TWO RESULTS ON ZFC:

(1) IF ZFC IS CONSISTENT THEN IT IS DEDUCTIVELY INCOMPLETE,

(2) ZFC IS INCONSISTENT

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Abstract

The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) system of axioms for set theory appears to be inconsistent. A step in developing this proof is the observation that ZFC would be deductively incomplete if it were consistent. Both points are proven by means of the singleton. The axioms are still too lax on the notion of a well-defined set.

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1. Introduction

1.1. The problem

The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) system of axioms for set theory is studied here with a focus on the singleton. Section 2 defines the case. Section 3 shows deductive incompleteness, i.e. that there is a truth that cannot be derived. Section 4 derives this truth and thus shows inconsistency. Section 5 discusses the results. Section 6 concludes.

This introduction proceeds with basic definitions and theorems.

1.2. Definition of ZFC

We take the definition of ZFC from a matricola course in set theory at Leiden and Delft.

Definition (Coplakova et al. (2011:18), I.4.7): Let A be a set. The power set of A is the set of all subsets of A. Notation: $P[A]$. Another notation is 2^A, whence its name.

Definition (Coplakova et al. (2011:144-145)): ZFC.

Remark. This includes the axiom that each set has a power set. (POW)

Definition of the *Axiom of Separation* (Coplakova et al. (2011:145), inserting here a by-line on freedom): If A is a set and $\gamma[x]$ is a formula with variable x, while B is not free in $\gamma[x]$, then there exists a set B that consists of the elements of A that satisfy $\gamma[x]$:

$$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in A) \wedge \gamma[x])) \qquad \text{(SEP)}$$

Remark. This is also called an axiom-schema since there is no quantifier on γ.

1.3. Cantorian sets in ZFC

The following conjectures are commonly accepted "theorems of ZFC". Below we will show that ZFC is inconsistent, so that it proves everything. Thus these aren't "theorems" but *conjectures.* Below we will use "theorem" for what this paper proves itself.

Conjecture. (E), (*Existence of strictly Cantorian sets*). Let A be a set, $P[A]$ its power set. For every function $f : A \to P[A]$ there is a set $\Psi = \{x \in A \mid x \notin f[x]\}$.

Proof. (a) $P[A]$ exists because of the Axiom of the Power Set. (b) f can be regarded as a subset of A x $P[A]$, and f exists because of Axiom of Pairing. (c) Ψ exists because of the Axiom of Separation. Q.E.D.

Remark. Find $\Psi \subseteq A$, thus $\Psi \in P[A]$. Observe that Ψ depends upon f, i.e. $\Psi = \Psi[f]$.

When $x \in A$ then we can use $(x \in \Psi) \Leftrightarrow (x \notin f[x])$.

Definition of a Cantorian set. Above $\Psi = \{x \in A \mid x \notin f[x]\}$ is called a *strictly* Cantorian set. A generalised Cantorian set has $x \notin f[x]$ as part of its definition. The meaning of 'Cantorian set' without qualification depends upon the context.

Remark. The Cantorian set clearly has some kind of self-reference.

Conjecture. (W), (*Weakest conjecture on strictly Cantorian sets.*) Let (domain) A be a set, and (range) $B \subseteq P[A]$. For every $f : A \to B$ there is a $\Psi \in P[A]$ such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$.

Proof. Define $\Psi = \{x \in A \mid x \notin f[x]\}$. Take $\alpha \in A$. Check the two possibilities.

Case 1: $\alpha \in \Psi$. In that case $\alpha \notin f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in \Psi \setminus f[\alpha]$.)

Case 2: $\alpha \notin \Psi$. In that case $\alpha \in f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in f[\alpha] \setminus \Psi$.) Q.E.D.

*Remark.*This conjecture combines the definition of strictly Cantorian sets, the existence Conjecture E and an identification of their key property. It is essentially a rewrite of:

(i) $\forall \alpha \in A$: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ and application of extensionality (EXT),

(ii) which is deducible from $\forall \alpha$: $(\alpha \in \Psi) \Leftrightarrow ((\alpha \in A) \wedge (\alpha \notin f[\alpha]))$, i.e. the definition of Ψ.

1.4. Appendices

This paper leans more on logic than set theory. The author is no expert on ZFC but wrote a book on logic, see Colignatus (1981 unpublished, 2007, 2011) *A Logic of Exceptions* (**ALOE**). See for background Colignatus (2013). This present paper has been taken from pages 61-72 of Colignatus (2015a) (**FMNAI**). **Appendix A** discusses the versions of ALOE, for proper reference. **Appendix B** has more on the genesis of this paper.

