equivalence implication. However, the relativistic mass being equal to $\gamma(v) \times m_o$ contradicts the actual Special Relativity prediction of the longitudinal (as well as transverse) mass based on the Lorentz transformation.

It is customary to conclude the relativistic mass as being $\gamma(v) \times m_o$ from the conservation of momentum principle applied to colliding particles from the perspective of two inertial frames in relative motion. In the present simplified approach, the transverse velocity of a body moving transversally relative to the "traveling" frame is reduced by a factor of $\gamma(v)$ in the "stationary" frame, according to the relativistic velocity addition—or as a consequence of the time dilation—although there is no relative motion in the transverse direction between the frames. This will result in unjustified transverse momentum decrease (by a factor of γ) in the stationary frame relative to the moving one. Hence, by the means of the conservation of momentum law, the mass should be scaled up by a factor of $\gamma(v)$ in the stationary frame to compensate for the momentum loss. The adopted relativistic mass equation $m = \gamma(v) \times m_o$ is therefore an ad-hoc implemented to reconcile the conservation of momentum law that would otherwise be violated by the Special Relativity; it is not a natural prediction of the Special Relativity, and inconsistent with both the transverse and the longitudinal mass predicted by the Lorentz transformation.

2. The "moving" mass in the Special Relativity

Consider two inertial frames with coordinate systems $K(x,y,z,t)$ and $k\left(\xi,\eta,\zeta,\tau\right)$ in relative motion with velocity v. The frame K is considered to be the "stationary" one (i.e., the other frame is being observed from it). Assume the frames are under the influence of a uniform electromagnetic field. Let (X,Y,Z) and (X',Y',Z') be the electric field [vector] components as measured in the systems K and K', respectively, while (L,M,N) and (L',M',N') are the corresponding magnetic field components. Let there be an electrically charged particle in motion within the field. Assume at an instant of time set as $t = \tau = 0$, the particle is at the K system origin moving along with k origin at the same relative velocity v. At this initial time, the particle is at rest relative to the system k. For an infinitesimal elapse of time, the motion of the particle can be described from the k perspective by the following equations.

$$m_o \frac{d^2\xi}{dt^2} = \varepsilon X', \tag{1}$$

$$m_o \frac{d^2\eta}{dt^2} = \varepsilon Y', \tag{2}$$

$$m_o \frac{d^2\zeta}{dt^2} = \varepsilon Z', \tag{3}$$

where m_o and ε are the rest mass and the charge of the particle, respectively.

According to Einstein's 1905 paper,[1] with the help of the Lorentz transformation for the space-time coordinates,

$\xi = \gamma(x - vt);\ \eta = y;\ \zeta = z;\ \tau = \gamma(t - vx/c^2),$

and for the electromagnetic field components,

$X' = X;\ Y' = \gamma(Y - vN/c);$
$Z' = \gamma(Z + vM/c),$

Eqs. (1)–(3) are transformed in the system K as

$$\frac{d^2x}{dt^2} = \frac{\varepsilon}{m_o\gamma^3} X,$$

$$\frac{d^2y}{dt^2} = \frac{\varepsilon}{m_o\gamma}\left(Y - \frac{v}{c}N\right),$$

$$\frac{d^2z}{dt^2} = \frac{\varepsilon}{m_o\gamma}\left(Z + \frac{v}{c}M\right),$$

which can be written in the form

$$m_o\gamma^3\frac{d^2x}{dt^2}=\varepsilon X=\varepsilon X', \quad (4)$$

$$m_o\gamma^2\frac{d^2y}{dt^2}=\varepsilon\gamma\left(Y-\frac{v}{c}N\right)=\varepsilon Y', \quad (5)$$

$$m_o\gamma^2\frac{d^2z}{dt^2}=\varepsilon\gamma\left(Z+\frac{v}{c}M\right)=\varepsilon Z'. \quad (6)$$

$\varepsilon X'$, $\varepsilon Y'$, and $\varepsilon Z'$ are, as Einstein put it, the components of the ponderomotive force acting upon the charged particle, or simply the "force acting upon a material ponderable point". Therefore, maintaining the equation $mass\times acceleration=force$, Eqs.(4)–(6), in which the second derivative terms are the particle acceleration components as measured from the stationary system, imply the particle's "traveling" mass measured from the stationary system can be given by

