Non-archimedean analysis on the extended hyperreal line ${}^{*}\mathbb{R}_{\mathbf{d}}$ and the solution of some very old transcendence conjectures over the field $\mathbb{Q}$.

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Abstract

In 1980 F. Wattenberg constructed the Dedekind completion ${}^{*}\mathbb{R}_{\mathbf{d}}$ of the Robinson non-archimedean field ${}^{*}\mathbb{R}$ and established basic algebraic properties of ${}^{*}\mathbb{R}_{\mathbf{d}}$ [6]. In 1985 H. Gonshor established further fundamental properties of ${}^{*}\mathbb{R}_{\mathbf{d}}$ [7].In [4] important construction of summation of countable sequence of Wattenberg numbers was proposed and corresponding basic properties of such summation were considered. In this paper the important applications of the Dedekind completion ${}^{*}\mathbb{R}_{\mathbf{d}}$ in transcendental number theory were considered. We dealing using set theory $ZFC + \neg\exists$(ω-model of ZFC).Given an class of analytic functions of one complex variable $f \in \mathbb{Q}[[z]]$, we investigate the arithmetic nature of the values of $f(z)$ at transcendental points $e^n, n \in \mathbb{N}$. Main results are: (i) the both numbers $e+\pi$ and $e \times \pi$ are irrational, (ii) number e^e is transcendental. Nontrivial generalization of the Lindemann- Weierstrass theorem is obtained.

Keywords

Non-archimedean analysis,**Robinson transfer**, **Robinson non-archimedian field**,**Dedekind completion**, **Dedekind hyperreals**,**Wattenberg embeding**,**Gonshor idempotent theory**,**Gonshor transfer**.

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1.Introduction.

1.1.

In 1873 French mathematician, Charles Hermite, proved that e is transcendental. Coming as it did 100 years after Euler had established the significance of e, this meant that the issue of transcendence was one mathematicians could not afford to ignore.Within 10 years of Hermite's breakthrough,his techniques had been extended by Lindemann and used to add π to the list of known transcendental numbers. Mathematician then tried to prove that other numbers such as $e+\pi$ and $e \times \pi$ are transcendental too,but these questions were too difficult and so no further examples emerged till today's time. The transcendence of e^{π} has been proved in1929 by A.O.Gel'fond.

Conjecture 1. Whether the both numbers $e+\pi$ and $e\times\pi$ are irrational.

Conjecture 2. Whether the numbers e and π are algebraically independent.

However, the same question with e^{π} and π has been answered:

Theorem.(Nesterenko,1996 [1]) The numbers e^{π} and π are algebraically independent.

Throughout of 20-th century,a typical question: whether $f(\alpha)$ is a transcendental number for each algebraic number α has been investigated and answered many authors.Modern result in the case of entire functions satisfying a linear differential equation provides the strongest results, related with Siegel's E-functions [1],[2].Reference [1] contains references to the subject before 1998, including Siegel E and G functions.

Theorem.(Siegel C.L.) Suppose that $\lambda\in\mathbb{Q},\lambda\neq-1,-2,\ldots,\alpha\neq0$.

$$\varphi_{\lambda}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}. \quad (1.1)$$

Then $\varphi_{\lambda}(\alpha)$ is a transcendental number for each algebraic number $\alpha\neq0$.

Let f be an analytic function of one complex variable $f\in\mathbb{Q}[[z]]$.

Conjecture 3.Whether $f(\alpha)$ is an irrational number for given transcendental number α.

Conjecture 4.Whether $f(\alpha)$ is a transcendental number for given transcendental number α.

Remark 1.1. In classical analysis usually one dealing using set theory

$$ZFC_{\omega}\triangleq ZFC+\exists(\omega\text{-model of }ZFC)$$

under assumption:

Assumption 1.1. $Con(ZFC_{\omega})$.

However in this paper we dealing using strictly weaker than set theory ZFC_{ω},set theory

$$ZFC_{\neg\omega}\triangleq ZFC+\neg\exists(\omega\text{-model of }ZFC)$$

under assumption:

Assumption 1.2. Let $M_{\neg\omega}^{Nst}$ by any countable nonstandard model of $ZFC_{\neg\omega}$,let $\mathbf{PRA}$ be a

primitive recursive arithmetic and let $M_{\mathbf{PRA}}^{st}$ be a standard model of $\mathbf{PRA}$. Then

(i) there exist an countable nonstandard model $\overline{M}_{\neg\omega}^{Nst}$ of $ZFC_{\neg\omega}$,

(ii) there exist standard model $M_{\mathbf{PRA}}^{st}$ and (iii) $M_{\mathbf{PRA}}^{st}\subset M_{\neg\omega}^{Nst}$,

(iii) $Con(ZFC_{\neg\omega})$.

Remark 1.1. In this paper using set theory $ZFC_{\neg\omega}$ we investigate the arithmetic nature of

the values of $f(z)$ at transcendental points $e^{n},n\in\mathbb{N}$.

Definition 1.1. Let $g(x):\mathbb{R}\to\mathbb{R}$ be any real analytic function such that: (i)

$$g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n, |x| < r, \forall n[a_n \in \mathbb{Q}], \qquad (1.2)$$

and where (ii) the sequence $\{a_n\}_{n\in\mathbb{N}}$ is primitive recursive (constructive).

We will call any function given by Eq.(1.2) constructive $\mathbb{Q}$-analytic function and denoted such function by $g_{\mathbb{Q}}(x)$.

Definition 1.2.[3],[4]. A transcendental number $z \in \mathbb{R}$ is called #-transcendental number over field $\mathbb{Q}$, if there does not exist constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$, i.e. for every constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ the inequality $g_{\mathbb{Q}}(z) \neq 0$ is satisfied.

Definition 1.3.[3],[4].A transcendental number z is called w-transcendental number over field $\mathbb{Q}$,if z is not #-transcendental number over field $\mathbb{Q}$,i.e.there exists an constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$.

Notation 1.1.We will call for a short any constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ simply $\mathbb{Q}$-analytic function.

Example 1.1. Number π is transcendental but number π is not #-transcendental number over field $\mathbb{Q}$ as

(**1**) function $\sin x$ is a $\mathbb{Q}$-analytic and

(**2**) $\sin\left(\frac{\pi}{2}\right) = 1$, i.e.

$$-1 + \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \ldots + \frac{(-1)^{2n+1}\pi^{2n+1}}{2^{2n+1}(2n+1)!} + \ldots = 0. \qquad (1.3)$$

Note that the sequence $a_n = \frac{(-1)^{2n+1}\pi^{2n+1}}{2^{2n+1}(2n+1)!}, n = 0,1,2.\ldots.$ obviously is primitive recursive.

Example 1.2. Let $v_0 = 1$. For each $n > 0$ choose an rational number v_n inductively such that

$$1 - \sum_{k=1}^{n-1} v_k e^k - (n!)^{-1} < v_n e^n < 1 - \sum_{k=1}^{n-1} v_k e^k.$$

The rational number v_n exists because the rational numbers are dense. Now the power series $f(x) = 1 - \sum_{n=1}^{\infty} v_n e^n$ has the radius of convergence ∞ and $f(e) = 0$. However any sequence $\{v_n\}_{n\in\mathbb{N}}$ obviously is not recursive.

Main results are.

Theorem 1.1.[18,19].Set theory $ZFC_\omega \triangleq ZFC + \exists(\omega$-model of $ZFC)$ is inconsistent.

Remark 1.1.Note that theorem 1.1was proved in [18,19] by using an non trivial generalization of the classical Löb’s theorem. In this paper we prove $\neg Con(ZFC_\omega)$ directly.

Theorem 1.2.Assume that set theory $ZFC_{\neg\omega} \triangleq ZFC + \neg\exists(\omega$-model of $ZFC)$ is consistent.

Let $\{v_n^{st}\}_{n\in\mathbb{N}}$ be standard sequence defined above and let $M_{\neg\omega}$ by any countable nonstandard model of $ZFC_{\neg\omega}$. Then $\{v_n^{st}\}_{n\in\mathbb{N}} \notin M_{\neg\omega}^{Nst}$.

Remark 1.2. Note that a statement $\{v_n\}_{n\in\mathbb{N}} \notin M_{\neg\omega}^{Nst}$ is a seems in contradiction with standard intuition, however that arises from the downward Löwenheim-Skolem theorem and presents an form of known Skolem's "paradox".

Theorem 1.3.[3],[4].Number e is #-transcendental over $\mathbb{Q}$.

From theorem 1.1 immediately follows.

Theorem 1.4.Number e^e is transcendental.

Theorem 1.5.[3],[4]. The both numbers $e+\pi$ and $e-\pi$ are irrational.

Theorem 1.6.For any $\xi \in \mathbb{Q}$ number e^{ξ} is #-transcendental over the field $\mathbb{Q}$.

Theorem 1.7.[3],[4]. The both numbers $e\times\pi$ and $e^{-1}\times\pi$ are irrational.

Theorem 1.8.[4] Let $f_l(z), l = 1,2,\ldots$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l}$, $k_l \geq 1, l = 1,2,\ldots$ form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, \tag{1.4}$$

and $a_l \in \mathbb{Q}, l = 1,2,\ldots; a_0 \neq 0,$ where the sequence $\{a_l\}_{l\in\mathbb{N}}$ is primitive recursive. Assume that

$$a_0 + \sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{1.5}$$

Then

$$a_0 + \sum_{l=1}^{\infty} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{1.6}$$

Remark 1.3. Note that Theorem 1.3-1.8 can be proven (instead set theory $ZFC_{\neg\omega}$) using a theory $\mathbf{RCA}_0 + \exists\mathbf{S}_{\omega}$. Here (i) $\mathbf{RCA}_0$ is the subsystem of second-order arithmetic whose axioms are the axioms of Robinson arithmetic, induction for Σ_1^0 formulas, and comprehension for Δ_1^0 formulas and (ii) S_{ω} a minimal ω-model of **RCA**$_0$ where $\mathbf{S}_{\omega}$ consists of the all recursive subsets of ω [15]. In this case one can use constructive Moerdijk's approach instead nonconstructive ultrapower construction acepted in this paper; see Remarks 1.4-1.5 below.

Remark 1.4.The subsystem $\mathbf{RCA}_0$ is the one most commonly used as a base system for reverse mathematics. The initials "**RCA**" stand for "recursive comprehension axiom", where "recursive" means "computable", as in recursive function.This name is used because **RCA**$_0$ corresponds informally to "computable mathematics". In particular, any set of natural numbers that can be proven to exist in $\mathbf{RCA}_0$ is computable, and thus any theorem which implies that noncomputable sets exist is not provable in $\mathbf{RCA}_0$. To this extent, $\mathbf{RCA}_0$ is a constructive system.

Remark 1.5. As is well known, elementary nonarchimedean extensions of the real number structure $\mathbb{R}$ can be obtained in essentially two different ways, both nonconstructive: one is to use an ultrapower construction [5], the other is to use the compactness theorem.

As is known from topos theory the category of sheaves over a generalised topological space is a universe of variable sets a Grothendieck topos which obeys the laws of intuitionistic logic rather than classical logic We briefly review Moerdijk's sheaf model construction from [9]-[10].To motivate the construction recall Los's fundamental theorem for ultrapowers which states that for any ultrafilter U on a set I any $\mathcal{L}$-structure M and any $\mathcal{L}$-formula $\Phi(x_1,\ldots,x_n)$,

$$M^I/U \models \Phi(\alpha_1,\ldots,\alpha_n) \iff \{i \in I | M \models \Phi(\alpha_1(i),\ldots,\alpha_n(i))\} \tag{1.7}$$

That the filter is an ultrafilter i.e. maximal among proper filters on I is crucial for proving the equivalence when involves the logical constants and Moerdijk gave a constructive analogue of Los's fundamental theorem, by changing the notion of model to a sheaf model over a category of filters The idea of shifting to a nonstandard semantics occurs also in Martin Löf [11].

A filter base $\mathcal{F} = (F, \{F_i\}_{i \in I})$ consists of a nonvoid index set I and a family $\{F_i\}_{i \in I}$ of subsets of an underlying set F called base sets satisfying the filtering condition for all $i,j \in I$ there exists $k \in I$ such that $F_k \subseteq F_i \cap F_j$.The filter generated is then $\{S \subseteq F : (\exists i \in I) F_i \subseteq S\}$ For constructive reasons it will be better to work only withthe bases of filters. So in the sequel we shall abuse the language and simply call them filters. Let $\mathcal{F} = (F, \{F_i\}_{i \in I})$ and $\Im = (G, \{G_i\}_{i \in I})$ be filters. A continuous map α from $\mathcal{F}$ to $\Im$ in symbols $\alpha : \mathcal{F} \to \Im$ is a partial function $\alpha : F \to G$ which is totally defined on some base set of $\mathcal{F}$ and satisfies the continuity condition $(\forall j \in J)(\exists i \in I)\ \{\alpha[F_i] \subseteq G_j\}$ Two such morphisms are equivalent if they agree on some base set of $\mathcal{F}$. The filters together with the continuous maps then form a category $\mathbf{B}$ with terminal object and all pullbacks see [9] and [12]. For each set A there is a trivial filter $\overline{A} = (A\{A\})$. In this way the category of sets can be considered as a full subcategory of $\mathbf{B}$. Note that the set of morphisms from $\mathcal{F}$ to $\overline{A}$, $Hom_{\mathbf{B}}(F,\overline{A})$ can be identified with the reduced power $A^F/\mathcal{F}$ As for any category the hom-sets give a contravariant functor

$${}^*A \triangleq Hom_{\mathbf{B}}(-,\overline{A}) : \mathbf{B}^{op} \to \mathbf{Sets}. \tag{1.8}$$

We now define a Grothendieck topology K on $\mathbf{B}$ so that ${}^*A$ becomes a sheaf Let $\Im^{(k)} = \left\{G^{(k)}, \left\{G^{(k)}_{j \in J^{(k)}}\right\}\right\}$, $k = 1,\ldots,n$ and $\mathcal{F} = (F, \{F_i\}_{i \in I})$ be filters. A finite set of continuous maps $\{\beta_k : \Im^{(k)} \to \mathcal{F}\}_{k=1}^n$ is called a K-cover of $\mathcal{F}$ if for all $j_1 \in J^{(1)},\ldots,j_n \in J^{(n)}$ there exists $i \in I$ for which

$$\beta_1\left[G^{(1)}_{j_1}\right] \cup \ldots \cup \beta_n\left[G^{(n)}_{j_n}\right] \supseteq F_i \tag{1.9}$$

In case $\{\beta : \Im \to \mathcal{F}\}$ g is a cover, we say that β is a covering map which is the same as an epimorphism in $\mathbf{B}$. The category of sheaves over the topology $(\mathbf{B},K)$ denoted $\widetilde{N}$ for nonstandard universe $\mathbf{U}^\#$ is a universe containing both standard and nonstandard objects. With this topology each presheaf of the form ${}^*A$ becomes a sheaf. This sheaf is the nonstandard version of A. The sheaf of locally constant functions $\Delta(A)$ is a subsheaf of ${}^*A$ denoted by ${}^\sigma A$ which constitute the standard elements of ${}^*A$. In particular each constant element

${}^*a = x \mapsto a \in {}^*A(\mathcal{F})$ is standard. For each relation $R \subseteq A_1 \times \cdots A_n$ define a subsheaf ${}^*R$ of ${}^*A_1 \times \cdots \times {}^*A_n$ by

$$(\alpha_1,\ldots,\alpha_n) \in {}^*R(\mathcal{F}) \iff (\exists i \in I)(\forall u \in F_i)[(\alpha_1,\ldots,\alpha_n) \in R], \tag{1.10}$$

where $\mathcal{F} = (F, \{F_i\}_{i\in I})$. We assume now that $\mathcal{L}$ is a first order language including symbols for all sets relations functions and constants of interest to us here this can be made precise using universes of sets. ${}^*\mathcal{L}$ denotes the language where all symbols have been decorated with $*$. For any $\mathcal{L}$ formula Φ we define its * transform ${}^*\Phi$ to be the ${}^*\mathcal{L}$ formula where all symbols have been replaced by their starred counterparts. A formula which is a transform of an $\mathcal{L}$ formula is called internal. The language ${}^*\mathcal{L}$ can be regarded as a sublanguage of the language $\mathcal{L}\left(\widetilde{N}\right)$ of the topos $\widetilde{N}$. We now use ordinary sheaf semantics to interpret $\mathcal{L}\left(\widetilde{N}\right)$. Corresponding to the fundamental theorem for ultrapowers we have the following:

Theorem (Moerdijk) Let $\Phi(x_1,\ldots,x_n)$ be an $\mathcal{L}$ formula where $x_1,\ldots,x_n$ vary over $S_1,\ldots,S_n$ respectively Then for $\alpha_1 \in {}^*S_1(\mathcal{F}),\ldots,\alpha_n \in {}^*S_n(\mathcal{F})$:

$$\mathcal{F} \Vdash {}^*\Phi(\alpha_1,\ldots,\alpha_n) \text{ iff } (\exists i \in I)(\forall u \in F_i)[\Phi(\alpha_1,\ldots,\alpha_n)]. \quad (1.11)$$

We list the main principles valid for the model $\widetilde{N}$ but refer to [12-13].

Theorem. (Transfer principle). For any $\mathcal{L}$ formula Φ : Φ is true iff ${}^*\Phi$ holds in $\widetilde{N}$.

Theorem. (Idealization). The following is true in $\widetilde{N}$ for any $\mathcal{L}$-formula Φ : If for any standard
n and any sequence $a_0,\ldots,a_n \in {}^\sigma S$ there exists $z \in {}^*T$ such that ${}^*\Phi(x,a_k,z)$ for $k = 0,\ldots,n,$ then there is some $z \in {}^*T$ such that for all $\in {}^\sigma S$: $\Phi(x,y,z)$.

Theorem. (Underspill). The following holds in $\widetilde{N}$ for any $\mathcal{L}$-formula Φ : If ${}^*\Phi(x,n)$ for all in finite $n \in {}^*\mathbb{N}$ then there is some standard n with ${}^*\Phi(x,n)$.

In paper [14] the x_1-saturation principle was established, see [14] Theorem 3.1. Thereby all the main principles of nonstandard analysis are available to us Note however that the transfer principle is weaker than the usual since the interpretation of the logical constants is nonstandard in $\widetilde{N}$. As a consequence the standard part map does not take its customary form (see [14] section 4). Moreover induction and dependent choice is valid in $\widetilde{N}$ for the set of standard natural numbers ${}^\sigma\mathbb{N}$; see [12]. This means that the results of constructive analysis [16-17] can be reused within the model. To prove results in constructive analysis one need to use the transfer principle For some examples of this, see [14] Section 5.

1.2. Preliminaries.Short outline of Dedekind tipe hyperreals

Let $\mathbb{R}$ be the set of real numbers and ${}^*\mathbb{R}$ a nonstandard model of $\mathbb{R}$ [5].Of course ${}^*\mathbb{R}$ is not Dedekind complete.For example, $\mu(0) = \left\{x \in {}^*\mathbb{R}_+ | \, x \approx 0\right\} \triangleq \varepsilon_{\mathbf{d}}$ and $\mathbb{R}$ are bounded subsets of ${}^*\mathbb{R}$ which have no suprema or infima in ${}^*\mathbb{R}$.Possible completion of the field ${}^*\mathbb{R}$ can be constructed by Dedekind sections [6],[7]. In [6] Wattenberg constructed the Dedekind completion of a nonstandard model of the real numbers and applied the construction to obtain certain kinds of special measures on the set of integers. Thus was established that the Dedekind completion ${}^*\mathbb{R}_{\mathbf{d}}$ of the field ${}^*\mathbb{R}$ is a structure of interest not for its own sake only and we establish further important applications here. Important concept introduced by Gonshor [7] is that of the **absorption number** of an element $\mathbf{a} \in {}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures the degree to which the cancellation law

$\mathbf{a}+b=\mathbf{a}+c \Rightarrow b=c$ fails for $\mathbf{a}$.

We remind that there exist natural imbedding

$$j:\ \mathbb{R} \hookrightarrow {}^{*}\mathbb{R}, \tag{1.2.1}$$

see for example [5].We will be denoted the image of this imbedding $j(\mathbb{R})$ by ${}^{*}\mathbb{R}_{\mathbf{st}}$.

Definition 1.2.1. Let $\mathbf{n} \in {}^{*}\mathbb{N}\backslash\mathbb{N}$ and $\mathbf{m} \in {}^{*}\mathbb{N}\backslash\mathbb{N}$ relatively prime hypernaturals, i.e. the only positive integer that divides both of them is 1, and let $\epsilon \in {}^{*}\mathbb{R},\ \epsilon \approx 0,$ We will say that hyperrational number $\mathbf{n}/\mathbf{m}$ is ϵ-near-standard iff there exists $\eta(\epsilon) \approx 0, \eta(\epsilon) > 0$ such that

$$\left|\mathbf{st}\left(\frac{\mathbf{n}}{\mathbf{m}}\right) - \frac{\mathbf{n}}{\mathbf{m}}\right| \leq \eta(\epsilon), \tag{1.2.2}$$

where $\eta(\epsilon) \in \epsilon \times \varepsilon'_{\mathbf{d}}$ and $\varepsilon'_{\mathbf{d}} = \varepsilon_{\mathbf{d}} \cap {}^{*}\mathbb{R}_{+}$.

We will be denoted a set of the all ϵ-near-standard hyperrational numbers by ${}^{*}\mathbb{R}_{\mathbf{n.st}}(\epsilon)$.

Definition 1.2.2.Let ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$ be a set such that

$$\begin{gathered}\forall x\left(x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)\right) \Leftrightarrow \left[\left(y \in {}^{*}\mathbb{R}_{\mathbf{n.st}}(\epsilon)\right)\right] \vee \\ \left\{\left[\exists y\left(y \in {}^{*}\mathbb{R}_{\mathbf{st}}\right)\right] \wedge [\exists z(z \in \epsilon \times \varepsilon'_{\mathbf{d}})] \wedge [x = y+z]\right\} \vee \\ \left\{\left[\exists y\left(y \in {}^{*}\mathbb{R}_{\mathbf{n.st}}\right)\right] \wedge [\exists z(z \in \epsilon \times \varepsilon'_{\mathbf{d}})] \wedge [x = y+z]\right\}.\end{gathered} \tag{1.2.3}$$

Note that ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$ is a subring of ${}^{*}\mathbb{R}$.

Definition 1.2.3. Let $\mathbf{N} \in {}^{*}\mathbb{N}\backslash\mathbb{N}$ and let $j_{\mathbf{N}}$ be morphism $j_{\mathbf{N}} : {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon) \mapsto {}^{*}\mathbb{R}$ such that $\forall x\left(x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)\right)$

$$j_{\mathbf{N}}(x) = \mathbf{N} \times x. \tag{1.2.4}$$

We will be denoted the image $j_{\mathbf{N}}({}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon))$ of this morphism by ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$

Note that ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$ is a submodule of ${}^{*}\mathbb{R}$.

2.1 The Dedekind hyperreals ${}^{*}\mathbb{R}_{\mathbf{d}}$ and ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

Definition 2.1.Let ${}^{*}\mathbb{R}$ be a nonstandard model of $\mathbb{R}$ and $P({}^{*}\mathbb{R})$ the power set of ${}^{*}\mathbb{R}$.

(i) A Dedekind hyperreal $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}, \alpha \notin {}^{*}\mathbb{R}$ is an ordered pair $\{U,V\} \in P({}^{*}\mathbb{R}) \times P({}^{*}\mathbb{R})$ that

satisfies the next conditions:

1.$\exists x \exists y(x \in U \wedge y \in V)$. 2. $U \cap V = \varnothing$. 3.$\forall x(x \in U \Leftrightarrow \exists y(y \in V \wedge x < y))$.

4. $\forall x(x \in V \Leftrightarrow \exists y(y \in V \wedge x < y))$. 5. $\forall x \forall y(x < y \Rightarrow x \in U \vee y \in V)$.

(ii) A Dedekind hyperreal $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon), \alpha \notin {}^{*}\mathbb{R}$ is an ordered pair $\{U_\epsilon, V_\epsilon\} \in P({}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)) \times P({}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon))$ that satisfies the next conditions:

1.$\exists x \exists y(x \in U_\epsilon \wedge y \in V_\epsilon)$. 2. $U_\epsilon \cap V_\epsilon = \varnothing$. 3.$\forall x(x \in U_\epsilon \Leftrightarrow \exists y(y \in V_\epsilon \wedge x < y))$.

4. $\forall x(x \in V_\epsilon \Leftrightarrow \exists y(y \in V_\epsilon \wedge x < y))$. 5. $\forall x \forall y(x < y \Rightarrow x \in U_\epsilon \vee y \in V_\epsilon)$.

(iii) A Dedekind hyperreal $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N}), \alpha \notin {}^{*}\mathbb{R}$ is an ordered pair $\{U^{\epsilon,\mathbf{N}}, V^{\epsilon,\mathbf{N}}\} \in P({}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})) \times P({}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N}))$ that satisfies the next conditions:

1.$\exists x \exists y(x \in U^{\epsilon,\mathbf{N}} \wedge y \in V^{\epsilon,\mathbf{N}})$. 2. $U^{\epsilon,\mathbf{N}} \cap V^{\epsilon,\mathbf{N}} = \varnothing$. 3.$\forall x(x \in U^{\epsilon,\mathbf{N}} \Leftrightarrow \exists y(y \in V^{\epsilon,\mathbf{N}} \wedge x < y))$.

4. $\forall x(x \in V^{\epsilon,\mathbf{N}} \Leftrightarrow \exists y(y \in V^{\epsilon,\mathbf{N}} \wedge x < y))$. 5. $\forall x \forall y(x < y \Rightarrow x \in U^{\epsilon,\mathbf{N}} \vee y \in V^{\epsilon,\mathbf{N}})$.

Compare the Definition 2.1 with original Wattenberg definition [6],(see [6] def.II.1).

Designation 2.1. (i) Let $\{U,V\} \triangleq \alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$. We have designate in this paper

$$U \triangleq \mathbf{cut}_{-}(\alpha), V \triangleq \mathbf{cut}_{+}(\alpha)$$
$$\alpha \triangleq \{\mathbf{cut}_{-}(\alpha), \mathbf{cut}_{+}(\alpha)\}. \tag{2.1.1}$$

(ii) Let $\{U_\epsilon, V_\epsilon\} \triangleq \alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. We have designate in this paper

$$U_\epsilon \triangleq \mathbf{cut}_{-}(\alpha,\epsilon), V_\epsilon \triangleq \mathbf{cut}_{+}(\alpha,\epsilon)$$
$$\alpha \triangleq \{\mathbf{cut}_{-}(\alpha,\epsilon), \mathbf{cut}_{+}(\alpha,\epsilon)\}. \tag{2.1.2}$$

(iii) Let $\{U^{\epsilon,\mathbf{N}}, V^{\epsilon,\mathbf{N}}\} \triangleq \alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$. We have designate in this paper

$$U^{\epsilon,\mathbf{N}} \triangleq \mathbf{cut}_{-}(\alpha,\epsilon,\mathbf{N}), V^{\epsilon,\mathbf{N}} \triangleq \mathbf{cut}_{+}(\alpha,\epsilon,\mathbf{N})$$
$$\alpha \triangleq \{\mathbf{cut}_{-}(\alpha,\mathbf{N}), \mathbf{cut}_{+}(\alpha,\mathbf{N})\} \tag{2.1.3}$$

Designation 2.2. (i) Let $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. We have designate in this paper

$$\alpha_{\#} \triangleq \mathbf{cut}_{-}(\alpha), \alpha^{\#} \triangleq \mathbf{cut}_{+}(\alpha)$$
$$\alpha \triangleq \{\alpha_{\#}, \alpha^{\#}\}. \tag{2.1.4}$$

(ii) Let $\alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. We have designate in this paper

$$\alpha_{\#_\epsilon} \triangleq \mathbf{cut}_{-}(\alpha,\epsilon), \alpha^{\#_\epsilon} \triangleq \mathbf{cut}_{+}(\alpha,\epsilon)$$
$$\alpha \triangleq \{\alpha_{\#_\epsilon}, \alpha^{\#_\epsilon}\}. \tag{2.1.5}$$

(iii) Let $\alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$. We have designate in this paper

$$\alpha_{\#_{\epsilon,\mathbf{N}}} \triangleq \mathbf{cut}_{-}(\alpha\epsilon,\mathbf{N}), \alpha^{\#_{\epsilon,\mathbf{N}}} \triangleq \mathbf{cut}_{+}(\alpha\epsilon,\mathbf{N})$$
$$\alpha \triangleq \{\alpha_{\#_{\epsilon,\mathbf{N}}}, \alpha^{\#_{\epsilon,\mathbf{N}}}\}. \tag{2.1.6}$$

Remark 2.1. (i) The monad of $\alpha \in {}^*\mathbb{R}$ is the set: $\left\{x \in {}^*\mathbb{R} | \, x \approx \alpha\right\}$ is denoted by $\mu(\alpha)$.
Supremum of $\mu(0)$ in ${}^*\mathbb{R}_{\mathbf{d}}$ is denoted by $\varepsilon_{\mathbf{d}}$. Supremum of $\mathbb{R}$ is denoted by $\Delta_{\mathbf{d}}$. Note that
[6]

$$\varepsilon_{\mathbf{d}} = {}^*(-\infty,0] \cup \mu(0),$$
$$\Delta_{\mathbf{d}} = \bigcup_{n\in\mathbb{N}} [{}^*(-\infty,n)].$$

(ii) The monad of $\alpha \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ is the set: $\left\{x \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) | \, x \approx \alpha\right\}$ is denoted by $\mu_\epsilon(\alpha)$ or $\mu(\alpha,\epsilon)$. Supremum of $\mu_\epsilon(0)$ in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is denoted by $\varepsilon_{\mathbf{d}}(\epsilon)$ or $\varepsilon_{\mathbf{d}}^{\epsilon}$. Note that

$$\varepsilon_{\mathbf{d}}(\epsilon) = \{({}^*(-\infty,0]) \cap ({}^*\mathbb{R}_{\mathbf{st}}(\epsilon))\} \cup \mu_\epsilon(0),$$
$$\varepsilon_{\mathbf{d}}(\epsilon) = \epsilon \times \varepsilon'_{\mathbf{d}}.$$

(iii) Let A be a subset of ${}^*\mathbb{R}$ bounded above. Then $\sup(A)$ exists in ${}^*\mathbb{R}_{\mathbf{d}}$ [6].
(iv) Let A be a subset of ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ bounded above. Then $\sup(A)$ exists in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.
(v) Let A be a subset of ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$ bounded above. Then $\sup(A)$ exists in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$.
Example 2.1. (i) $\Delta_{\mathbf{d}} = \sup(\mathbb{R}_+) \in {}^*\mathbb{R}_{\mathbf{d}} \backslash {}^*\mathbb{R}$, (ii) $\varepsilon_{\mathbf{d}} = \sup\left(\mu(0)\right) \in {}^*\mathbb{R}_{\mathbf{d}} \backslash {}^*\mathbb{R}$.
Remark 2.2. Note that the set ${}^*\mathbb{R}_{\mathbf{d}}$ inherits some but by no means all of the algebraic structure on ${}^*\mathbb{R}$. For example: (i) ${}^*\mathbb{R}_{\mathbf{d}}$ is not a group with respect to addition since if $x +_{{}^*\mathbb{R}_{\mathbf{d}}} y$ denotes the addition in ${}^*\mathbb{R}_{\mathbf{d}}$ then: $\varepsilon_{\mathbf{d}} +_{{}^*\mathbb{R}_{\mathbf{d}}} \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}} +_{{}^*\mathbb{R}_{\mathbf{d}}} 0_{{}^*\mathbb{R}_{\mathbf{d}}} = \varepsilon_{\mathbf{d}}$. Thus ${}^*\mathbb{R}_{\mathbf{d}}$ is not even a ring but pseudo-ring only.

(ii) ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is not a group with respect to addition since if $x +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} y$ denotes the addition

in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ then:$\varepsilon_{\mathbf{d}}(\epsilon) +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \varepsilon_{\mathbf{d}}(\epsilon) = \varepsilon_{\mathbf{d}}(\epsilon) +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} = \varepsilon_{\mathbf{d}}(\epsilon)$. Thus ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is not even a ring but pseudo-ring only.

Definition 2.2 We define:

1.(a)The additive identity in ${}^{*}\mathbb{R}_{\mathbf{d}}$ (zero cut in ${}^{*}\mathbb{R}$) $0_{{}^{*}\mathbb{R}_{\mathbf{d}}}$, often denoted by $0^{\#}$ or simply 0 is

$0_{{}^{*}\mathbb{R}_{\mathbf{d}}} \triangleq \left\{x \in {}^{*}\mathbb{R} | \ x < 0_{{}^{*}\mathbb{R}}\right\}$.

1.(b) The additive identity in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ (zero cut in ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$) $0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$, often denoted by $0^{\#_{\epsilon}}$ or

simply 0_{ϵ} is $0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \triangleq \left\{x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon) | \ x < 0_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)}\right\}$.

1.(c) The additive identity in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$ (zero cut in ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$) $0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$, often denoted by

$0^{\#_{\epsilon}}$ or simply 0_{ϵ} is $0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \triangleq \left\{x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N}) | \ x < 0_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})}\right\}$.

2.(a) The multiplicative identity in ${}^{*}\mathbb{R}_{\mathbf{d}}$: $1_{{}^{*}\mathbb{R}_{\mathbf{d}}}$, often denoted by $1^{\#}$ or simply 1 is

$1_{{}^{*}\mathbb{R}_{\mathbf{d}}} \triangleq \left\{x \in {}^{*}\mathbb{R} | \ x <_{{}^{*}\mathbb{R}} 1_{{}^{*}\mathbb{R}}\right\}$.

2.(b) The multiplicative identity in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$: $1_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$, often denoted by $1^{\#_{\epsilon}}$ or simply 1_{ϵ} is

$1_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \triangleq \left\{x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon) | \ x <_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)} 1_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)}\right\}$.

2.(c) The multiplicative identity in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,)$: $1_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})}$, often denoted by $1^{\#_{\epsilon,\mathbf{N}}}$ or simply $1_{\epsilon,\mathbf{N}}$

is $1_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \triangleq \left\{x \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N}) | \ x <_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)} 1_{{}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})}\right\}$.

3.(a) Given two Dedekind hyperreal numbers $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ and $\beta \in {}^{*}\mathbb{R}_{\mathbf{d}}$ we define:

Addition $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta$ of α and β often denoted by $\alpha + \beta$ is

$\alpha + \beta \triangleq \left\{x + y | \ x \in \alpha, y \in \beta\right\}$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} 0_{{}^{*}\mathbb{R}_{\mathbf{d}}} = \alpha$ for all $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta$ is again a cut in ${}^{*}\mathbb{R}$ and $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta = \beta +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \alpha$.

Another fundamental property of cut addition is associativity:

$(\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta) +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \gamma = \alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \gamma)$.

This follows from the corresponding property of ${}^{*}\mathbb{R}$.

3.(b) Given two Dedekind hyperreal numbers $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ and $\beta \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ we define:

Addition $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$ of α and β often denoted by $\alpha +_{\epsilon} \beta$ is

$\alpha +_{\epsilon} \beta \triangleq \left\{x + y | \ x \in \alpha, y \in \beta\right\}$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} = \alpha$ for all $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$ is again a cut in ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$ and $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta = \beta +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha$.