2. The singleton

2.1. The singleton with a nutshell link between Russell and Cantor

Let A be a set with a single element, $A = \{\alpha\}$. Thus $P[A] = \{\varnothing, A\}$. Let $f: A \to P[A]$.

If $f[\alpha] = \varnothing$ then $\alpha \notin f[\alpha]$. If $f[\alpha] = A$ then $\alpha \in f[\alpha]$.

Thus $(f[\alpha] = \varnothing) \Leftrightarrow (\alpha \notin f[\alpha])$.

Consider:

(1) In steps: define $\Psi = \{x \in A \mid x \notin f[x]\}$, then try $f[\alpha] = \Psi$.

(2) Directly: $f[\alpha] = \{x \in A \mid x \notin f[x]\}$.

(3) Either directly or indirectly via (1) or (2): $\Psi = \{x \in A \mid x \notin \Psi\}$.

The latter is a variant of Russell's paradox: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$. Thus (1) - (3) are only consistent when $\Psi \neq f[\alpha]$. This is an instance of Conjecture W.

Choosing $f[\alpha] = \Psi$ in (1) assumes freedom that conflicts with the other properties. We have liberty to choose $f[\alpha] = \varnothing$ or $f[\alpha] = A$. This choice defines f and we should write is $\Psi = \Psi[f]$ indeed. This shows why (2) with $f[\alpha] = \Psi[f]$ is tricky. If (2) is an implicit definition of f then it doesn't exist. If it exists then this $f[\alpha]$ will not be in its definition.

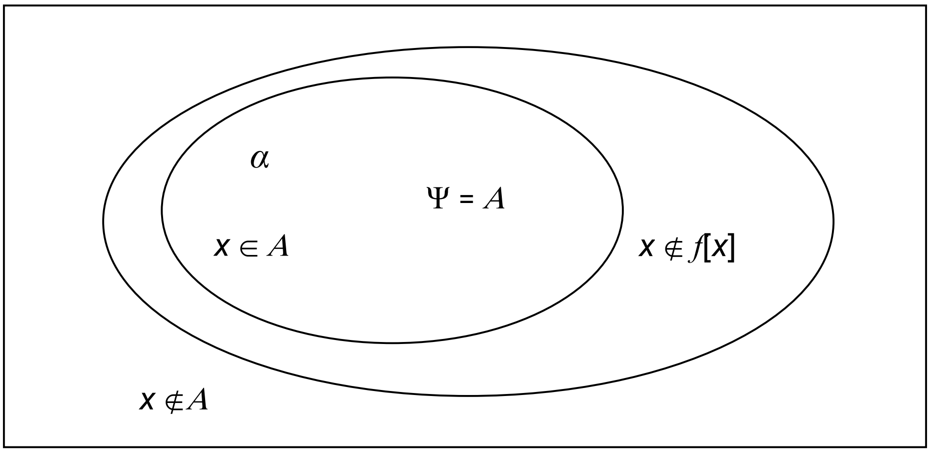
2.2. Possibilities for the singleton

Checking all possibilities in the former subsection gives Table 1. The cells are labeled with Δ-*case-numbers*. The Δ refers to a *difference analysis* when a set is extended with single element. Because of $\alpha \notin f[\alpha]$, the row $f[\alpha] = \varnothing$ is important for us. The case of Δ2 is depicted in a Venn-diagram in Figure 1.

Table 1. Possibilities for $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha])$ **given that** $\alpha \in A$

For all cases: $\alpha \in A$	$\Psi = \varnothing$, $\alpha \notin \Psi$	$\Psi = A$, $\alpha \in \Psi$
$f[\alpha] = \varnothing$ $\alpha \notin f[\alpha]$	**Δ1**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin \varnothing$ $f[\alpha] = \Psi$, **impossible**	**Δ2**: $\alpha \in A \Leftrightarrow \alpha \notin \varnothing$ $f[\alpha] \neq \Psi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	**Δ3**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin A$ $f[\alpha] \neq \Psi$, possible	**Δ4**: $\alpha \in A \Leftrightarrow \alpha \notin A$ $f[\alpha] = \Psi$, **impossible**

Figure 1. The strictly Cantorian set for the singleton, case Δ2: f[α] = Ø ≠ Ψ



3. Deductive incompleteness

3.1. Existence of Δ1

An idea is that Ψ in Conjecture E and Conjecture W or Table 1 covers all $\alpha \notin f[\alpha]$. This appears to be false: it doesn't cover Δ1. The cell is declared *impossible*. Let us first verify that it exists as a truth (outside of ZFC), and then accept deductive incompleteness.