$Longitudinal\ mass=m_l=\gamma^3 m_o$;

$$m_l=m_o\Big/\left(\sqrt{1-v^2/c^2}\right)^3 \quad (7)$$

$Transverse\ mass=m_t=\gamma^2 m_o$

$$m_t=m_o\Big/\left(1-v^2/c^2\right). \quad (8)$$

Equations (7) and (8) can be verified using simple dimensional analysis. In fact, a longitudinal force f_l in the system k acting upon a resting mass m_o, has the dimensional form of $mass\times length\big/time^2$,

$$f_l\equiv\frac{m_o\times d}{\tau^2},$$

which, by the help of the Special Relativity prediction of length contraction and time dilatation, is viewed from the system K as

$$F_l\equiv\frac{m_l\times D}{t^2}\equiv\frac{m_l\times d/\gamma}{(\gamma\times\tau)^2}.$$

Satisfying the requirement $f_l=F_l$ of Eq. (4), we get $m_l=\gamma^3 m_o$, verifying Eq. (7).

Similarly, a transverse force f_t in the system k acting upon a resting mass m_o has the dimensional form of $mass\times length\big/time^2$,

$$f_t\equiv\frac{m_o\times h}{\tau^2},$$

which, by the aid of the Special Relativity prediction of time dilatation and invariant transverse length, is viewed from the system K as

$$F_t\equiv\frac{m_t\times h}{t^2}\equiv\frac{m_t\times h}{(\gamma\times\tau)^2}.$$

Satisfying the requirement $f_t=F_t$ of Eq. (5) or (6), we get $m_t=\gamma^2 m_o$, verifying Eq.(8).

3. Relativistic mass predicaments

3.1 Relativistic mass by conservation of momentum

Considering our systems $K(x,y,z,t)$ and $k\left(\xi,\eta,\zeta,\tau\right)$, let there be a ponderable material point, or a body of mass m_o, traveling at velocity w_η in the transverse direction with respect to the system k. The body is at rest in the longitudinal direction relative to k $(w_\xi=w_\zeta=0)$. Suppose the momentum $p_\eta=m_o w_\eta$ of the body is just sufficient to break a glass chip at rest in k and intercepting the body's motion. According to the relativistic velocity addition, the transverse component W_y of the body's velocity with respect to K is given by w_η/γ. If the mass was invariant, the body's transverse momentum relative to K would be $P_y=m_oW_y=m_o\,w_\eta/\gamma=p_\eta/\gamma$, insufficient to break the chip, creating an absurdity. Thus the body's momentum relative to K must be greater than or equal to its momentum in k . Hence, its mass must be $m \geq m_o \gamma$. Now, assuming the Body's momentum in k is just below the threshold required to break a different glass chip, the corresponding momentum P_y relative to K must then be equal to or less than its momentum p_η in k. Hence, its mass must be $m \leq m_o \gamma$. It follows that the same body's mass measured from the system K could be either $m \geq m_o \gamma$ or $m \leq m_o \gamma$, depending on some careful selection of the glass chip in either case. Hence, $m = m_o \gamma$ must hold in order for the two results to agree on the mass, making the body's transverse momentum relative to K equal to its momentum in k. Hence, the mass invariance assumption can't hold, and the body's mass in K must be converted to

$$m = m_o \Big/ \sqrt{1 - v^2 / c^2}\,. \tag{9}$$

The above conclusion could be arrived at using the following argument. The body's transverse velocity W_y as viewed from the system K is reduced by the factor γ compared to its velocity w_η in k, due to the time dilation predicted by the Special Relativity. Accordingly, the body's transverse momentum would be equally reduced by the same factor, had the mass been invariant. However, this change in transverse momentum can't be justified with the absence of any relative motion in the transverse direction. Hence, the body's mass must increase by the factor γ in order to conserve its transverse momentum. In other words, the time dilation should result in a mass increase—in order to compensate for the incurred momentum loss. What an absurdity!