Another fundamental property of cut addition is associativity:

$(\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta) +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma = \alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} (\beta +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma)$.

This follows from the corresponding property of ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$.

3.(c) Given two Dedekind hyperreal numbers $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$ and $\beta \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$ we define:

Addition $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta$ of α and β often denoted by $\alpha +_{\epsilon,\mathbf{N}} \beta$ is

$\alpha +_{\epsilon,\mathbf{N}} \beta \triangleq \left\{x + y | \ x \in \alpha, y \in \beta\right\}$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} 0_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} = \alpha$ for all $\alpha \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$.

It is easy to see that $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta$ is again a cut in ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$ and

$\alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta = \beta +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \alpha.$

Another fundamental property of cut addition is associativity:

$(\alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta) +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma = \alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} (\beta +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma).$

This follows from the corresponding property of $^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N})$.

4.(a) The opposite $-_{^*\mathbb{R}_{\mathbf{d}}} \alpha$ of α, often denoted by $(-\alpha)^{\#_\epsilon}$ or simply by $-\alpha$, is $-\alpha \triangleq \left\{x \in {}^*\mathbb{R} \mid -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{R}\backslash\alpha\right\}$.

4.(b) The opposite $-_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha$ of α, often denoted by $(-\alpha)^{\#_\epsilon}$ or simply by $-_\epsilon \alpha$, is $-_\epsilon \alpha \triangleq \left\{x \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) \mid -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)\backslash\alpha\right\}$.

4.(c) The opposite $-_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \alpha$ of α, often denoted by $(-\alpha)^{\#_{\epsilon,\mathbf{N}}}$ or simply by $-_{\epsilon,\mathbf{N}} \alpha$, is $-_{\epsilon,\mathbf{N}} \alpha \triangleq \left\{x \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{N}) \mid -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)\backslash\alpha\right\}$.

5.(a)We say that the $\mathbf{cut}_-(\alpha)$ is positive if $0_{^*\mathbb{R}_{\mathbf{d}}} <_{^*\mathbb{R}_{\mathbf{d}}} \alpha$ i.e., $0_\# \subsetneqq \alpha_\#$, or negative if $\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}$ i.e., $\alpha_\# \subsetneqq 0_\#$. The absolute value of α, denoted $|\alpha|$, is $|\alpha| \triangleq \alpha$, if $\alpha_{^*\mathbb{R}_{\mathbf{d}}} \geq 0_{^*\mathbb{R}_{\mathbf{d}}}$ and

$|\alpha| \triangleq -_{^*\mathbb{R}_{\mathbf{d}}} \alpha$, if $\alpha \leq_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}$.

5.(b)We say that the $\mathbf{cut}_-(\alpha,\epsilon)$ is positive if $0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha$ i.e., $0_{\#_\epsilon} \subsetneqq \alpha_{\#_\epsilon}$, or negative if $\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ i.e., $\alpha_{\#_\epsilon} \subsetneqq 0_{\#_\epsilon}$. The absolute value of α, denoted $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$, or $|\alpha|_\epsilon$ is $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \triangleq \alpha$, if $\alpha_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \geq 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ and $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \triangleq -_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha$, if $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$.

5.(c)We say that the $\mathbf{cut}_-(\alpha,\epsilon,\mathbf{N})$ is positive if $0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \alpha$ i.e., $0_{\#_{\epsilon,\mathbf{N}}} \subsetneqq \alpha_{\#_{\epsilon,\mathbf{N}}}$, or negative if $\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})}$ i.e., $\alpha_{\#_{\epsilon,\mathbf{N}}} \subsetneqq 0_{\#_{\epsilon,\mathbf{N}}}$. The absolute value of α, denoted $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})}$, or $|\alpha|_{\epsilon,\mathbf{N}}$ is $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \triangleq \alpha$, if $\alpha_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \geq 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})}$ and

$|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \triangleq -_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \alpha$, if $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})}$.

6.(a) If $\alpha,\beta > 0$ then multiplication $\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta$ of α and β often denoted $\alpha \times \beta$ is $\alpha \times \beta \triangleq \left\{z \in {}^*\mathbb{R} \mid z = x \times y \text{ for some } x \in \alpha, y \in \beta \text{ with } x,y > 0\right\}$.

In general, $\alpha \times \beta = 0$ if $\alpha = 0$ or $\beta = 0$,

$\alpha \times \beta \triangleq |\alpha| \times |\beta|$ if $\alpha > 0, \beta > 0$ or $\alpha < 0, \beta < 0$,

$\alpha \times \beta \triangleq -(|\alpha| \cdot |\beta|)$ if $\alpha > 0, \beta < 0$, or $\alpha < 0, \beta > 0$.

7.(a) The cut order enjoys on $^*\mathbb{R}_{\mathbf{d}}$ the standard additional properties of:

(**i**) transitivity: $\alpha \leq_{^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{^*\mathbb{R}_{\mathbf{d}}} \gamma \Rightarrow \alpha \leq_{^*\mathbb{R}_{\mathbf{d}}} \gamma$.

(**ii**) trichotomy: eizer $\alpha <_{^*\mathbb{R}_{\mathbf{d}}} \beta, \beta <_{^*\mathbb{R}_{\mathbf{d}}} \alpha$ or $\alpha =_{^*\mathbb{R}_{\mathbf{d}}} \beta$ but only one of the three

(**iii**) translation: $\alpha \leq_{^*\mathbb{R}_{\mathbf{d}}} \beta \Rightarrow \alpha +_{^*\mathbb{R}_{\mathbf{d}}} \gamma \leq_{^*\mathbb{R}_{\mathbf{d}}} \beta +_{^*\mathbb{R}_{\mathbf{d}}} \gamma$.

7.(b) The cut order enjoys on $^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ the standard additional properties of:

(**i**) transitivity: $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma \Rightarrow \alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma$.

(**ii**) trichotomy: eizer $\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta, \beta <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha$ or $\alpha =_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$ but only one of the three

(**iii**) translation: $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \Rightarrow \alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma \leq \beta +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \gamma$.

7.(c) The cut order enjoys on $^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})$ the standard additional properties of:

(**i**) transitivity: $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma \Rightarrow \alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma$.

(**ii**) trichotomy: eizer $\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta, \beta <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \alpha$ or $\alpha =_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta$ but only one of the three

(**iii**) translation: $\alpha \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \beta \Rightarrow \alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma \leq \beta +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{N})} \gamma$.

2.2 The Wattenberg embeding $^*\mathbb{R}$ into $^*\mathbb{R}_{\mathbf{d}}$. The

Wattenberg embeding ${}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon)$ into ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

Definition 2.3.[6]. Wattenberg hyperreal or #-hyperreal is a nonepty subset $\alpha \subsetneq {}^{*}\mathbb{R}$ such
that:
(**i**) For every $a \in \alpha$ and $b < a,\ b \in \alpha$.
(**ii**) $\alpha \neq \varnothing, \alpha \neq {}^{*}\mathbb{R}$.
(**iii**) α has no greatest element.

Definition 2.4.[6].In paper [6] Wattenberg embed ${}^{*}\mathbb{R}$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$ by following way:
if $\alpha \in {}^{*}\mathbb{R}$ the corresponding element, $\alpha^{\#}$, of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is

$$\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \middle| x < \alpha\right\} \tag{2.1}$$

Remark 2.3.[6]. In paper [6] Wattenberg pointed out that condition (iii) above is included
only to avoid nonuniqueness. Without it $\alpha^{\#}$ would be represented by both $\alpha^{\#}$and $\alpha^{\#} \cup \{\alpha\}$.

Remark 2.4.[7]. However in paper [7] H. Gonshor pointed out that the definition (2.1) in Wattenberg paper [6] is technically incorrect. Note that Wattenberg [6] defines $-\alpha$ in general by

$$-\alpha = \left\{a \in {}^{*}\mathbb{R} \middle| -a \notin \alpha\right\}. \tag{2.2}$$

If $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ i.e. ${}^{*}\mathbb{R}_{\mathbf{d}} \backslash \alpha$ has no mininum, then there is no any problem with definitions (2.1) and (2.2). However if $\alpha = a^{\#}$ for some $a \in {}^{*}\mathbb{R}$, i.e. $\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \middle| x < a\right\}$ then according to the latter definition (2.2)

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \middle| x \leq -a\right\} \tag{2.3}$$

whereas the definition of ${}^{*}\mathbb{R}_{\mathbf{d}}$ requires that:

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \middle| x < -a\right\}, \tag{2.4}$$

but this is a contradiction.

Remark 2.5.Note that in the usual treatment of Dedekind cuts for the ordinary real numbers both of the latter sets are regarded as equivalent so that no serious problem arises [7].

Remark 2.6.H.Gonshor [7] defines $-\alpha^{\#}$ by

$$-\alpha^{\#} = \left\{x \in {}^{*}\mathbb{R} \middle| \exists b[b > a \wedge -b \notin a]\right\}, \tag{2.5}$$

Definition 2.5. (**Wattenberg embeding**) We embed ${}^{*}\mathbb{R}$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$ of the following way:
if $\alpha \in {}^{*}\mathbb{R}$, the corresponding element $\alpha^{\#}$ of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is

$$\alpha^{\#} \triangleq \left\{x \in {}^{*}\mathbb{R} | x \leq_{{}^{*}\mathbb{R}} \alpha\right\} \tag{2.6}$$

and

$$(-\alpha)^{\#} = \left\{a \in {}^{*}\mathbb{R} \middle| -a \notin \alpha\right\} \cup \{\alpha\}. \tag{2.7}$$

or in the equivalent way,i.e. if $\alpha \in {}^{*}\mathbb{R}$ the corresponding element $\alpha_{\#}$ of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is

$$\alpha_{\#} \triangleq \left\{x \in {}^{*}\mathbb{R} | x \ {}_{{}^{*}\mathbb{R}} \geq \alpha\right\} \tag{2.8}$$

Thus if $\alpha \in {}^*\mathbb{R}$ then $\alpha^\# \triangleq A|B$ where

$$A = \left\{x \in {}^*\mathbb{R} | x \leq_{^*\mathbb{R}} \alpha\right\}, B = \left\{y \in {}^*\mathbb{R} | y_{\ ^*\mathbb{R}} \geq \alpha\right\}. \quad (2.9)$$

Such embeding ${}^*\mathbb{R}$ into ${}^*\mathbb{R}_\mathbf{d}$ Such embeding we will name Wattenberg embeding and to designate by ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_\mathbf{d}$.

Definition 2.5$'$.(**i**) We embed ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ into ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ of the following way: (i) if $\alpha \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$,

the corresponding element $\alpha^{\#_\epsilon}$ of ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is

$$\alpha^{\#_\epsilon} \triangleq \left\{x \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) \middle| x \leq_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha\right\} \quad (2.6.a)$$

and

$$(-\alpha)^{\#_\epsilon} \triangleq \left\{a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) \middle| -a \notin \alpha\right\} \cup \{\alpha\}. \quad (2.7.a)$$

(**ii**) We embed ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}), \mathbf{n} \in {}^*\mathbb{N}\backslash\mathbb{N}$ into ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})$ of the following way: if $\alpha \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})$, the corresponding element $\alpha^{\#_\epsilon}$ of ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is

$$\alpha^{\#_\epsilon} \triangleq \left\{x \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}) \middle| x \leq_{^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha\right\} \quad (2.6.b)$$

and

$$(-\alpha)^{\#_\epsilon} \triangleq \left\{a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}) \middle| -a \notin \alpha\right\} \cup \{\alpha\}. \quad (2.7.b)$$

Lemma 2.1.(**I**) [6].

(**i**) Addition $(\circ +_{^*\mathbb{R}_\mathbf{d}} \circ)$ is commutative and associative in ${}^*\mathbb{R}_\mathbf{d}$.

(**ii**) $\forall \alpha \in {}^*\mathbb{R}_\mathbf{d} : \alpha +_{^*\mathbb{R}_\mathbf{d}} 0_{^*\mathbb{R}_\mathbf{d}} =_{^*\mathbb{R}_\mathbf{d}} \alpha$.

(**iii**) $\forall \alpha, \beta \in {}^*\mathbb{R} : \alpha^\# +_{^*\mathbb{R}_\mathbf{d}} \beta^\# = (\alpha +_{^*\mathbb{R}} \beta)^\#$.

Proof. [6]. (**i**) is clear from definitions,

(**ii**) $\alpha +_{^*\mathbb{R}_\mathbf{d}} 0_{^*\mathbb{R}_\mathbf{d}} \subseteqq \alpha$ is clear from definitions.

Now, suppose $a \in \alpha$. Since α has no greatest element $\exists b_{^*\mathbb{R}} > a$, i.e. $b \in \alpha$, therefore $a -_{^*\mathbb{R}} b \in 0_{^*\mathbb{R}_\mathbf{d}}$ and $a =_{^*\mathbb{R}} (a -_{^*\mathbb{R}} b) +_{^*\mathbb{R}} b \in \alpha +_{^*\mathbb{R}_\mathbf{d}} 0_{^*\mathbb{R}_\mathbf{d}}$.

(**iii**) $\alpha^\# +_{^*\mathbb{R}_\mathbf{d}} \beta^\# \subseteqq (\alpha +_{^*\mathbb{R}} \beta)^\#$ is clear since $x <_{^*\mathbb{R}} \alpha, y <_{^*\mathbb{R}} \beta$ implies $x +_{^*\mathbb{R}} y <_{^*\mathbb{R}} \alpha +_{^*\mathbb{R}} \beta$.

Now, suppose $x <_{^*\mathbb{R}} (\alpha +_{^*\mathbb{R}} \beta)$, then

$$\alpha -_{^*\mathbb{R}} \frac{(\alpha +_{^*\mathbb{R}} \beta) -_{^*\mathbb{R}} x}{2} <_{^*\mathbb{R}} \alpha, \beta -_{^*\mathbb{R}} \frac{(\alpha +_{^*\mathbb{R}} \beta) -_{^*\mathbb{R}} x}{2} <_{^*\mathbb{R}} \beta.$$

Therefore

$$x =_{^*\mathbb{R}} \left[\alpha -_{^*\mathbb{R}} \frac{(\alpha +_{^*\mathbb{R}} \beta) -_{^*\mathbb{R}} x}{2}\right] +_{^*\mathbb{R}} \left[\beta -_{^*\mathbb{R}} \frac{(\alpha +_{^*\mathbb{R}} \beta) -_{^*\mathbb{R}} x}{2}\right] \in \alpha^\# +_{^*\mathbb{R}_\mathbf{d}} \beta^\#.$$

This completes the proof.

Lemma 2.1.(**II**)

(**i**) Addition $(\circ +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \circ)$ is commutative and associative in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

(**ii**) $\forall \alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon) : \alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} = \alpha$.

(**iii**) $\forall \alpha, \beta \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) : \alpha^{\#_\epsilon} +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta^{\#_\epsilon} = (\alpha +_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta)^{\#_\epsilon}$.

Proof.Similarly as above.

(**i**) is clear from definitions,

(**ii**) $\alpha +_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \subseteqq \alpha$ is clear from definitions.

Now, suppose $a \in \alpha$. Since α has no greatest element $\exists b_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} > a$, i.e. $b \in \alpha$, therefore $a -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} b \in 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ and $a =_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} (a -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} b) +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} b \in \alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$.

(iii) $\alpha^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta^{\#_\epsilon} \subseteqq (\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta)^{\#_\epsilon}$ is clear since $x <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha, y <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta$ implies $x +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} y <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta$.

Now, suppose $x <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} (\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta)$, then

$$\alpha -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} x}{2} <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha, \beta -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} x}{2} <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta.$$

Therefore

$$x =_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \left[\alpha -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} x}{2}\right] +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)}$$

$$+_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \left[\beta -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} x}{2}\right] \in \alpha^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta^{\#_\epsilon}.$$

This completes the proof.

Lemma 2.1.(III)

(i) Addition $(\circ +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \circ)$ is commutative and associative in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})$.

(ii) $\forall \alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n}) : \alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} = \alpha$.

(iii) $\forall \alpha, \beta \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}) : \alpha^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta^{\#_{\epsilon,\mathbf{n}}} = (\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta)^{\#_{\epsilon,\mathbf{n}}}$

Proof. Similarly as above.

(i) is clear from definitions,

(ii) $\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \subseteqq \alpha$ is clear from definitions.

Now, suppose $a \in \alpha$. Since α has no greatest element $\exists b_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} > a$, i.e. $b \in \alpha$, therefore

$a -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} b \in 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}$ and $a =_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} (a -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} b) +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} b \in \alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}$.

(iii) $\alpha^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta^{\#_{\epsilon,\mathbf{n}}} \subseteqq (\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta)^{\#_{\epsilon,\mathbf{n}}}$ is clear since $x <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha, y <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta$ implies $x +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} y <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta$.

Now, suppose $x <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} (\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta)$, then

$$\alpha -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} x}{2} <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha,$$

$$\beta -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} x}{2} <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta.$$

Therefore

$$x =_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \left[\alpha -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} x}{2}\right] +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})}$$

$$+_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \left[\beta -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \frac{(\alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta) -_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} x}{2}\right] \in \alpha^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta^{\#_{\epsilon,\mathbf{n}}}.$$

This completes the proof.

Remark 2.7. Notice, here again something is lost going from ${}^*\mathbb{R}$ to ${}^*\mathbb{R}_{\mathbf{d}}$ since $a < \beta$ does

not imply $\alpha + \alpha < \beta + \alpha$ since $0 < \varepsilon_{\mathbf{d}}$ but $0 + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}} + \varepsilon_{\mathbf{d}} = \varepsilon_{\mathbf{d}}$.

Lemma 2.2.(I)[6].

(i) $\leq_{{}^*\mathbb{R}_{\mathbf{d}}}$ a linear ordering on ${}^*\mathbb{R}_{\mathbf{d}}$ often denoted $\leq$, which extends the usual ordering on ${}^*\mathbb{R}$.

(ii) $(\alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha') \wedge (\beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta') \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$.

(iii) $\left(\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha'\right) \wedge \left(\beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'\right) \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$.

(**iv**) ${}^*\mathbb{R}$ is dense in ${}^*\mathbb{R}_{\mathbf{d}}$. That is if $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ in ${}^*\mathbb{R}_{\mathbf{d}}$ there is an $a \in {}^*\mathbb{R}$ then

$\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$.

(**v**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ is bounded above then $\sup A = \sup_{\alpha \in A} \{\alpha\} \triangleq \bigcup_{\alpha \in A} \mathbf{cut}_{-}(\alpha)$

exist in ${}^*\mathbb{R}_{\mathbf{d}}$.

(**vi**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ is bounded below then $\inf A = \inf_{\alpha \in A} \{\alpha\} \triangleq \bigcup_{\alpha \in A} \mathbf{cut}_{+}(\alpha)$

exist in ${}^*\mathbb{R}_{\mathbf{d}}$.

Proof.Note that (i), (ii), (iv), (v) are clear.

(iii) $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha', \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$ imply $\exists\, a, a', b, b' \in {}^*\mathbb{R}$

s.t. $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} a'^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha'$ and $\beta <_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} b'^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$. Therefore

$\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} a'^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}} b'^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta' \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$

completing the proof .

Lemma 2.2.(**II**)

(**i**) $\leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ a linear ordering on ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ often denoted $\leq$, which extends the usual ordering

on ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$.

(**ii**) $(\alpha \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha') \wedge (\beta \leq_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta') \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta \leq_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \beta'$.

(**iii**) $(\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha') \wedge (\beta <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta') \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta'$.

(**iv**) ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ is dense in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. That is if $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$ in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ there is an $a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$

then $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} a^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$.

(**v**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is bounded above then $\sup A = \sup_{\alpha \in A} \alpha \triangleq \bigcup_{\alpha \in A} \mathbf{cut}_{-}(\alpha, \epsilon)$

exist in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

(**vi**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ is bounded below then $\inf A = \inf_{\alpha \in A} \alpha \triangleq \bigcup_{\alpha \in A} \mathbf{cut}_{+}(\alpha, \epsilon)$

exist in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.

Proof. Similarly as above. Note that (i), (ii), (iv), (v) are clear.

(iii) $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha', \beta <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta'$ imply $\exists\, a, a', b, b' \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$

s.t. $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} a^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{d}}} a'^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha'$ and

$\beta <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} b^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} b'^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta'$. Therefore

$\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} a^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} b^{\#_\epsilon} \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} a'^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} b'^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta' \Rightarrow$

$\Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta'$ completing the proof .

Lemma 2.2.(**III**)

(**i**) $\leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}$ a linear ordering on ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})$ often denoted $\leq$, which extends the usual ordering on ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon, \mathbf{n})$.

(**ii**) $(\alpha \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha') \wedge (\beta \leq_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta') \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta \leq_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta'$.

(**iii**) $\left(\alpha <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha'\right) \wedge \left(\beta <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta'\right) \Rightarrow \alpha +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta <_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \alpha' +_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})} \beta'$.

(**iv**) ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon, \mathbf{n})$ is dense in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})$. That is if $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$ in ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ there is an

$a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon, \mathbf{n})$ then $\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} a^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta$.

(**v**) Suppose that $A \subsetneqq {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})$ is bounded above then $\sup A =$

$\sup_{\alpha\in A} \alpha \triangleq \bigcup_{\alpha\in A} \mathbf{cut}_{-}(\alpha,\epsilon,\mathbf{n})$
exist in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})$.

(**vi**) Suppose that $A \subsetneqq {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})$ is bounded below then $\inf A = \inf_{\alpha\in A} \alpha \triangleq \bigcup_{\alpha\in A} \mathbf{cut}_{+}(\alpha,\mathbf{n})$
exist in ${}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})$.

Proof.Similarly as above. Note that (i), (ii), (iv), (v) are clear.

(iii) $\alpha <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha', \beta <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta'$ imply $\exists\, a, a', b, b' \in {}^{*}\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})$
s.t. $\alpha <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} a^{\#_{\epsilon,\mathbf{n}}} <_{{}^{*}\mathbb{R}_{\mathbf{d}}} a'^{\#_{\epsilon,\mathbf{n}}} <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha'$ and
$\beta <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} b^{\#_{\epsilon,\mathbf{n}}} <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} b'^{\#_{\epsilon,\mathbf{n}}} <_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta'$. Therefore
$\alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta \leq_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} a^{\#_{\epsilon,\mathbf{n}}} +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} b^{\#_{\epsilon,\mathbf{n}}} \leq_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} a'^{\#_{\epsilon,\mathbf{n}}} +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} b'^{\#_{\epsilon,\mathbf{n}}}$
$<_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha' +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta' \Rightarrow$
$\Rightarrow \alpha +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \leq_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha' +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \beta'$ completing the proof .

Remark 2.8.Note that in general case $\inf A = \inf_{\alpha\in A} \alpha \neq \bigcap_{\alpha\in A} \mathbf{cut}_{-}(\alpha)$. In particular the formula for $\inf A$ given in [6] on the top of page 229 is not quite correct [7], see Example 2.2. However by Lemma 2.2 (vi) this is no problem.

Example 2.2.[7].The formula $\inf A = \inf_{\alpha\in A} = \bigcap_{\alpha\in A} \mathbf{cut}_{-}(\alpha)$ says

$$\inf_{\alpha\in A} = \left\{ a \,\middle|\, \exists d(d > 0)\left[a + d \in \bigcap_{\alpha\in A} \mathbf{cut}_{-}(\alpha) \right] \right\}$$

Let A be the set $A = \{a + d\}$ where d runs through the set of all positive numbers in ${}^{*}\mathbb{R}$, then $\inf A = a = \{x | x < a\}$. However $\bigcap_{\alpha\in A} \mathbf{cut}_{-}(\alpha) = \{x | x \leq a\}$.

Lemma 2.3.(**I**)[6].

(**i**) If $\alpha \in {}^{*}\mathbb{R}$ then $-_{{}^{*}\mathbb{R}_{\mathbf{d}}} (\alpha^{\#}) = (-_{{}^{*}\mathbb{R}}\, \alpha)^{\#}$.

(**ii**) $-_{{}^{*}\mathbb{R}_{\mathbf{d}}}(-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\, \alpha) = \alpha$.

(**iii**) $\alpha \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta \Longleftrightarrow -_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} -_{{}^{*}\mathbb{R}_{\mathbf{d}}} \alpha$.

(**iv**) $(-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\alpha) +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(-_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right) \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} -_{{}^{*}\mathbb{R}_{\mathbf{d}}}\left(\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$.

(**v**) $\forall a \in {}^{*}\mathbb{R} : (-_{{}^{*}\mathbb{R}}a)^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(-_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right) = -_{{}^{*}\mathbb{R}_{\mathbf{d}}} (a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta)$.

(**vi**) $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} (-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\, \alpha) \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} 0_{{}^{*}\mathbb{R}_{\mathbf{d}}}$.

Proof.(**i**)-(**iii**) are clear.

(**iv**) Suppose $a \in (-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\alpha)$ and $b \in \left(-_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$. Then $\exists a_1, a_2$ s.t. $a <_{{}^{*}\mathbb{R}_{\mathbf{d}}} a_1 \in (-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\alpha)$ and $b <_{{}^{*}\mathbb{R}_{\mathbf{d}}} b_1 \in (-_{{}^{*}\mathbb{R}_{\mathbf{d}}}\beta)$. Thus we have: (1) $-_{{}^{*}\mathbb{R}_{\mathbf{d}}}a_1 \notin \alpha, -_{{}^{*}\mathbb{R}_{\mathbf{d}}}b_1 \notin \beta$, (2) $\alpha <_{{}^{*}\mathbb{R}_{\mathbf{d}}} (-_{{}^{*}\mathbb{R}}\, a_1)^{\#}$, $\beta <_{{}^{*}\mathbb{R}_{\mathbf{d}}} (-_{{}^{*}\mathbb{R}}\, b_1)^{\#}$, (3) $\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta <_{{}^{*}\mathbb{R}_{\mathbf{d}}} (-_{{}^{*}\mathbb{R}}\, a_1)^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} (-_{{}^{*}\mathbb{R}}\, b_1)^{\#} = (-_{{}^{*}\mathbb{R}}a_1 -_{{}^{*}\mathbb{R}} b_1)^{\#}$. Finally one obtains $-_{{}^{*}\mathbb{R}_{\mathbf{d}}}a_1 -_{{}^{*}\mathbb{R}_{\mathbf{d}}} b_1 \notin \alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta$, since $a +_{{}^{*}\mathbb{R}} b <_{{}^{*}\mathbb{R}} a_1 +_{{}^{*}\mathbb{R}} b_1, a +_{{}^{*}\mathbb{R}} b \in -_{{}^{*}\mathbb{R}_{\mathbf{d}}}\left(\alpha +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$.

(**v**) By (iv): $(-_{{}^{*}\mathbb{R}}a)^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(-_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right) \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} -_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$.

(1) Suppose now $c \in -_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$ this means

(2) $\exists c_1\left[c <_{{}^{*}\mathbb{R}} c_1 \in -_{{}^{*}\mathbb{R}_{\mathbf{d}}} \left(a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)\right]$ and therefore

(3) $-_{{}^{*}\mathbb{R}}\, c_1 \notin \left(a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta\right)$.

(4) Note that: $-_{{}^{*}\mathbb{R}}\, c -_{{}^{*}\mathbb{R}}\, a \notin \beta$ (since $-_{{}^{*}\mathbb{R}}\, c -_{{}^{*}\mathbb{R}}\, a \in \beta$ and $a -_{{}^{*}\mathbb{R}} (c -_{{}^{*}\mathbb{R}} c_1) \in a^{\#}$ imply $-_{{}^{*}\mathbb{R}}\, c_1 =_{{}^{*}\mathbb{R}} a -_{{}^{*}\mathbb{R}} (c -_{{}^{*}\mathbb{R}} c_1) +_{{}^{*}\mathbb{R}} (-_{{}^{*}\mathbb{R}}\, c -_{{}^{*}\mathbb{R}}\, a) \in a^{\#} +_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta$ but this is a contradiction)

(5) Thus $-_{{}^*\mathbb{R}}\ c -_{{}^*\mathbb{R}}\ a \in \beta$ and therefore $c +_{{}^*\mathbb{R}}\ a \in -_{{}^*\mathbb{R}_{\mathbf{d}}}\ \beta$.

(6) By similar reasoning one obtain: $c_1 +_{{}^*\mathbb{R}}\ a \in -_{{}^*\mathbb{R}_{\mathbf{d}}}\ \beta$.

(7) Note that: $-a - (c_1 - c) \in a^{\#}$ and therefore

$c =_{{}^*\mathbb{R}}\ -_{{}^*\mathbb{R}}\ a -_{{}^*\mathbb{R}}\ (c_1 -_{{}^*\mathbb{R}}\ c) +_{{}^*\mathbb{R}}\ (c_1 +_{{}^*\mathbb{R}}\ a) \in (-_{{}^*\mathbb{R}}a)^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}}\ \left(-_{{}^*\mathbb{R}_{\mathbf{d}}}\ \beta\right)$.

Lemma 2.3.(**II**)

(**i**) If $\alpha \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ then $-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ (\alpha^{\#_\epsilon}) = (-_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)}\ \alpha)^{\#_\epsilon}$.

(**ii**) $-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \alpha) = \alpha$.

(**iii**) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta \Leftrightarrow -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \alpha$.

(**iv**) $(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\alpha) +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \left(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta\right) \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\left(\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta\right)$.

(**v**) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$: $(-_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon)}a)^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \left(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta\right) = -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ (a^{\#_\epsilon} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \beta)$.

(**vi**) $\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ (-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \alpha) \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$.

Proof.Immediately from definitions.

Lemma 2.3.(**III**)

(**i**) If $\alpha \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})$ then $-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (\alpha^{\#_{\epsilon,\mathbf{n}}}) = (-_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})}\ \alpha)^{\#_{\epsilon,\mathbf{n}}}$.

(**ii**) $-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}\ \alpha) = \alpha$.

(iii) $\alpha \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta \Leftrightarrow -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \alpha$.

(iv) $(-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\alpha) +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta) \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta)$.

(v) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})$: $(-_{{}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n})}a)^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta) = -_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (a^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta)$.

(vi) $\alpha +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (-_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \alpha) \leq_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ 0_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}$.

Proof.Immediately from definitions.

Lemma 2.4.(I) (i) $\forall a \in {}^*\mathbb{R}, \forall \beta \in {}^*\mathbb{R}_{\mathbf{d}}, \mu \in {}^*\mathbb{R}, \mu \geq 0$: $(-\mu a)^{\#} + (-\mu^{\#}\beta) = -\mu^{\#}(a^{\#} + \beta)$,

(ii) $\forall a \in {}^*\mathbb{R}, \forall \beta \in {}^*\mathbb{R}_{\mathbf{d}}, \mu \in {}^*\mathbb{R}, \mu \geq 0$: $(\mu a)^{\#} + \mu^{\#}\beta = \mu^{\#}(a^{\#} + \beta)$.

Proof.(i) For $\mu = 0$ the statement is clear. Suppose now without loss of generality $\mu > 0$. By Lemma 2.3.(iv): $(-\mu a)^{\#} + (-\mu^{\#}\beta) \leq -(\mu^{\#}a^{\#} + \mu^{\#}\beta)$.

(1) Suppose $c \in -\mu^{\#}(a^{\#} + \beta)$ and therefore $\frac{c}{\mu} \in -(a^{\#} + \beta)$, but this means

(2) $\exists c_1\left[\frac{c}{\mu} < \frac{c_1}{\mu} \in -(a^{\#} + \beta)\right]$ and therefore

(3) $-\frac{c_1}{\mu} \notin (a^{\#} + \beta)$.

(4) Note that: $-\frac{c}{\mu} - a \notin \beta$ (since $-\frac{c}{\mu} - a \in \beta$ and $a - \left(\frac{c}{\mu} - \frac{c_1}{\mu}\right) \in a^{\#}$ imply

$-\frac{c_1}{\mu} = a - \left(\frac{c}{\mu} - \frac{c_1}{\mu}\right) + \left(-\frac{c}{\mu} - a\right) \in a^{\#} + \beta$ but this is a contradiction)

(5) Thus $-\frac{c}{\mu} - a \in \beta$ and therefore $c + \mu a \in -\mu^{\#}\beta$.

(6) By similar reasoning one obtain: $c_1 + \mu a \in -\mu^{\#}\beta$.

(7) Note that: $-\mu a - (c_1 - c) \in \mu^{\#}a^{\#}$ and therefore

$c = -\mu a - (c_1 - c) + (c_1 + \mu a) \in (-\mu a)^{\#} + (-\mu^{\#}\beta)$.

(ii) Immediately follows from (i) by Lemma 2.3.

Lemma 2.4.(II) (i) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \forall \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon), \mu \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \mu \geq 0$:

$(-\mu a)^{\#_\epsilon} + (-\mu^{\#_\epsilon}\beta) = -\mu^{\#_\epsilon}(a^{\#_\epsilon} + \beta)$,

(ii) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \forall \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon), \mu \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \mu \geq 0$: $(\mu a)^{\#_\epsilon} + \mu^{\#_\epsilon}\beta = \mu^{\#_\epsilon}(a^{\#_\epsilon} + \beta)$.

Proof.Immediately from definitions similarly as Lemma 2.4.(I) above.

Lemma 2.4.(**III**) (i) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \forall \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n}), \mu \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon), \mu \geq 0$:

$(-\mu a)^{\#_{\epsilon,\mathbf{n}}} + (-\mu^{\#_\epsilon}\beta) = -\mu^{\#_\epsilon}(a^{\#_{\epsilon,\mathbf{n}}} + \beta)$,

(ii) $\forall a \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}), \forall \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n}), \mu \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon,\mathbf{n}), \mu \geq 0$:

$(\mu a)^{\#_{\epsilon,\mathbf{n}}} + \mu^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta = \mu^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ (a^{\#_{\epsilon,\mathbf{n}}} +_{{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})}\ \beta)$.