Theorem. (ExT1) Case Δ1 exists as a possibility with $\alpha \notin f[\alpha]$.

Proof. We consider the case $f[\alpha]$ = Ø, so that ($\alpha \notin f[\alpha]$).

Take q = ($\alpha \notin f[\alpha]$) and use tautology T1: $q \Rightarrow (p \Leftrightarrow (q \wedge p))$ for any p, see Table 2.

We are free to take p = ($\alpha \in A$) for Ψ = A, which would give Δ2, or not-p = ($\alpha \in$ Ø) for Ψ = Ø, which would give Δ1. Take the latter, apply modus ponens on q and tautology T1, and find $\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \wedge \alpha \in \varnothing)$. The equivalence reduces into $\alpha \notin$ Ø or $\alpha \in A$. The equivalence is by itself consistent, so that it is possible for $\alpha \in \{ \alpha \}$. Case Δ1 with both $f[\alpha]$ = Ø and Ψ = Ø, fits this equivalence: $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$. We merely establish possibility, and thus the deduction stops here. Q.E.D.

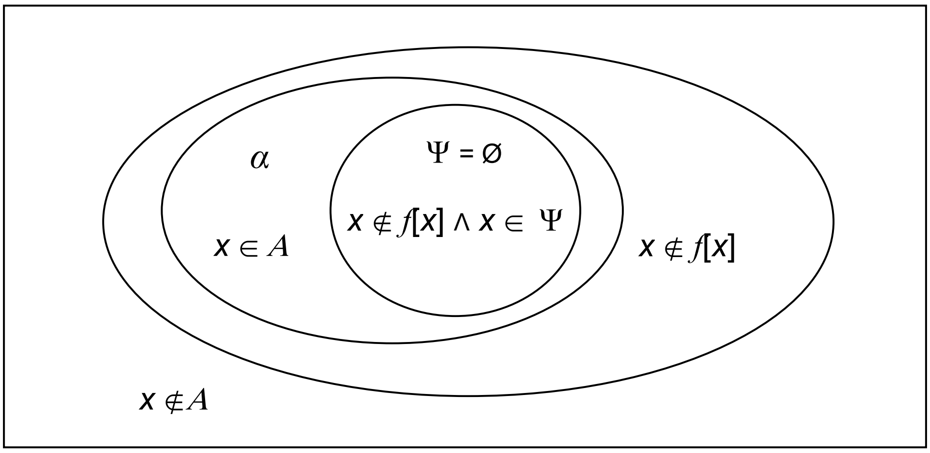
Table 2. Truthtable for T1: $q \Rightarrow (p \Leftrightarrow (q \wedge p))$ with $q = (\alpha \notin f[\alpha])$

Case	α ∉ f[α]	⇒	(p	⇔	(α ∉ f[α]	∧	p))
Δ2	1	1	1	**1**	1	1	1
Δ4	0	1	1	**0**	0	0	1
Δ1	1	1	0	**1**	1	0	0
Δ3	0	1	0	**1**	0	0	0

Remark. Δ1 can exist outside of ZFC based upon other axioms than ZFC, see FMNAI. We are tempted to derive inconsistency now, but for understanding of the situation it is better to formally establish deductive incompleteness.

The case shows up by requiring that *all* properties of the case hold, thus jointly ($\alpha \notin \varnothing \wedge \alpha \in \varnothing$) and not only ($\alpha \notin \varnothing$). See Figure 2. The diagram uses that $\varnothing \subseteq A$. In this approach for Δ1 or in the figure: when we test $\alpha \in \Psi$, then we test $\alpha \notin f[\alpha] \wedge \alpha \in \Psi$ jointly. From this joint test it follows that $\alpha \notin \Psi$, or $\alpha \in A$.

Figure 2. The strictly Cantorian set for the singleton, case Δ1: $f[\alpha] = \varnothing = \Psi$



3.2. Definition, theorem and proof

Definition. (DeLong (1971:132)): "A formal system is deductively complete if under the intended interpretation there is no truth which is not also a theorem."

Theorem. (D) If ZFC is consistent then it is deductively incomplete.

Proof. Let $A = \{\alpha\}$ have a single element. Thus $P[A] = \{\varnothing, A\}$.