Equation (9), required to reconcile the Special Relativity with the conservation of momentum law, is in contradiction with the prediction of the longitudinal and transverse mass obtained from the Lorentz transformation, as given by Eq. (8).

3.2 Relativistic mass by conservation of energy

On the other hand, assume the considered body is set up in transverse motion in k $(\Delta\xi = 0)$ by exerting a constant force f_η in the transverse direction. Let Δp_η be the transverse momentum picked up by the body within an interval of time $\Delta\tau$. The exerted force can then be written as $f_\eta = \Delta p_\eta / \Delta\tau$. If we let the transverse distance travelled during that time interval be $\Delta\eta$, the work done on the body by the exerted force will be given by

$$\mathrm{w} = \frac{\Delta p_\eta}{\Delta\tau}\Delta\eta. \tag{10}$$

From the perspective of K, the work done on the body is given by

$$\mathrm{W} = \frac{\Delta P_y}{\Delta t}\Delta y, \tag{11}$$

where ΔP_y is the picked up transverse momentum relative to K, Δt the elapsed time, and Δy the corresponding transverse distance with respect o K.

As seen earlier, the body momentum must be the same relative to both frames. Due to the invariance of the transverse spatial dimension, and the time dilation obtained from the Lorentz transformation $\Delta t = \gamma(\Delta\tau + v\Delta\xi / c^2) = \gamma\Delta\tau$, since $\Delta\xi = 0$, Eq. (11) leads to

$$\mathrm{W} = \frac{\Delta p_\eta}{\gamma\Delta\tau}\Delta\eta = \frac{\mathrm{w}}{\gamma}. \tag{12}$$

Hence, the work done on the body, or the absorbed energy, depends on the reference frame, which is in contradiction with the relativity principle and the conservation of energy law.

It follows that the invariance of the transverse momentum leads to the relativistic mass equation. Yet, this invariance leads to the violation of the energy conservation. To maintain the energy conservation, we must have, using Eqs. (11) and (12), $\Delta P = \gamma\Delta p$, or $mW_y = \gamma m_o w_\eta$, yielding $m w_\eta / \gamma = \gamma m_o w_\eta$, or

$$m = \gamma^2 m_o = \frac{m_o}{1 - v^2 / c^2}, \tag{13}$$

contradicting the relativistic mass $m = \gamma m_o$, obtained in connection with the momentum conservation, yet confirming the transverse mass eq.(8) presented by Einstein (1905) as obtained by the means of the Lorentz transformation.

4. Kinetic energy equations under force = mass x acceleration

Let there be a ponderable material point, call it a body, of rest mass m_o acted upon by a constant force F in the transverse direction. Suppose at the instant of time $t_o = 0$ the body is at the system K origin. Let v be the velocity of the body at the time t, and m its mass, all with respect to K. Relative to the system k traveling at the velocity v, the mass of the body at the time instant t would be m_o (body is at rest in k at this instant). The kinetic energy acquired by the body at this time is given by $E_k = \int F dx$. Applying Newton's second law of motion $\left(F = m\, d^2x/dt^2 = m\, dv/dt\right)$, the kinetic energy can be written as $E_k = \int m\left(dv/dt\right)dx$. Since $dx = vdt$, we get

$$E_k = \int mvdv. \qquad (14)$$

Evidently, if the mass was constant, Eq. (14) will result in the classical kinetic energy equation $E_k = 1/2(m_o v^2)$.

4.1 Using the longitudinal mass obtained from Lorentz transformation

As the force is impressed in the motion direction, applying Eq.(7) for the longitudinal mass $\left(m = \gamma^3 m_o\right)$ obtained in Einstein's 1905 paper, we get

$$E_k = \int_0^v \frac{m_o}{\left(\sqrt{1 - v^2/c^2}\right)^3} vdv =$$

$$= m_o \frac{c^2}{\sqrt{1 - v^2/c^2}}\Bigg|_0^v .$$

Hence,

$$E_k = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} - m_o c^2; \qquad (15)$$

$$E_k = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) m_o c^2, \qquad \mathbf{(16)}$$

which is the relativistic kinetic energy formula obtained in Einstein's 1905 paper, based on the longitudinal mass $\left(m = \gamma^3 m_o\right)$ obtained from the Lorentz transformation, and the definition of force being $mass \times acceleration$.