Proof.Immediately from definitions similarly as Lemma 2.4.(I) above.
Definition 2.6.(I) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. The absolute value of α written $|\alpha|$ is defined as follows:

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \ {}_{^*\mathbb{R}_{\mathbf{d}}} \geq 0_{^*\mathbb{R}_{\mathbf{d}}} \\ -_{^*\mathbb{R}_{\mathbf{d}}} \alpha & \text{if } \alpha \ \leq_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \end{cases}$$

Definition 2.**6**.(**II**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. The absolute value of α written $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ or $|\alpha|$ is defined as follows:

$$|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} = \begin{cases} \alpha & \text{if } \alpha \ {}_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \geq 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \\ -_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha & \text{if } \alpha \ \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \end{cases}$$

Definition 2.6.(III) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. The absolute value of α written $|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$ or $|\alpha|$ is defined as follows:

$$|\alpha|_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} = \begin{cases} \alpha & \text{if } \alpha \ {}_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \geq 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \\ -_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \alpha & \text{if } \alpha \ \leq_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})} \end{cases}$$

Definition 2.**7**.(**I**).Suppose $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$.The product $\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta$, is defined as follows: Case (1) $\alpha, \beta \ {}_{^*\mathbb{R}_{\mathbf{d}}} > 0_{^*\mathbb{R}_{\mathbf{d}}}$:

$$\begin{gathered}\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq \\ \{a \times_{^*\mathbb{R}} b | (0_{^*\mathbb{R}_{\mathbf{d}}} <_{^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{^*\mathbb{R}_{\mathbf{d}}} \alpha) \wedge (0_{^*\mathbb{R}_{\mathbf{d}}} <_{^*\mathbb{R}_{\mathbf{d}}} b^{\#} <_{^*\mathbb{R}_{\mathbf{d}}} \beta)\} \cup \left\{({}^* -\infty, {}^*0)^{\#}\right\}.\end{gathered} \tag{2.10}$$

Case (2) $\alpha =_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \vee \beta =_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}$: $\alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq 0_{^*\mathbb{R}_{\mathbf{d}}}$.
Case (3) $(\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \wedge \beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}})$

$$\begin{cases} \alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq |\alpha| \times_{^*\mathbb{R}_{\mathbf{d}}} |\beta| \text{ iff } \alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}} \wedge \beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}, \\ \alpha \times_{^*\mathbb{R}_{\mathbf{d}}} \beta \triangleq -_{^*\mathbb{R}_{\mathbf{d}}} (|\alpha| \times_{^*\mathbb{R}_{\mathbf{d}}} |\beta|) \text{ iff } (\alpha <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}) \vee (\beta <_{^*\mathbb{R}_{\mathbf{d}}} 0_{^*\mathbb{R}_{\mathbf{d}}}). \end{cases} \tag{2.11}$$

Definition 2.**7**.(**II**).Suppose $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$.The product $\alpha \times_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta$, is defined as follows: Case (1) $\alpha_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} > 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}, \beta \ {}_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} > 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$:

$$\begin{gathered}\alpha \times_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \triangleq \\ \left\{a \times_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)} b \middle| \left(0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} a^{\#_\epsilon} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \alpha\right) \wedge \left(0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} b^{\#_\epsilon} <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta\right)\right\} \\ \bigcup \left\{\left(-_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)} \infty_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)}, 0_{^*\mathbb{R}_{\mathbf{st}}(\epsilon)}\right)^{\#_\epsilon}\right\}.\end{gathered} \tag{2.10.a}$$

Case (2) $\alpha =_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \vee \beta =_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$: $\alpha \times_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \beta \triangleq 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}$.
Case (3) $(\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}) \vee (\beta <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)}) \vee$
$\vee(\alpha <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \wedge \beta <_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)} 0_{^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)})$.

$$\begin{cases} \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta \triangleq |\alpha| \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} |\beta| \text{ iff } \alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} 0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \wedge \beta <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} 0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}}, \\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta \triangleq -_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} (|\alpha| \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} |\beta|) \text{ iff } (\alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} 0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}}) \vee \\ \vee(\beta <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} 0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}}). \end{cases} \quad (2.11.a)$$

Definition 2.7.(III).Suppose $\mathbf{n} \in^* \mathbb{N}\backslash\mathbb{N}$ and $\beta \in {}^*\mathbb{R}_{\mathbf{d}}$. The $\ltimes_{{}^*\mathbb{R}}$ -product

$$\mathbf{n} \ltimes_{{}^*\mathbb{R}} \beta \in {}^*\mathbb{R}_{\mathbf{d}},$$

is defined as follows :

$$\mathbf{n} \ltimes_{{}^*\mathbb{R}} \beta \triangleq \mathbf{n} \times_{{}^*\mathbb{R}} \{\beta_{\#}\} = \mathbf{n} \times_{{}^*\mathbb{R}} \{\mathbf{cut}_{-}(\beta)\} = \{\mathbf{n} \times_{{}^*\mathbb{R}} b | b \in \mathbf{cut}_{-}(\beta)\}. \quad (2.11.b)$$

Definition 2.7.(IV).Suppose $\mathbf{n} \in^* \mathbb{N}\backslash\mathbb{N}$ and $\beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$. The $\ltimes_{{}^*\mathbb{R}}$ -product

$$\mathbf{n} \ltimes_{{}^*\mathbb{R}} \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n}),$$

is defined as follows :

$$\mathbf{n} \ltimes_{{}^*\mathbb{R}} \beta \triangleq \mathbf{n} \times_{{}^*\mathbb{R}} \{\beta_{\#_\epsilon}\} = \mathbf{n} \times_{{}^*\mathbb{R}} \{\mathbf{cut}_{-}(\beta, \epsilon)\} = \{\mathbf{n} \times_{{}^*\mathbb{R}} b | b \in \mathbf{cut}_{-}(\beta, \epsilon)\}, \quad (2.11.c)$$

see Designation 2.2.(ii).

Definition 2.7.(V).Suppose $\mathbf{n} \in^* \mathbb{N}\backslash\mathbb{N}$ and $\beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})$. The product

$$\mathbf{n}^{-1} \ltimes_{{}^*\mathbb{R}} \beta \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon),$$

is defined as follows :

$$\begin{gathered} \mathbf{n}^{-1} \ltimes_{{}^*\mathbb{R}} \beta \triangleq \mathbf{n}^{-1} \times_{{}^*\mathbb{R}} \{\beta_{\#_{\epsilon,\mathbf{n}}}\} = \mathbf{n}^{-1} \times_{{}^*\mathbb{R}} \{\mathbf{cut}_{-}(\beta, \epsilon, \mathbf{n})\} = \\ \{\mathbf{n}^{-1} \times_{{}^*\mathbb{R}} b | b \in \mathbf{cut}_{-}(\beta, \epsilon, \mathbf{n})\}. \end{gathered} \quad (2.11.d)$$

Lemma 2.5.(I)[6]. (**i**) $\forall a, b \in {}^*\mathbb{R} : (a \times_{{}^*\mathbb{R}} b)^{\#} = a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#}$.

(**ii**) Multiplication $(\cdot \times_{{}^*\mathbb{R}_{\mathbf{d}}} \cdot)$ is associative and commutative:

$$(\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma), \ \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta = \beta \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha. \quad (2.12)$$

(**iii**) $1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = \alpha$; $-_{{}^*\mathbb{R}_{\mathbf{d}}} 1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = -_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$, where $1_{{}^*\mathbb{R}_{\mathbf{d}}} = (1_{{}^*\mathbb{R}})^{\#}$.

(**iv**) $|\alpha|_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} |\beta|_{{}^*\mathbb{R}_{\mathbf{d}}} = |\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta|_{{}^*\mathbb{R}_{\mathbf{d}}}$.

(**v**)

$$\begin{gathered} [(\alpha_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}}) \wedge (\beta_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}}) \wedge (\gamma_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}})] \Rightarrow \\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma. \end{gathered} \quad (2.13)$$

and

$$[(\alpha_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}}) \wedge (\beta_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}})] \Rightarrow \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma. \quad (2.13.a)$$

(**vi**)

$$0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha', 0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta' \Rightarrow \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'. \quad (2.14)$$

Proof.(**i**) First, suppose $a_{{}^*\mathbb{R}} > 0_{{}^*\mathbb{R}}, b_{{}^*\mathbb{R}} > 0_{{}^*\mathbb{R}}$ then clearly $a^{\#} \times_{{}^*\mathbb{R}} b^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} (a \times_{{}^*\mathbb{R}} b)^{\#}$ since $0_{{}^*\mathbb{R}} <_{{}^*\mathbb{R}} x <_{{}^*\mathbb{R}} a$, $0_{{}^*\mathbb{R}} <_{{}^*\mathbb{R}} y <_{{}^*\mathbb{R}} b$ implies $x \times_{{}^*\mathbb{R}} y <_{{}^*\mathbb{R}} a \times_{{}^*\mathbb{R}} b$. Now suppose $0_{{}^*\mathbb{R}} <_{{}^*\mathbb{R}} c <_{{}^*\mathbb{R}} a \times_{{}^*\mathbb{R}} b$ and let $a' = a \times_{{}^*\mathbb{R}} \sqrt{c/a \times_{{}^*\mathbb{R}} b} <_{{}^*\mathbb{R}} a, b' = b \times_{{}^*\mathbb{R}} \sqrt{c/a \times_{{}^*\mathbb{R}} b} <_{{}^*\mathbb{R}} b$, thus $c = a' \times_{{}^*\mathbb{R}} b' \in a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#}$, so $(a \times_{{}^*\mathbb{R}} b)^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} \times_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#}$.

The other cases follow from (**iv**).

(**iv**) Immediate from Definition **2.7**.(**I**) and Lemma **2.3**.(**I**).(**ii**).

(**ii**) is immediate from the definition for $\alpha,\beta,\gamma\ {}_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}}$ and (**iv**) otherwise.

(**iii**) We may assume $\alpha\ {}_{{}^*\mathbb{R}_{\mathbf{d}}} \geq 0_{{}^*\mathbb{R}_{\mathbf{d}}}$. Clearly $1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}} \alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$. Now suppose $a \in \alpha$ then $\exists a' \in \alpha,\ a <_{{}^*\mathbb{R}} a'$. Thus $a/a' <_{{}^*\mathbb{R}} 1$ so $a/a' \in 1_{{}^*\mathbb{R}_{\mathbf{d}}}$ and $a' \times_{{}^*\mathbb{R}} (a/a') = a \in 1_{{}^*\mathbb{R}_{\mathbf{d}}}$.

By the definition $(-_{{}^*\mathbb{R}_{\mathbf{d}}} 1_{{}^*\mathbb{R}_{\mathbf{d}}}) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha = -_{{}^*\mathbb{R}_{\mathbf{d}}} (1_{{}^*\mathbb{R}_{\mathbf{d}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha) = -_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$.

(**v**) 1.Suppose: $d \in \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma)$, thus $d = a \times_{{}^*\mathbb{R}} (b +_{{}^*\mathbb{R}} c)$, where $a \in \alpha, b \in \beta, c \in \gamma$. Note that

$$d = a \times_{{}^*\mathbb{R}} (b +_{{}^*\mathbb{R}} c) = a \times_{{}^*\mathbb{R}} b +_{{}^*\mathbb{R}} a \times_{{}^*\mathbb{R}} c \in \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma$$

and therefore $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma$.

2.Suppose now: $d \in \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma$, thus $d = a \times_{{}^*\mathbb{R}} b +_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} c$ where $a, a' \in \alpha, b \in \beta, c \in \gamma$. Without loss of generality we may assume $a \leq_{{}^*\mathbb{R}} a'$. Hence

$$d = a \times_{{}^*\mathbb{R}} b +_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} c \leq_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} b +_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} c$$

since $0_{{}^*\mathbb{R}} \leq_{{}^*\mathbb{R}} a',\ 0_{{}^*\mathbb{R}} \leq_{{}^*\mathbb{R}} b$. Thus finally one obtains

$$d \leq_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} b +_{{}^*\mathbb{R}} a' \times_{{}^*\mathbb{R}} c = a' \times_{{}^*\mathbb{R}} (b +_{{}^*\mathbb{R}} c) \in \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma),$$

i.e. $d \in \alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma)$.

(**vi**) Note that $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha'$ and $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$ implies $\exists\ a, a', b, b'$ s.t. $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} a^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} a'^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha'$ and $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} b^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} b'^{\#} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$. Thus $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}} (a \times_{{}^*\mathbb{R}} b)^{\#} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$ and therefore $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha' \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta'$. This completes the proof.

Lemma 2.5.(**II**). (**i**) $\forall a, b \in {}^*\mathbb{R}_{\mathbf{st}}(\epsilon) : (a \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b)^{\#_\epsilon} = a^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} b^{\#_\epsilon}$.

(**ii**) Multiplication $(\cdot \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \cdot)$ is associative and commutative:

$$(\alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta) \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \gamma = \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} (\beta \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \gamma),\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta = \beta \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha. \qquad (2.12')$$

(**iii**) $1_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha = \alpha;\ -_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} 1_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha = -_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha$, where $1_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} = (1_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}})^{\#_\epsilon}$.

(**iv**) $|\alpha| \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} |\beta| = |\beta| \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} |\alpha|$.

(**v**)

$$\begin{gathered}[(\alpha_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \geq 0) \wedge (\beta_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \geq 0) \wedge (\gamma_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \geq 0)] \Rightarrow \\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} (\beta +_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \gamma) = \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta +_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \gamma.\end{gathered} \qquad (2.13')$$

(**vi**)

$$\begin{gathered}0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha', 0_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta' \Rightarrow \\ \alpha \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \alpha' \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} \beta'.\end{gathered} \qquad (2.14')$$

Proof.(**i**) First, suppose $a_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} > 0_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}}, b_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} > 0_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}}$ then clearly

$$a^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} b^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} (a \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b)^{\#_\epsilon}$$

since $0_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} x <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} a,\ 0_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} y <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b$ implies $x \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} y <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} a \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b$. Now suppose $0_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} c <_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} a \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b$ and let

$$a' = a \times_{{}^*\mathbb{R}} \sqrt{c/a \times_{{}^*\mathbb{R}} b} <_{{}^*\mathbb{R}} a, b' = b \times_{{}^*\mathbb{R}} \sqrt{c/a \times_{{}^*\mathbb{R}} b} <_{{}^*\mathbb{R}} b,$$

thus $c = a' \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b' \in a^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} b^{\#_\epsilon}$, so

$$(a \times_{{}^*\mathbb{R}_{\mathbf{st}(\epsilon)}} b)^{\#_\epsilon} <_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} a^{\#_\epsilon} \times_{{}^*\mathbb{R}_{\mathbf{st.d}(\epsilon)}} b^{\#_\epsilon}.$$

The other cases follow from (**iv**).

Lemma 2.6.(**I**).Suppose $\mu \in {}^*\mathbb{R}$ and $\beta, \gamma \in {}^*\mathbb{R}_{\mathbf{d}}$. Then

$$[(\mu^\# \geq 0) \wedge (\beta \geq 0)] \Rightarrow \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma. \quad (2.15)$$

Proof. We choose now: (1) $a \in {}^*\mathbb{R}$ such that: $-_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma +_{{}^*\mathbb{R}_{\mathbf{d}}} a^\# \ {}_{{}^*\mathbb{R}_{\mathbf{d}}} > 0_{{}^*\mathbb{R}_{\mathbf{d}}}$.
(2) Note that $\mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) +_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\# -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\#$.
Then from (2) by Lemma 2.4.I.(ii) one obtains
(3) $\mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} [(\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) +_{{}^*\mathbb{R}_{\mathbf{d}}} a^\#] -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\#$. Therefore
(4) $\mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} [\beta +_{{}^*\mathbb{R}_{\mathbf{d}}} (a^\# -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma)] - \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\#$.
(5) Then from (4) by Lemma 2.5.(v) one obtains
(6) $\mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (a^\# -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\#$.
Then from (6) by Lemma 2.4.(ii) one obtains
(7) $\mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} (\beta -_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma) = \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta +_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\# -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma -_{{}^*\mathbb{R}_{\mathbf{d}}} \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} a^\# =$
$= \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta - \mu^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \gamma$.

Definition 2.8.(**I**). Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, 0_{{}^*\mathbb{R}_{\mathbf{d}}} \neq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$ then $\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$ is defined as follows:
(**i**) $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \triangleq \inf\left\{a_\#^{-1_{{}^*\mathbb{R}}} | 0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} a \in \alpha\right\}$, see Designation 2.2. (i),
(**ii**) $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}}} 0 : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \triangleq -_{{}^*\mathbb{R}_{\mathbf{d}}} (-_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.

Definition 2.8.(**II**). Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}(\epsilon), 0_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} \neq_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} \alpha$ then $\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}}}$ is defined as follows:
(**i**)
$0_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} <_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} \alpha : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}}} \triangleq \inf\left\{a_{\#_\epsilon}^{-1_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}}} | 0_{{}^*\mathbb{R}(\epsilon)} <_{{}^*\mathbb{R}(\epsilon)} a \in \alpha\right\}$, see Designation 2.2. (ii),
(**ii**) $\alpha <_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} 0 : \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}}} \triangleq -_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} (-_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}} \alpha)^{-1_{{}^*\mathbb{R}_{\mathbf{d}(\epsilon)}}}$.

Lemma 2.7.(**I**).[6].
(**i**) $\forall a \in {}^*\mathbb{R} : (a^\#)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} =_{{}^*\mathbb{R}_{\mathbf{d}}} (a^{-1_{{}^*\mathbb{R}}})^\#$.
(**ii**) $\left(\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} =_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$.
(**iii**) $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \Rightarrow \beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.
(**iv**) $[(0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha) \wedge (0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)] \Rightarrow$
$\Rightarrow \left(\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) \times_{{}^*\mathbb{R}_{\mathbf{d}}} \left(\beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) \leq_{{}^*\mathbb{R}_{\mathbf{d}}} (\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$
(**v**) $\forall a \in {}^*\mathbb{R} : a \neq_{{}^*\mathbb{R}} 0_{{}^*\mathbb{R}} \Rightarrow (a^\#)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \times_{{}^*\mathbb{R}_{\mathbf{d}}} \left(\beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right) = (a^\# \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.
(**vi**) $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} 1_{{}^*\mathbb{R}_{\mathbf{d}}}$.

Lemma 2.7.(**II**).

(**i**) $\forall a \in {}^*\mathbb{R} : (a^\#)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} =_{{}^*\mathbb{R}_{\mathbf{d}}} (a^{-1_{{}^*\mathbb{R}}})^\#$.
(**ii**) $\left(\alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}\right)^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} =_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$.
(**iii**) $0_{{}^*\mathbb{R}_{\mathbf{d}}} <_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \beta \Rightarrow \beta^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}} \leq_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha^{-1_{{}^*\mathbb{R}_{\mathbf{d}}}}$.

(iv) $[(0_{*\mathbb{R}_\mathbf{d}} <_{*\mathbb{R}_\mathbf{d}} \alpha) \wedge (0_{*\mathbb{R}_\mathbf{d}} <_{*\mathbb{R}_\mathbf{d}} \beta)] \Rightarrow$
$\Rightarrow \left(\alpha^{-1_{*\mathbb{R}_\mathbf{d}}}\right) \times_{*\mathbb{R}_\mathbf{d}} \left(\beta^{-1_{*\mathbb{R}_\mathbf{d}}}\right) \leq_{*\mathbb{R}_\mathbf{d}} (\alpha \times_{*\mathbb{R}_\mathbf{d}} \beta)^{-1_{*\mathbb{R}_\mathbf{d}}}$

(v) $\forall a \in {}^*\mathbb{R} : a \neq_{*\mathbb{R}} 0_{*\mathbb{R}} \Rightarrow (a^{\#})^{-1_{*\mathbb{R}_\mathbf{d}}} \times_{*\mathbb{R}_\mathbf{d}} \left(\beta^{-1_{*\mathbb{R}_\mathbf{d}}}\right) = (a^{\#} \times_{*\mathbb{R}_\mathbf{d}} \beta)^{-1_{*\mathbb{R}_\mathbf{d}}}$.

(vi) $\alpha \times_{*\mathbb{R}_\mathbf{d}} \alpha^{-1_{*\mathbb{R}_\mathbf{d}}} \leq_{*\mathbb{R}_\mathbf{d}} 1_{*\mathbb{R}_\mathbf{d}}$.

Lemma 2.8.[6]. Suppose that $a \in {}^*\mathbb{R}, a > 0, \beta, \gamma \in {}^*\mathbb{R}_\mathbf{d},\ \beta > 0, \gamma > 0$. Then
$a^{\#} \times_{*\mathbb{R}_\mathbf{d}} (\beta +_{*\mathbb{R}_\mathbf{d}} \gamma) = a^{\#} \times_{*\mathbb{R}_\mathbf{d}} \beta +_{*\mathbb{R}_\mathbf{d}} a^{\#} \times_{*\mathbb{R}_\mathbf{d}} \gamma$.

Theorem 2.1.(**i**) Suppose that $\mathbf{S}$ is a non-empty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded from above, i.e. $\sup(\mathbf{S})$ exists in ${}^*\mathbb{R}_\mathbf{d}$ and suppose that $\xi \in {}^*\mathbb{R}, \xi > 0$. Then

$$\sup\{\xi^{\#} \times \mathbf{S}\} = \xi^{\#} \times (\sup \mathbf{S}). \qquad (2.16.a)$$

(**ii**) Suppose that $\mathbf{S}$ is a non-empty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded from above, i.e. $\sup(\mathbf{S})$
exist and suppose that: (a) $\mathbf{S}$ >0 and (b) $\xi \in {}^*\mathbb{R}, \xi \neq 0$. Then

$$\sup\{\xi^{\#} + \mathbf{S}\} = \xi^{\#} + (\sup \mathbf{S}). \qquad (2.16.b)$$

(**iii**) Suppose that $\mathbf{S}$ is a non-empty subset of ${}^*\mathbb{R}_\mathbf{d}$ which is bounded from above, i.e. $\sup(\mathbf{S})$
exist and suppose that: (a) $\mathbf{S}$ >0 and (b) $\xi \in {}^*\mathbb{R}, \xi \neq 0, \alpha \in {}^*\mathbb{R}, \alpha > 0$. Then

$$\alpha^{\#}[\xi^{\#} + (\sup \mathbf{S})] = \alpha^{\#}\xi^{\#} + \alpha^{\#}(\sup \mathbf{S}). \qquad (2.16.c)$$

Proof: (i) Let $B = \sup \mathbf{S}$. Then B is the smallest number in ${}^*\mathbb{R}_\mathbf{d}$ such that, for any $x \in \mathbf{S}, x \leq B$. Let $\mathbf{T} = \{\xi^{\#} \times x | x \in \mathbf{S}\}$. Since $\xi^{\#} > 0, \xi^{\#} \times x \leq \xi^{\#} \times B$ for any $x \in \mathbf{S}$. Hence $\mathbf{T}$ is bounded above by $\xi^{\#} \times B$. Hence $\mathbf{T}$ has a supremum $C_\mathbf{T} = \sup \mathbf{T}$. Now we have to prove that $C_\mathbf{T} = \xi^{\#} \times B = = \xi^{\#} \times (\sup \mathbf{S})$. Since $\xi^{\#} \times B = \xi^{\#} \times (\sup \mathbf{S})$ is an upper bound for $\mathbf{T}$and C is the smallest upper bound for $\mathbf{T}, C_\mathbf{T} \leq \xi^{\#} \times B$. Now we repeat the argument above with the roles of $\mathbf{S}$ and $\mathbf{T}$ reversed. We know that $C_\mathbf{T}$ is the smallest number in ${}^*\mathbb{R}_\mathbf{d}$ such that, for any $y \in \mathbf{T}, y \leq C_\mathbf{T}$. Since $\xi^{\#} > 0$ it follows that $(\xi^{\#})^{-1} \times y \leq (\xi^{\#})^{-1} \times C_\mathbf{T}$ for any $y \in \mathbf{T}$. But $\mathbf{S} = \left\{(\xi^{\#})^{-1} \times y | y \in \mathbf{T}\right\}$. Hence $(\xi^{\#})^{-1} \times C_\mathbf{T}$ is an upper bound for $\mathbf{S}$. But B is a supremum for $\mathbf{S}$. Hence $B \leq (\xi^{\#})^{-1} \times C_\mathbf{T}$ and $\xi^{\#} \times B \leq C_\mathbf{T}$. We have shown that $C_\mathbf{T} \leq \xi^{\#} \times B$ and also that $\xi^{\#} \times B \leq C_\mathbf{T}$. Thus $\xi^{\#} \times B = C_\mathbf{T}$.

(ii) $\xi^{\#} + (\sup \mathbf{S}) \subseteq \sup(\xi^{\#} + \mathbf{S})$ is clear since for any $x, y, s \in {}^*\mathbb{R}$ such that

$$x < \xi, y < s, s^{\#} \in \mathbf{S}$$

implies

$$x^{\#} + y^{\#} < \xi^{\#} + s^{\#} \in \sup(\xi^{\#} + \mathbf{S}).$$

Now suppose $x^{\#} \in \sup(\xi^{\#} + \mathbf{S})$ and therefore $x^{\#} < \xi^{\#} + s^{\#}, s^{\#} \in \mathbf{S}$. Then

$$\xi^{\#} - \frac{(\xi^{\#} + s^{\#}) - x^{\#}}{2} < \xi^{\#}$$

and

$$s^{\#} - \frac{(\xi^{\#} + s^{\#}) - x^{\#}}{2} < s^{\#}.$$

So

$$x = \left[\xi^{\#} - \frac{(\xi^{\#} + s^{\#}) - x^{\#}}{2}\right] + \left[s^{\#} - \frac{(\xi^{\#} + s^{\#}) - x^{\#}}{2}\right] \in \xi^{\#} + \sup(\mathbf{S}).$$

This completes the proof.

(iii) By (ii) one obtains

$$\alpha^{\#}[\xi^{\#} + (\sup \mathbf{S})] = \alpha^{\#}[\sup\{\xi^{\#} + \mathbf{S}\}] .$$

By (i) one obtains

$$\alpha^{\#}[\sup\{\xi^{\#} + \mathbf{S}\}] = [\sup(\alpha^{\#}\{\xi^{\#} + \mathbf{S}\})] = [\sup\{\alpha^{\#}\xi^{\#} + \alpha^{\#}\mathbf{S}\}] .$$

By (ii) one obtains

$$\sup\{\alpha^{\#}\xi^{\#} + \alpha^{\#}\mathbf{S}\} = \alpha^{\#}\xi^{\#} + \sup\{\alpha^{\#}\mathbf{S}\}.$$

By (i) one obtains

$$\alpha^{\#}\xi^{\#} + \sup\{\alpha^{\#}\mathbf{S}\} = \alpha^{\#}\xi^{\#} + \alpha^{\#} \sup \mathbf{S}.$$

This completes the proof.

2.3 Absorption numbers in ${}^*\mathbb{R}_{\mathbf{d}}$.

One of standard ways of defining the completion of ${}^*\mathbb{R}$ involves restricting oneself to subsets, which have the following property $\forall \varepsilon_{\varepsilon>0} \exists x_{x\in\alpha} \exists y_{y\in\alpha}\left[y - x < \varepsilon\right]$. It is well known that in this case we obtain a field. In fact the proof is essentially the same as the one used in the case of ordinary Dedekind cuts in the development of the standard real numbers, $\varepsilon_{\mathbf{d}}$, of course, does not have the above property because no infinitesimal works.This suggests the introduction of the concept of absorption part $\mathbf{ab.p.}(\alpha)$ of a number α for an element α of ${}^*\mathbb{R}_{\mathbf{d}}$ which, roughly speaking, measures how much α departs from having the above property [7].

Definition 2.9.[7]. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$, then

$$\mathbf{ab.p.}(\alpha) \triangleq \{d \geq 0 | \forall x_{x\in\alpha}[x + d \in \alpha]\}. \qquad (2.17)$$

Example 2.5.

(i) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#}) = 0,$

(ii) $\mathbf{ab.p.}(\varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(iii) $\mathbf{ab.p.}(-\varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(iv) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#} + \varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}},$

(v) $\forall \alpha \in {}^*\mathbb{R} : \mathbf{ab.p.}(\alpha^{\#} - \varepsilon_{\mathbf{d}}) = \varepsilon_{\mathbf{d}}.$

Lemma 2.9.[7].

(i) $c < \mathbf{ab.p.}(\alpha)$ and $0 \leq d < c \Rightarrow d \in \mathbf{ab.p.}(\alpha)$

(ii) $c \in \mathbf{ab.p.}(\alpha)$ and $d \in \mathbf{ab.p.}(\alpha) \Rightarrow c + d \in \mathbf{ab.p.}(\alpha).$

Remark 2.9. By Lemma 2.7 $\mathbf{ab.p.}(\alpha)$ may be regarded as an element of ${}^*\mathbb{R}_{\mathbf{d}}$ by adding on all negative elements of ${}^*\mathbb{R}_{\mathbf{d}}$ to $\mathbf{ab.p.}(\alpha)$. Of course if the condition $d \geq 0$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x < y \in \alpha \Rightarrow x \in \alpha$.The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha) = \{0\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with 0. [$\mathbf{ab.p.}(\alpha)$ becomes $\{x|x < 0\}$ which by our early convention is not in ${}^*\mathbb{R}_{\mathbf{d}}$].

Remark 2.10. By Lemma 2.7(ii), $\mathbf{ab.p.}(\alpha)$ is additive idempotent.

Lemma 2.10.[7].

(**i**) $\mathbf{ab.p.}(\alpha)$ is the maximum element $\beta \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\alpha + \beta = \alpha$.

(**ii**) $\mathbf{ab.p.}(\alpha) \leq \alpha$ for $\alpha > 0$.

(**iii**) If α is positive and idempotent then $\mathbf{ab.p.}(\alpha) = \alpha$.

Lemma 2.11.[7]. Let $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ satsify $\alpha > 0$. Then the following are equivalent. In what follows assume $a,b > 0$.

(**i**) α is idempotent,

(**ii**) $a,b \in \alpha \Rightarrow a + b \in \alpha$,

(**iii**) $a \in \alpha \Rightarrow 2a \in \alpha$,

(**iv**) $\forall n_{n\in\mathbb{N}}[a \in \alpha \Rightarrow n \cdot a \in \alpha]$,

(**v**) $a \in \alpha \Rightarrow r \cdot a \in \alpha$, for all finite $r \in {}^*\mathbb{R}$.

Theorem 2.2.[7]. $(-\alpha) + \alpha = -[\mathbf{ab.p.}(\alpha)]$.

Theorem 2.3.[7]. $\mathbf{ab.p.}(\alpha + \beta) \geq \mathbf{ab.p.}(\alpha)$.

Theorem 2.4.[7].

(**i**) $\alpha + \beta \leq \alpha + \gamma \Rightarrow -\mathbf{ab.p.}(\alpha) + \beta \leq \gamma$.

(**ii**) $\alpha + \beta = \alpha + \gamma \Rightarrow -[\mathbf{ab.p.}(\alpha)] + \beta = -[\mathbf{ab.p.}(\alpha)] + \gamma$.

Theorem 2.5.[7].Suppose $\alpha,\beta \in {}^*\mathbb{R}_{\mathbf{d}}$, then

(**i**) $\mathbf{ab.p.}(-\alpha) = \mathbf{ab.p.}(\alpha)$,

(**ii**) $\mathbf{ab.p.}(\alpha + \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$

Theorem 2.6.[7]. Assume $\beta > 0$. If α absorbs $-\beta$ then α absorbs β.

Theorem 2.7.[7]. Let $0 < \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then the following are equivalent

(**i**) α is an idempotent,

(**ii**) $(-\alpha) + (-\alpha) = -\alpha$,

(**iii**) $(-\alpha) + \alpha = -\alpha$.

(**iv**) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 > \Delta_1$. Then $\Delta_2 + (-\Delta_1) = \Delta_2$.

2.4 Gonshor types of α with given $\mathbf{ab.p.}(\alpha)$.

Among elements of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ such that $\mathbf{ab.p.}(\alpha) = \Delta$ one can distinguish two many different types following [7].

Definition 2.10.[7].Assume $\Delta > 0$.

(**i**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 1 if $\exists x(x \in \alpha)\forall y[x + y \in \alpha \Rightarrow y \in \Delta]$,

(**ii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 if $\forall x(x \in \alpha)\exists y(y \notin \Delta)[x + y \in \alpha]$, i.e. $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type 2 iff α does not have type 1.

(**iii**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ if $\exists x(x \notin \alpha)\forall y[x - y \notin \alpha \Rightarrow y \in \Delta]$,

(**iv**) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $2\mathbf{A}$ if $\forall x(x \notin \alpha)\exists y(y \notin \alpha)[x - y \notin \alpha]$.

2.5 Robinson Part $\Re\mathbf{p}\{\alpha\}$ of absorption number $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$

Theorem 2.8.[6].Suppose $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$. Then there is a unique standard $x \in \mathbb{R}$, called Wattenberg standard part of α and denoted by $\mathbf{Wst}(\alpha)$, such that:

(i) $({}^{*}x)^{\#} \in [\alpha - \varepsilon_{\mathbf{d}}, \alpha + \varepsilon_{\mathbf{d}}]$.
(ii) $\alpha \leq_{{}^{*}\mathbb{R}_{\mathbf{d}}} \beta$ implies $\mathbf{Wst}(\alpha) \leq \mathbf{Wst}(\beta)$.
(iii) The map $\mathbf{Wst}(\cdot) : {}^{*}\mathbb{R}_{\mathbf{d}} \to \mathbb{R}$ is continuous.
(iv) $\mathbf{Wst}(\alpha + \beta) = \mathbf{Wst}(\alpha) + \mathbf{Wst}(\beta)$.
(v) $\mathbf{Wst}(\alpha \times \beta) = \mathbf{Wst}(\alpha) \times \mathbf{Wst}(\beta)$.
(vi) $\mathbf{Wst}(-\alpha) = -\mathbf{Wst}(\alpha)$.
(vii) $\mathbf{Wst}(\alpha^{-1}) = [\mathbf{Wst}(\alpha)]^{-1}$ if $\alpha \notin [-\varepsilon_{\mathbf{d}}, \varepsilon_{\mathbf{d}}]$.

Theorem 2.9.[7].
(i) $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ has type 1 iff $-\alpha$ has type $1\mathbf{A}$,
(ii) $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.
(iii) Suppose $\mathbf{ab.p.}(\alpha) = \Delta > 0.$ Then α has type 1 iff α has the form $a^{\#} + \Delta$ for some $a \in {}^{*}\mathbb{R}$.
(iv) Suppose $\mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0. \alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ iff α has the form $a^{\#} + (-\Delta)$ for some $a \in {}^{*}\mathbb{R}$.
(v) If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 1 iff α has type 1.
(vi) If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 2 iff either α or β has type 2.

Proof (iii) Let $\alpha = a + \Delta.$ Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta > 0, a \in a + \Delta$ (we chose $d \in \Delta$ such that $0 < d$ and write a as $(a - d) + d$).
It is clear that a works to show that α has type 1.
Conversely, suppose α has type 1 and choose $a \in \alpha$ such that:
$\forall y[a + y \in \alpha \Rightarrow y \in \Delta]$. Then we claim that: $\alpha = a + \Delta$.
By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a + \Delta \leq \alpha$. On the other hand by choice of a, every element of α has the form $a + d$ with $d \in \Delta$.
Choose $d' \in \Delta$ such that $d' > d,$ then $a + d = \left[a - \left(d' - d\right)\right] + d' \in a + \Delta$.
Hence $\alpha \leq a + \Delta$. Therefore $\alpha = a + \Delta$.