Let $f: A \to P[A]$ with $f[\alpha] = \varnothing$. Then $\alpha \notin \varnothing$ and $\alpha \notin f[\alpha]$.

Under the intended interpretation, there is the case Δ1 that has $\alpha \notin f[\alpha]$.

Ψ is formulated such that it should contain *all cases* with $\alpha \notin f[\alpha]$.

However, trying to prove that Δ1 fits Ψ, causes a shift to Δ2 or $\Psi = A$ (Conjecture W).

If ZFC is consistent then there is no path to reach Δ1. Q.E.D.

Remark. If there is such a path then ZFC becomes inconsistent.

4. Inconsistency

4.1. An implication for the singleton Cantorian set

For the singleton we have $\alpha \in A$, and thus we have $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. It is possible to weaken this by means of another tautology T2: $\forall p,q$: $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \wedge p))$. The truthtable for the singleton Cantorian set is in Table 3. The truthtable holds for every f while $\Psi = \Psi[f]$.

Table 3. Truthtable for T2: $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \wedge p))$ **for the singleton Cantorian set**

Case	$(\alpha \in \Psi$	$\Leftrightarrow$	$\alpha \notin f[\alpha])$	$\Rightarrow$	$(\alpha \in \Psi$	$\Leftrightarrow$	$(\alpha \notin f[\alpha]$	$\wedge$	$\alpha \in \Psi))$
Δ2	1	**1**	1	1	1	**1**	1	1	1
Δ4	1	**0**	0	1	1	**0**	0	0	1
Δ1	0	**0**	1	1	0	**1**	1	0	0
Δ3	0	**1**	0	1	0	**1**	0	0	0

Consider again $f[\alpha] = \varnothing$. The equivalence on the LHS only allows $\Psi = A$. Look at row Δ1 in Table 3. On the LHS we have Δ1 with $(\alpha \notin \Psi) \wedge (\alpha \notin f[\alpha])$, and the equivalence would declare this combination impossible. However, there is also the relaxed condition on the RHS, that we already encountered in Theorem ExT1.

The crucial step is to distill the RHS from the table. Conjecture E and Conjecture W establish the LHS. Modus ponens with T2 gives the RHS as a separate expression FT2 ('from T2') - provided that we maintain $\Psi = \Psi[f]$:

$$\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi) \qquad \text{(FT2)}$$

For FT2 we get get Table 4. The same Δ-*case-numbers* apply. Now Δ1 is allowed too: a possible $f[\alpha] = \Psi$ rather than an impossible $f[\alpha] = \Psi$.

Table 4. Possibilities for $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$**, given** $\alpha \in A$

For all cases: $\alpha \in A$	$\Psi = \varnothing$, $\alpha \notin \Psi$	$\Psi = A$, $\alpha \in \Psi$
$f[\alpha] = \varnothing$ $\alpha \notin f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing$ & $\alpha \in \varnothing)$ $f[\alpha] = \Psi$, **possible**: $\alpha \notin \varnothing$	$\alpha \in A \Leftrightarrow (\alpha \notin \varnothing$ & $\alpha \in A)$ $f[\alpha] \neq \Psi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin A$ & $\alpha \in \varnothing)$ $f[\alpha] \neq \Psi$, possible	$\alpha \in A \Leftrightarrow (\alpha \notin A$ & $\alpha \in A)$ $f[\alpha] = \Psi$, **impossible**: $\alpha \notin A$

Note that $f[\alpha] = \varnothing$ doesn't give a unique Ψ now. Both $\Psi = A$ (Figure 1) and $\Psi = \varnothing$ (Figure 2) are possible. Note that f is still a function and no correspondence.

PM. We can also gain access to Δ4 by another relaxing condition but we are interested in the $\alpha \notin f[\alpha]$ case.

4.2. A counterexample for Conjecture W

Let us make the latter observations formal. The discovery of Δ1 and tautology T2 gives a contradiction to Conjecture W. While Theorem D stopped looking for a path towards Δ1, we now found that path, namely tautology T2, which gives Theorem Not-W. When Δ1 not merely exists as a truth outside of ZFC (using tautology T1) but also can been proven from Ψ (using tautology T2), then it becomes a counterexample for Conjecture W, which gives Theorem Not-ZFC.

Theorem. (Not-W) For the strictly Cantorian case there are a f and Ψ with $f[\alpha] = \Psi$.

Proof. Let $A = \{\alpha\}$ have a single element. Thus $P[A] = \{\varnothing, A\}$.