Since the first term of the right hand side of Eq.(15) is not equal to the longitudinal mass in the used context of Eq. (7), the latter equation has no implication of mass-energy equivalence. It is just a formula for the kinetic energy. For $v \ll c$, Eq. (16) can be written to the second order as

$$E_k = \left[\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) - 1\right] m_o c^2 = \frac{1}{2} m_o v^2,$$

which is the classical formula for the kinetic energy.

4.2 Using the relativistic mass obtained from the conservation of momentum

Now, had we used Eq. (9) for the relativistic mass $\left(m = \gamma m_o\right)$, an ad-hoc to satisfy the conservation of momentum, Eq. (14) would give

$$E_k = \int_0^v \frac{m_o}{\sqrt{1 - v^2/c^2}} vdv =$$

$$= -m_o c^2 \sqrt{1 - v^2/c^2}\Big|_0^v ;$$

$$E_k = \left(1 - \sqrt{1 - v^2/c^2}\right) m_o c^2. \qquad \mathbf{(17)}$$

This is another formula (contradicting the one obtained earlier) of the relativistic kinetic energy based on the relativistic mass $\left(m = \gamma m_o\right)$, and on the force being defined as $mass \times acceleration$.

Again, for $v \ll c$, Eq. (17) can be written to the second order as

$$E_k = \left[1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\right] m_o c^2 = \frac{1}{2} m_o v^2,$$

which is the classical formula for the kinetic energy.

5. Kinetic energy equations under force = momentum change rate

If the force acting upon the body is defined as $F = d(mv)/dt$—which would be equivalent to our earlier definition $(F = m\, dv/dt)$, had the mass been invariant—the kinetic energy acquired by the body becomes $E_k = \int (dmv/dt) dx$. Hence

$$E_k = \int \left(m dv/dt\right) dx + \int \left(v\, dm/dt\right) dx;$$

$$E_k = \int_0^v mvdv + \int_{m_o}^m v^2 dm. \qquad \textbf{(18)}$$

Evidently, if the mass was constant, Eq.(18) will result in the classical kinetic energy equation $E_k = 1/2(m_o v^2)$.

5.1 Using the relativistic mass obtained from the conservation of momentum

Using Eq. (9) for the relativistic mass $(m = \gamma m_o)$ and solving it for v^2 as a function of m, we obtain

$$v^2 = c^2 - \frac{m_o^2 c^2}{m^2}.$$

Hence, Eq. (18) leads to

$$E_k = \int_0^v \frac{m_o v}{\sqrt{1 - v^2/c^2}} dv + \int_{m_o}^m c^2 dm - \int_{m_o}^m \frac{m_o^2 c^2}{m^2} dm;$$

$$E_k = -m_o c^2 \sqrt{1 - v^2/c^2}\Big|_0^v + mc^2\Big|_{m_o}^m + \frac{m_o^2 c^2}{m}\Big|_{m_o}^m;$$

$$E_k = -\frac{m_o c^2}{\gamma} + m_o c^2 + mc^2 - m_o c^2 + \frac{m_o}{m} m_o c^2 - m_o c^2;$$

$$E_k = mc^2 - m_o c^2; \qquad \textbf{(19)}$$

$$E_k = (\gamma - 1) m_o c^2.$$

Equation (19) has the implication of energy-mass equivalence, with $E_o = m_o c^2$ being the body rest energy, and $E = mc^2$ its total energy (rest + kinetic energy). As shown earlier, for $v \ll c$, the latter equation leads to the classical kinetic energy equation $(E_k = 1/2\, m_o v^2)$.