Examples. **(i)** $\varepsilon_{\mathbf{d}}$ has type 1 and therefore $-\varepsilon_{\mathbf{d}}$ has type $1\mathbf{A}$. Note that also $-\varepsilon_{\mathbf{d}}$ has type 2. **(ii)** Suppose $\varepsilon \approx 0, \varepsilon \in {}^{*}\mathbb{R}.$ Then $\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$ has type 1 and therefore $-\varepsilon^{\#} \times \varepsilon_{\mathbf{d}}$ has type $1\mathbf{A}$.
(ii) Suppose $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \varepsilon_{\mathbf{d}} > 0,$ i.e. α has type 1 and therefore by Theorem 2.9 α has the form $({}^{*}a)^{\#} + \varepsilon_{\mathbf{d}}$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$. Then, we define unique Robinson part $\mathfrak{R}\mathbf{p}[\alpha]$ of absorption number α by formula

$$\begin{cases} \mathfrak{R}\mathbf{p}\{\alpha\} \triangleq ({}^{*}a)^{\#}, \\ \mathfrak{R}\mathbf{p}\{\alpha\} = ({}^{*}\mathbf{Wst}(\alpha))^{\#}. \end{cases} \tag{2.18}$$

(ii) Suppose $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = -\varepsilon_{\mathbf{d}},$ i.e. α has type $1\mathbf{A}$ and therefore by Theorem 2.9

α

has the form $({}^{*}a)^{\#} - \varepsilon_{\mathbf{d}}$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$. Then we define unique Robinson part $\mathfrak{R}\mathbf{p}[\alpha]$ of absorption number α by formula

$$\begin{cases} \mathfrak{R}\mathbf{p}\{\alpha\} \triangleq ({}^{*}a)^{\#}, \\ \mathfrak{R}\mathbf{p}\{\alpha\} = ({}^{*}\mathbf{Wst}(\alpha))^{\#}. \end{cases} \tag{2.19}$$

(iii) Suppose $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \Delta, \Delta > 0$ and α has type $1\mathbf{A}$, i.e. α has the form $a^{\#} + \Delta$ for some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption number α by formula

$$\Re\mathbf{p}\{\alpha\} \triangleq a^{\#}. \tag{2.20}$$

(**iv**) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0$ and α has type $1\mathbf{A}$, i.e. α has the form $a^{\#} + (-\Delta)$ for some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption number α by formula

$$\Re\mathbf{p}\{\alpha\} \triangleq a^{\#}. \tag{2.21}$$

Remark 2.11. Note that in general case,i.e. if $\alpha \notin (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption number α is not unique.

Remark 2.12. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ and $\alpha \in (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ has type 1or type $1\mathbf{A}$. Then by definitions
above one obtain the representation

$$\alpha = \Re\mathbf{p}\{\alpha\} + \mathbf{ab.p.}(\alpha).$$

2.6 The pseudo-ring of Wattenberg hyperintegers ${}^*\mathbb{Z}_{\mathbf{d}}$

Lemma 2.12. [6].Suppose that $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then the following two conditions on α are equivalent:

(i) $\alpha = \sup\left\{\nu^{\#} | \left(\nu \in {}^*\mathbb{Z}\right) \wedge (\nu^{\#} \leq \alpha)\right\}$,

(ii) $\alpha = \inf\left\{\nu^{\#} | \left(\nu \in {}^*\mathbb{Z}\right) \wedge (\alpha \leq \nu^{\#})\right\}$.

Definition 2.11.[6].If α satisfies the conditions mentioned above α is said to be the Wattenberg hyperinteger. The set of all Wattenberg hyperintegers is denoted by ${}^*\mathbb{Z}_{\mathbf{d}}$.

Lemma 2.13. [6]. Suppose $\alpha, \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$. Then

(i) $\alpha + \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$.

(ii) $-\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$.

(iii) $\alpha \times \beta \in {}^*\mathbb{Z}_{\mathbf{d}}$.

The set of all positive Wattenberg hyperintegers is called the Wattenberg hypernaturals and
is denoted by ${}^*\mathbb{N}_{\mathbf{d}}$.

Definition 2.12.Suppose that (i) $\lambda \in {}^*\mathbb{N}, \nu \in {}^*\mathbb{Z}_{\mathbf{d}}$, (ii) $\widehat{\lambda} = \lambda^{\#}, \widehat{\nu} = \nu^{\#}$ and (iii) $\lambda | \nu$.

If $\widehat{\lambda} \in {}^*\mathbb{N}_{\mathbf{d}}$ and $\widehat{\nu} \in {}^*\mathbb{Z}_{\mathbf{d}}$ satisfies these conditions then we say that $\widehat{\nu}$ is divisible by $\widehat{\lambda}$ and we
denote this by $\lambda^{\#} | \nu^{\#}$.

Definition 2.13.Suppose that (i) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ and (ii) there exists $\lambda^{\#} \in {}^*\mathbb{N}_{\mathbf{d}}$ such that

(1) $\alpha = \sup\left\{\nu^{\#} | \left(\nu \in {}^*\mathbb{Z}\right) \wedge (\lambda | \nu) \wedge (\nu^{\#} \leq \alpha)\right\}$ or

(2) $\alpha = \inf\left\{\nu^{\#} | \left(\nu \in {}^*\mathbb{Z}\right) \wedge (\lambda | \nu) \wedge (\alpha \leq \nu^{\#})\right\}$.

If α satisfies the conditions mentioned above is said α is divisible by $\lambda^{\#}$ and we denote this by$\lambda^{\#} | \alpha$.

Theorem 2.10. (**i**) Let $\mathbf{p} \in {}^*\mathbb{N}$, $M(\mathbf{p}) \in {}^*\mathbb{N}$, be a prime hypernaturals such that (i) $\mathbf{p} \nmid M(\mathbf{p})$.

Let $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ be a Wattenberg hypernatural such that (i) $p | \alpha$. Then

$$\left|(M(\mathbf{p}))^{\#} + \alpha\right| > 1.$$

(**ii**) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ has type 1 iff $-\alpha$ has type $1\mathbf{A}$,

(**iii**) $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.

(**iv**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}, \mathbf{ab.p.}(\alpha) = \Delta > 0.$ Then α has type 1 iff α has the form $a^{\#} + \Delta$ for some $a \in \alpha, a \in {}^*\mathbb{Z}$.
(**v**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}},\ \mathbf{ab.p.}(\alpha) = -\Delta, \Delta > 0. \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ iff α has the form $a^{\#} + (-\Delta)$ for some $a \in \alpha, a \in {}^*\mathbb{Z}$.
(**vi**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$. If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 1 iff α has type 1.
(**vii**) Suppose $\alpha \in {}^*\mathbb{Z}_{\mathbf{d}}$. If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha + \beta$ has type 2 iff either α or β has type 2.
Proof. (**i**) Immediately follows from definitions (2.12)-(2.13).
(**iv**) Let $\alpha = a + \Delta$. Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta > 0, a \in a + \Delta$ (we chose $d \in \Delta$ such that $0 < d$ and write a as $(a - d) + d$).
It is clear that a works to show that α has type 1.
Conversely, suppose α has type 1 and choose $a \in \alpha$ such that:
$\forall y[a + y \in \alpha \Rightarrow y \in \Delta]$. Then we claim that: $\alpha = a + \Delta$.
By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a + \Delta \leq \alpha$. On the other hand by choice of a, every element of α has the form $a + d$ with $d \in \Delta$.
Choose $d' \in \Delta$ such that $d' > d$, then $a + d = \left[a - \left(d' - d\right)\right] + d' \in a + \Delta$.
Hence $\alpha \leq a + \Delta$. Therefore $\alpha = a + \Delta$.

2.7 The integer part $\mathbf{Int.p}(\alpha)$ of Wattenberg hyperreals $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$

Definition 2.14. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha \geq 0$. Then, we define $\mathbf{Int.p}(\alpha) = [\alpha] \in {}^*\mathbb{N}_{\mathbf{d}}$ by formula

$$[\alpha] \triangleq \sup\left\{v^{\#} | \left(v \in {}^*\mathbb{N}\right) \wedge (v^{\#} \leq \alpha)\right\}.$$

Obviously there are two possibilities:
1. A set $\left\{v^{\#} | \left(v \in {}^*\mathbb{N}\right) \wedge (v^{\#} \leq \alpha)\right\}$ has no greatest element. In this case valid only the
Property I: $[\alpha] = \alpha$
since $[\alpha] < \alpha$ implies $\exists a \in {}^*\mathbb{R}$ such that $[\alpha] < a^{\#} < \alpha$. But then $[a^{\#}] < \alpha$ which implies $[a^{\#}] + 1 < \alpha$ contradicting $[\alpha] < a^{\#} < [a^{\#}] + 1$.
2. A set $\left\{v^{\#} | \left(v \in {}^*\mathbb{N}\right) \wedge (v^{\#} \leq \alpha)\right\}$ has a greatest element, $\upsilon \in {}^*\mathbb{N}$. In this case valid the
Property II: $[\alpha] = \upsilon$
and obviously $\upsilon = [\alpha] \leq \alpha < [\alpha] + 1 = \upsilon + 1$.
Definition 2.15. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}$. Then, we define $\mathbf{Int.p}(\alpha) \in {}^*\mathbb{Z}_{\mathbf{d}}$ by formula

$$\mathbf{Int.p}(\alpha) = \begin{cases} [\alpha] \text{ for } \alpha \geq 0 \\ -[\alpha] \text{ for } \alpha < 0. \end{cases}$$

Note that obviously: $\mathbf{Int.p}(-\alpha) = -\mathbf{Int.p}(\alpha)$.

2.8 External sum of the countable infinite series in ${}^*\mathbb{R}_{\mathbf{d}}$, ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon), {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{p})$.

This subsection contains key definitions and properties of sum of countable sequence of Wattenberg tipe hyperreals.

Definition 2.16.(I)[4]. Let $\{\mathbf{s}_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}.$ such that
(i) $\forall n(s_n \geq 0)$ or (ii) $\forall n(s_n < 0)$ or
(iii) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0],$
$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2.$
Then external sum over ${}^*\mathbb{R}_{\mathbf{d}}$ of the sequence $\{\mathbf{s}_n\}_{n=1}^{\infty}$ (#-sum)

$$\#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n\in\mathbb{N}} s_n^{\#}$$

or #-sum of the corresponding countable sequence ${}^*s_n : \mathbb{N} \to {}^*\mathbb{R}$ is defined by

$$\begin{cases} (\mathbf{i}) \quad \forall n(s_n \geq 0) : \\ \#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k} ({}^*s_n)^{\#}\right\}, \\ (\mathbf{ii}) \quad \forall n(s_n < 0) : \\ \#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} s_n^{\#}\right\} = -\sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k} (|{}^*s_n|)^{\#}\right\}, \\ (\mathbf{iii}) \quad \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \\ \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0], \mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 : \\ \#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#} + \#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#}. \end{cases} \quad (2.22)$$

Abbreviation 2.8.A. We often abbreviate $\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#}$ instead $\#Ext\text{-}{}^*\mathbb{R}_{\mathbf{d}}\text{-}\sum_{n\in\mathbb{N}} s_n^{\#}$

Definition 2.16.(II). Let $\{\mathbf{s}_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}.$ such that
(i) $\forall n(s_n \geq 0)$ or (ii) $\forall n(s_n < 0)$ or
(iii) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0],$
$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2.$
Then external sum over ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ of the sequence $\{\mathbf{s}_n\}_{n=1}^{\infty}$ ($\#_\epsilon$-sum)

$$\#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon}$$

or $\#_\epsilon$-sum of the corresponding countable sequence ${}^*s_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{st}}(\epsilon)$ is defined by

$$\begin{cases} (\mathbf{i}) \quad \forall n(s_n \geq 0): \\ \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k}({}^*s_n)^{\#_\epsilon}\right\}, \\ (\mathbf{ii}) \quad \forall n(s_n < 0): \\ \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} s_n^{\#_\epsilon}\right\} = -\sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k}(|{}^*s_n|)^{\#_\epsilon}\right\}, \\ (\mathbf{iii}) \quad \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \\ \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0], \mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2: \\ \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#_\epsilon} + \\ \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#_\epsilon}. \end{cases} \quad (2.22.a)$$

Abbreviation 2.8.B. We often abbreviate $\#_\epsilon Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon}$ instead $\#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon}$.

Definition 2.16.(III). Let $\{\mathbf{s}_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}.$ such that
(i) $\forall n(s_n \geq 0)$ or (ii) $\forall n(s_n < 0)$ or
(iii) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0],$
$\forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0], \mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2.$
Then external sum over ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})$ of the sequence $\{\mathbf{s}_n\}_{n=1}^{\infty}$ ($\#_{\epsilon,\mathbf{n}}$-sum)

$$\#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon, \mathbf{n})\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_{\epsilon,\mathbf{n}}}$$

or $\#_{\epsilon,\mathbf{n}}$-sum of the corresponding countable sequence ${}^*s_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{st}}(\epsilon, \mathbf{n})$ is defined by

$$\begin{cases} (\mathbf{i}) \quad \forall n(s_n \geq 0): \\ \#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_{\epsilon,\mathbf{n}}} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k}({}^*s_n)^{\#_{\epsilon,\mathbf{n}}}\right\}, \\ (\mathbf{ii}) \quad \forall n(s_n < 0): \\ \#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k} s_n^{\#_{\epsilon,\mathbf{n}}}\right\} = -\sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k}(|{}^*s_n|)^{\#_{\epsilon,\mathbf{n}}}\right\}, \\ (\mathbf{iii}) \quad \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \\ \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0], \mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2: \\ \#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#_{\epsilon,\mathbf{n}}} + \\ \#_{\epsilon,\mathbf{n}} Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#_{\epsilon,\mathbf{n}}}. \end{cases} \quad (2.22.b)$$

Abbreviation 2.8.C. We often abbreviate $\#_{\epsilon,\mathbf{n}} Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_{\epsilon,\mathbf{n}}}$ instead

$\#_{\epsilon,\mathbf{n}}Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_{\epsilon,\mathbf{n}}}$.

Theorem 2.11.(**I**).(**i**) Let $\{s_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_{n+1} > s_n]$ and $\lim_{n\to\infty} s_n = \eta$. Then

$$\sup_{n\in\mathbb{N}} \left\{({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}}.$$

(**ii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_{n+1} < s_{n+1}]$ and $\lim_{n\to\infty} s_n = \eta$. Then

$$\inf_{n\in\mathbb{N}} \left\{({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}}.$$

(**iii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n \geq 0]$, $\sum_{n=1}^{\infty} s_n = \eta < \infty$ and infinite series $\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{\sum_{n\leq k}({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \qquad (2.23)$$

(**iv**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n < 0]$, $\sum_{n=1}^{\infty} s_n = \eta > -\infty$ and infinite series$\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{\sum_{n\leq k}({}^*s_n)^{\#}\right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \qquad (2.24)$$

(**v**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that (1) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0], \forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0]$, $\mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$ and (2) $\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1} = \eta_1 < \infty, \sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2} = \eta_2 > -\infty$. Then

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} s_n^{\#} \triangleq \#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#} + \#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#} = ({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \quad (2.25)$$

Proof. (**i**) Let $\forall n(n \in \mathbb{N})[s_{n+1} > s_n]$ and $\lim_{n\to\infty} s_n = \eta$. Then obviously: $\forall n(n \in \mathbb{N})[s_n < \eta]$.

Thus $\forall\varepsilon \in \mathbb{R}$ there exists $M \in \mathbb{N}$ such that (1)

$$(1) \quad \forall k \in \mathbb{N} : \eta - \varepsilon < s_{M+k} < \eta.$$

Therefore from (1) by Robinson transfer one obtains (2)

$$(2) \quad \forall\varepsilon \in \mathbb{R}, \forall k \in \mathbb{N} : ({}^*\eta) - ({}^*\varepsilon) < ({}^*s_{M+k}) < ({}^*\eta).$$

Using now Wattenberg embedding from (2) we obtain (3)

$$(3) \quad \forall\varepsilon \in \mathbb{R}, \forall k \in \mathbb{N} : ({}^*\eta^{\#}) - ({}^*\varepsilon^{\#}) < ({}^*s_{M+k}^{\#}) < ({}^*\eta^{\#}).$$

From (3) one obtains (4)

$$(4) \quad \forall\varepsilon \in \mathbb{R} : ({}^*\eta^{\#}) - ({}^*\varepsilon^{\#}) < \sup_{k\in\mathbb{N}} ({}^*s_{M+k}^{\#}) < ({}^*\eta^{\#}).$$

Note that $\forall\delta[(\delta \in \mathbb{R}) \wedge (\delta \approx 0)]$ obviously

$$(5) \quad \sup_{n\in\mathbb{N}} ({}^*s_n^{\#}) < ({}^*\eta^{\#}) - \delta^{\#}.$$

From (4) and (5) one obtains (6)

$$(6)\ \ \forall\varepsilon(\varepsilon\in\mathbb{R})\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx0)]\left\{({}^*\eta^\#)-({}^*\varepsilon^\#)<\sup_{n\in\mathbb{N}}\ ({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#\right\}.$$

Thus (**i**) immediately from (6) and from definition of the idempotent $-\varepsilon_{\mathbf{d}}$.

Proof.(**ii**) Immediately from (**i**) by Lemma 2.3 (**v**).

Proof.(**iii**) Let $\eta_m=\sum_{n=1}^m s_n$. Then obviously: $\eta_m<\eta$ and $\lim_{m\to\infty}\eta_m=\eta$. Thus $\forall\varepsilon\in\mathbb{R}$ there exists $M\in\mathbb{N}$ such that (1)

$$(1)\ \ \ \forall k\in\mathbb{N}:\eta-\varepsilon<\eta_{M+k}<\eta.$$

Therefore from (1) by Robinson transfer one obtains (2)

$$(2)\ \ \forall\varepsilon\in\mathbb{R},\forall k\in\mathbb{N}:({}^*\eta)-({}^*\varepsilon)<({}^*\eta_{M+k})<({}^*\eta).$$

Using now Wattenberg embedding from (2) we obtain (3)

$$(3)\ \ \forall\varepsilon\in\mathbb{R},\forall k\in\mathbb{N}:({}^*\eta^\#)-({}^*\varepsilon^\#)<({}^*\eta^\#_{M+k})<({}^*\eta^\#).$$

From (3) one obtains (4)

$$(4)\ \ \forall\varepsilon\in\mathbb{R}:({}^*\eta^\#)-({}^*\varepsilon^\#)<\sup_{k\in\mathbb{N}}\ ({}^*\eta^\#_{M+k})<({}^*\eta^\#).$$

From (4) by Definition 2.16 (i) one obtains

$$(5)\ \ \forall\varepsilon\in\mathbb{R}:({}^*\eta^\#)-({}^*\varepsilon^\#)<\ \#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#).$$

Note that $\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx0)]$ obviously

$$(6)\ \ \ \ \#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#.$$

From (5)-(6) follows (7)

$$(7)\ \ \forall\varepsilon(\varepsilon\in\mathbb{R})\forall\delta[(\delta\in\mathbb{R})\wedge(\delta\approx0)]\left\{({}^*\eta^\#)-({}^*\varepsilon^\#)<\ \#Ext\text{-}\sum_{n\in\mathbb{N}}({}^*s_n^\#)<({}^*\eta^\#)-\delta^\#\right\}.$$

Thus Eq.(2.23) immediately from (7) and from definition of the idempotent $-\varepsilon_{\mathbf{d}}$.

Proof.(**iv**) Immediately from (**iii**) by Lemma 2.3 (**v**).

Proof.(**v**) From Definition 2.16.(iii) and Eq.(2.23)-Eq.(2.24) by Theorem 2.7.(iii) one obtain

$$\left\{\begin{array}{l}\#Ext\text{-}\sum_{n\in\mathbb{N}}s_n^\#\triangleq\#Ext\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1}s^\#_{n_1}+\#Ext\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2}s^\#_{n_2}=({}^*\eta_1)^\#-\varepsilon_{\mathbf{d}}+\left(({}^*\eta_2)^\#+\varepsilon_{\mathbf{d}}\right)=\\ \quad=({}^*\eta_1)^\#+({}^*\eta_2)^\#-\varepsilon_{\mathbf{d}}+\varepsilon_{\mathbf{d}}=({}^*\eta_1)^\#+({}^*\eta_2)^\#-\varepsilon_{\mathbf{d}}\in{}^*\mathbb{R}_{\mathbf{d}}.\end{array}\right.$$

Theorem 2.11.(**II**).(**i**) Let $\{s_n\}_{n=1}^\infty$ be a countable sequence $s_n:\mathbb{N}\to\mathbb{R}$ such that $\forall n(n\in\mathbb{N})[s_{n+1}>s_n]$ and $\lim_{n\to\infty}s_n=\eta$. Then

$$\sup_{n\in\mathbb{N}}\left\{({}^*s_n)^{\#_\epsilon}\right\}=({}^*\eta)^{\#_\epsilon}-\epsilon\times\varepsilon_{\mathbf{d}}.$$

(**ii**) Let $\{s_n\}_{n=1}^\infty$ be an countable sequence $s_n:\mathbb{N}\to\mathbb{R}$ such that $\forall n(n\in\mathbb{N})[s_{n+1}<s_{n+1}]$ and $\lim_{n\to\infty}s_n=\eta$. Then

$$\inf_{n\in\mathbb{N}}\left\{({}^*s_n)^{\#_\epsilon}\right\}=({}^*\eta)^{\#_\epsilon}+\epsilon\times\varepsilon_{\mathbf{d}}.$$

(**iii**) Let $\{s_n\}_{n=1}^\infty$ be an countable sequence $s_n:\mathbb{N}\to\mathbb{R}$ such that $\forall n(n\in\mathbb{N})[s_n\geq0]$,

$\sum_{n=1}^{\infty} s_n = \eta < \infty$ and infinite series $\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#_{\epsilon}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \sup_{k\in\mathbb{N}}\left\{\sum_{n\leq k}({}^{*}s_n)^{\#_\epsilon}\right\} = ({}^{*}\eta)^{\#_\epsilon} - \epsilon \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon), \qquad (2.23.a)$$

(**iv**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n < 0]$, $\sum_{n=1}^{\infty} s_n = \eta > -\infty$ and infinite series$\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#_{\epsilon}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \inf_{k\in\mathbb{N}}\left\{\sum_{n\leq k}({}^{*}s_n)^{\#_\epsilon}\right\} = ({}^{*}\eta)^{\#_\epsilon} + \epsilon \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon), \quad (2.24.a)$$

(**v**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that (1) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1\in\mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2\in\mathbb{N}_2}^{\infty}, \forall n_1\left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0], \forall n_2\left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0]$, $\mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$ and (2) $\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1} = \eta_1 < \infty, \sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2} = \eta_2 > -\infty$. Then

$$\#_{\epsilon}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}} s_n^{\#_\epsilon} \triangleq \#_{\epsilon}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_1\in\widehat{\mathbb{N}}_1} s_{n_1}^{\#_\epsilon} + \#_{\epsilon}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_2\in\widehat{\mathbb{N}}_2} s_{n_2}^{\#_\epsilon} = ({}^{*}\eta_1)^{\#_\epsilon} + ({}^{*}\eta_2)^{\#_\epsilon} - \epsilon \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon). \qquad (2.25.a)$$

Theorem 2.11.(**III**).(**i**) Let $\mathbf{n} \in {}^{*}\mathbb{N}\backslash\mathbb{N}$. Let $\{s_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such
that $\forall n(n \in \mathbb{N})[s_{n+1} > s_n]$ and $\lim_{n\to\infty} s_n = \eta$. Then

$$\sup_{n\in\mathbb{N}}\left\{(\mathbf{n} \times {}^{*}s_n)^{\#_{\epsilon,\mathbf{n}}}\right\} = (\mathbf{n} \times {}^{*}\eta)^{\#_{\epsilon,\mathbf{n}}} - \mathbf{n} \times \epsilon \times \varepsilon_{\mathbf{d}}.$$

(**ii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_{n+1} < s_{n+1}]$ and
$\lim_{n\to\infty} s_n = \eta$. Then

$$\inf_{n\in\mathbb{N}}\left\{(\mathbf{n} \times {}^{*}s_n)^{\#_{\epsilon,\mathbf{n}}}\right\} = (\mathbf{n} \times {}^{*}\eta)^{\#_{\epsilon,\mathbf{n}}} + \mathbf{n} \times \epsilon \times \varepsilon_{\mathbf{d}}.$$

(**iii**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n \geq 0]$,

$\sum_{n=1}^{\infty} s_n = \eta < \infty$ and infinite series $\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#_{\epsilon,\mathbf{n}}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n})\text{-}\sum_{n\in\mathbb{N}}(\mathbf{n} \times {}^{*}s_n)^{\#_{\epsilon,\mathbf{n}}} \triangleq \sup_{k\in\mathbb{N}}\left\{\sum_{n\leq k}(\mathbf{n} \times {}^{*}s_n)^{\#_{\epsilon,\mathbf{n}}}\right\} = (\mathbf{n} \times {}^{*}\eta)^{\#_{\epsilon,\mathbf{n}}} - \mathbf{n} \times \epsilon \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n}), \qquad (2.23.b)$$

(**iv**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(n \in \mathbb{N})[s_n < 0]$, $\sum_{n=1}^{\infty} s_n = \eta > -\infty$ and infinite series$\sum_{n=1}^{\infty} s_n$ absolutely converges to η in $\mathbb{R}$. Then

$$\#_{\epsilon,\mathbf{n}}Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n\in\mathbb{N}}(\mathbf{n} \times s_n)^{\#_{\epsilon,\mathbf{n}}} \triangleq \inf_{k\in\mathbb{N}}\left\{\sum_{n\leq k}(\mathbf{n} \times {}^{*}s_n)^{\#_{\epsilon,\mathbf{n}}}\right\} = (\mathbf{n} \times {}^{*}\eta)^{\#_{\epsilon,\mathbf{n}}} + \mathbf{n} \times \epsilon \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{n}), \qquad (2.24.b)$$

(**v**) Let $\{s_n\}_{n=1}^{\infty}$ be an countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that (1)

$\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1 \left(n_1 \in \widehat{\mathbb{N}}_1\right)[s_{n_1} \geq 0], \forall n_2 \left(n_2 \in \widehat{\mathbb{N}}_2\right)[s_{n_2} < 0],$
$\mathbb{N} = \widehat{\mathbb{N}}_1 \cup \widehat{\mathbb{N}}_2$ and (2) $\sum_{n_1 \in \widehat{\mathbb{N}}_1} s_{n_1} = \eta_1 < \infty, \sum_{n_2 \in \widehat{\mathbb{N}}_2} s_{n_2} = \eta_2 > -\infty.$ Then

$$\#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n \in \mathbb{N}} s_n^{\#_\epsilon} \triangleq \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_1 \in \widehat{\mathbb{N}}_1} s_{n_1}^{\#_\epsilon} + \#_\epsilon Ext\text{-}{}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)\text{-}\sum_{n_2 \in \widehat{\mathbb{N}}_2} s_{n_2}^{\#_\epsilon} = ({}^*\eta_1)^{\#_\epsilon} + ({}^*\eta_2)^{\#_\epsilon} - \epsilon \times \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon). \quad (2.25.b)$$

Theorem 2.12.Let $\{a_n\}_{n=1}^{\infty}$be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$ such that $\forall n(a_n \geq 0)$ and infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$. Let $s = \#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#}$ be external sum of the corresponding countable sequence $\{{}^*a_n\}_{n=1}^{\infty}$. Let $\{b_n\}_{n=1}^{\infty}$ be a countable sequence where $b_n = a_{m(n)}$ is any rearrangement of terms of the sequence $\{a_n\}_{n=1}^{\infty}$. Then external sum $\sigma = \#Ext\text{-}\sum_{n \in \mathbb{N}} b_n^{\#}$ of the corresponding countable sequence $\{{}^*b_n\}_{m=1}^{\infty}$ has the same value s as external sum of the countable sequence $\{{}^*a_n\}$, i.e. $\sigma = s - \varepsilon_{\mathbf{d}}$.

Theorem 2.13.(**i**) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}_{\mathbf{d}}$, such that (1) $\forall n(a_n \geq 0)$, (2) infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges to $\eta \neq +\infty$ in $\mathbb{R}$ and let $\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#}$ be external sum of the corresponding sequence $\{{}^*a_n\}_{n=1}^{\infty}$. Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#}\right) = \#Ext\text{-}\sum_{n \in \mathbb{N}} c^{\#} \times a_n^{\#} = \\ = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.26)$$

(**ii**) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that (1) $\forall n(a_n < 0)$, (2) infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges to $\eta \neq -\infty$ in $\mathbb{R}$ and let $\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#}$ be external sum of the corresponding sequence $\{{}^*a_n\}_{n=1}^{\infty}$. Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied:

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#}\right) = \#Ext\text{-}\sum_{n \in \mathbb{N}} c^{\#} \times a_n^{\#} = \\ = c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.27)$$

(**iii**) Let $\{s_n\}_{n=1}^{\infty}$ be a countable sequence $s_n : \mathbb{N} \to \mathbb{R}$ such that

(1) $\{s_n\}_{n=1}^{\infty} = \{s_{n_1}\}_{n_1 \in \mathbb{N}_1}^{\infty} \cup \{s_{n_2}\}_{n_2 \in \mathbb{N}_2}^{\infty}, \forall n_1(n_1 \in \mathbb{N}_1)[s_{n_1} \geq 0], \forall n_2(n_2 \in \mathbb{N}_2)[s_{n_2} < 0]$, $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$,
(2) infinite series $\sum_{n=1}^{\infty} s_{n_1}$ absolutely converges to $\eta_1 \neq +\infty$ in $\mathbb{R}$,
(3) infinite series $\sum_{n=1}^{\infty} s_{n_2}$ absolutely converges to $\eta_2 \neq -\infty$ in $\mathbb{R}$.
Then the equality is satisfied:

$$\begin{cases} c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^{\#} \right) = \\ = \#Ext\text{-}\sum_{n_1 \in \widehat{\mathbb{N}}_1} c^{\#} \times s_{n_1}^{\#} + \#Ext\text{-}\sum_{n_2 \in \widehat{\mathbb{N}}_2} c^{\#} \times s_{n_2}^{\#} = \\ = c^{\#} \times \left(({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} \right) - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \tag{2.28}$$

Proof.(i) From Definition 2.16.(i) by Theorem 2.1,Theorem 2.11.(i) and Lemma (2.4) (ii) one obtain

$$\begin{cases} \#Ext\text{-}\sum_{n \in \mathbb{N}} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#} \right) = \\ = c^{\#} \times \left(({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \right) = c^{\#} \times ({}^*\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases}$$

(ii) Straightforward from Definition 2.16.(i) and Theorem 2.1,Theorem 2.11.(iii) and Lemma (2.4) (ii) one obtain

$$\left(\#Ext\text{-}\sum_{n \in \mathbb{N}} c^{\#} \times a_n^{\#} \right) = c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} a_n^{\#} \right) =$$
$$= c^{\#} \times \left(({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \right) = c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}}.$$

(iii) By Theorem 2.11.(iv) and Lemma (2.4).(ii) one obtain

$$c^{\#} \times \left(\#Ext\text{-}\sum_{n \in \mathbb{N}} s_n^{\#} \right) = c^{\#} \times \left(({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} - \varepsilon_{\mathbf{d}} \right) =$$
$$= c^{\#} \times \left(({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} \right) - c^{\#} \times \varepsilon_{\mathbf{d}}.$$

But other side from (i) and (ii) follows

$$\#Ext\text{-}\sum_{n_1 \in \widehat{\mathbb{N}}_1} c^{\#} \times s_{n_1}^{\#} + \#Ext\text{-}\sum_{n_2 \in \widehat{\mathbb{N}}_2} c^{\#} \times s_{n_2}^{\#} =$$
$$= c^{\#} \times ({}^*\eta_1)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}}) + c^{\#} \times ({}^*\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}} =$$
$$c^{\#} \times \left(({}^*\eta_1)^{\#} + ({}^*\eta_2)^{\#} \right) - c^{\#} \times \varepsilon_{\mathbf{d}}.$$

Definition 2.17. Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta \neq \pm\infty$. We assume now that:

(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > \eta$, or
(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < \eta$, or
(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that
(a) $\forall i, m : \sum_{i=1}^{m} a_{n_i} > \eta$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to η
and
(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j, m : \sum_{j=1}^{m} a_{n_j} < \eta$ and infinite series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to η.
Then: (i) external upper sum (#-upper sum) of the corresponding countable sequence ${}^*a_n : \mathbb{N} \to \mathbb{R}$ is defined by

$$\begin{cases} \quad\quad\quad\quad \mathbf{(i)} \\ \#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^*a_n)^{\#} \right\}, \\ \quad\quad\quad\quad \mathbf{(ii)} \\ \#Ext\text{-}\overset{\vee}{\sum_{i\in\mathbb{N}}} a_{n_i}^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{i\leq k} ({}^*a_{n_i})^{\#} \right\}, \end{cases} \tag{2.29}$$

(ii) external lower sum (#-lower sum) of the corresponding countable sequence ${}^*a_n : \mathbb{N} \to \mathbb{R}$
is defined by

$$\begin{cases} \quad\quad\quad\quad \mathbf{(i)} \\ \#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^*a_n)^{\#} \right\}, \\ \quad\quad\quad\quad \mathbf{(ii)} \\ \#Ext\text{-}\overset{\wedge}{\sum_{j\in\mathbb{N}}} a_{n_j}^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{j\leq k} ({}^*a_{n_j})^{\#} \right\}. \end{cases} \tag{2.30}$$

Theorem 2.14.Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta \neq \pm\infty$. We assume now that:
(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > \eta$, or
(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < \eta$, or
(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that
(a) $\forall i, m : \sum_{i=1}^{m} a_{n_i} > \eta$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to η
and
(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j, m : \sum_{j=1}^{m} a_{n_j} < \eta$ and infinite series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to η. Then

$$\begin{cases} \#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \inf_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^*a_n)^{\#} \right\} = ({}^*\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} \triangleq \sup_{k\in\mathbb{N}} \left\{ \sum_{n\leq k} ({}^*a_n)^{\#} \right\} = ({}^*\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^*\mathbb{R}_{\mathbf{d}}. \end{cases} \tag{2.31}$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#} \triangleq \inf\limits_{k\in\mathbb{N}} \left\{\sum_{i\leq k}({}^{*}a_{n_i})^{\#}\right\} = ({}^{*}\eta)^{\#} + \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#} \triangleq \sup\limits_{k\in\mathbb{N}} \left\{\sum_{j\leq k}({}^{*}a_{n_j})^{\#}\right\} = ({}^{*}\eta)^{\#} - \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.32)$$

Proof. straightforward from definitions and by Theorem 2.11 (i)-(ii).

Theorem 2.15. (1) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta \neq \pm\infty$. We assume now that:

(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > \eta$, or

(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < \eta$, or

(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that

(a) $\forall i,m : \sum_{i=1}^{m} a_{n_i} > \eta$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to η and

(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j,m : \sum_{j=1}^{m} a_{n_j} < \eta$ and infinite series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to η.

Then for any $c \in {}^{*}\mathbb{R}_{+}$ the equalities are satisfied

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#}\right) = c^{\#} \times ({}^{*}\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#}\right) = c^{\#} \times ({}^{*}\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.33)$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#} \times a_{n_i}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#}\right) = c^{\#} \times ({}^{*}\eta)^{\#} + c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#} \times a_{n_j}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#}\right) = c^{\#} \times ({}^{*}\eta)^{\#} - c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.34)$$

Proof. Copy the proof of the Theorem 2.13.

Theorem 2.16. (1) Let $\{a_n\}_{n=1}^{\infty}$ be a countable sequence $a_n : \mathbb{N} \to \mathbb{R}$, such that infinite series $\sum_{n=1}^{\infty} a_n$ absolutely converges in $\mathbb{R}$ to $\eta = 0$. We assume now that:

(i) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n > 0$, or

(ii) there exists $m > 1$ such that $\forall k \geq m : \sum_{n=1}^{k} a_n < 0$, or

(iii) there exists infinite sequence $n_i, i = 1,2,\ldots$ such that

(a) $\forall i,m : \sum_{i=1}^{m} a_{n_i} > 0$ and infinite series $\sum_{i=1}^{\infty} a_{n_i}$ absolutely converges in $\mathbb{R}$ to $\eta = 0$ and

(b) there exists infinite sequence $n_j, j = 1,2,\ldots$ such that $\forall j,m : \sum_{j=1}^{m} a_{n_j} < 0$ and infinite series $\sum_{j=1}^{\infty} a_{n_j}$ absolutely converges in $\mathbb{R}$ to $\eta = 0$.