Let $f: A \to P[A]$. (*Remark* 1.) Let $f[\alpha] = \varnothing$. Then $\alpha \notin \varnothing$ and $\alpha \notin f[\alpha]$.

Consider $\Psi = \{x \in A \mid x \notin f[x]\}$. With $\alpha \in A$ we use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. (*Remark 2.*)

Look at Table 3. Use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ and tautology T2 $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \wedge p))$, and apply modus ponens to find FT2: $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$. In this deduction we have maintained the definition of Ψ. The modus ponens is independent of the possibility that also $\Psi = A$ might be derived via another route. The formula FT2 *stands as a separate relation* for $\Psi = \{x \in A \mid x \notin f[x]\}$.

A substitution of $f[\alpha] = \varnothing$ and $\Psi = \varnothing$ into FT2 gives Δ1: $\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \wedge \alpha \in \varnothing)$ that we saw above in Theorem ExT1 and subsequently inTable 4, which reduces to $\alpha \notin \varnothing$, or $\alpha \in A$. The case is consistent by itself. And we have established a path to it. The case has $f[\alpha] = \varnothing = \Psi$. Q.E.D.

Remark 1. In an algebraic version of the proof, we would not assign a value to f but first derive FT2 and only then assign such value. This might however run into the objection that such algebraic application is not possible, and that application of the Axiom of Separation requires a definition of f beforehand.

Remark 2. At this point in the proof we do not follow the deductive path that leads to a deduction of $\Psi = A$, since this has been already done in Conjecture W. Making this deduction here again confuses the two different proofs. We are now interested in the other path that follows from T2. Inconsistency of ZFC can better be established by the separate Theorem Not-ZFC.

Theorem. (Not-ZFC) ZFC is inconsistent.

Proof. For the singleton, Conjecture E and Conjecture W generate that $\Psi = A$. Theorem Not-W generates the possibility that $\Psi = \varnothing$. Thus it is possible that $A = \varnothing$. This is a clear contradiction. Q.E.D.

5. Discussion

5.1. Nominalism versus realism

This paper deals with self-reference and derives a contradiction. It may thus be difficult to follow. The reader can maintain clarity by holding on to the key notion of *freedom of definition.* When a restriction on that freedom generates a consistent framework, while release of that restriction generates confusion, then the restriction is to be preferred above too much freedom. Amendment of ZFC thus will tend to reduce the freedom of definition, unless one allows for a three-valued logic that is strong enough to recognise nonsense.

We can look at Table 1 and Table 4 in horizontal or vertical direction. This reflects the schism in philosophy between *nominalism* and *realism*. See William of Ockham (Occam).

(1) The horizontal view gives the **realists** who take predicates as 'real': $\alpha \notin f[\alpha]$ versus $\alpha \in f[\alpha]$. They are also sequentialist: Δ1 & Δ2 versus Δ3 & Δ4.

(2) The vertical view gives the **nominalists** (Occam) who regard the horizontal properties as mere names or stickers, and who more realistically look at Ψ = Ø versus $\Psi = A$. They see the table in *even* versus *uneven* fashion: Δ2 & Δ4 versus Δ1 & Δ3.

The nominalist reasoning is: The sets Ø and *A* exist. We are merely discussing how they are referred to. The expression for Ψ is not a *defining* statement but an *inferential* observation. Once the functions have been mapped out, the criteria can be used to see whether the underlying sets may get also another sticker Ψ. We are discussing *'consistent referring'* and not existence.

At issue is now whether ZFC has sufficient logical strength to block nonsensical situations. ZFC has a realist bend. It translates predicates into sets (their extensions). Instead it can be better to *only test* whether a predicate is useful. Merely cataloguing differently what already exists should not be confused with existence itself. The freedom of definition can be a mere illusion and then should not be abused to create nonsense.

A defining equivalence for Ψ on the LHS of Table 3 results via the tautology into a weaker relation on the RHS that contradicts that definition.

The problem with Conjecture E and Conjecture W is that they *impose* the equivalence on the LHS. This assumes a freedom of definition, whence this assumes that the truthtable on the LHS is true, whence Δ1 is forbidden. But that freedom of definition does not exist. Something exists, that is infringed upon by the definition. When Ψ is the empty set, as in the singleton possibility of Figure 2, then one no longer has the freedom to switch from Ø to *A,* see Figure 1.