Nevertheless, Eq. (19) obtained based on the definition of force $F = dP/dt$ and relativistic mass $m = \gamma m_o$, is not in agreement with the longitudinal mass $(m = \gamma^3 m_o)$ and force definition $(F = mass \times acceleration)$ adopted in the Special Relativity original paper.[1]

5.2 Using the longitudinal mass obtained from Lorentz transformation

Had we used Eq.(7) for the longitudinal mass $(m = \gamma^3 m_o)$ as obtained in Einstein's 1905 paper, Eq.(18) would yield

$$E_k = \int_0^v \frac{m_o v}{\left(\sqrt{1 - v^2/c^2}\right)^3} dv + \int_{m_o}^m c^2 dm - \int_{m_o}^m \frac{m_o^3 c^2}{m^3} dm;$$

$$E_k = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}}\Bigg|_0^v + mc^2\Big|_{m_o}^m + \frac{m_o^3 c^2}{2m^2}\Bigg|_{m_o}^m;$$

$$E_k = \left(\gamma^3 + \gamma + \frac{1}{2\gamma^6} - \frac{5}{2}\right) m_o c^2; \qquad \textbf{(20)}$$

$$E_k = \begin{pmatrix} \left(1-\frac{v^2}{c^2}\right)^{-3/2} + \left(1-\frac{v^2}{c^2}\right)^{-1/2} + \\ +\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{3} - \frac{5}{2} \end{pmatrix} m_o c^2,$$

which, for $v \ll c$, can be written to the second order as

$$E_k = \begin{pmatrix} \left(1+\frac{3}{2}\frac{v^2}{c^2}\right) + \left(1+\frac{1}{2}\frac{v^2}{c^2}\right) + \\ +\frac{1}{2}\left(1-3\frac{v^2}{c^2}\right) - \frac{5}{2} \end{pmatrix} m_o c^2;$$

$$E_k = \frac{1}{2} m_o v^2,$$

which is the classical formula for the kinetic energy.

Equation (20), being based on the Special Relativity longitudinal mass derivation from the Lorentz transformation, and on the more general definition of force as $F = dP/dt$ (rather than $F = m\,dv/dt$), it is the most representative of the kinetic energy in the context of the Special Relativity. Yet, it is far off from implying the general energy equation $E = mc^2$, boasted as being the most remarkable prediction of the special relativity theory!

Conclusion

The Special Relativity transformation equations result in longitudinal and transverse mass formulae contradicting the relativistic mass equation $(m = \gamma m_o)$ required to maintain the conservation of momentum under the Special Relativity predictions. This relativistic mass contradicts in turn the undesired relative mass obtained in satisfying the conservation of energy law under the Special Relativity, in conformance with its transverse mass equation derived from the Lorentz transformation. In defining the force as $mass \times acceleration$, each of the aforementioned longitudinal and relativistic mass formulae results in a different equation for the kinetic energy. The formula obtained under the Special Relativity assumptions doesn't actually imply the claimed equivalence of mass and energy. On the other hand, if the force was defined as being the momentum change rate, another two different formulae for the relativistic kinetic energy will be obtained; one for each of the two aforementioned longitudinal and relativistic mass formulae. All these formulae lead to the classical kinetic energy equation when $v << c$. The relativistic kinetic energy obtained under the relativistic mass $(m = \gamma m_o)$ and the force as being the momentum change rate, is given by the formula $E_k = \Delta mc^2$, implying the famous equation for the total energy: $E = mc^2$. However, it is not in agreement with the genuine Special Relativity predictions, and imprecisely attributed to it, not forgetting that the other three different obtained formulae are all validated from the Special Relativity perspective, creating a detrimental incoherence in the Special Relativity. In addition, the relativistic kinetic energy equation is based on the relativistic mass ensuing from the conservation of momentum law in the Special Relativity framework, but resulting in its violation of the law of conservation of energy.

1 A. Einstein, "Zur elektrodynamik bewegter Körper," Annalen der Physik **322** (10), 891–921 (1905).

2 A. Einstein, "Does the Inertia of a Body Depend Upon Its Energy Content?," in A. Einstein et al. (1952), pp. 69–71 (1905); A. Einstein, H.A. Lorentz, H. Minkowski, and H. Weyl, *The Principle of Relativity*. (Dover, New York, 1952).

3 A. Einstein, "Elementary Derivation of the Equivalence of Mass and Energy," American Mathematical Society Bulletin **4**, 223—230 (1935).