Then for any $c \in {}^{*}\mathbb{R}_{+}$ the equalities are satisfied

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#}\right) = c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#}\right) = -c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.35)$$

and

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#} \times a_{n_i}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#}\right) = c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#} \times a_{n_j}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\wedge} a_{n_j}^{\#}\right) = -c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.36)$$

Proof. (1) From Eq.(2.31) we obtain

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#} = +\varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} a_n^{\#} = -\varepsilon_{\mathbf{d}}. \end{cases} \quad (2.37)$$

From Eq.(2.37) by Theorem 2.1 we obtain directly

$$\begin{cases} \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\vee} a_n^{\#}\right) = c^{\#} \times \varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} \times a_n^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{n\in\mathbb{N}}^{\wedge} c^{\#} a_n^{\#}\right) = -c^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (2.38)$$

(2) From Eq.(2.32) we obtain

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#} = +\varepsilon_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} a_{n_j}^{\#} = -\varepsilon_{\mathbf{d}}. \end{cases} \quad (2.39)$$

From Eq.(2.39) by Theorem 2.1 we obtain directly

$$\begin{cases} \#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} c^{\#} \times a_{n_i}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\vee} a_{n_i}^{\#}\right) = c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}, \\ \#Ext\text{-}\sum\limits_{j\in\mathbb{N}}^{\wedge} c^{\#} \times a_{n_j}^{\#} = c^{\#} \times \left(\#Ext\text{-}\sum\limits_{i\in\mathbb{N}}^{\wedge} a_{n_i}^{\#}\right) = -c^{\#} \times \varepsilon_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}. \end{cases} \quad (2.40)$$

Remark 2.13. Note that we have proved Eq.(2.35) and Eq.(2.36) without any reference to the Lemma 2.4.

Definition 2.18. (**i**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, such that

$$\forall n(n \geq m > 0)[\alpha_n > 0] \text{ and } \forall n(n \leq m-1)\left[(\alpha_n = a_n^{\#}) \wedge \left(\alpha_n \in {}^*\mathbb{R}\right)\right] \qquad (2.41)$$

Then external countable lower sum (#-lower sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \alpha_n = \sum_{n=0}^{m-1} \alpha_n + \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n$$
$$\#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \sup_{k\in\mathbb{N}} \sum_{n=m}^{k} \alpha_n. \qquad (2.42)$$

In particular if $\{\alpha_n\}_{n=1}^{\infty} = \{a_n^{\#}\}_{n=1}^{\infty}$, where $\forall n \in \mathbb{N}\left[a_n \in {}^*\mathbb{R}\right]$ the external countable lower sum (#-lower sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \alpha_n = \sum_{n=0}^{m-1} a_n^{\#} + \#Ext\text{-}\sum_{n=m}^{\infty} a_n^{\#},$$
$$\#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \sup_{k\in\mathbb{N}} \sum_{n=m}^{k} a_n^{\#}. \qquad (2.43)$$

(**ii**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, such that

$$\forall n(n \geq m > 0)[\alpha_n < 0] \text{ and } \forall n(n \leq m-1)\left[(\alpha_n = a_n^{\#}) \wedge \left(a_n \in {}^*\mathbb{R}\right)\right] \qquad (2.44)$$

Then external countable upper sum (#-upper sum) of the countable sequence $a_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee} \alpha_n = \sum_{n=0}^{m-1} \alpha_n + \#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n$$
$$\#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \inf_{k\in\mathbb{N}} \sum_{n=m}^{k} \alpha_n. \qquad (2.45)$$

In particular if $\{\alpha_n\}_{n=1}^{\infty} = \{a_n^{\#}\}_{n=1}^{\infty}$, where $\forall n \in \mathbb{N}\left[a_n \in {}^*\mathbb{R}\right]$ the external countable upper sum (#-upper sum) of the countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee} \alpha_n = \sum_{n=0}^{m-1} a_n^{\#} + \#Ext\text{-}\sum_{n=m}^{\infty} a_n^{\#},$$
$$\#Ext\text{-}\sum_{n=m}^{\infty} \alpha_n \triangleq \inf_{k\in\mathbb{N}} \sum_{n=m}^{k} a_n^{\#}. \qquad (2.46)$$

Theorem 2.17. (**i**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a countable sequence $\alpha_n : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$, such that valid the property (2.41). Then for any $c \in {}^*\mathbb{R}_+$ the equality is satisfied

$$c^{\#}\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}\alpha_n\right)=\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}c^{\#}\times\alpha_n=$$
$$=\sum_{n=0}^{m-1}c^{\#}\times a_n^{\#}+\#Ext\text{-}\sum_{n=m}^{\infty}c^{\#}\times a_n^{\#}. \tag{2.47}$$

(**ii**) Let $\{\alpha_n\}_{n=1}^{\infty}$ be an countable sequence $\alpha_n:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{d}}$, such that valid the property (2.44).

Then for any $c\in{}^*\mathbb{R}_+$ the equality is satisfied

$$c^{\#}\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}\alpha_n\right)=\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}c^{\#}\times\alpha_n=$$
$$=\sum_{n=0}^{m-1}c^{\#}\times a_n^{\#}+\#Ext\text{-}\sum_{n=m}^{\infty}c^{\#}\times a_n^{\#}. \tag{2.48}$$

Proof. Immediately from Definition 2.18 by Theorem 2.1.

Definition 2.19. Let $\{z_n\}_{n=1}^{\infty}=\{a_n+ib_n\}_{n=1}^{\infty}$ be a countable sequence $z_n=a_n+ib_n:\mathbb{N}\to\mathbb{C}$ such that infinite series $\sum_{n=1}^{\infty}z_n$ absolutely converges in $\mathbb{C}$ to $z,|z|\neq\infty$. Then: (i) external complex sum (complex #-sum), (ii) external upper complex sum (upper complex #-sum) and (iii) external lover complex sum (lover complex #-sum) of the corresponding countable sequence ${}^*z_n:\mathbb{N}\to{}^*\mathbb{C}$ is defined by

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}z_n^{\#}=\#Ext\text{-}\sum_{n\in\mathbb{N}}a_n^{\#}+i\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}b_n^{\#}\right),$$
$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}z_n^{\#}=\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}a_n^{\#}+i\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\vee}b_n^{\#}\right) \tag{2.49}$$
$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}z_n^{\#}=\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}a_n^{\#}+i\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge}b_n^{\#}\right).$$

correspondingly.

Note that any properties of this sum immediately follow from the properties of the real external sum.

Definition 2.20. (i) We define now Wattenberg complex plane ${}^*\mathbb{C}_{\mathbf{d}}$ by ${}^*\mathbb{C}_{\mathbf{d}}={}^*\mathbb{R}_{\mathbf{d}}\oplus i\times{}^*\mathbb{R}_{\mathbf{d}}$ with $i^2=1$. Thus for any $z\in{}^*\mathbb{C}_{\mathbf{d}}$ we obtain $z=x+iy$, where $x,y\in{}^*\mathbb{R}_{\mathbf{d}}$, (ii) for any $z\in{}^*\mathbb{C}_{\mathbf{d}}$ such that $z=x+iy$ we define $|z|^2$ by $|z|^2=x^2+y^2\in{}^*\mathbb{R}_{\mathbf{d}}$.

Theorem 2.18. Let $\{z_n\}_{n=1}^{\infty}=\{a_n+ib_n\}_{n=1}^{\infty}$ be a countable sequence $z_n=a_n+ib_n:\mathbb{N}\to\mathbb{C}$ such that infinite series $\sum_{n=1}^{\infty}z_n$ absolutely converges in $\mathbb{C}$ to $z=\zeta_1+i\zeta_2$ and $|z|\neq\infty$. Then

(**i**)

$$\#Ext\text{-}\sum_{n\in\mathbb{N}} z_n^{\#} = \#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#} + i\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n^{\#}\right) =$$

$$\left[({}^{*}\zeta_1)^{\#} - \varepsilon_{\mathbf{d}}\right] + i\left[({}^{*}\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}\right] = ({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)$$

$$\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} z_n^{\#} =$$

$$\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i\times\left(\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right) = ({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)$$

$$\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} z_n^{\#} =$$

$$\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i\times\left(\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right) = ({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)$$

(ii)

$$\left|\#Ext\text{-}\sum_{n\in\mathbb{N}} z_n^{\#}\right|^2 =$$

$$= \left|\#Ext\text{-}\sum_{n\in\mathbb{N}} a_n^{\#} + i\times\left(\#Ext\text{-}\sum_{n\in\mathbb{N}} b_n^{\#}\right)\right|^2 = \left|({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} - \varepsilon_{\mathbf{d}}(1+i)\right|^2,$$

$$\left|\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} z_n^{\#}\right|^2 =$$

$$\left|\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i\times\left(\#Ext\text{-}\overset{\vee}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right)\right|^2 = \left|({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)\right|^2,$$

$$\left|\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} z_n^{\#}\right|^2 =$$

$$\left|\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} a_n^{\#} + i\times\left(\#Ext\text{-}\overset{\wedge}{\sum_{n\in\mathbb{N}}} b_n^{\#}\right)\right|^2 = \left|({}^{*}\zeta_1)^{\#} + i({}^{*}\zeta_2)^{\#} + \varepsilon_{\mathbf{d}}(1+i)\right|^2.$$

2.9 Gonshor transfer

Definition 2.21.[7]. Let $[S]_{\mathbf{d}} = \{x | \exists y(y \in S)[x \leq y]\}$.

Note that $[S]_{\mathbf{d}}$ satisfies the usual axioms for a closure operator,i.e. if (i) $S \neq \varnothing, S' \neq \varnothing$ and

(ii) S has no maximum, then $[S]_{\mathbf{d}} \in {}^{*}\mathbb{R}_{\mathbf{d}}$.

Let f be a continuous strictly increasing function in each variable from a subset of $\mathbb{R}^n$ into $\mathbb{R}$. Specifically, we want the domain to be the cartesian product $\prod_{i=1}^{n} A_i$, where $A_i = \{x | x > a_i\}$ for some $a_i \in \mathbb{R}$.By Robinson transfer f extends to a function ${}^{*}f : {}^{*}\mathbb{R}^n \to {}^{*}\mathbb{R}$ from the corresponding subset of ${}^{*}\mathbb{R}^n$ into ${}^{*}\mathbb{R}$ which is also strictly increasing in each variable and continuous in the Q topology (i.e. ε and δ range over arbitrary positive elements in ${}^*\mathbb{R}$).We now extend ${}^*f$ to $[{}^*f]_{\mathbf{d}}$

$$[{}^*f]_{\mathbf{d}} : {}^*\mathbb{R}^n_{\mathbf{d}} \to {}^*\mathbb{R}_{\mathbf{d}}. \qquad (2.50)$$

Definition 2.22.[7]. Let $\alpha_i \in {}^*\mathbb{R}_{\mathbf{d}},\ \alpha_i > a_i\ ,\ b_i \in {}^*\mathbb{R},$ then

$$[{}^*f]_{\mathbf{d}}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left[\left\{{}^*f(b_1,b_2,\ldots,b_n)|\ a_i < b_i \in \alpha_i\right\}\right]_{\mathbf{d}}. \qquad (2.51)$$

Theorem 2.20.[7]. If f and g are functions of one variable then

$$[{}^*(f\cdot g)]_{\mathbf{d}}(\alpha) = ([{}^*f]_{\mathbf{d}}(\alpha))\cdot([{}^*g]_{\mathbf{d}}(\alpha)). \qquad (2.52)$$

Theorem 2.21.[7].Let f be a function of two variables. Then for any $\alpha \in {}^*\mathbb{R}$ and $a \in {}^*\mathbb{R}$

$$[{}^*f]_{\mathbf{d}}(\alpha,a) = [{}^*f(b,c)|b \in \alpha, c < a]. \qquad (2.53)$$

Theorem 2.22.[7].Let f and g be any two terms obtained by compositions of strictly increasing continuous functions possibly containing parameters in ${}^*\mathbb{R}$. Then any relation ${}^*f = {}^*g$ or ${}^*f < {}^*g$ valid in ${}^*\mathbb{R}$ extends to ${}^*\mathbb{R}_{\mathbf{d}}$, i.e.

$$[{}^*f]_{\mathbf{d}}(\alpha) = [{}^*g]_{\mathbf{d}}(\alpha) \text{ or } [{}^*f]_{\mathbf{d}}(\alpha) < [{}^*g]_{\mathbf{d}}(\alpha). \qquad (2.54)$$

Remark 2.14. For any function ${}^*f : {}^*\mathbb{R}^n \to {}^*\mathbb{R}$ we often write for short $f^\#$ instead of $[{}^*f]_{\mathbf{d}}$.

Theorem 2.23.[7].(1) For any $a,b \in {}^*\mathbb{R}_+$

$$\begin{cases} \exp^\#(a^\# + b^\#) = \exp^\#(a^\#)\exp^\#(b^\#), \\ (\exp^\#(a^\#))^{b^\#} = \exp^\#(b^\# a^\#). \end{cases} \qquad (2.55)$$

For any $\alpha,\beta \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha,\beta > 0$

$$\begin{cases} \exp^\#(\alpha+\beta) = \exp^\#(\alpha)\exp^\#(\beta), \\ (\exp^\#(\alpha))^{\beta} = \exp^\#(\beta\alpha). \end{cases} \qquad (2.56)$$

(2) For any $a,b \in {}^*\mathbb{R}$

$$(a^b)^\# = (a^\#)^{b^\#}. \qquad (2.57)$$

(3) For any $\alpha,\beta,\gamma \in {}^*\mathbb{R}_{\mathbf{d}}, \alpha,\beta,\gamma > 0$

$$(\alpha^\beta)^\gamma = \alpha^{\gamma\beta} \qquad (2.58)$$

(4) For any $a \in {}^*\mathbb{R}$

$$\begin{aligned} \ln^\#(\exp^\#(a^\#)) &= a^\#, \\ \exp^\#(\ln^\#(a^\#)) &= a^\#. \end{aligned} \qquad (2.59)$$

Note that we must always beware of the restriction in the domain when it comes to multiplication

Theorem 2.24.[7].The map $\alpha \mapsto [\exp]_{\mathbf{d}}(\alpha)$ maps the set of additive idempotents onto the set of all multiplicative idempotents other than 0.

2.10.The rings ${}^*\mathbb{R}^{\approx}(\varepsilon)$ and ${}^*\mathbb{Q}^{\approx}(\varepsilon)$.

For the remainder of this paper note that:
(i) we use the canonical embeding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ by formula $a \mapsto {}^*a = (a, a, \ldots)$;
(ii) we use the canonical notation $a \approx b$, for a infinitely close to b and $St(\alpha)$ for the unique standard number infinitely close to a finite nonstandard number α;
(iii) the monad of a, the set $\left\{x \in {}^*\mathbb{R} \,\middle|\, x \approx a\right\}$ is denoted, $\mu(a)$;
(iv) the subset of the all finite numbers in ${}^*\mathbb{R}$ is denoted, ${}^*\mathbb{R}^{\mathbf{fin}}$;
(v) the subset of the all finite numbers in ${}^*\mathbb{Q}$ is denoted, ${}^*\mathbb{Q}^{\mathbf{fin}}$;
(vi) the subset of the all finite numbers in ${}^*\mathbb{R}_{\mathbf{d}}$ is denoted, ${}^*\mathbb{R}_{\mathbf{d}}^{\mathbf{fin}}$;
(vii) the subset of the all finite numbers in ${}^*\mathbb{Q}_{\mathbf{d}}$ is denoted, ${}^*\mathbb{Q}_{\mathbf{d}}^{\mathbf{fin}}$;

Definition 2.23.Standard number $a \in {}^*\mathbb{R}$ that is a number such that $a = {}^*b, b \in {}^*\mathbb{R}$. The set of all standard numbers is denoted ${}^*\mathbb{R}_{\mathbf{st}}$.

Definition 2.24.Let $\varepsilon \approx 0$. The ε-monad of 0, the set $\varepsilon \cdot \mu(0)$ is denoted, $\mu_\varepsilon(0)$.

Definition 2.25.Let $a \in \mathbb{R}, \varepsilon \approx 0$. We will say that ${}^*a$ infinitely ε-close to $b \in {}^*\mathbb{R}$ iff ${}^*a - b \in \mu_\varepsilon(0)$.

Definition 2.26.Let $a \in \mathbb{R}, \varepsilon \approx 0$. The ε-monad of a, the set $\left\{x \in {}^*\mathbb{R} \,\middle|\, x \approx {}^*a\right\}$ is denoted, $\mu_\varepsilon({}^*a)$.

Definition 2.27.Let $a \in {}^*\mathbb{R}, \exists St(a), \varepsilon \approx 0$. We will say that a is ε-near-standard number iff

$$\exists \eta_\varepsilon(\eta_\varepsilon \in \mu_\varepsilon(0)) \; \left[a - ({}^*St(a)) \leq \eta_\varepsilon\right]. \tag{2.60}$$

Definition 2.28.The set of the all ε-near-standard numbers is denoted, ${}^*\mathbb{R}^{\approx}(\varepsilon)$.

Theorem 2.25.The set ${}^*\mathbb{R}^{\approx}(\varepsilon)$ as algebraic structure in a natural way is an ordered ring,
i.e.,a structure of the form

$$\langle {}^*\mathbb{R}^{\approx}(\varepsilon), +_{{}^*\mathbb{R}^{\approx}(\varepsilon)}, \cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)}, <_{{}^*\mathbb{R}^{\approx}(\varepsilon)}, {}^*0, {}^*1 \rangle, \tag{2.61}$$

where ${}^*\mathbb{R}^{\approx}(\varepsilon)$ is the set of elements of the structure,where
(i) $+_{{}^*\mathbb{R}^{\approx}(\varepsilon)} \triangleq +_{{}^*\mathbb{R}} \restriction_{{}^*\mathbb{R}^{\approx}(\varepsilon) \times {}^*\mathbb{R}^{\approx}(\varepsilon)}$
(ii) $\cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)} \triangleq \cdot_{{}^*\mathbb{R}} \restriction_{{}^*\mathbb{R}^{\approx}(\varepsilon) \times {}^*\mathbb{R}^{\approx}(\varepsilon)}$
are the binary operations of additions and multiplication, and
(iii) $<_{{}^*\mathbb{R}^{\approx}(\varepsilon)} \triangleq <_{{}^*\mathbb{R}} \restriction_{{}^*\mathbb{R}^{\approx}(\varepsilon) \times {}^*\mathbb{R}^{\approx}(\varepsilon)}$
is the ordering relation induced from corresponding operations and relation on ${}^*\mathbb{R}$, and
(iv) ${}^*0, {}^*1 \in {}^*\mathbb{R}$ are distinguished canonical elements of the domain.

Proof.Immediately from definitions.

Definition 2.29.Let

$$a = \frac{\mathbf{m}}{\mathbf{n}} \in {}^*\mathbb{Q}, \mathbf{m} \in {}^*\mathbb{Z} \backslash \mathbb{Z}, \mathbf{n} \in {}^*\mathbb{N} \backslash \mathbb{N}, \exists St(a), \varepsilon \approx 0.$$

We will say that a is ε-near-standard hyper rational number iff

$$\exists \eta_\varepsilon(\eta_\varepsilon \in \mu_\varepsilon(0))\left[\left|\frac{\mathbf{m}}{\mathbf{n}} - ({}^*St(a))\right| \leq \eta_\varepsilon\right].$$

Definition 2.30.The set of the all ε-near-standard hyper rational numbers is denoted, ${}^*\mathbb{Q}^{\approx}(\varepsilon)$.

Theorem 2.26.The set ${}^*\mathbb{Q}^{\approx}(\varepsilon)$ as algebraic structure in a natural way is an ordered ring,i.e., a structure of the form

$$\langle {}^*\mathbb{Q}^{\approx}(\varepsilon), +_{{}^*\mathbb{Q}^{\approx}(\varepsilon)}, \cdot_{{}^*\mathbb{Q}^{\approx}(\varepsilon)}, <_{{}^*\mathbb{Q}^{\approx}(\varepsilon)}, {}^*0, {}^*1 \rangle, \tag{2.62}$$

where ${}^{*}\mathbb{Q}^{\approx}(\varepsilon)$ is the set of elements of the structure, $+_{{}^{*}\mathbb{Q}^{\approx}(\varepsilon)}$ and $\bullet_{{}^{*}\mathbb{Q}^{\approx}(\varepsilon)}$ are the binary operations of additions and multiplication, $<_{{}^{*}\mathbb{Q}^{\approx}(\varepsilon)}$ is the ordering relation induced from corresponding operations and relation on ${}^{*}\mathbb{R}$, and ${}^{*}0, {}^{*}1$ are distinguished canonical elements of the domain.

Proof.Immediately from definitions.

2.11.The semirings ${}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$.

2.11.1.The semiring ${}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.

Definition 2.31. (**Wattenberg embeding**) We embed ${}^{*}\mathbb{R}^{\approx}(\varepsilon)$ into ${}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ of the following way:

(i) If $\alpha \in {}^{*}\mathbb{R}^{\approx}(\varepsilon)$, the corresponding element $\alpha^{\#^{\approx}_{\varepsilon}}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} = \alpha^{\#^{\approx}_{\varepsilon}}$ of ${}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ is

$$\alpha^{\#^{\approx}_{\varepsilon}}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} = \alpha^{\#^{\approx}_{\varepsilon}} \triangleq \left\{x \in {}^{*}\mathbb{R}^{\approx}(\varepsilon) | x \leq_{{}^{*}\mathbb{R}^{\approx}(\varepsilon)} \alpha\right\} \tag{2.63}$$

and

$$-_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha^{\#^{\approx}_{\varepsilon}} = \left\{a \in {}^{*}\mathbb{R}^{\approx}(\varepsilon) \middle| -_{{}^{*}\mathbb{R}} a \notin \alpha^{\#^{\approx}_{\varepsilon}}\right\} \cup \{\alpha\}. \tag{2.64}$$

(ii) If $\alpha,\beta \in {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ we define the sum $\alpha +_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ by

$$\alpha +_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq \{a +_{{}^{*}\mathbb{R}} b | a \in \alpha, b \in \beta\}. \tag{2.65}$$

(iii) If $\alpha \in {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \notin {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \in {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$ we define the sum $\alpha \check{+}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta$ by

$$\alpha \check{+}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq \{a +_{{}^{*}\mathbb{R}} b | a \in \alpha, b \in \beta\}. \tag{2.66}$$

(iv) Suppose $\alpha,\beta \in {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.Then we define the ordering relations $\alpha \leq_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ and $\alpha <_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ and equivalence relation $\alpha =_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ by

$$\begin{gathered} \alpha \leq_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \alpha \subseteq \beta, \\ \alpha <_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \alpha \subsetneq \beta, \\ \alpha =_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \left(\alpha \leq_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta\right) \wedge \left(\beta \leq_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha\right). \end{gathered} \tag{2.67}$$

(v) Suppose $\alpha \in {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$, $\beta \notin {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \in {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$.Then we define the ordering relations $\alpha \check{\leq}_{1{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \subset {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \times {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$ and $\alpha \check{<}_{1{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \subset {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \times {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$ and equivalence relation $\alpha \check{=}_{1{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \subset {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \times {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$ by

$$\begin{gathered} \alpha \check{\leq}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^{*}\mathbb{R}} b], \\ \alpha \check{<}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^{*}\mathbb{R}} \bar{b}], \\ \alpha \check{=}_{1{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \Leftrightarrow \left(\alpha \check{\leq}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta\right) \wedge \forall b(b \in \beta)\exists a(a \in \alpha)[b <_{{}^{*}\mathbb{R}} a]. \end{gathered} \tag{2.68}$$

(vi) Suppose $\alpha \in {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$, $\beta \notin {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}, \beta \in {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.Then we define the ordering relations $\alpha \check{\leq}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \subset {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}} \times {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and $\alpha \check{<}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \subset {}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}} \times {}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ by

$$\begin{gathered} \alpha \check{\leq}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^{*}\mathbb{R}} b], \\ \alpha \check{<}_{{}^{*}\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^{*}\mathbb{R}} \bar{b}]. \end{gathered} \tag{2.69}$$

(vii) If $A \subset {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ is bounded above in ${}^*\mathbb{R}^{\mathbf{fin}}$ then we define

$$\sup A = \bigcup_{\alpha \in A} \alpha \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \quad (2.70)$$

and

$$\inf A = \bigcap_{\alpha \in A} \alpha \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon). \quad (2.71)$$

(viii) Suppose $\alpha, \beta \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$. The product $\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$, is defined as follows.
Case (1) $\alpha_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}, \beta_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq \{a \bullet_{{}^*\mathbb{R}} b | 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha, 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} b^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta\} \cup \{{}^*\mathbb{R}^{\approx}_{-}(\varepsilon) \cup \{{}^*0\}\}. \quad (2.72)$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \quad (2.73)$$

Case (3) $\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$ or $\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$

$$\begin{gathered}
\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq |\alpha| \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } (\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}) \wedge \left(\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right), \\
\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } (\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}) \wedge \left(\beta_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right), \\
\alpha \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } (\alpha_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}) \wedge (\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}).
\end{gathered} \quad (2.74)$$

(ix) Suppose $\alpha \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \notin^* \mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \in {}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}$ The product $\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta$, is defined as follows.
Case (1) $\alpha_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}, \beta_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} > 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq \left\{a \bullet b | 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha, 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} b^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta\right\} \cup \{{}^*\mathbb{R}^{\mathbf{fin}}_{-} \cup^* 0\}. \quad (2.75)$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \quad (2.76)$$

Case (3) $\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$ or $\beta <_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} 0^{\#^{\approx}_{\varepsilon}}$

$$\begin{gathered}
\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq |\alpha| \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} |\beta| \text{ iff } (\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}) \wedge \left(\beta <_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} 0^{\#^{\approx}_{\varepsilon}}\right), \\
\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq -_{\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} |\beta| \text{ iff } (\alpha <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}) \wedge \left(\beta_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} > 0^{\#^{\approx}_{\varepsilon}}\right), \\
\alpha \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} \beta \triangleq -_{\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \breve{\bullet}_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} |\beta| \text{ iff } \left(\alpha_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta <_{{}^*\mathbb{R}^{\mathbf{fin}}_{\mathbf{d}}} 0^{\#^{\approx}_{\varepsilon}}\right).
\end{gathered} \quad (2.77)$$

Such embeding ${}^*\mathbb{R}^{\approx}(\varepsilon)$ into ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ as required above we will name Wattenberg embeding
and is denoted by ${}^*\mathbb{R}^{\approx}(\varepsilon) \overset{\#^{\approx}_{\varepsilon}}{\hookrightarrow} {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.

Theorem 2.27. ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ is complete ordered semiring.

Proof. Immediately from definitions.

Remark 2.15. The following element of ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ will be particularly useful for examples,

$$\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}) \triangleq \{{}^*\mathbb{R}^{\approx}_{-}(\varepsilon)\} \cup \mu_{\varepsilon}(0). \quad (2.78)$$

Examples. Note, for importent examples, that:

$$\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}) + \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}),$$
$$\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}) +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} (-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx})) = -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}). \tag{2.79}$$

2.11.2.The semiring ${}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$.

Definition 2.32. (**Wattenberg embeding**) We embed ${}^*\mathbb{Q}^{\approx}(\varepsilon)$ into ${}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ of the following way: (i) if $\alpha \in {}^*\mathbb{Q}^{\approx}(\varepsilon)$, the corresponding element $\alpha^{\#_{\varepsilon}^{\approx}}$ of ${}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ is

$$\alpha^{\#_{\varepsilon}^{\approx}}_{\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} = \alpha^{\#_{\varepsilon}^{\approx}} \triangleq \left\{x \in {}^*\mathbb{Q}^{\approx}(\varepsilon) | x \leq_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} \alpha\right\} \tag{2.80}$$

and

$$-_{\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha^{\#_{\varepsilon}^{\approx}} = \left\{a \in {}^*\mathbb{Q}^{\approx}(\varepsilon) \,\middle|\, -_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} a \notin \alpha^{\#_{\varepsilon}^{\approx}}\right\} \cup \{\alpha\}. \tag{2.81}$$

(ii) If $\alpha,\beta \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ we define the sum $\alpha +_{\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ by

$$\alpha +_{\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \triangleq \{a +_{{}^*\mathbb{Q}} b | a \in \alpha, b \in \beta\}. \tag{2.82}$$

(iii) If $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon), \beta \notin {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon), \beta \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ we define the sum $\alpha \,\breve{+}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ by

$$\alpha \,\breve{+}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \triangleq \{a +_{{}^*\mathbb{R}} b | a \in \alpha, b \in \beta\}. \tag{2.83}$$

(iv) Suppose $\alpha,\beta \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$. Then we define the ordering relations $\alpha \leq_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ and $\alpha <_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ by

$$\alpha \leq_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \alpha \subset \beta,$$
$$\alpha <_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \alpha \subsetneqq \beta. \tag{2.84}$$

(v) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$, $\alpha \notin {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon), \beta \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$.Then we define the ordering relations $\alpha \,\breve{\leq}_{1{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \subset {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon) \times {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ and $\alpha \,\breve{<}_{1{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \subset {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon) \times {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$
by

$$\alpha \,\breve{\leq}_{1{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^*\mathbb{R}} b],$$
$$\alpha \,\breve{<}_{1{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^*\mathbb{R}} \bar{b}] \tag{2.85}$$

(vi) Suppose $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$, $\beta \notin {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon), \beta \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$.Then we define the ordering relations $\alpha \,\breve{\leq}_{2{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \subset {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon) \times {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ and $\alpha \,\breve{<}_{2{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \subset {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon) \times {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$
by

$$\alpha \,\breve{\leq}_{2{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^*\mathbb{R}} b],$$
$$\alpha \,\breve{<}_{2{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^*\mathbb{R}} \bar{b}] \tag{2.86}$$

(vii) If $A \subset {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ is bounded above in ${}^*\mathbb{R}^{\mathbf{fin}}$ then we define

$$\sup A = \bigcup_{\alpha \in A} \alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon) \tag{2.87}$$

and

$$\inf A = \bigcap_{\alpha \in A} \alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon). \tag{2.88}$$

(viii) Suppose $\alpha,\beta \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$. The product $\alpha \bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$, is defined as follows.

Case (1) $\alpha\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}, \beta\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq \{a \bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} b | 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha, 0^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} b^{\#^{\approx}_{\varepsilon}} <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta\} \cup \{{}^*\mathbb{Q}^{\approx}_{-}(\varepsilon), \{{}^*0\}\}. \quad (2.89)$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \quad (2.90)$$

Case (3) $\alpha < 0^{\#^{\approx}_{\varepsilon}}$ or $\beta < 0^{\#^{\approx}_{\varepsilon}}$

$$\begin{gathered}\alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq |\alpha| \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right),\\ \alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right),\\ \alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}).\end{gathered} \quad (2.91)$$

(ix) Suppose $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon), \beta \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ The product $\alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$, is defined as follows.

Case (1) $\alpha\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}, \beta\ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$:

$$\begin{gathered}\alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq\\ \left\{a \bullet_{{}^*\mathbb{Q}} b | 0^{\#^{\approx}_{\varepsilon}} <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a^{\#^{\approx}_{\varepsilon}} <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha, 0^{\#^{\approx}_{\varepsilon}} <\ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} b^{\#^{\approx}_{\varepsilon}} <\ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta\right\} \bigcup\\ \bigcup \{{}^*\mathbb{R}^{\approx}_{-}(\varepsilon) \cup \{{}^*0\}\}.\end{gathered} \quad (2.92)$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$:

$$\alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \quad (2.93)$$

Case (3) $\alpha <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$ or $\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$:

$$\begin{gathered}\alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq |\alpha| \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}),\\ \alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta\ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right),\\ \alpha \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \check{\bullet}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}).\end{gathered} \quad (2.94)$$

Such embeding ${}^*\mathbb{Q}^{\approx}(\varepsilon)$ into ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ as required above we will name Wattenberg embeding and is denoted by ${}^*\mathbb{Q}^{\approx}(\varepsilon) \overset{\#^{\approx}_{\varepsilon}}{\hookrightarrow} {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$

Theorem 2.27.${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ is complete ordered semiring.

Proof.Immediately from definitions.

Remark 2.15.The following element of ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ will be particularly useful for examples,

$$\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}) \triangleq \{{}^*\mathbb{Q}^{\approx}_{-}(\varepsilon)\} \cup \mu_{\varepsilon}(0). \quad (2.95)$$

Examples.Note,for importent examples, that:

$$\begin{gathered}\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}),\\ \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} (-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})) = -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}).\end{gathered} \quad (2.96)$$

2.12.Absorption numbers in ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and in ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$.

Absorption numbers in ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$

Definition 2.33. Suppose $\alpha \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$, then

$$\mathbf{ab.p.}(\alpha) \triangleq \{d \geq 0^{\#\tilde{\tilde{\varepsilon}}} | \forall x_{x \in \alpha}[x +_{{}^*\mathbb{R}^{\approx}(\varepsilon)} d \in \alpha]\}. \quad (2.97)$$

Examples.

(i) $\forall a \in {}^*\mathbb{R}^{\approx}(\varepsilon) : \mathbf{ab.p.}(a^{\#\tilde{\tilde{\varepsilon}}}) = 0^{\#\tilde{\tilde{\varepsilon}}}$,

(ii) $\mathbf{ab.p.}(\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$,

(iii) $\mathbf{ab.p.}(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$,

(iv) $\forall \alpha \in {}^*\mathbb{R}^{\approx}(\varepsilon) : \mathbf{ab.p.}(\alpha^{\#\tilde{\tilde{\varepsilon}}} +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$,

(v) $\forall \alpha \in {}^*\mathbb{R}^{\approx}(\varepsilon) : \mathbf{ab.p.}(\alpha^{\#\tilde{\tilde{\varepsilon}}} -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$.

Theorem 2.28.

(i) $c <_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \mathbf{ab.p.}(\alpha)$ and $0 \leq_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d <_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} c \Rightarrow d \in \mathbf{ab.p.}(\alpha)$

(ii) $c \in \mathbf{ab.p.}(\alpha)$ and $d \in \mathbf{ab.p.}(\alpha) \Rightarrow c +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d \in \mathbf{ab.p.}(\alpha)$.

Remark 2.16. By Theorem 2.28 $\mathbf{ab.p.}(\alpha)$ may be regarded as an element of ${}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ by adding on all negative elements of ${}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ to $\mathbf{ab.p.}(\alpha)$. Of course if the condition $d \geq 0^{\#\tilde{\tilde{\varepsilon}}}$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x <_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha \Rightarrow x \in \alpha$. The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha) = \{0^{\#\tilde{\tilde{\varepsilon}}}\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with $0^{\#\tilde{\tilde{\varepsilon}}}$.

Remark 2.17. By Theorem 2.28 (ii), $\mathbf{ab.p.}(\alpha)$ is additive idempotent.

Theorem 2.29.

(i) $\mathbf{ab.p.}(\alpha)$ is the maximum element $\beta \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ such that $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta = \alpha$.

(ii) $\mathbf{ab.p.}(\alpha) \leq_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha$ for $\alpha > 0^{\#\tilde{\tilde{\varepsilon}}}$.

(iii) If α is positive and idempotent then $\mathbf{ab.p.}(\alpha) = \alpha$.

Theorem 2.30.Let $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ satsify $\alpha \ {}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\!> 0^{\#\tilde{\tilde{\varepsilon}}}$. Then the following are equivalent. In what follows assume $a, b \ {}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\!> 0^{\#\tilde{\tilde{\varepsilon}}}$.