The discussion is not without consequence, see FMNAI:79:

> The logical construction $x \notin f[x]$ and only a single problematic element, in badly understood self-reference, should not be abused to draw conclusions on the infinite. There are ample reasons to look for ways how this can be avoided.

It would not be a solution to repair ZFC in such a way that the transfinites would be saved, since they are a figment of $x \notin f[x]$ confusions, and there is no intended interpretation for them outside of those Cantorian confusions.

If one would hold that ZFC is consistent, against all logic, then one also accepts the transfinites – which makes one wonder what ZFC is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense would no longer be mathematics.

5.2. Diagnosis, and an axiom for a solution set

ZFC blocks Russell's paradox (essentially) by the *Axiom of Separation* (SEP). When the paradoxical $\gamma[x] = (x \notin x)$ is separated from *A* to create some set *R* then the conclusion follows that $R \notin A$, so that the separation cannot be achieved. For the Cantorian set we use $\alpha \in A$, or for the singleton $\alpha \in \{\alpha\}$, so that we can use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$, and there is no separation escape anymore. Separation is an irrelevant solution concept here, and what is at issue is self-reference that requires a fundamental solution.

The diagnosis is that Ψ is a variable (name) rather than a constant. There is a solution set $\Psi^* = \{Ø, A\}$, and Ψ is a variable that runs over Ψ^*. Compare to algebra, when one uses a variable x with value $x = 0$ in one case and $x = 1$ in another case: then one might derive $0 = x$

= 1, but this goes against the notion of a variable. The inconsistency in ZFC is caused by that it does not allow for that Ψ is such a variable.

The following is not in ZFC but will help to understand ZFC.

Definition. An *Axiom for a Solution Set* might be:

$$(\forall A)\ (\exists Z)\ (\exists B)\ ((B \in Z) \Leftrightarrow (\forall x)\ ((x \in B) \Leftrightarrow ((x \in A) \wedge \gamma[x]))) \qquad \text{(SOL)}$$

This SOL could reduce to the Axiom of Separation (SEP). A way is to eliminate $B \in Z$ as superfluous, with $Z = \{B\}$, or self-evident (which it apparently isn't). Another way is to replace $B \in Z$ by $B = Z$. This imposes uniqueness. When $\gamma[x]$ has more solutions, then a contradiction arises when SEP requires that a single solution B is also the whole set Z. ZFC has the latter effect.

For the singleton $A = \{\alpha\}$ with $f[\alpha] = \varnothing$, Conjecture W finds $B = \Psi = A$ but we find $Z = \{\varnothing, A\} = P[A]$. In itself it is true that $\Psi \in P[A]$, but when the solution set $Z = P[A]$ then it is erroneous to require $Z \in P[A]$.

It is not just an issue of notation. It is not sufficient to suggest to read Conjecture W now as generating a value for the variable, rather than restricting the solution set to that value. For, this reads something into SEP which it does not do: for it really restricts that solution set.

In ZFC Ψ creates the illusion of a unique set, and thus we need amendment of ZFC to correct that. One might hold that Conjecture E and Conjecture W are not necessarily wrong, since one can find for any f a $\Psi[f]$ such that for all α: $f[\alpha] \neq \Psi$. (For the singleton $f[\alpha] = \varnothing$ gives $\Psi = A$.) But the formula of Ψ allows Theorem ExT1 to also find another case with $f[\alpha] = \varnothing$. (For the singleton $f[\alpha] = \varnothing$ also gives $\Psi = \varnothing$.) One can conceive that the two options co-exist, but Conjecture W does not allow for $\Psi = \varnothing$. Thus Theorem Not-W is a real counterexample for Conjecture W.

The freedom of definition used in Conjecture W depends upon the existence Conjecture E. Then something is wrong with Conjecture E, that proved the existence of what Conjecture W uses. The Conjectures were derived in ZFC. Thus ZFC has a counterexample and thus is inconsistent.

Again, consider $f[\alpha] = \varnothing = \Psi$ ($\Delta 1$). This is consistent, but cannot be seen *'easily'* by Ψ, even though it is covered in Table 3 by the falsehood of $\alpha \in \Psi$. In a realist mode of thought, we deduce from $f[\alpha] = \varnothing$ that $\Psi = A$, which is the only possibility on the LHS for $\alpha \notin f[\alpha]$ that is recognised (row $\Delta 2$). This is not necessarily the proper response. The problem with ZFC is that it focuses on the LHS and neglects the RHS. We can derive a relaxed condition FT2, and then Theorem Not-W allows to recover $\Delta 1$. The latter deductions are actually within ZFC and thus there is scope to argue that Conjecture W presents only part of the picture. However, that part is formulated in such manner that it causes the contradiction in Theorem Not-ZFC. We must switch to a better axiomatic system that covers the *intended interpretation* and that blocks the paradoxical Ψ. The better system blocks the LHS and allows only the RHS.

While this analysis has a destructive flavour on ZFC, it is actually constructive since it indicates what the improvement will be. See FMNAI for possible new axioms.

5.3. Logical structure of this paper

The inconsistency shows itself in Table 3 with two cases on the LHS for Conjecture W and three cases on the RHS for Theorem Not-W. In itself it might be possible to use only this table and forget about deductive incompleteness. However, it is useful to build up understanding by first explaining such existence by the use of tautology T1.

The idea that Ψ in Conjecture E and Conjecture W or Table 1 covers all $\alpha \notin f[\alpha]$ appears to be false: it doesn't cover Δ1. Thus when $\alpha \notin f[\alpha]$ then there still exists a case of $\alpha \notin f[\alpha]$. Now, isn't Ψ *supposed* to cover *all* such cases ? The conclusion is: Conjecture E and Conjecture W do not cover the *intended interpretation* (DeLong (1971)). However: since Theorem Not-W deducts this neglected truth, and still is in ZFC, ZFC becomes inconsistent, can prove everything, and the notion of deductive incompleteness loses meaning.

Thus, looking at consistency only would lose sight of Theorem ExT1 and Theorem D.

Since Conjecture E and Conjecture W are well accepted in the literature and Theorem Not-W is new, there is great inducement to find error in it. Indeed, Theorem Not-W allows the deduction of a contradiction in Theorem Not-ZFC, and thus one might hold that it should go. However, its steps are correct. It is more productive for the reader to accept inconsistency of ZFC.

A discussion about self-reference that identifies a contradiction is always difficult to follow. The problem lies not in the identification of the logical framework of the situation but in the inconsistency of ZFC. Potentially the distinction between constant and variable has most effect for clarity. But it also helps to see the distinct roles of the two tautologies T1 and T2.

5.4. Further reading

This paper has been taken from FMNAI:61-72. See there for a longer discussion and a link to Cantor's Conjecture on the Power Set, and the transfinites, and suggestions for new axioms for set theory. Inconsistency of ZFC deproves Cantor's Conjecture on the Power Set. FMNAI also deproves Cantor's original proofs of 1874 and 1890/91, and defines the notion of *bijection by abstraction* between natural numbers $\mathbb{N}$ and real numbers $\mathbb{R}$, so that $\mathbb{N} \sim \mathbb{R}$.

Though ALOE presents elementary logic, it contains various innovations that are relevant but little-known. The common self-referential logical paradoxes have solution methods: (1) the theory of types (levels), (2) proof theory, (3) three-valued logic with true, false and nonsense. ALOE shows that the latter is superior, with a way to deal with the three-valued-Liar as well. An alternative for ZFC can be developed for two-valued logic (ZFC-PV) but it is likely more interesting to develop the alternative in three-valued logic (BST).

6. Conclusion

1. If ZFC is consistent then it is deductively incomplete (via tautology T1).
2. ZFC is inconsistent (via tautology T2). See FMNAI for alternatives.

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Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981 unpublished, 2007, 2011) (ALOE) existed in 1981 as *In memoriam Philetas of Cos,* then was rebaptised and self-published in 2007. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica.*

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. This edition rejects Cantor's standard set-theoretic argument. See ALOE:129 on Russell and ALOE:239 on Cantor, and observe that the same consistency condition is used, inspired by Paul of Venice (1369-1429).

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. This edition also refers to Cantor's original argument on the natural numbers $\mathbb{N}$ and real numbers $\mathbb{R}$, and suggests that $\mathbb{N} \sim \mathbb{R}$. The discussion of this is not in ALOE but in Colignatus (2012, 2013) (CCPO-PCWA), that also presents the notion of *bijection by abstraction.* This is superseded now by FMNAI so that CCPO-PCWA gets legacy status.

(4) ALOE is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory.

Appendix B: On the genesis of this paper

Colignatus (2013) explains my background and **Appendix A** explains about ALOE. It is a joy to see that the application of basic propositional logic helps to resolve the issue of this paper. It is quite conceivable that ZFC theorists simply don't have this affinity with logic – and the methodology of science – that I can advise to every student.