(i) α is idempotent,

(ii) $a, b \in \alpha \Rightarrow a +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} b \in \alpha$,

(iii) $a \in \alpha \Rightarrow 2 \bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} a \in \alpha$,

(iv) $\forall n_{n \in \mathbb{N}}[a \in \alpha \Rightarrow n \bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} a \in \alpha]$,

(v) $a \in \alpha \Rightarrow r \bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} a \in \alpha$, for all finite $r \in {}^*\mathbb{R}^{\approx}(\varepsilon)$.

Theorem 2.31. $(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha) +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha = -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} [\mathbf{ab.p.}(\alpha)]$.

Theorem 2.32. $\mathbf{ab.p.}(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta) \ {}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\!\geq \mathbf{ab.p.}(\alpha)$.

Theorem 2.33.

(i) $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \gamma \Rightarrow -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \mathbf{ab.p.}(\alpha) +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta \leq_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \gamma$.

(ii) $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta = \alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \gamma \Rightarrow -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} [\mathbf{ab.p.}(\alpha)] +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta = -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} [\mathbf{ab.p.}(\alpha)] +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \gamma$.

Theorem 2.34.Suppose $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$, then

(i) $\mathbf{ab.p.}(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha) = \mathbf{ab.p.}(\alpha)$,

(ii) $\mathbf{ab.p.}(\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$

Theorem 2.35. Assume $\beta \ {}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\!> 0$. If α absorbs $-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ then α absorbs β.

Theorem 2.36. Let $0^{\#\tilde{\tilde{\varepsilon}}} < \alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$. Then the following are equivalent

(i) α is an idempotent,

(ii) $\left(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha\right) +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \left(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha\right) = -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha$,

(iii) $\left(-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha\right) +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha = -\alpha$.

(iv) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2 \ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > \Delta_1$.

Then $\Delta_2 +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \left(-_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \Delta_1\right) = \Delta_2$.

d

Absorption numbers in ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$

Definition 2.34. Suppose $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$, then

$$\mathbf{ab.p.}(\alpha) \triangleq \left\{d \ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \geq 0^{\#^{\approx}_{\varepsilon}} | \forall x_{x\in\alpha}[x +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d \in \alpha]\right\}. \quad (2.98)$$

Examples.

(i) $\forall a \in {}^*\mathbb{Q}^{\approx}(\varepsilon) : \mathbf{ab.p.}(a^{\#^{\approx}_{\varepsilon}}) = 0^{\#^{\approx}_{\varepsilon}}$,

(ii) $\mathbf{ab.p.}(\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$,

(iii) $\mathbf{ab.p.}(-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$,

(iv) $\forall \alpha \in {}^*\mathbb{Q}^{\approx}(\varepsilon) : \mathbf{ab.p.}(\alpha^{\#^{\approx}_{\varepsilon}} +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$,

(v) $\forall \alpha \in {}^*\mathbb{Q}^{\approx}(\varepsilon) : \mathbf{ab.p.}(\alpha^{\#^{\approx}_{\varepsilon}} -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$.

Theorem 2.37.

(i) $c <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{ab.p.}(\alpha)$ and $0 \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} c \Rightarrow d \in \mathbf{ab.p.}(\alpha)$

(ii) $c \in \mathbf{ab.p.}(\alpha)$ and $d \in \mathbf{ab.p.}(\alpha) \Rightarrow c +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d \in \mathbf{ab.p.}(\alpha)$.

Remark 2.18. By Theorem 2.37 $\mathbf{ab.p.}(\alpha)$ may be regarded as an element of ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ by adding on all negative elements of ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ to $\mathbf{ab.p.}(\alpha)$. Of course if the condition $d \ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \geq 0^{\#^{\approx}_{\varepsilon}}$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x < y \in \alpha \Rightarrow x \in \alpha$. The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha) = \{0^{\#^{\approx}_{\varepsilon}}\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with $0^{\#^{\approx}_{\varepsilon}}$.

Remark 2.19. By Theorem 2.37(ii), $\mathbf{ab.p.}(\alpha)$ is additive idempotent.

Theorem 2.38.

(i) $\mathbf{ab.p.}(\alpha)$ is the maximum element $\beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ such that $\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta = \alpha$.

(ii) $\mathbf{ab.p.}(\alpha) \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha$ for $\alpha \ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$.

(iii) If α is positive and idempotent then $\mathbf{ab.p.}(\alpha) = \alpha$.

Theorem 2.39.Let $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ satsify $\alpha > 0^{\#^{\approx}_{\varepsilon}}$. Then the following are equivalent. In what follows assume $a, b \ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$.

(i) α is idempotent,

(ii) $a, b \in \alpha \Rightarrow a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} b \in \alpha$,

(iii) $a \in \alpha \Rightarrow 2 \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a \in \alpha$,

(iv) $\forall n_{n\in\mathbb{N}}[a \in \alpha \Rightarrow n \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a \in \alpha]$,

(v) $a \in \alpha \Rightarrow q \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a \in \alpha$, for all finite $q \in {}^*\mathbb{Q}^{\approx}(\varepsilon)$.

Theorem 2.40. $(-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha = -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} [\mathbf{ab.p.}(\alpha)]$.

Theorem 2.41. $\mathbf{ab.p.}(\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta) \ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \geq \mathbf{ab.p.}(\alpha)$.

Theorem 2.42.

(i) $\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \gamma \Rightarrow -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{ab.p.}(\alpha) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \gamma$.

(ii) $\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta = \alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \gamma \Rightarrow -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} [\mathbf{ab.p.}(\alpha)] +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta = -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} [\mathbf{ab.p.}(\alpha)] +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \gamma$.

Theorem 2.43.Suppose $\alpha, \beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$, then

(i) $\mathbf{ab.p.}(-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha) = \mathbf{ab.p.}(\alpha)$,

(ii) $\mathbf{ab.p.}(\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$

Theorem 2.44. Assume $\beta > 0$. If α absorbs $-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \beta$ then α absorbs β.
Theorem 2.45. Let $0 < \alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$. Then the following are equivalent
(i) α is an idempotent,
(ii) $(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha) +_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} (-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha) = -_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha,$
(iii) $(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha) +_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha = -_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha.$
(iv) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} > \Delta_1$.
Then $\Delta_2 +_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} (-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta_1) = \Delta_2$.

2.13.Gonshor's types of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ and $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ with given $\mathbf{ab.p.}(\alpha)$.

2.13.1.Gonshor's types of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ with given $\mathbf{ab.p.}(\alpha)$.

Among elements of $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ such that $\mathbf{ab.p.}(\alpha) = \Delta$ one can distinguish two many different types following Gonshor's paper [7].
Definition 2.35.Assume $\Delta > 0^{\#\tilde{\tilde{\varepsilon}}}$.
(i) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 1 if $\exists x(x \in \alpha)\forall y[x +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha \Rightarrow y \in \Delta]$,
(ii) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 2 if $\forall x(x \in \alpha)\exists y(y \notin \Delta)[x +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha]$, i.e.
$\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 2 iff α does not have type 1.
(iii) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type $1\mathbf{A}$ if $\exists x(x \notin \alpha)\forall y[x -_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} y \notin \alpha \Rightarrow y \in \Delta]$,
Theorem 2.46.
(i) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 1 iff $-_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha$ has type $1\mathbf{A}$,
(ii) $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.
(iii) Suppose $\mathbf{ab.p.}(\alpha) = \Delta > 0^{\#\tilde{\tilde{\varepsilon}}}$. Then α has type 1 iff α has the form
$a^{\#\tilde{\tilde{\varepsilon}}} +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$ for some $a \in {}^*\mathbb{R}^{\approx}(\varepsilon)$
(iv) Suppose $\mathbf{ab.p.}(\alpha) = -_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta, \Delta > 0^{\#\tilde{\tilde{\varepsilon}}}. \alpha \in {}^*\mathbb{R}_{\mathbf{d}}$ has type $1\mathbf{A}$ iff α has the form
$a^{\#\tilde{\tilde{\varepsilon}}} +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ (-_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta)$ for some $a \in {}^*\mathbb{R}^{\approx}(\varepsilon)$.
(v) If $\mathbf{ab.p.}(\alpha) > \mathbf{ab.p.}(\beta)$ then $\alpha +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ has type 1 iff α has type 1.
(vi) If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \beta$ has type 2 iff either α or β
has type 2.
Proof. (iii) Let $\alpha = a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$. Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta > 0, a \in a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$
(we chose $d \in \Delta$ such that $0 < d$ and write a as $(a -_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d) +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d$).
It is clear that a works to show that α has type 1.
Conversely, suppose α has type 1 and choose $a \in \alpha$ such that:
$\forall y[a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha \Rightarrow y \in \Delta]$.Then we claim that: $\alpha = a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$.
By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta \leq_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \alpha$. On the other hand by choice of $a,$every element of α has the form $a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d$ with $d \in \Delta$.
Choose $d' \in \Delta$ such that $d'\ {}_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} > d$, then $a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d =$
$\left[a -_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \left(d' -_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d\right)\right] +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} d' \in a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$.
Hence $\alpha \leq_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$.Therefore $\alpha = a +_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$.
Examples. (i) $\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$ has type 1 and therefore $-_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$ has type $1\mathbf{A}$.Note that
also $-_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R}^{\approx})$ has type 2. (ii) Suppose $\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$. Then $\varepsilon^{\#\tilde{\tilde{\varepsilon}}} \bullet_{^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{R})$ has

type 1 and therefore $-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \varepsilon^{\#_{\varepsilon}^{\approx}}\ \cdot_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx})$ has type $1\mathbf{A}$.

(ii) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon), \mathbf{ab.p.}(\alpha) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx}) > 0$, i.e. α has type 1 and therefore by Theorem 2.46 α has the form $({}^*a)^{\#_{\varepsilon}^{\approx}} + \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx})$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$.

Then, we define unique Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α by formula

$$\Re\mathbf{p}[\alpha] \triangleq ({}^*a)^{\#_{\varepsilon}^{\approx}}, \alpha = \ ({}^*\mathbf{Wst}(\alpha))^{\#_{\varepsilon}^{\approx}}.$$

(iii) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon), \mathbf{ab.p.}(\alpha) = -\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx})$, i.e. α has type $1\mathbf{A}$ and therefore by Theorem 2.46 α has the form $({}^*a)^{\#} - \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{R}^{\approx})$ for some unique $a \in \mathbb{R}, a = \mathbf{Wst}(\alpha)$.

Then we define unique Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α by formula

$$\Re\mathbf{p}[\alpha] \triangleq ({}^*a)^{\#_{\varepsilon}^{\approx}}, \alpha = \ ({}^*\mathbf{Wst}(\alpha))^{\#_{\varepsilon}^{\approx}}.$$

(iv) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon), \mathbf{ab.p.}(\alpha) = \Delta, \Delta\ {}_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} > 0^{\#_{\varepsilon}^{\approx}}$ and α has type $1\mathbf{A}$, i.e. α has the form

$a^{\#} +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$ for some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}\{\alpha\}$ of absorption number

α by formula

$$\Re\mathbf{p}[\alpha] \triangleq a^{\#_{\varepsilon}^{\approx}}.$$

(v) Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon), \mathbf{ab.p.}(\alpha) = -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta, \Delta_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} > 0$ and α has type $1\mathbf{A}$, i.e. α has the

form $a^{\#_{\varepsilon}^{\approx}} +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ (-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta)$ for some $a \in {}^*\mathbb{R}$. Then, we define Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α by formula

$$\Re\mathbf{p}[\alpha] \triangleq a^{\#_{\varepsilon}^{\approx}}.$$

Remark. Note that in general case,i.e. if $\alpha \notin (-\Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ Robinson part $\Re\mathbf{p}[\alpha]$ of absorption number α is not unique.

Remark. Suppose $\alpha \in {}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ and $\alpha \in (-_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta_{\mathbf{d}}, \Delta_{\mathbf{d}})$ has type 1or type $1\mathbf{A}$. Then by definitions above one obtains the representation

$$\alpha = \Re\mathbf{p}[\alpha] +_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \mathbf{ab.p.}(\alpha).$$

2.13.2.Gonshor's types of $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ with given $\mathbf{ab.p.}(\alpha)$.

Among elements of $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ such that $\mathbf{ab.p.}(\alpha) = \Delta$ one can distinguish two many different types following Gonshor's paper [7].

Definition 2.36.Assume $\Delta > 0^{\#_{\varepsilon}^{\approx}}$.

(i) $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 1 if $\exists x(x \in \alpha)\forall y[x +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha \Rightarrow y \in \Delta]$,

(ii) $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 2 if $\forall x(x \in \alpha)\exists y(y \notin \Delta)[x +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} y \in \alpha]$, i.e.

$\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 2 iff α does not have type 1.

(iii) $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type $1\mathbf{A}$ if $\exists x(x \notin \alpha)\forall y\left[x -_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} y \notin \alpha \Rightarrow y \in \Delta\right]$

Theorem 2.47.

(i) $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type 1 iff $-_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \alpha$ has type $1\mathbf{A}$,

(ii) $\alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.

(iii) Suppose $\mathbf{ab.p.}(\alpha) = \Delta\ {}_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} > 0^{\#_{\varepsilon}^{\approx}}$. Then α has type 1 iff α has the form

$a^{\#_{\varepsilon}^{\approx}} +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta$ for some $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon)$

(iv) Suppose $\mathbf{ab.p.}(\alpha) = -_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\ \Delta, \Delta\ {}_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)} > 0^{\#_{\varepsilon}^{\approx}}. \alpha \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)$ has type $1\mathbf{A}$ iff α has the form

$a^{\#_{\tilde{\varepsilon}}} + (-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\,\Delta)$ for some $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon)$.

(v) If $\mathbf{ab.p.}(\alpha)\ {}_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > \mathbf{ab.p.}(\beta)$ then $\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ has type 1 iff α has type 1.

(vi) If $\mathbf{ab.p.}(\alpha) = \mathbf{ab.p.}(\beta)$ then $\alpha +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ has type 2 iff either α or β has type 2.

Proof. (iii) Let $\alpha = a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta$. Then $\mathbf{ab.p.}(\alpha) = \Delta$.Since $\Delta_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0, a \in a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta$ (we chose $d \in \Delta$ such that $0 < d$ and write a as $(a -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d$).

It is clear that a works to show that α has type 1.

Conversely, suppose α has type 1 and choose $a \in \alpha$ such that:

$\forall y[a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} y \in \alpha \Rightarrow y \in \Delta]$.Then we claim that: $\alpha = a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta$.

By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha$. On the other hand by choice of a, every element of α has the form $a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d$ with $d \in \Delta$.

Choose $d' \in \Delta$ such that $d'_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} > d$, then $a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d =$
$\left[a -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \left(d' -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d\right)\right] +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} d' \in a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta$.

Hence $\alpha \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} a +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \Delta$.Therefore $\alpha = a + \Delta$.

Examples. (i) $\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$ has type 1 and therefore $-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$ has type $1\mathbf{A}$.Note that also $-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$ has type 2. (ii) Suppose $\varepsilon \approx 0, \varepsilon \in {}^*\mathbb{R}$. Then $\varepsilon^{\#_{\tilde{\varepsilon}}} \cdot_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$ has type 1 and therefore $-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)}\ \varepsilon^{\#_{\tilde{\varepsilon}}} \cdot \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx})$ has type $1\mathbf{A}$.

2.14.The Special Kinds of Idempotents in ${}^*\mathbb{R}_{\mathbf{d}}$.

Let $a \in {}^*\mathbb{R}, a > 0$.Then a gives rise to two idempotents $\mathbf{A}_a, \mathbf{B}_a$ in a natural way [7]:

$$\mathbf{A}_a \triangleq \{x \in ({}^*\mathbb{R}) | \exists n(n \in \mathbb{N})[x \leq ({}^*n) \cdot a]\}, \tag{2.99}$$

and

$$\mathbf{B}_a \triangleq \{x \in ({}^*\mathbb{R}) | \forall r(r \in \mathbb{R}_+)[x \leq ({}^*r) \cdot a]\}. \tag{2.100}$$

Remark 2.18. It is immediate that $\mathbf{A}_a$ and $\mathbf{B}_a$ are idempotents.It is also clear that $\mathbf{A}_a$ is the smallest idempotent containing a and $\mathbf{B}_a$ is the largest idempotent not containing a. It follows that $\mathbf{B}_a$ and $\mathbf{A}_a$ are consecutive idempotents.Note that $\mathbf{B}_1 = \boldsymbol{\varepsilon}_{\mathbf{d}}$.

Theorem 2.48.[7].(i) No idempotent of the form $\mathbf{A}_a$ has an immediate successor.

(ii) All consecutive pairs of idempotents have the form $\mathbf{B}_a$ and $\mathbf{A}_a$ for some $a \in {}^*\mathbb{R}$, $a > 0$.

Proof.(i) Let $\mathbf{A}_a \subsetneq \Delta$. Suppose $x \in \Delta$ but $x \notin \mathbf{A}_a$. Then $x > n \cdot a$ for all positive standard integers n.Let $y = \sqrt{x \cdot a}$ which is defined since ${}^*\mathbb{R}$ is a nonstandard model of $\mathbb{R}$. Then $y \geq a\sqrt{n}$ for all positive standard integers n so that $y \notin \mathbf{A}_a$. So $\mathbf{A}_y > \mathbf{A}_a$. Similarly $x > y\sqrt{n}$ so $x \notin \mathbf{A}_y$. Hence $\mathbf{A}_y < \Delta$.Thus $\mathbf{A}_a$ and Δ are not consecutive.

(ii) Let C and D be consecutive idempotents such that $C < D$. Let $a \in D$ with $a \notin C$. Then $C < \mathbf{B}_a < \mathbf{A}_a < D$.Hence $C = \mathbf{B}_a$ and $D = \mathbf{A}_a$.

Theorem 2.49.[7].If $\mathbf{ab.p.}(\alpha)$ has the form $\mathbf{B}_\alpha$ then α has type 1 or $1\mathbf{A}$.

Proof.Incidentally, we already know that in general $\mathbf{ab.p.}(\alpha)$ cannot have type 1 and $1\mathbf{A}$ simultaneously. Now $a \notin \mathbf{B}_a$ and therefore $\exists b(b \in \alpha)\exists c(c \notin \alpha)[a^{\#} = c - b]$.We now define an ordinary Dedekind cut L_b for the real numbers $\mathbb{R}$, where L_b is the set of lower elements, as follows. Let $r \in L_b$ iff $b + r^{\#} \cdot a^{\#} \in \alpha$.It is immediate that $0 \in L_b, 1 \notin L_b, z < y \in L_b \Rightarrow z \in L_b$. So we have a Dedekind cut. Then L_b has a maximum or L_b' has a minimum. Suppose first that L_b has a maximum $\bar{r} = r_{\max}$. Then $b + \bar{r}^{\#} \cdot a^{\#} \in \alpha$ but for any real $s > r,\ b + s^{\#} \cdot a^{\#} \notin \alpha.$ We now claim that $b + \bar{r}^{\#} \cdot a^{\#}$ works to show that α has type $1.$ In fact, suppose $b + \bar{r}^{\#} \cdot a^{\#} - x \in \alpha.$ Let $s > \bar{r}.$ Since $b + s^{\#}a^{\#} \notin \alpha,$ $b + s^{\#} \cdot a^{\#} > b + r^{\#} \cdot a + x.$ Therefore $x < (s^{\#} - \bar{r}^{\#}) \cdot a^{\#}.$ Thus $x < \delta^{\#} \cdot \alpha$ for every positive real $\delta \in \mathbb{R}_+;$ i.e. $x \in \mathbf{B}_a.$ A similar argument shows that α has type $1\mathbf{A}$ if L_b' has a minimum.

Examples.(i)The result applies to $\mathbf{B}_1 = \boldsymbol{\varepsilon}_{\mathbf{d}}$. It follows from Theorem 2.49 that every α with $\mathbf{ab.p.}(\alpha) = \boldsymbol{\varepsilon}_{\mathbf{d}}$ must be either of the form $a^{\#} + \boldsymbol{\varepsilon}_{\mathbf{d}}$ or $a^{\#} + (-\boldsymbol{\varepsilon}_{\mathbf{d}})$ with $a \in {}^*\mathbb{R},\ a > 0^{\#}$.

(ii)The result applies to $\mathbf{B}_{\varepsilon}, \varepsilon \approx 0;$ i.e. $\mathbf{B}_{\varepsilon} = \varepsilon \cdot \boldsymbol{\varepsilon}_{\mathbf{d}}.$ It follows from Theorem 2.49 that every α with $\mathbf{ab.p.}(\alpha) = \varepsilon \cdot \boldsymbol{\varepsilon}_{\mathbf{d}}$ must be either of the form $a^{\#} + \varepsilon \cdot \boldsymbol{\varepsilon}_{\mathbf{d}}$ or $a^{\#} + (-\varepsilon \cdot \boldsymbol{\varepsilon}_{\mathbf{d}})$ with $a \in {}^*\mathbb{R},\ a > 0^{\#}$.

2.15.The Special Kinds of Idempotents in ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and in ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$.

2.15.1.The Special Kinds of Idempotents in ${}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.

Let $a \in {}^*\mathbb{R}^{\approx}(\varepsilon), a > 0.$ Then a gives rise to two idempotents $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)), \mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ in a natural way :

$$\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) \triangleq \{x \in ({}^*\mathbb{R}^{\approx}(\varepsilon)) | \exists n (n \in \mathbb{N})[x \leq_{{}^*\mathbb{R}^{\approx}(\varepsilon)} ({}^*n) \cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)} a]\}, \quad (2.101)$$

and

$$\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) \triangleq \{x \in ({}^*\mathbb{R}^{\approx}(\varepsilon)) | \forall r (r \in \mathbb{R}_+)[x \leq_{{}^*\mathbb{R}^{\approx}(\varepsilon)} ({}^*r) \cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)} a]\}. \quad (2.102)$$

Remark 2.19. It is immediate that $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and $\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ are idempotents.It is also clear that $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ is the smallest idempotent containing a and $\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ is the largest idempotent not containing a. It follows that $\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ are consecutive idempotents.Note that $\mathbf{B}^{\approx}_1(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$.

Theorem 2.50.(i) No idempotent of the form $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ has an immediate successor.

(ii) All consecutive pairs of idempotents have the form $\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ for

some $a \in {}^*\mathbb{R}^{\approx}(\varepsilon), a > 0$.

Proof.(i) Let $\mathbf{A}_a \subsetneq \Delta$. Suppose $x \in \Delta$ but $x \notin \mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)).$ Then $x > n \cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)} a$ for all positive standard integers $n.$ Let $y = \sqrt{x \cdot_{{}^*\mathbb{R}^{\approx}(\varepsilon)} a}$ which is defined since ${}^*\mathbb{R}^{\approx}(\varepsilon)$ is a subset of nonstandard model of $\mathbb{R}$. Then $y_{{}^*\mathbb{R}^{\approx}(\varepsilon)} \geq a\sqrt{n}$ for all positive standard integers n so that $y \notin \mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$. So $\mathbf{A}_y\ {}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > \mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$. Similarly $x\ {}_{{}^*\mathbb{R}^{\approx}(\varepsilon)} > y\sqrt{n}$ so $x \notin \mathbf{A}_y$. Hence $\mathbf{A}_y <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \Delta.$ Thus $\mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and Δ are not consecutive.

(ii) Let $\mathbf{C}$ and $\mathbf{D}$ be consecutive idempotents such that $\mathbf{C} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{D}.$ Let $a \in D$ with $a \notin \mathbf{C}$.

Then $\mathbf{C} <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{D}.$ Hence $\mathbf{C} = \mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and

$\mathbf{D} = \mathbf{A}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$.

Theorem 2.51.[7].If $\mathbf{ab.p.}(\alpha)$ has the form $\mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ then α has type 1 or $1\mathbf{A}$.

Proof.Incidentally, we already know that in general $\mathbf{ab.p.}(\alpha)$ cannot have type 1 and $1\mathbf{A}$ simultaneously. Now $a \notin \mathbf{B}^{\approx}_a(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ and therefore $\exists b (b \in \alpha) \exists c (c \notin \alpha)[a = c -_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} b]$. We now define an ordinary Dedekind cut L_b for the real numbers $\mathbb{R}$, where L_b is the set of lower elements, as follows. Let $r \in L_b$ iff $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} r^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \in \alpha$. It is immediate that $0 \in L_b, 1 \notin L_b,\ z < y \in L_b \Rightarrow z \in L_b$. So we have a Dedekind cut. Then L_b has a maximum or L'_b has a minimum. Suppose first that L_b has a maximum $\bar{r} = r_{\max}$. Then $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \in \alpha$ but for any real $s > r$, $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} s^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \notin \alpha$. We now claim that $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a$ works to show that α has type 1. In fact, suppose $b + \bar{r}^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} x \in \alpha$. Let $s > \bar{r}$. Since $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} s^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \notin \alpha,\ b + s^{\#^{\approx}_{\varepsilon}} \cdot a > b + r^{\#^{\approx}_{\varepsilon}} \cdot a + x$. Therefore $x < (s^{\#^{\approx}_{\varepsilon}} - \bar{r}^{\#^{\approx}_{\varepsilon}}) \cdot a$. Thus $x <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \delta^{\#^{\approx}_{\varepsilon}} \cdot_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \alpha$ for every positive real $\delta \in \mathbb{R}_+$; i.e. $x \in \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$. A similar argument shows that α has type $1\mathbf{A}$ if L'_b has a minimum.

Examples.(i)The result applies to $\mathbf{B}_1 = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R})$. It follows from Theorem 2.51 that every α with $\mathbf{ab.p.}(\alpha) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ must be either of the form $a^{\#^{\approx}_{\varepsilon}} + \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ or $a^{\#^{\approx}_{\varepsilon}} + (-\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)))$ with $a \in {}^*\mathbb{R}^{\approx}(\varepsilon),\ a > 0$.

(ii)The result applies to $\mathbf{B}^{\approx}_{\varepsilon_1}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)), \varepsilon_1 \approx 0$; i.e. $\mathbf{B}^{\approx}_{\varepsilon_1}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)) = \varepsilon_1 \cdot \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$. It follows from Theorem 2.51 that every α with $\mathbf{ab.p.}(\alpha) = \varepsilon_1 \cdot \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R})$ must be either of the form $a^{\#} + \varepsilon \cdot \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon))$ or $a^{\#} + (-\varepsilon \cdot \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{R}^{\approx}(\varepsilon)))$ with $a \in {}^*\mathbb{R}^{\approx}(\varepsilon),\ a > 0$.

2.15.2.The Special Kinds of Idempotents in ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$

Let $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon), a > 0^{\#^{\approx}_{\varepsilon}}$. Then a gives rise to two idempotents $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)), \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ in a natural way

$$\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) \triangleq \{x \in ({}^*\mathbb{Q}^{\approx}(\varepsilon)) | \exists n(n \in \mathbb{N})[x \leq_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} ({}^*n) \cdot_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} a]\}, \qquad (2.103)$$

and

$$\mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) \triangleq \{x \in ({}^*\mathbb{Q}^{\approx}(\varepsilon)) | \forall r(r \in \mathbb{Q}_+)[x \leq_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} ({}^*r) \cdot_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} a]\}. \qquad (2.104)$$

Remark 2.20. It is immediate that $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ and $\mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ are idempotents.It is also clear that $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ is the smallest idempotent containing a and $\mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ is the largest idempotent not containing a. It follows that $\mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ and $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ are consecutive idempotents.Note that $\mathbf{B}^{\approx}_{1}(\varepsilon; {}^*\mathbb{Q}) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$.

Theorem 2.53.(i) No idempotent of the form $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ has an immediate successor.

(ii) All consecutive pairs of idempotents have the form $\mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q})$ and $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ for some $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon), a > 0$.

Proof.(i) Let $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) \subsetneq \Delta$. Suppose $x \in \Delta$ but $x \notin \mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$. Then $x > n \cdot a$ for
all positive standard integers n. Let $y = \sqrt{x \cdot a}$ which is defined since ${}^*\mathbb{R}$ is a nonstandard
model of $\mathbb{R}$. Then $y \geq a\sqrt{n}$ for all positive standard integers n so that $y \notin \mathbf{A}_a$. So $\mathbf{A}_y > \mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$. Similarly $x > y\sqrt{n}$ so $x \notin \mathbf{A}_y$. Hence $\mathbf{A}_y < \Delta$. Thus $\mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ and Δ are
not consecutive.

(ii) Let $\mathbf{C}$ and $\mathbf{D}$ be consecutive idempotents such that $\mathbf{C} < \mathbf{D}$. Let $a \in D$ with $a \notin \mathbf{C}$. Then $\mathbf{C} < \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) < \mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) < \mathbf{D}$. Hence $\mathbf{C} = \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ and $\mathbf{D} = \mathbf{A}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$.

Theorem 2.54.If $\mathbf{ab.p.}(\alpha)$ has the form $\mathbf{B}_\alpha$ then α has type 1 or $1\mathbf{A}$.

Proof.Incidentally, we already know that in general $\mathbf{ab.p.}(\alpha)$ cannot have type 1 and $1\mathbf{A}$ simultaneously. Let $d \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and $d \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ then we write

$$d \,\breve{\in}\, \alpha \Leftrightarrow \exists q(q \in \alpha)[d <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} q]. \tag{2.105}$$

Now $a \notin \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ and therefore $\exists b(b \in \alpha)\exists c(c \notin \alpha)[a = c -_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} b]$. We now define an ordinary Dedekind cut L_b for the real numbers $\mathbb{R}$, where L_b is the set of lower elements, as follows. Let $r \in L_b$ iff $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} r^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \,\breve{\in}\, \alpha$. It is immediate that $0 \in L_b, 1 \notin L_b,\ z < y \in L_b \Rightarrow z \in L_b$. So we have a Dedekind cut. Then L_b has a maximum or L'_b has a minimum. Suppose first that L_b has a maximum $\bar{r} = r_{\max}$. Then $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \,\breve{\in}\, \alpha$ and therefore by definition (2.105) there exist $q \triangleq q(\bar{r}) \in \alpha$ such that

$$b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} q(\bar{r}), \tag{2.106}$$

but for any real $s > r,\ b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} s^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a \,\breve{\notin}\, \alpha$. We now claim that $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a$ works to show that α has type 1. In fact, suppose $q(\bar{r}) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} x \in \alpha$ then from enequality (2.106) follows that

$$b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} x <_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} q(\bar{r}) +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} x. \tag{2.107}$$

and therefore $b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} x \,\breve{\in}\, \alpha$. Let $s > \bar{r}$. Since $b +_{+{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} s^{\#^{\approx}_{\varepsilon}} a \,\breve{\notin}\, \alpha$,

$$b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} s^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a > q(\bar{r}) +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} x > b +_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \bar{r}^{\#^{\approx}_{\varepsilon}} \bullet_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} a +_{+{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} x. \tag{2.108}$$

Therefore $x < (s^{\#^{\approx}_{\varepsilon}} - \bar{r}^{\#^{\approx}_{\varepsilon}}) \bullet a$. Thus $x < \delta^{\#^{\approx}_{\varepsilon}} \bullet \alpha$ for every positive real $\delta \in \mathbb{Q}_{+}$; i.e. $x \in \mathbf{B}^{\approx}_{a}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$. A similar argument shows that α has type $1\mathbf{A}$ if L'_b has a minimum.

Examples.(i)The result applies to $\mathbf{B}^{\approx}_{1}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$. It follows from Theorem 2.54 that every α with $\mathbf{ab.p.}(\alpha) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ must be either of the form $a^{\#^{\approx}_{\varepsilon}} + \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ or $a^{\#^{\approx}_{\varepsilon}} +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} (-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)))$ with $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon),\ a > 0$.

(ii)The result applies to $\mathbf{B}^{\approx}_{\varepsilon_1}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)), \varepsilon_1 \approx 0$; i.e. $\mathbf{B}^{\approx}_{\varepsilon}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)) = \varepsilon \bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$. It follows from Theorem 2.54 that every α with $\mathbf{ab.p.}(\alpha) = \varepsilon \bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q})$ must be either of the form $a^{\#^{\approx}_{\varepsilon}} +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \varepsilon \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon))$ or $a^{\#^{\approx}_{\varepsilon}} +_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} (-_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \varepsilon \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon)))$ with $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon),\ a > 0$.

2.16. The semirings ${}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p})$ and ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon, \mathbf{p})$.

2.16.1. The semiring ${}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p})$.

Definition 2.37.Let $a = \dfrac{\mathbf{m}}{k \bullet \mathbf{n}} \in {}^*\mathbb{Q}, \mathbf{m} \in {}^*\mathbb{Z}\backslash\mathbb{Z}, \mathbf{n} \in {}^*\mathbb{N}\backslash\mathbb{N}, k \in \mathbb{N}, \exists St(a), \varepsilon \approx 0,\ \mathbf{p} \in {}^*\mathbb{N}\backslash\mathbb{N}$, where $\mathbf{p}$ is an *given* infinite prime number. We will say that a is ε-near-standard hyper rational $\mathbf{p}$-number iff:

(i) $\exists \eta_{\varepsilon}(\eta_{\varepsilon} \in \mu_{\varepsilon}(0))\left[\left|\dfrac{\mathbf{m}}{k \bullet \mathbf{n}} - \left({}^*St\left(\dfrac{\mathbf{m}}{k \bullet \mathbf{n}}\right)\right)\right| \leq \eta_{\varepsilon}\right]$,

(ii) $\mathbf{m}|\mathbf{p}, \mathbf{m} \not| k$, and

(iii) $\mathbf{n} \not| \mathbf{p}$.

Definition 2.38.The set of the all ε-near-standard hyper rational $\mathbf{p}$-numbers is denoted,

${}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p})$.

Theorem 2.55.The set ${}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p})$ as algebraic structure in a natural way is an ordered semiring,i.e., a structure of the form

$$\langle {}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p}), +_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}, \bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}, <_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}, {}^*0, {}^*1\rangle, \tag{2.109}$$

where ${}^*\mathbb{Q}^{\approx}(\varepsilon, \mathbf{p})$ is the set of elements of the structure, $+_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}$ and $\bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}$ are the binary operations of additions and multiplication, $<_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}$ is the ordering relation,and ${}^*0, {}^*1$ are

distinguished elements of the domain.

Proof.Immediately from definitions.