While I was satisfied with ALOE and its objectives on the logical paradoxes, it was only via CCPO-PCWA that I finally decided in 2014-2015 to look at ZFC. This became Colignatus (2014a), then (2014b, 2015) (PV-RP-CDA-ZFC), and then (2015a) (FMNAI). The latter FMNAI causes that PV-RP-CDA-ZFC now gets legacy status too.

Conjecture E is a reformulation of the addendum provided by B. Edixhoven, statement in Colignatus (2014a), its appendix D.

Conjecture W for $B = P[A]$ was given by K.P. Hart (TU Delft), 2012, in Colignatus (2015b). This form has the advantage of avoiding notions of injection, surjection and bijection.

A visit to a restaurant in October 27 2014 and subsequent e-mail exchange with Edixhoven (Leiden), co-author of Coplakova et al. (2011), led to the memos Colignatus (2014ab), and the inspiration to write about ZFC. My analysis uses ZFC and no other system since Edixhoven said that he restricted his attention to this. Originally I asked Edixhoven the question on the relation between Cantorian Ψ and Pauline Φ, see (2014ab) or FMNAI. Edixhoven agreed that the Pauline consistency condition should have no effect, and I asked him to explain that it could have an effect. Since November 2014 I have not received a response even though the question was clear and articulate. Hart (Delft), who has invested deeply into the transfinites, apparently rejects the usefulness of these questions. Having seen ZFC more often in the course of these exchanges, I decided on the morning of Wednesday May 27 2015 to provide for the answers myself, and established the key relation for the singleton case before noon. The result is the book FMNAI, from which this article has been taken. For a breach in scientific integrity, see Colignatus (2015c).

References

Colignatus, Th. (1981 unpublished, 2007, 2011), *A logic of exceptions,* (ALOE) 2nd edition, Thomas Cool Consultancy & Econometrics, Scheveningen, http://thomascool.eu/Papers/ALOE/Index.html (PDF online)

Colignatus, Th. (2012, 2013), *Contra Cantor Pro Occam - Proper constructivism with abstraction*, paper, http://thomascool.eu/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf (legacy paper)

Colignatus, Th. (2013), *What a mathematician might wish to know about my work*, http://thomascool.eu/Papers/Math/2013-03-26-WAMMWTKAMW.pdf

Colignatus, Th. (2014a), *Logical errors in the standard "diagonal argument" proof of Cantor for the power set",* memo, http://thomascool.eu/Papers/ALOE/2014-10-29-Cantor-Edixhoven-02.pdf

Colignatus, Th. (2014b, 2015), *A condition by Paul of Venice (1369-1429) solves Russell's Paradox, blocks Cantor's Diagonal Argument, and provides a challenge to ZFC,* abbreviated PV-RP-CDA-ZFC, versions at http://vixra.org/abs/1412.0235, latest version at same link http://thomascool.eu/Papers/ALOE/2014-11-14-Paul-of-Venice.pdf (legacy paper)

Colignatus, Th. (2015a), *Foundations of Mathematics. A Neoclassical Approach to Infinity,* MijnBestseller.nl, http://thomascool.eu/Papers/FMNAI/Index.html

Colignatus, Th. (2015b), *Review of the email exchange between Colignatus and K.P. Hart (TU Delft) in 2011-2015 on Cantor's diagonal argument and his original argument of 1874,* May 6 (thus limited to up to then), http://thomascool.eu/Papers/ALOE/KPHart/2015-05-06-Review-emails-Colignatus-KPHart-2011-2015.pdf

Colignatus, Th. (2015c), *A breach of scientific integrity since 1980 on the common logical paradoxes*, http://thomascool.eu/Papers/ALOE/2015-05-21-A-breach-of-integrity-on-paradoxes.pdf

Coplakova, E., B. Edixhoven, L. Taelman, M. Veraar (2011), *Wiskundige Structuren,* dictaat 2011/2012, Universiteit van Leiden and TU Delft, http://ocw.tudelft.nl/courses/technische-wiskunde/wiskundige-structuren/literatuur

DeLong, H. (1971), *A profile of mathematical logic*, Addison-Wesley

Gill, R.D. (2008), 'Book reviews. Thomas Colignatus. A Logic of Exceptions: Using the Economics Pack Applications of Mathematica for Elementary Logic", *Nieuw Archief voor Wiskunde,* 5/9 nr. 3, pp. 217-219, http://www.nieuwarchief.nl/serie5/pdf/naw5-2008-09-3-217.pdf

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