2.16.2 The semiring ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$.

Definition 2.39. (**Wattenberg embeding**) We embed ${}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$ into ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ of the following way: (i) if $\alpha \in {}^*\mathbb{Q}^{\approx}(\varepsilon)$, the corresponding element $\alpha^{\#^{\approx}_{\varepsilon}}$ of ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ is

$$\alpha^{\#^{\approx}_{\varepsilon}}_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} = \alpha^{\#^{\approx}_{\varepsilon}} \triangleq \left\{x \in {}^*\mathbb{Q}^{\approx}(\varepsilon) | x \leq_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})} \alpha\right\} \quad (2.110)$$

and

$$-_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha^{\#^{\approx}_{\varepsilon}} = \left\{a \in {}^*\mathbb{Q}^{\approx}(\varepsilon) \,\middle|\, -_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})} a \notin \alpha^{\#^{\approx}_{\varepsilon}}\right\} \cup \{\alpha\}. \quad (2.111)$$

(ii) If $\alpha,\beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ we define the sum $\alpha +_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta$ by

$$\alpha +_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq \{a +_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})} b | a \in \alpha, b \in \beta\}. \quad (2.112)$$

(iii) If $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ we define the sum $\alpha \check{+}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$ by

$$\alpha \check{+}_{{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq \{a +_{{}^*\mathbb{R}^{\approx}(\varepsilon)} b | a \in \alpha, b \in \beta\}. \quad (2.113)$$

(iv) Suppose $\alpha,\beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$. Then we define the ordering relations $\alpha \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta$ and $\alpha <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta$ by

$$\begin{gathered}\alpha \leq_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \Leftrightarrow \alpha \subset \beta,\\ \alpha <_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \Leftrightarrow \alpha \subsetneqq \beta.\end{gathered} \quad (2.114)$$

(v) Suppose $\alpha \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon),\ \alpha \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$.Then we define the ordering relations

$\alpha \check{\leq}_{1{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \subset {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \times {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ and $\alpha \check{<}_{1{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \subset {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon) \times {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$

by

$$\begin{gathered}\alpha \check{\leq}_{1{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^*\mathbb{R}} b],\\ \alpha \check{<}_{1{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^*\mathbb{R}} \bar{b}]\end{gathered} \quad (2.115)$$

(vi) Suppose $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}),\ \beta \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$.Then we define the ordering relations $\alpha \check{\leq}_{2{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \subset {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}) \times {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ and $\alpha \check{<}_{2{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \subset {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}) \times {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$

by

$$\begin{gathered}\alpha \check{\leq}_{2{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \forall a(a \in \alpha)\exists b(b \in \beta)[a <_{{}^*\mathbb{R}} b],\\ \alpha \check{<}_{2{}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \Leftrightarrow \exists \bar{b}(\bar{b} \in \beta)\forall a(a \in \alpha)[a <_{{}^*\mathbb{R}} \bar{b}]\end{gathered} \quad (2.116)$$

(vii) If $A \subset {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ is bounded above in ${}^*\mathbb{R}^{\mathbf{fin}}$ then we define

$$\sup A = \bigcup_{\alpha \in A} \alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}) \quad (2.117)$$

and

$$\inf A = \bigcap_{\alpha \in A} \alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}). \quad (2.118)$$

(viii) Suppose $\alpha,\beta \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$. The product $\alpha \bullet_{{}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta$, is defined as follows.

Case (1) $\alpha\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}, \beta\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq$$
$$\{a \bullet_{^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})} b | 0^{\#^{\approx}_{\varepsilon}} <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} a^{\#^{\approx}_{\varepsilon}} <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha, 0^{\#^{\approx}_{\varepsilon}} <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} b^{\#^{\approx}_{\varepsilon}} <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta\} \cup \tag{2.119}$$
$$\cup \{^*\mathbb{Q}^{\approx}_{-}(\varepsilon,\mathbf{p}), \{^*0\}\}.$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$

$$\alpha \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \tag{2.120}$$

Case (3) $\alpha < 0^{\#^{\approx}_{\varepsilon}}$ or $\beta < 0^{\#^{\approx}_{\varepsilon}}$

$$\begin{gathered}
\alpha \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq |\alpha| \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\beta| \text{ iff } \left(\alpha <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}\right),\\
\alpha \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq -_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} |\alpha| \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\beta| \text{ iff } \left(\alpha <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}\right),\\
\alpha \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \triangleq -_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\alpha| \bullet_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\beta| \text{ iff } \left(\alpha\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}).
\end{gathered} \tag{2.121}$$

(ix) Suppose $\alpha \in {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \notin {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p}), \beta \in {}^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)$ The product $\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta$, is defined as
follows.

Case (1) $\alpha\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}, \beta\ _{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}$:

$$\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq$$
$$\left\{a \bullet_{^*\mathbb{Q}} b | 0^{\#^{\approx}_{\varepsilon}} <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} a^{\#^{\approx}_{\varepsilon}} <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha, 0^{\#^{\approx}_{\varepsilon}} <\ _{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} b^{\#^{\approx}_{\varepsilon}} <\ _{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta\right\} \bigcup \tag{2.122}$$
$$\bigcup \{^*\mathbb{R}^{\approx}_{-}(\varepsilon,\mathbf{p}) \cup \{^*0\}\}.$$

Case (2) $\alpha = 0^{\#^{\approx}_{\varepsilon}}$ or $\beta = 0^{\#^{\approx}_{\varepsilon}}$:

$$\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq 0^{\#^{\approx}_{\varepsilon}}. \tag{2.123}$$

Case (3) $\alpha <_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}$ or $\beta <_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}$:

$$\begin{gathered}
\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq |\alpha| \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}),\\
\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\alpha| \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha <\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} 0^{\#^{\approx}_{\varepsilon}}\right) \wedge \left(\beta\ _{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} > 0^{\#^{\approx}_{\varepsilon}}\right),\\
\alpha \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \beta \triangleq -_{\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} |\alpha| \breve{\bullet}_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} |\beta| \text{ iff } \left(\alpha\ _{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#^{\approx}_{\varepsilon}}\right) \wedge (\beta <_{^*\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} 0^{\#^{\approx}_{\varepsilon}}).
\end{gathered} \tag{2.124}$$

Such embeding ${}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$ into ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ as required above we will name Wattenberg embeding and is denoted by ${}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}) \overset{\#^{\approx}_{\varepsilon}}{\hookrightarrow} {}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$

Theorem 2.56.${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ is complete ordered semiring.

Proof.Immediately from definitions.

Remark 2.21.The following element of ${}^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ will be particularly useful for examples,

$$\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) \triangleq \{^*\mathbb{Q}^{\approx}_{-}(\varepsilon,\mathbf{p})\} \cup \mu_{\varepsilon}(0). \tag{2.125}$$

Examples.Note,for importent examples, that:

$$\begin{gathered}
\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) +_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})),\\
\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) +_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} (-_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))) = -_{^*\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})).
\end{gathered} \tag{2.126}$$

2.16.3. Absorption numbers in ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$

Definition 2.34. Suppose $\alpha \in {}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$, then

$$\mathbf{ab.p.}(\alpha) \triangleq \left\{d\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \geq 0^{\#\tilde{\varepsilon}} | \forall x_{x\in\alpha}[x +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} d \in \alpha]\right\}. \quad (2.127)$$

Examples.

(i) $\forall a \in {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}) : \mathbf{ab.p.}(a^{\#\tilde{\varepsilon}}) = 0^{\#\tilde{\varepsilon}}$,

(ii) $\mathbf{ab.p.}(\breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$,

(iii) $\mathbf{ab.p.}(-_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx})) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx})$,

(iv) $\forall \alpha \in {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}) : \mathbf{ab.p.}(\alpha^{\#\tilde{\varepsilon}} +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$,

(v) $\forall \alpha \in {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}) : \mathbf{ab.p.}(\alpha^{\#\tilde{\varepsilon}} -_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))) = \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$.

Theorem 2.57.

(i) $c <_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \mathbf{ab.p.}(\alpha)$ and $0 \leq_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} d <_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} c \Rightarrow d \in \mathbf{ab.p.}(\alpha)$

(ii) $c \in \mathbf{ab.p.}(\alpha)$ and $d \in \mathbf{ab.p.}(\alpha) \Rightarrow c +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} d \in \mathbf{ab.p.}(\alpha)$.

Remark 2.22. By Theorem 2.57 $\mathbf{ab.p.}(\alpha)$ may be regarded as an element of ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ by adding on all negative elements of ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ to $\mathbf{ab.p.}(\alpha)$. Of course if the condition $d\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \geq 0^{\#\tilde{\varepsilon}}$ in the definition of $\mathbf{ab.p.}(\alpha)$ is deleted we automatically get all the negative elements to be in $\mathbf{ab.p.}(\alpha)$ since $x <\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} y \in \alpha \Rightarrow x \in \alpha$. The reason for our definition is that the real interest lies in the non-negative numbers. A technicality occurs if $\mathbf{ab.p.}(\alpha) = \{0^{\#\tilde{\varepsilon}}\}$. We then identify $\mathbf{ab.p.}(\alpha)$ with $0^{\#\tilde{\varepsilon}}$.

Remark 2.23. By Theorem 2.57(ii), $\mathbf{ab.p.}(\alpha)$ is additive idempotent.

Theorem 2.58.

(i) $\mathbf{ab.p.}(\alpha)$ is the maximum element $\beta \in {}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$ such that $\alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta = \alpha$.

(ii) $\mathbf{ab.p.}(\alpha) \leq_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha$ for $\alpha\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#\tilde{\varepsilon}}$.

(iii) If α is positive and idempotent then $\mathbf{ab.p.}(\alpha) = \alpha$.

Theorem 2.59.Let $\alpha \in {}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon)$ satsify $\alpha > 0^{\#\tilde{\varepsilon}}$. Then the following are equivalent. In what follows assume $a,b\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} > 0^{\#\tilde{\varepsilon}}$.

(i) α is idempotent,

(ii) $a,b \in \alpha \Rightarrow a +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} b \in \alpha$,

(iii) $a \in \alpha \Rightarrow 2 \bullet_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} a \in \alpha$,

(iv) $\forall n_{n\in\mathbb{N}}[a \in \alpha \Rightarrow n \bullet_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} a \in \alpha]$,

(v) $a \in \alpha \Rightarrow q \bullet_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} a \in \alpha$, for all finite $q \in {}^{*}\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$.

Theorem 2.60. $(-_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha) +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha = -_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} [\mathbf{ab.p.}(\alpha)]$.

Theorem 2.61. $\mathbf{ab.p.}(\alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta)\ _{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \geq \mathbf{ab.p.}(\alpha)$.

Theorem 2.62.

(i) $\alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \leq_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \gamma \Rightarrow -_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \mathbf{ab.p.}(\alpha) +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta \leq_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \gamma$.

(ii) $\alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta = \alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \gamma \Rightarrow -_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} [\mathbf{ab.p.}(\alpha)] +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta =$
$= -_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} [\mathbf{ab.p.}(\alpha)] +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \gamma$.

Theorem 2.63.Suppose $\alpha,\beta \in {}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$, then

(i) $\mathbf{ab.p.}(-_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \alpha) = \mathbf{ab.p.}(\alpha)$,

(ii) $\mathbf{ab.p.}(\alpha +_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta) = \max\{\mathbf{ab.p.}(\alpha), \mathbf{ab.p.}(\beta)\}$

Theorem 2.44. Assume $\beta > 0$. If α absorbs $-_{{}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \beta$ then α absorbs β.

Theorem 2.45. Let $0 < \alpha \in {}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$. Then the following are equivalent

(i) α is an idempotent,
(ii) $(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha)+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha)=-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha,$
(iii) $(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha)+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\alpha=-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha.$
(iv) Let Δ_1 and Δ_2 be two positive idempotents such that $\Delta_2\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>\Delta_1$.
Then $\Delta_2+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\Delta_1)=\Delta_2.$

2.17.Gonshor's types of $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ with given $\mathbf{ab.p.}(\alpha)$.

Among elements of $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ such that $\mathbf{ab.p.}(\alpha)=\Delta$ one can distinguish two many different types.

Definition 2.36.Assume $\Delta\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>0^{\#_{\varepsilon}^{\approx}}$.
(i) $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type 1 if $\exists x(x\in\alpha)\forall y[x+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}y\in\alpha\Rightarrow y\in\Delta]$,
(ii) $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type 2 if $\forall x(x\in\alpha)\exists y(y\notin\Delta)[x+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}y\in\alpha]$,i.e. $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type 2 iff α does not have type 1.
(iii) $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type $1\mathbf{A}$ if $\exists x(x\notin\alpha)\forall y[x-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}y\notin\alpha\Rightarrow y\in\Delta]$

Theorem 2.47.
(i) $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type 1 iff $-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\,\alpha$ has type $1\mathbf{A}$,
(ii) $\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ cannot have type 1 and type $1\mathbf{A}$ simultaneously.
(iii) Suppose $\mathbf{ab.p.}(\alpha)=\Delta\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>0^{\#_{\varepsilon}^{\approx}}$. Then α has type 1 iff α has the form $a^{\#_{\varepsilon}^{\approx}}+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$ for some $a\in{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$
(iv) Suppose $\mathbf{ab.p.}(\alpha)=-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta,\Delta\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}>0^{\#_{\varepsilon}^{\approx}}.\alpha\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ has type $1\mathbf{A}$ iff α has the form $a^{\#_{\varepsilon}^{\approx}}+(-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta)$ for some $a\in{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$.
(v) If $\mathbf{ab.p.}(\alpha)\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>\mathbf{ab.p.}(\beta)$ then $\alpha+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\beta$ has type 1 iff α has type 1.
(vi) If $\mathbf{ab.p.}(\alpha)=\mathbf{ab.p.}(\beta)$ then $\alpha+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\beta$ has type 2 iff either α or β has type 2.

Proof. (iii) Let $\alpha=a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$. Then $\mathbf{ab.p.}(\alpha)=\Delta$.Since $\Delta{}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>0,a\in a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$ (we chose $d\in\Delta$ such that $0<_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d$ and write a as $(a-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d)+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d$).
It is clear that a works to show that α has type 1.
Conversely, suppose α has type 1 and choose $a\in\alpha$ such that:
$\forall y[a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}y\in\alpha\Rightarrow y\in\Delta]$.Then we claim that: $\alpha=a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$.
By definition of $\mathbf{ab.p.}(\alpha)$ certainly $a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta\leq_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\alpha$. On the other hand by choice of a,every element of α has the form $a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d$ with $d\in\Delta$.
Choose $d'\in\Delta$ such that $d'{}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>d$, then $a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d=$
$\left[a-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\left(d'-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}d\right)\right]+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}d'\in a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$.
Hence $\alpha\leq_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}a+_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$.Therefore $\alpha=a+{}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\Delta$.

Examples. (i) $\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has type 1 and therefore $-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has type $1\mathbf{A}$. Note that also $-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has type 2. (ii) Suppose $\varepsilon\approx0,\varepsilon\in{}^*\mathbb{R}$. Then $\varepsilon^{\#_{\varepsilon}^{\approx}}\bullet_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has type 1 and therefore $-_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\varepsilon^{\#_{\varepsilon}^{\approx}}\bullet\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has type $1\mathbf{A}$.

2.18.The Special Kinds of Idempotents in ${}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$

Let $a\in{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}),a\ {}_{^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>0^{\#_{\varepsilon}^{\approx}}$.Then a gives rise to two idempotents $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$, $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ in a natural way

$$\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))\triangleq\{x\in({}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))|\exists n(n\in\mathbb{N})[x\leq_{^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}({}^*n)\bullet_{^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}a]\},\qquad(2.128)$$

and

$$\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) \triangleq \{x\in({}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))|\forall r(r\in\mathbb{Q}_+)[x\leq_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}({}^*r)\bullet_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})} a]\}. \quad (2.129)$$

Remark 2.20. It is immediate that $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ and $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ are idempotents.It is also clear that $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ is the smallest idempotent containing a and $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ is the largest idempotent not containing a. It follows that $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ and $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ are consecutive idempotents.Note that $\mathbf{B}_1^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))=\breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$.

Theorem 2.53.(i) No idempotent of the form $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ has an immediate successor.

(ii) All consecutive pairs of idempotents have the form $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ and $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ for some $a\in{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}), a\ {}_{{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})}>0^{\#\widetilde{\varepsilon}}$.

Proof.(i) Let $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))\subsetneqq\Delta$. Suppose $x\in\Delta$ but $x\notin\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$. Then $x_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}>n\bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} a$ for all positive standard integers n.Let $y=\sqrt{x\bullet a}$ which is defined since ${}^*\mathbb{R}$ is a nonstandard model of $\mathbb{R}$. Then $y\geq a\sqrt{n}$ for all positive standard integers n so that $y\notin\mathbf{A}_a$. So $\mathbf{A}_y>\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$. Similarly $x>y\sqrt{n}$ so $x\notin\mathbf{A}_y$. Hence $\mathbf{A}_y<\Delta$.Thus $\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ and Δ are not consecutive.

(ii) Let $\mathbf{C}$ and $\mathbf{D}$ be consecutive idempotents such that $\mathbf{C}<_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\mathbf{D}$. Let $a\in D$ with $a\notin\mathbf{C}$. Then $\mathbf{C}<_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))<_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))<_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}\mathbf{D}$.Hence $\mathbf{C}=\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ and $\mathbf{D}=\mathbf{A}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$.

Theorem 2.54.If $\mathbf{ab.p.}(\alpha)$ has the form $\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ then α has type 1 or $1\mathbf{A}$.

Proof.Incidentally, we already know that in general $\mathbf{ab.p.}(\alpha)$ cannot have type 1 and $1\mathbf{A}$ simultaneously. Let $d\in{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)$ and $d\notin{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$, then we write

$$d\ \breve{\in}\ \alpha\Leftrightarrow\exists q(q\in\alpha)[d<_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}q]. \quad (2.130)$$

Now $a\notin\mathbf{B}_a^{\approx}(\varepsilon;{}^*\mathbb{Q}^{\approx}(\varepsilon))$ and therefore $\exists b(b\in\alpha)\exists c(c\notin\alpha)[a=c-_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})}b]$.We now define an ordinary Dedekind cut L_b for the real numbers $\mathbb{R}$, where L_b is the set of lower elements, as follows. Let $r\in L_b$ iff $b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}r^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a\ \breve{\in}\ \alpha$.It is immediate that $0\in L_b, 1\notin L_b,\ z<y\in L_b\Rightarrow z\in L_b$. So we have a Dedekind cut. Then L_b has a maximum or L_b' has a minimum. Suppose first that L_b has a maximum $\bar{r}=r_{\max}$. Then $b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a\ \breve{\in}\ \alpha$ and therefore by definition 2.130, there exist $q\triangleq q(\bar{r})\in\alpha$ such that

$$b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a<_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}q(\bar{r}), \quad (2.131)$$

but for any real $s>r,\ b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}s^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a\ \breve{\notin}\ \alpha$.We now claim that $b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a$ works to show that α has type 1.In fact, suppose $x\in{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$ and $q(\bar{r})+_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}x\in\alpha$ then from enequality (2.131) follows that

$$b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}x<_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}q(\bar{r})+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}x. \quad (2.132)$$

and therefore $b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}x\ \breve{\in}\ \alpha$.Let $s>\bar{r}$. Since $b+_{+{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}s^{\#\widetilde{\varepsilon}}a\ \breve{\notin}\ \alpha$,

$$b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}s^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a>q(\bar{r})+_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon)}x>b+_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}\bar{r}^{\#\widetilde{\varepsilon}}\bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}a+_{+{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)}x. \quad (2.133)$$

Therefore $x < (s^{\#_{\tilde{\varepsilon}}^{\approx}} -_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} \bar{r}^{\#_{\tilde{\varepsilon}}^{\approx}}) \bullet_{{}^*\mathbb{R}_{\mathbf{d}}^{\approx}(\varepsilon)} a.$ Thus $x < \delta^{\#_{\tilde{\varepsilon}}^{\approx}} \bullet \alpha$ for every positive real $\delta \in \mathbb{Q}_+$; i.e. $x \in \mathbf{B}_a^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$, since $x \in {}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})$. A similar argument shows that α has type $1\mathbf{A}$ if L_b' has a minimum.

Examples.(i)The result applies to $\mathbf{B}_{\varepsilon}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$. It follows from Theorem 2.54 that every α with $\mathbf{ab.p.}(\alpha) = \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ must be either of the form $a^{\#_{\tilde{\varepsilon}}^{\approx}} +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ or $a^{\#_{\tilde{\varepsilon}}^{\approx}} +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} (-_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})))$ with $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$, $a > 0$.

(ii)The result applies to $\mathbf{B}_{\varepsilon_1}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})), \varepsilon_1 \approx 0$; i.e. $\mathbf{B}_{\varepsilon}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})) = \varepsilon_1 \bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$. It follows from Theorem 2.54 that every α with $\mathbf{ab.p.}(\alpha) = \varepsilon_1 \bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q})$ must be either of the form $a^{\#_{\tilde{\varepsilon}}^{\approx}} +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \varepsilon_1^{\#_{\tilde{\varepsilon}}^{\approx}} \bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))$ or $a^{\#_{\tilde{\varepsilon}}^{\approx}} +_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \left(-_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \varepsilon_1^{\#_{\tilde{\varepsilon}}^{\approx}} \bullet_{{}^*\mathbb{Q}_{\mathbf{d}}^{\approx}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}_{\mathbf{d}}^{\approx}(\varepsilon; {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p}))\right)$ with $a \in {}^*\mathbb{Q}^{\approx}(\varepsilon,\mathbf{p})$, $a > 0$.

3. The proof of the #-transcendence of the numbers $e^k, k \in \mathbb{N}$.

In this section we will prove the #-transcendence of the numbers $e^k, k \in \mathbb{N}$. Key idea of this proof reduction of the statement of e is #-transcendental number to equivalent statement in ${}^*\mathbb{Z}_{\mathbf{d}}$ by using pseudoring of Wattenberg hyperreals ${}^*\mathbb{R}_{\mathbf{d}} \supset {}^*\mathbb{Z}_{\mathbf{d}}$ [6] and Gonshor idempotent theory [7]. We obtain this reduction by three steps, see subsections 3.2.1-3.2.3.

3.1. The basic definitions of the Shidlovsky quantities

In this section we remind the basic definitions of the Shidlovsky quantities [8].Let $M_0(n,p), M_k(n,p)$ and $\varepsilon_k(n,p)$ be the Shidlovsky quantities:

$$M_0(n,p) = \int_0^{+\infty} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx \neq 0, \tag{3.1}$$

$$M_k(n,p) = e^k \int_k^{+\infty} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx, k = 1,2,\ldots \tag{3.2}$$

$$\varepsilon_k(n,p) = e^k \int_0^{k} \left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!} \right] dx, k = 1,2,\ldots \tag{3.3}$$

where $p \in \mathbb{N}$ this is any prime number.Using Eqs.(3.1)-(3.3.) by simple calculation one obtains:

$$M_k(n,p) + \varepsilon_k(n,p) = e^k M_0(n,p) \neq 0, k = 1,2,\ldots. \tag{3.4}$$

and consequently

$$\begin{cases} e^k = \dfrac{M_k(n,p) + \varepsilon_k(n,p)}{M_0(n,p)} \\ \quad k = 1,2,\ldots \end{cases} \tag{3.5}$$

Lemma 3.1.[8]. Let p be a prime number. Then $M_0(n,p) = (-1)^n (n!)^p + p\Theta_1, \Theta_1 \in \mathbb{Z}$.

Proof. ([8], p.128) By simple calculation one obtains the equality

$$\begin{cases} x^{p-1}[(x-1)\dots(x-n)]^p = (-1)^n (n!)^p x^{p-1} + \sum\limits_{\mu=p+1}^{(n+1)\times p} c_{\mu-1} x^{\mu-1}, \\ c_\mu \in \mathbb{Z}, \mu = p, p+1, \dots, [(n+1)\times p]-1, n>0, \end{cases} \quad (3.6)$$

where p is a prime. By using equality $\Gamma(\mu) = \int_0^\infty x^{\mu-1} e^{-x} dx = (\mu-1)!$, where $\mu \in \mathbb{N}$, from Eq.(3.1) and (3.6) one obtains

$$\begin{cases} M_0(n,p) = (-1)^n (n!)^p \frac{\Gamma(p)}{(p-1)!} + \sum\limits_{\mu=p+1}^{(n+1)\times p} c_{\mu-1} \frac{\Gamma(\mu)}{(p-1)!} = \\ = (-1)^n (n!)^p + c_p p + c_{p+1} p(p+1) + \dots = \\ = (-1)^n (n!)^p + p \times \Theta_1, \Theta_1 \in \mathbb{Z}. \end{cases} \quad (3.7)$$

Thus

$$M_0(n,p) = (-1)^n (n!)^p + p \cdot \Theta_1(n,p), \Theta_1(n,p) \in \mathbb{Z}. \quad (3.8)$$

Lemma 3.2.[8]. Let p be a prime number. Then $M_k(n,p) = p \cdot \Theta_2(n,p)$, $\Theta_2(n,p) \in \mathbb{Z}$, $k = 1, 2, \dots, n$.

Proof.([8], p.128) By subsitution $x = k + u \Rightarrow dx = du$ from Eq.(3.3) one obtains

$$\begin{cases} M_k(n,p) = \int\limits_0^{+\infty} \left[\frac{(u+k)^{p-1}[(u+k-1)\times\dots\times u\times\dots\times(u+k-n)]^p e^{-u}}{(p-1)!} \right] du \\ k = 1, 2, \dots \end{cases} \quad (3.9)$$

By using equality

$$\begin{cases} (u+k)^{p-1}[(u+k-1)\times\dots\times u\times\dots\times(u+k-n)]^p = \sum\limits_{\mu=p+1}^{(n+1)\times p} d_{\mu-1} u^{\mu-1}, \\ d_\mu \in \mathbb{Z}, \mu = p, p+1, \dots, [(n+1)\times p]-1, \end{cases} \quad (3.10)$$

and by subsitution Eq.(3.10) into RHS of the Eq.(3.9) one obtains

$$M_k(n,p) = \frac{1}{(p-1)!} \int\limits_0^{+\infty} \sum\limits_{\mu=p+1}^{(n+1)\times p} d_{\mu-1} u^{\mu-1} du = p \cdot \Theta_2(n,p), \quad (3.11)$$
$$\Theta_2(n,p) \in \mathbb{Z}, k = 1, 2, \dots .$$

Lemma 3.3.[8]. (i) There exists sequences $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ such that

$$|\varepsilon_k(n,p)| \leq \frac{n \cdot g(n) \cdot [a(n)]^{p-1}}{(p-1)!}, \quad (3.12)$$

where sequences $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ does not depend on number p. (ii) For any $n \in \mathbb{N}$: $\varepsilon_k(n,p) \to 0$ if $p \to \infty$.

Proof.([8], p.129) Obviously there exists sequences $a(n), n \in \mathbb{N}$ and $g(n), k \in \mathbb{N}, n \in \mathbb{N}$

such that $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ does not depend on number p

$$|x(x-1)\ldots(x-n)| < a(n), 0 \leq x \leq n \tag{3.13}$$

and

$$|(x-1)\ldots(x-n)e^{-x+k}| < g(n), 0 \leq x \leq n, k = 1,2,\ldots,n. \tag{3.14}$$

Substitution inequalities (3.13)-(3.14) into RHS of the Eq.(3.3) by simple calculation gives

$$\varepsilon_k(n,p) \leq g(n)\frac{[a(n)]^{p-1}}{(p-1)!}\int_0^k dx \leq \frac{n \cdot g(n) \cdot [a(n)]^{p-1}}{(p-1)!}. \tag{3.15}$$

Statement (i) follows from (3.15). Statement (ii) immediately follows from a statement (ii).

Lemma 3.4.[8]. For any $k \leq n$ and for any δ such that $0 < \delta < 1$ there exists $p \in \mathbb{N}$ such that

$$\left| e^k - \frac{M_k(n,p)}{M_0(n,p)} \right| < \delta. \tag{3.16}$$

Proof.From Eq.(3.5) one obtains

$$\left| e^k - \frac{M_k(n,p)}{M_0(n,p)} \right| = \frac{|\varepsilon_k(n,p)|}{M_0(n,p)}. \tag{3.17}$$

From Eq.(3.17) by using Lemma 3.3.(ii) one obtains (3.17).

Remark 3.1.We remind now the proof of the transcendence of e following Shidlovsky proof is given in his book [8].

Theorem 3.1. The number e is transcendental.

Proof.([8], pp.126-129) Suppose now that e is an algebraic number; then it satisfies some relation of the form

$$a_0 + \sum_{k=1}^{n} a_k e^k = 0, \tag{3.18}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ integers and where $a_0 > 0$. Having substituted RHS of the Eq.(3.5) into Eq.(3.18) one obtains

$$a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p) + \varepsilon_k(n,p)}{M_0(n,p)} = a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p)}{M_0(n,p)} + \sum_{k=1}^{n} a_k \frac{\varepsilon_k(n,p)}{M_0(n,p)} = 0. \tag{3.19}$$

From Eq.(3.19) one obtains

$$a_0 M_0(n,p) + \sum_{k=1}^{n} a_k M_k(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = 0. \tag{3.20}$$

We rewrite the Eq.(3.20) for short in the form

$$\begin{cases} a_0M_0(n,p) + \sum_{k=1}^{n} a_kM_k(n,p) + \sum_{k=1}^{n} a_k\varepsilon_k(n,p) = \\ = a_0M_0(n,p) + \Xi(n,p) + \sum_{k=1}^{n} a_k\varepsilon_k(n,p) = 0, \\ \Xi(n,p) = \sum_{k=1}^{n} a_kM_k(n,p). \end{cases} \quad (3.21)$$

We choose now the integers $M_1(n,p), M_2(n,p),\ldots,M_n(n,p)$ such that:

$$\begin{cases} p|M_1(n,p), p|M_2(n,p),\ldots,p|M_n(n,p) \\ \text{where } p > |a_0| \end{cases} \quad (3.22)$$

and $p \nmid M_0(n,p)$. Note that $p|\ \Xi(n,p)$. Thus one obtains

$$p \nmid a_0M_0(n,p) + \Xi(n,p) \quad (3.23)$$

and therefore

$$\begin{cases} a_0M_0(n,p) + \Xi(n,p) \in \mathbb{Z}, \\ \text{where} \\ a_0M_0(n,p) + \Xi(n,p) \neq 0. \end{cases} \quad (3.24)$$

By using Lemma 3.4 for any δ such that $0 < \delta < 1$ we can choose a prime number $p = p(\delta)$ such that:

$$\left|\sum_{k=1}^{n} a_k\varepsilon_k(n,p)\right| < \delta \sum_{k=1}^{n} |a_k| = \epsilon < 1. \quad (3.25)$$

From (3.25) and Eq.(3.21) we obtain

$$a_0M_0(n,p) + \Xi(n,p) + \epsilon = 0. \quad (3.26)$$

From (3.26) and Eq.(3.24) one obtains the contradiction.This contradiction finalized the proof.

3.2 The proof of the #-transcendence of the numbers $e^k, k \in \mathbb{N}$. We will divide the proof into four parts

3.2.1. Part I.The Robinson transfer of the Shidlovsky quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$

In this subsection we will replace using Robinson transfer the Shidlovsky quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$ by corresponding nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p})$, ${}^*\varepsilon_k(\mathbf{n},\mathbf{p})$. The properties of the nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p})$, ${}^*\varepsilon_k(\mathbf{n},\mathbf{p})$ one obtains directly from the properties of the standard quantities $M_0(n,p), M_k(n,p), \varepsilon_k(n,p)$ using Robinson transfer principle [4],[5].

1.Using Robinson transfer principle [4],[5] from Eq.(3.8) one obtains directly

$$\begin{cases} {}^*M_0(\mathbf{n},\mathbf{p}) = (-1)^{\mathbf{n}}(\mathbf{n}!)^{\mathbf{p}} + \mathbf{p} \times {}^*\Theta_1(\mathbf{n},\mathbf{p}), \\ {}^*\Theta_1(\mathbf{n},\mathbf{p}) \in {}^*\mathbb{Z}_\infty, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \\ \mathbb{N}_\infty \triangleq {}^*\mathbb{N}\backslash\mathbb{N}. \end{cases} \quad (3.27)$$

From Eq.(3.11) using Robinson transfer principle one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*M_k(\mathbf{n},\mathbf{p}) = \mathbf{p} \times \left({}^*\Theta_2(\mathbf{n},\mathbf{p})\right), \\ {}^*\Theta_2(\mathbf{n},\mathbf{p}) \in {}^*\mathbb{Z}_\infty, k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.28)$$

Using Robinson transfer principle from inequality (3.15) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*\varepsilon_k(\mathbf{n},\mathbf{p}) \leq \dfrac{\mathbf{n} \cdot ({}^*g(\mathbf{n})) \cdot \left([{}^*a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.29)$$

Using Robinson transfer principle, from Eq.(3.5) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*(e^k) = ({}^*e)^k = \dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} + \dfrac{{}^*\varepsilon_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.30)$$

Lemma 3.5. Let $\mathbf{n} \in {}^*\mathbb{N}_\infty$, then for any $k \in \mathbb{N}$ and for any $\delta \approx 0, \delta \in {}^*\mathbb{R}$ there exists $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\left| {}^*e^k - \frac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} \right| < \delta. \quad (3.31)$$

Proof. From Eq.(3.30) we obtain $\forall k(k \in \mathbb{N})$:

$$\begin{cases} \left| {}^*e^k - \dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} \right| = \dfrac{|{}^*\varepsilon_k(\mathbf{n},\mathbf{p})|}{|{}^*M_0(\mathbf{n},\mathbf{p})|}, \\ k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \quad (3.32)$$

From Eq.(3.32) and (3.29) we obtain (3.31).

3.2.2. Part II.The Wattenberg imbedding ${}^*(e^k)$ into ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$

In this subsection we will replace by using Wattenberg imbedding [6] and Gonshor tipe transfer of the nonstandard quantities ${}^*(e^k)$ and the nonstandard Shidlovsky quantities from ${}^*\mathbb{N}_\infty$ and from ${}^*\mathbb{R}$:

$${}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p}), {}^*\varepsilon_k(\mathbf{n},\mathbf{p})$$

by corresponding Wattenberg quantities from ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ and from ${}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathbf{q})$:

$$({}^*(e^k))^{\#_\epsilon}, ({}^*M_0(\mathbf{n},\mathbf{p}))^{\#_{\epsilon,\mathbf{q}}}, ({}^*M_k(\mathbf{n},\mathbf{p}))^{\#_{\epsilon,\mathbf{q}}}, ({}^*\varepsilon_k(\mathbf{n},\mathbf{p}))^{\#_\epsilon}.$$

The properties of the Wattenberg tipe quantities ${}^*(e^k)^{\#_\epsilon}, ({}^*M_0(\mathbf{n},\mathbf{p}))^{\#_{\epsilon,\mathbf{q}}}, ({}^*M_k(\mathbf{n},\mathbf{p}))^{\#_{\epsilon,\mathbf{q}}}$, $({}^*\varepsilon_k(\mathbf{n},\mathbf{p}))^{\#_\epsilon}$ one obtains directly from the properties of the corresponding nonstandard quantities ${}^*(e^k), {}^*M_0(\mathbf{n},\mathbf{p}), {}^*M_k(\mathbf{n},\mathbf{p}), {}^*\varepsilon_k(\mathbf{n},\mathbf{p})$ using Gonshor transfer principle [4],[7].

1.By using Wattenberg imbedding ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon) \overset{\#_\epsilon}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$, from Eq.(3.30) one obtains

$$\begin{cases} [{}^*(e^k)]^{\#_\epsilon} = \left[({}^*e)^{\#_\epsilon}\right]^k = \left[\dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right]^{\#_\epsilon} + \left[\dfrac{{}^*\varepsilon_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right]^{\#_\epsilon}, \\ \left[\dfrac{{}^*\varepsilon_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right]^{\#_\epsilon} \subsetneqq \epsilon \times \varepsilon_{\mathbf{d}}, \\ k = 1,2,\ldots; k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty, \epsilon = \epsilon(\mathbf{p}) \end{cases} \tag{3.33}$$

2.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer (see subsection 2.9 Theorem 2.19) from Eq.(3.27) one obtains

$$\begin{cases} [{}^*M_0(\mathbf{n},\mathbf{p})]^{\#_{\epsilon,\mathbf{q}}} = [(-1)^{\mathbf{n}}]^{\#} \times [(\mathbf{n}!)^{\mathbf{p}}]^{\#} + \mathbf{p}^{\#} \times [{}^*\Theta_1(\mathbf{n},\mathbf{p})]^{\#} = \\ = \left[(-1^{\#})^{\mathbf{n}^{\#}}\right] \times \left[\left((\mathbf{n}!)^{\#}\right)^{\mathbf{p}^{\#}}\right] + \mathbf{p}^{\#} \times [{}^*\Theta_1(\mathbf{n},\mathbf{p})]^{\#}, \\ {}^*\Theta_1(\mathbf{n},\mathbf{p}) \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \tag{3.34}$$

3.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} ({}^{\#}\mathbb{R})$, from Eq.(3.28) one obtains

$$\begin{cases} [{}^*M_k(\mathbf{n},\mathbf{p})]^{\#} = \mathbf{p}^{\#} \times \left[{}^*\Theta_2(\mathbf{n},\mathbf{p})\right]^{\#}, \\ [{}^*\Theta_2(\mathbf{n},\mathbf{p})]^{\#} \in {}^*\mathbb{Z}_{\infty,\mathbf{d}}, \\ k = 1,2,\ldots, k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \tag{3.35}$$

Lemma 3.6. Let $\mathbf{n} \in {}^*\mathbb{N}_\infty$, then for any $k \in \mathbb{N}$ and for any $\delta \approx 0, \delta \in {}^*\mathbb{R}$ there exist $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\left|({}^*e^k)^{\#_\epsilon} - \left(\frac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right)^{\#_\epsilon}\right| \leq \delta^{\#_\epsilon} = \left(\frac{{}^*\varepsilon_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right)^{\#_\epsilon} = \left(\frac{\mathbf{n}\cdot({}^*g(\mathbf{n}))\cdot\left([{}^*a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}\right)^{\#_\epsilon} \tag{3.36}$$

Proof. Inequality (3.36) immediately follows from inequality (3.31) by using Wattenberg imbedding ${}^*\mathbb{R}_{\mathbf{st}}(\epsilon) \overset{\#_\epsilon}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{st.d}}(\epsilon)$ and Gonshor transfer.

3.2.3.Part III.Reduction of the statement of e is #-transcendental number to equivalent statement in ${}^*\mathbb{Z}_{\mathbf{d}}(\epsilon,\mathbf{q})$ using Gonshor idempotent theory

To prove that e is #-transcendental number we must show that e is not w-transcendental, i.e., there does not exist real $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n$ with rational coefficients $a_0, a_1, \ldots, a_n, \ldots \in \mathbb{Q}$ such that

$$\begin{cases} \sum_{n=0}^{\infty} a_k e^n = 0, \\ \sum_{n=0}^{\infty} |a_k| e^n \neq \infty. \end{cases} \tag{3.37}$$

Suppose that e is w-transcendental, i.e., there exists an $\mathbb{Q}$-analytic function

$\breve{g}_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} \breve{a}_n x^n$, with rational coefficients:

$$\begin{cases} \breve{a}_0 = \frac{k_0}{m_0}, \breve{a}_1 = \frac{k_1}{m_1}, \ldots, \breve{a}_n = \frac{k_n}{m_n}, \ldots \in \mathbb{Q}, \\ \left| \breve{a}_0 \right| > 0, \end{cases} \tag{3.38}$$

such that the equality is satisfied:

$$\begin{cases} \sum_{n=0}^{\infty} \breve{a}_n e^n = 0. \\ \sum_{n=0}^{\infty} |a_k| e^n \neq \infty. \end{cases} \tag{3.39}$$

In this subsection we obtain an reduction of the equality given by Eq.(3.39) to equivalent equality given by Eq.(3.). The main tool of such reduction that external countable sum defined in subsection 2.8.

Lemma 3.7.Let $\Delta_{\leq}(k)$ and $\Delta_{>}(k)$ be the sum correspondingly

$$\begin{cases} \Delta_{\leq}(k) = \breve{a}_0 + \sum_{n=1}^{k \geq 1} \breve{a}_n e^n, \\ \Delta_{>}(k) = \sum_{n=k+1}^{\infty} \breve{a}_n e^n. \end{cases} \tag{3.40}$$

Then $\Delta_{>}(k) \neq 0, k = 1, 2, \ldots$

Proof. Suppose there exists k such that $\Delta_{>}(k) = 0$. Then from Eq.(3.39) follows $\Delta_{\leq}(k) = 0$. Therefore by Theorem 3.1 one obtains the contradiction.

Remark 3.2.Note that from Eq.(3.39) follows that in generel case there is a sequence $\{m_i\}_{i=1}^{\infty}$ such that

$$\begin{gathered} \lim_{i \to \infty} m_i = \infty, \\ \forall (i \in \mathbb{N}) \left[\sum_{n=1}^{m_i} \breve{a}_n e^n < 0 \right], \\ \breve{a}_0 + \lim_{i \to \infty} \left(\sum_{n=1}^{m_i} \breve{a}_n e^n \right) = 0, \end{gathered} \tag{3.41}$$

or there is a sequence $\{m_j\}_{j=1}^{\infty}$ such that

$$\begin{cases} \lim_{i \to \infty} m_j = \infty, \\ \forall (j \in \mathbb{N}) \left[\sum_{n=1}^{m_j} \breve{a}_n e^n > 0 \right], \\ \breve{a}_0 + \lim_{j \to \infty} \left(\sum_{n=1}^{m_j} \breve{a}_n e^n \right) = 0, \end{cases} \tag{3.42}$$

or both sequences $\{m_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$ with a property that is specified above exist.

Remark 3.3. We assume now for short but without loss of generelity that (3.41) is satisfied. Then from (3.41) by using Definition 2.17 and Theorem 2.14 (see subsection 2.8) one obtains the equality [4]

$$({}^{*}\breve{a}_0)^{\#_\epsilon} +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon)} \left[\#_\epsilon Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon) \sum_{n\in\mathbb{N}}^{\wedge} ({}^{*}\breve{a}_n)^{\#_\epsilon} \times ({}^{*}e^n)^{\#_\epsilon} \right] = -\epsilon \times \varepsilon_{\mathbf{d}}. \quad (3.43)$$

Remark 3.4.Let $\Delta^{\#}_{\leq}(k)$ and $\Delta^{\#}_{>}(k)$ be the upper external sum defined by

$$\begin{cases} \Delta^{\#_\epsilon}_{\leq}(k) = \breve{a}_0 + \#_\epsilon Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon) \sum_{n=1}^{k\geq 1} ({}^{*}\breve{a}_n)^{\#} \times ({}^{*}e^n)^{\#}, \\ \Delta^{\#_\epsilon}_{>}(k) = \#_\epsilon Ext\text{-}{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon) \sum_{\substack{n\in\mathbb{N} \\ n\geq k+1}}^{\wedge} \breve{a}_n e^n. \end{cases} \quad (3.44)$$

Note that from Eq.(3.43)-Eq.(3.44) follows that

$$\Delta^{\#}_{\leq}(k) + \Delta^{\#}_{>}(k) = -\varepsilon_{\mathbf{d}}. \quad (3.45)$$

Remark 3.5. Assume that $\alpha, \beta \in {}^{*}\mathbb{R}_{\mathbf{d}}$ and $\beta \notin {}^{*}\mathbb{R}$. In this subsection we will write for a short $\mathbf{ab}[\alpha|\beta]$ iff β absorbs α, i.e. $\beta + \alpha = \beta$.

Lemma 3.8. $\neg\mathbf{ab}[\Delta^{\#}_{\leq}(k)|\Delta^{\#}_{>}(k)], k = 1,2,\ldots$

Proof.Suppose there exists $k \in \mathbb{N}$ such that $\mathbf{ab}[\Delta^{\#}_{\leq}(k)|\Delta^{\#}_{>}(k)]$.Then from Eq.(3.45) one obtains

$$\Delta^{\#}_{>}(k) = -\varepsilon_{\mathbf{d}}. \quad (3.46)$$

From Eq.(3.46) by Theorem 2.11 follows that $\Delta_{>}(k) = 0$ and therefore by Lemma 3.7 one obtains the contradiction.

Theorem 3.2.[4] The equality (3.43) is inconsistent.

Proof.Let us consider hypernatural number $\mathfrak{I} \in {}^{*}\mathbb{N}_{\infty}$ defined by countable sequence

$$\mathfrak{I} = (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) \quad (3.47)$$

From Eq.(3.43) and Eq.(3.47) one obtains

$$\mathfrak{I} \times_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathfrak{I})} ({}^{*}\breve{a}_0)^{\#_\epsilon} +_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathfrak{I})} \mathfrak{I} \times_{{}^{*}\mathbb{R}_{\mathbf{st.d}}(\epsilon,\mathfrak{I})} \left[\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} ({}^{*}\breve{a}_n)^{\#_\epsilon} \times ({}^{*}e^n)^{\#_\epsilon} \right] = \quad (3.48)$$
$$-\mathfrak{I}^{\#} \times \epsilon \times \varepsilon_{\mathbf{d}}.$$

Remark 3.6.Note that from inequality (3.27) by Wattenberg transfer one obtains

$$[{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#} \leq \frac{\mathbf{n}^{\#} \cdot [g_n(\mathbf{n})]^{\#} \cdot \left[[a(\mathbf{n})]^{\mathbf{p}-1}\right]^{\#}}{[(\mathbf{p}-1)!]^{\#}}, \quad (3.49)$$
$$n \in \mathbb{N}, \mathbf{n}, \mathbf{p} \in {}^{*}\mathbb{N}_{\infty}.$$

Substitution Eq.(3.30) into Eq.(3.48) gives

$$\begin{cases} \Im_0^{\#} + \left[\#Ext\text{-}\sum_{n\in\mathbb{N}\setminus\{0\}}^{\wedge} (\Im_n)^{\#} \times ({}^{*}e^{n})^{\#} \right] = \\ \Im_0^{\#} + \left[\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} (\Im_n)^{\#} \times \dfrac{[{}^{*}M_n(\mathbf{n},\mathbf{p})]^{\#} + [{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#}}{[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#}} \right] = -\Im^{\#} \times \varepsilon_{\mathbf{d}}, \\ \Im_n^{\#} \triangleq \Im^{\#} \times ({}^{*}\breve{a}_n)^{\#}, n \in \mathbb{N}, \Im_0^{\#} = \Im^{\#} \times ({}^{*}\breve{a}_0)^{\#}. \end{cases} \quad (3.50)$$

Multiplying Eq.(3.50) by Wattenberg hyperinteger $[{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \in {}^{*}\mathbb{Z}_{\mathbf{d}}$ by Theorem 2.13 (see subsection 2.8) one obtains

$$\begin{cases} \Im_0^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \left\{ (\Im_n)^{\#} \times [{}^{*}M_n(\mathbf{n},\mathbf{p})]^{\#} + \Im_k^{\#} \times [{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#} \right\} = \\ = -\Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (3.51)$$

By using inequality (3.49) for a given $\delta \in {}^{*}\mathbb{R},\ \delta \approx 0$ we will choose infinite prime integer $\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$ such that:

$$\#Ext\text{-}\sum_{k\in\mathbb{N}}^{\wedge} (\Im_k)^{\#} \times [{}^{*}\varepsilon_k(\mathbf{n},\mathbf{p})]^{\#} \subseteq \Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \delta^{\#} \times \varepsilon_{\mathbf{d}} \quad (3.52)$$

Now using the inequality (3.49) we are free to choose a prime hyperinteger $\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$ and $\delta^{\#} \in {}^{*}\mathbb{R}_{\mathbf{d}},\ \delta^{\#} = \delta^{\#}(\mathbf{p}) \approx 0$ in the Eq.(3.51) for a given $\epsilon \in {}^{*}\mathbb{R}, \epsilon \approx 0$ such that:

$$\Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \delta^{\#}(\mathbf{p}) = \epsilon^{\#}. \quad (3.53)$$

Hence from Eq.(3.52) and Eq.(3.53) we obtain

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} (\Im_n)^{\#} \times [{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#} \subseteq -\epsilon^{\#} \times \varepsilon_{\mathbf{d}}. \quad (3.54)$$

Therefore from Eq.(3.51) and (3.54) by using definition (2.15) of the function $\mathbf{Int.p}(\alpha)$ given by Eq.(2.20)-Eq.(2.21) and corresponding basic property **I** (see subsection 2.7) of the function $\mathbf{Int.p}(\alpha)$ we obtain

$$\begin{cases} \mathbf{Int.p}\left(\Im_0^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \left\{ \Im_n^{\#} \times [{}^{*}M_n(\mathbf{n},\mathbf{p})]^{\#} + \Im_n^{\#} \times [{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})]^{\#} \right\} \right) = \\ \Im_0^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{k\in\mathbb{N}}^{\wedge} \left\{ \Im_n^{\#} \times [{}^{*}M_n(\mathbf{n},\mathbf{p})]^{\#} \right\} = \\ = -\mathbf{Int.p}\left(\Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}} \right) = -\Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}. \end{cases} \quad (3.55)$$

From Eq.(3.55) using basic property **I** of the function $\mathbf{Int.p}(\alpha)$ finally we obtain the main equality

$$\Im_0^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} + \#Ext\text{-}\sum_{n\in\mathbb{N}}^{\wedge} \left\{ (\Im_k)^{\#} \times [{}^{*}M_n(\mathbf{n},\mathbf{p})]^{\#} \right\} = \Im^{\#} \times [{}^{*}M_0(\mathbf{n},\mathbf{p})]^{\#} \times \varepsilon_{\mathbf{d}}. \quad (3.56)$$

We will choose now infinite prime integer $\mathbf{p}$ in Eq.(3.56) $\mathbf{p} = \widehat{\mathbf{p}} \in {}^{*}\mathbb{N}_{\infty}$ such that

$$\widehat{\mathbf{p}}^{\#} > \mathbf{max}(|\Im_0^{\#}|, \mathbf{n}^{\#}.) \quad (3.57)$$

Hence from Eq.(3.34) follows

$$\widehat{\mathbf{p}}^{\#} \nmid [{}^{*}M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#}. \tag{3.58}$$

Note that $[{}^{*}M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#} \neq 0^{\#}$. Using (3.57) and (3.58) one obtains:

$$\widehat{\mathbf{p}}^{\#} \nmid [{}^{*}M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#} \times (\Im_0)^{\#}. \tag{3.59}$$

Using Eq.(3.35) one obtains

$$\widehat{\mathbf{p}}^{\#} \mid [{}^{*}M_n(\mathbf{n},\widehat{\mathbf{p}})]^{\#}, n = 1,2,\ldots. \tag{3.60}$$

3.2.4.Part IV.The proof of the inconsistency of the main equality (3.56)

In this subsection we wil prove that main equality (3.56) is inconsistent. This prooff is based on the Theorem 2.10 (v), see subsection 2.6.

Lemma 3.9.The equality (3.56) under conditions (3.59)-(3.60) is inconsistent.

Proof. (**I**) Let us rewrite Eq.(3.56) in the short form

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}, \tag{3.61}$$

where

$$\begin{cases} \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N}\\ n\geq 1}}^{\wedge} \left\{(\Im_n)^{\#} \times [{}^{*}M_n(\mathbf{n},\widehat{\mathbf{p}})]^{\#}\right\}, \\ \Gamma(\mathbf{n},\widehat{\mathbf{p}}) = \Im_0^{\#} \times [{}^{*}M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#}, \\ \Lambda^{\#}(\widehat{\mathbf{p}}) = \Im^{\#} \times [{}^{*}M_0(\mathbf{n},\widehat{\mathbf{p}})]^{\#}. \end{cases} \tag{3.62}$$

From (3.59)-(3.60) follows that

$$\begin{cases} \widehat{\mathbf{p}}^{\#} \nmid \Gamma(\mathbf{n},\widehat{\mathbf{p}}), \\ \widehat{\mathbf{p}}^{\#} \mid \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}). \end{cases} \tag{3.63}$$

Remark 3.7.Note that $\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) \notin {}^{*}\mathbb{R}$. Otherwise we obtain that $\mathbf{ab.p}(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})) = \{\varnothing\}$. But the other hand from Eq.(3.61) follows that $\mathbf{ab.p}(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}$. But this is a contradiction. This contradiction completed the proof of the statement (**I**)

(**II**) Let $\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}}), \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}), \widetilde{\Delta}^{\#}_{\leq}(k_1,k_2,\mathbf{n},\widehat{\mathbf{p}})$ and $\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^{\#}), \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon_n^{\#})$, be the external sum correspondingly

$$\begin{cases} \widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \sum_{n=1}^{k\geq 1}\left\{\Im^{\#}_{n} \times [{}^{*}M_{n}(\mathbf{n},\widehat{\mathbf{p}})]^{\#}\right\}, \\ \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum_{\substack{n\in\mathbb{N} \\ n\geq k+1}}^{\wedge}\left\{\Im^{\#}_{n} \times [{}^{*}M_{n}(\mathbf{n},\widehat{\mathbf{p}})]^{\#}\right\}, \\ \widetilde{\Delta}^{\#}_{\leq}(k_1,k_2,\mathbf{n},\widehat{\mathbf{p}}) = \sum_{n=k_1}^{k_2}\left\{\Im^{\#}_{n} \times [{}^{*}M_{n}(\mathbf{n},\widehat{\mathbf{p}})]^{\#}\right\}, \\ \widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \sum_{n=1}^{k\geq 1}\left\{\Im^{\#}_{n} \times [{}^{*}M_{n}(\mathbf{n},\widehat{\mathbf{p}})]^{\#} + \Im^{\#}_{n} \times [{}^{*}\varepsilon_{n}(\mathbf{n},\mathbf{p})]^{\#}\right\}, \\ \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = \#Ext\text{-}\sum_{\substack{n\in\mathbb{N} \\ n\geq k+1}}^{\wedge}\left\{\Im^{\#}_{n} \times [{}^{*}M_{n}(\mathbf{n},\widehat{\mathbf{p}})]^{\#} + \Im^{\#}_{n} \times [{}^{*}\varepsilon_{n}(\mathbf{n},\mathbf{p})]^{\#}\right\}, \end{cases} \quad (3.64)$$

Note that from Eq.(3.61) and Eq.(3.64) follows that

$$\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}}) + \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \quad (3.65)$$

Lemma 3.10. (**i**) Under conditions (3.59)-(3.60)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) \middle| \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n})\right], k = 1,2,\ldots \quad (3.66)$$

And (**ii**) Under conditions (3.59)-(3.60)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}}) \middle| \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}})\right], k = 1,2,\ldots \quad (3.67)$$

Proof. (**i**) First note that under conditions (3.59)-(3.60) one obtains

$$\forall k\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) \neq 0\right] \quad (3.68)$$

Suppose that there exists a $k \geq 0$ such that $\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) \middle| \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n})\right]$. Then from Eq.(3.65) one obtains

$$\widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \quad (3.69)$$

From Eq.(3.69) by Theorem 2.17 one obtains

$$\begin{gathered} -\varepsilon_{\mathbf{d}} = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1} \times \widetilde{\Delta}^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}) = \\ = \Delta^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}). \end{gathered} \quad (3.70)$$

Thus

$$-\varepsilon_{\mathbf{d}} = \Delta^{\#}_{>}(k,\mathbf{n},\widehat{\mathbf{p}}). \quad (3.71)$$

From Eq.(3.71) by Theorem 2.11 follows that $\Delta_{>}(k) = 0$ and therefore by Lemma 3.7 one obtains the contradiction. This contradiction finalized the proof of the Lemma 3.10 (i).

Proof. (**ii**) This is immediate from the Definition 2.14 (**Property I**), see subsection 2.7.

Part (III)

Remark 3.8.(**i**) Note that from Eq.(3.62) by Theorem 2.10 (v) follws that $\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})$ has the form

$$\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \mathbf{q}^{\#} + \mathbf{ab}.\mathbf{p}(\Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}})) = \\ = \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}), \tag{3.72}$$

where

$$\mathbf{q}^{\#} \in \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \widetilde{\Delta}^{\#}_{>}(1,\mathbf{n},\widehat{\mathbf{p}}), \\ \mathbf{q} \in {}^{*}\mathbb{Z}_{\infty} \text{ and} \widehat{\mathbf{p}}|\mathbf{q}. \tag{3.73}$$

(**ii**) Substitution by Eq.(3.72) into Eq.(3.61) gives

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{n},\widehat{\mathbf{p}}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{3.74}$$

Remark 3.**9**. Note that from (3.74) by definitions follows that

$$\mathbf{ab}[(\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#})|(-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}})]. \tag{3.75}$$

From Eq.(3.74) follows that

$$\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \tag{3.79}$$

Therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) = -\varepsilon_{\mathbf{d}}. \tag{3.80}$$

From Eq.(3.80) obviously follows that

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \approx 0, \\ (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] \geq (\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1} \tag{3.81}$$

Now we dealing with semiring ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$,(see subsection 2.16.2).By consideration similarly

as above we obtain

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{n},\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#^{\approx}_{\varepsilon}}] \approx 0^{\#^{\approx}_{\varepsilon}}, \\ (\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{n},\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#}] \; {}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)1} \check{\geq} (\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \tag{3.82}$$

and

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{n},\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#^{\approx}_{\varepsilon}}] \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} (-_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{R}^{\approx}(\varepsilon))) = \\ = -_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{R}^{\approx}(\varepsilon)). \tag{3.83}$$

From inequality (3.36) follows that we willin to choose $\widehat{\mathbf{p}}$ and $\varepsilon \approx 0$ such that

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \notin \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon; {}^{*}\mathbb{R}^{\approx}(\varepsilon)). \tag{3.84}$$

But this is a contradiction. This contradiction completed the proof of the Lemma 3.9.

4.Generalized Lindemann-Weierstrass theorem

In this section we remind the basic definitions of the Shidlovsky quantities,see [8] p.132- 134.

Theorem 4.**1**.[8] Let $f_l(z), l = 1,2,\ldots,r$ be a polynomials with coefficients in $\mathbb{Z}$.Assume that

for any $l = 1,2,\ldots,r$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l}$,
$k_l \geq 1, l = 1,2,\ldots,r$

form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots,r \quad (4.1)$$

and $a_l \in \mathbb{Z}, l = 1,2,\ldots,r, a_0 \neq 0.$ Then

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \quad (4.2)$$

Let $f_r(z)$ be a polynomial such that

$$\begin{cases} f_r(z) = \prod_{l=1}^{r} f_l(z) = b_0 + b_1 z + \ldots + b_{N_r} z^{N_r} = \\ = b_{N_r} \prod_{l=1}^{r} \prod_{k=1}^{k_l} (z - \beta_{k,l}), b_0 \neq 0, b_N > 0, N_r = \sum_{l=1}^{r} k_l. \end{cases} \quad (4.3)$$

Let $M_0(N_r,p), M_{k,l}(N_r,p)$ and $\varepsilon_{k,l}(N_r,p)$ be the quantities [8]:

$$M_0(N_r,p) = \int_0^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \quad (4.4)$$

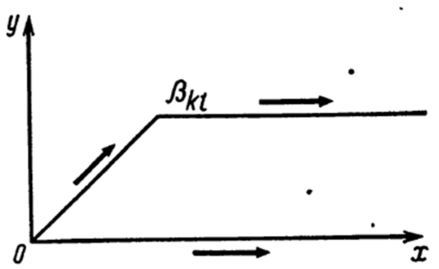
where in (4.4) we integrate in complex plane $\mathbb{C}$ along line $[0,+\infty]$, see Pic.1.

$$M_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_{\beta_{k,l}}^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \quad (4.5)$$

where $k = 1,\ldots,k_l$ and where in (4.5) we integrate in complex plane $\mathbb{C}$ along line with initial point $\beta_{k,l} \in \mathbb{C}$ and which are parallel to real axis of the complex plane $\mathbb{C}$, see Pic.1.

$$\varepsilon_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \quad (4.6)$$

where $k = 1,\ldots,k_l$ and where in (4.6) we integrate in complex plane $\mathbb{C}$ along contour $[0,\beta_{k,l}]$, see Pic.1.



Pic.1.Contour $[0,\beta_{k,l}]$ in complex plane $\mathbb{C}$.

From Eq.(4.3) one obtains

$$b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) = b_{N_r}^{(N_r-1)p-1} b_0^p z^{p-1} + \sum_{s=p+1}^{(N_r+1)p} c_{s-1} z^{s-1}, \quad (4.7)$$

where $b_{N_r}b_0 \neq 0, c_s \in \mathbb{Z}, s = p,\ldots,(N_r-1)p-1.$ Now from Eq.(4.4) and Eq.(4.7) using formula

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx = (s-1)!, s \in \mathbb{N}$$

one obtains

$$\begin{cases} M_0(N_r,p) = \dfrac{b_{N_r}^{(N_r-1)p-1}b_0^p}{(p-1)!}\displaystyle\int_0^{+\infty} z^{p-1}e^{-z}dz + \sum_{s=p+1}^{(N_r+1)p}\dfrac{c_{s-1}}{(p-1)!}\int_0^{+\infty} z^{s-1}e^{-z}dz = \\ b_{N_r}^{(N_r-1)p-1}b_0^p + \displaystyle\sum_{s=p+1}^{(N_r-1)p}\dfrac{(s-1)!}{(p-1)!}c_{s-1} = b_{N_r}^{(N_r-1)p-1}b_0^p + pC, \end{cases} \quad (4.8)$$

where $b_{N_r}b_0 \neq 0, C \in \mathbb{Z}.$ We choose now a prime p such that $p > \max(|a_0|, b_{N_r}, |b_0|).$ Then from Eq.(4.8) follows that

$$p \nmid a_0M_0(N_r,p). \quad (4.9)$$

From Eq.(4.3) and Eq.(4.5) one obtains

$$M_{k,l}(N_r,p) = \frac{e^{\beta_{k,l}}}{(p-1)!}\int_{\beta_{k,l}}^{+\infty}\left\{b_{N_r}^{N_rp-1}z^{p-1}z^{p-1}\left[\prod_{j=1}^{r}\prod_{i=1}^{k_j}(z-\beta_{i,j})^p\right]\right\}e^{-z+\beta_{k,l}}dz, \quad (4.10)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ By change of the variable integration $z = u + \beta_{k,l}$ in RHS of the Eq.(4.10) we obtain

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!}\int_0^{+\infty}\left\{b_{N_r}^{N_rp-1}(u+\beta_{k,l})^{p-1}u^pe^{-u}\left[\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(z+\beta_{k,l}-\beta_{i,j})^p\right]\right\}du, \quad (4.11)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ Let us rewrite now Eq.(4.11) in the following form

$$\begin{cases} M_{k,l}(N_r,p) = \\ \dfrac{1}{(p-1)!}\displaystyle\int_0^{+\infty}\left\{(b_{N_r}u+b_{N_r}\beta_{k,l})^{p-1}u^pe^{-u}\left[\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{r_i=1\\ i\neq k}}^{k_j}(b_{N_r}u+b_{N_r}\beta_{k,l}-b_{N_r}\beta_{i,j})^p\right]\right\}du \end{cases} \quad (4.12)$$

Let $\mathbb{Z}_{\mathbf{A}}$ be a ring of the all algebraic integers. Note that [8]

$$\alpha_{i,j} = b_{N_r}\beta_{i,j} \in \mathbb{Z}_{\mathbf{A}}, i = 1,\ldots,k_j, j = 1,\ldots,r. \quad (4.13)$$

Let us rewrite now Eq.(4.12) in the following form

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!}\int_0^{+\infty}(b_{N_r}u+\alpha_{k,l})^{p-1}u^pe^{-u}\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(b_{N_r}u+\alpha_{k,l}-\alpha_{i,j})^pdu \quad (4.14)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ From Eq.(4.14) one obtains

$$\begin{cases} \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int_0^\infty \frac{u^p e^{-u}\Phi_r(u)}{(p-1)!} du, \\ \Phi_r(u) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} (b_{N_r}u + \alpha_{k,l})^{p-1} u^p e^{-u} \prod_{\substack{j=1 \\ j\neq l}}^{r} \prod_{\substack{i=1 \\ i\neq k}}^{k_j} (b_{N_r}u + \alpha_{k,l} - \alpha_{i,j})^p \end{cases} \quad (4.15)$$

The polynomial $\Phi_r(u)$ is a symmetric polynomial on any system Δ_l of variables $\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l}$, where

$$\Delta_l = \{\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l}\}, l = 1,\ldots,r.$$
$$\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l} \in \mathbb{Z}_{\mathbf{A}}, l = 1,\ldots,r. \quad (4.16)$$

It well known that $\Phi_r(u) \in \mathbb{Z}[u]$ (see [8] p.134) and therefore

$$u^p \Phi_r(u) = \sum_{s=p+1}^{(N_r+1)p} c_{s-1} u^{s-1}, c_s \in \mathbb{Z}. \quad (4.17)$$

From Eq.(4.15) and Eq.(4.17) one obtains

$$\begin{cases} \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int_0^\infty \frac{u^p e^{-u}\Phi_r(u)}{(p-1)!} du = \\ \sum_{s=p+1}^{(N_r+1)p} \frac{c_{s-1}}{(p-1)!} \int_0^\infty u^{s-1} e^{-u} du = \sum_{s=p+1}^{(N_r+1)p} c_{s-1} \frac{(s-1)!}{(p-1)!} = pC, C \in \mathbb{Z}. \end{cases} \quad (4.18)$$

Therefore

$$\Xi(N_r,p) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) \in \mathbb{Z},$$
$$p|\Xi(N_r,p). \quad (4.19)$$

Let $O_R \subset \mathbb{C}$ be a circle wth the centre at point $(0,0)$. We assume now that $\forall k \forall l(\beta_{k,l} \in O_R)$. We will designate now

$$g_{k,l}(r) = \max_{|z|\leq R} |b_{N_r}^{-1} f_r(z) e^{-z+\beta_{k,l}}|,$$
$$g_0(r) = \max_{1\leq k\leq k_l, 1\leq l\leq r} g_{k,l}(r), g(r) = \max_{|z|\leq R} |b_{N_r}^{-1} z f_r(z)|. \quad (4.20)$$

From Eq.(4.6) and Eq.(4.20) one obtains

$$|\varepsilon_{k,l}(N_r,p)| = \left| \int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z+\beta_{k,l}} dz}{(p-1)!} \right| \leq$$
$$\frac{1}{(p-1)!} \int_0^{\beta_{k,l}} |b_{N_r}^{-1} f(z) e^{-z+\beta_{k,l}}| [|b_{N_r}^{-1} z f_r(z)|]^{p-1} dz \leq \frac{g_0(r) g^{p-1}(r) |\beta_{k,l}|}{(p-1)!} \leq \frac{g_0(r) g^{p-1}(r) R}{(p-1)!}, \quad (4.21)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. Note that

$$\frac{g_0(r)g^{p-1}(r)R}{(p-1)!} \to 0 \text{ if } p \to \infty. \tag{4.22}$$

From (4.22) follows that for any $\epsilon \in [0,\delta]$ there exists a prime number p such that

$$\sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = \epsilon(p) < 1. \tag{4.23}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ From Eq.(4.4)-Eq.(4.6) follows

$$e^{\beta_{k,l}} = \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} \tag{4.24}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ Assume now that

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} = 0. \tag{4.25}$$

Having substituted RHS of the Eq.(4.24) into Eq.(4.25) one obtains

$$\begin{cases} a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N,p)} = \\ a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p)}{M_0(N_r,p)} + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{\varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} = 0. \end{cases} \tag{4.26}$$

From Eq.(4.26) by using Eq.(4.19) one obtains

$$a_0 + \Xi(N_r,p) + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = 0. \tag{4.27}$$

We choose now a prime $p \in \mathbb{N}$ such that $p > \max(|a_0|,|b_0|,|b_{N_r}|)$ and $\epsilon(p) < 1.$ Note that $p|\Xi(N_r,p)$ and therefore from Eq.(4.19) and Eq.(4.27) one obtains the contradiction. This contradiction completed the proof.

5.Generalized Lindemann-Weierstrass theorem

Theorem 5.1.[4] Let $f_l(z), l = 1,2,\ldots,$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots$ form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots \tag{5.1}$$

and $a_l \in \mathbb{Q}, a_0 \neq 0, l = 1,2,\ldots,$. We assume now that

$$\sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{5.2}$$

Then

$$a_0 + \sum_{l=1}^{\infty} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{5.3}$$

We will divide the proof into three parts

Part I.The Robinson transfer

Let $f(z) = f_{\mathbf{r}}(z) \in {}^*\mathbb{Z}[z], z \in {}^*\mathbb{C}, l = 1,2,\ldots,\mathbf{r}, \mathbf{r} \in {}^*\mathbb{N}_\infty$ be a nonstandard polynomial such that

$$\begin{cases} f(z) = f_{\mathbf{r}}(z) = \prod_{l=1}^{\mathbf{r}} f_l(z) = \mathbf{b}_0 + \mathbf{b}_1 z + \ldots + \mathbf{b}_{\mathbf{N}} z^{\mathbf{N}} = \\ = \mathbf{b}_{\mathbf{N}} \prod_{l=1}^{\mathbf{r}} \prod_{k=1}^{{}^*k_l} (z - ({}^*\beta_{k,l})), \mathbf{b}_0 \neq 0, \mathbf{b}_{\mathbf{N}} > 0, \\ \mathbf{N} = \mathbf{N}_{\mathbf{r}} = \sum_{l=1}^{\mathbf{r}} ({}^*k_l) \in {}^*\mathbb{N}_\infty. \end{cases} \quad (5.4)$$

Let ${}^*M_0(\mathbf{N},\mathbf{p}), {}^*M_{k,l}(\mathbf{N},\mathbf{p})$ and ${}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})$ be the quantities:

$${}^*M_0(\mathbf{N},\mathbf{p}) = \int_0^{{}^*(+\infty)} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \quad (5.5)$$

$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where in (5.5) we integrate in nonstandard complex plaine ${}^*\mathbb{C}$ along line ${}^*[0,+\infty]$, see Pic.1.

$${}^*M_{k,l}(\mathbf{N},\mathbf{p}) = ({}^*e^{{}^*\beta_{k,l}}) \int_{{}^*\beta_{k,l}}^{{}^*(+\infty)} \frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \quad (5.6)$$

$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where $k = 1,\ldots,{}^*k_l$ and where in (5.6) we integrate in nonstandard complex plain ${}^*\mathbb{C}$ along line with initial point ${}^*\beta_{k,l} \in {}^*\mathbb{C}$ and which are parallel to real axis of the complex plane ${}^*\mathbb{C}$, see Pic.1.

$${}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}) = ({}^*e^{{}^*\beta_{k,l}}) \int_0^{{}^*\beta_{k,l}} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}, \quad (5.7)$$

$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where $k = 1,\ldots,{}^*k_l$ and where in (5.7) we integrate in nonstandard complex plain ${}^*\mathbb{C}$ along contour ${}^*[0,{}^*\beta_{k,l}]$, see Pic.1.

1.Using Robinson transfer principle [4],[5],[6] from Eq.(5.5) and Eq.(4.8) one obtains directly

$${}^*M_0(\mathbf{N},\mathbf{p}) = b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} \mathbf{b}_0^{\mathbf{p}} + \mathbf{p}\mathbf{C}, \quad (5.8)$$

where $\mathbf{b}_{\mathbf{N}}\mathbf{b}_0 \neq 0, \mathbf{C} \in {}^*\mathbb{Z}_\infty$.We choose now infinite prime $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\{ \mathbf{p} > \max(|\mathbf{a}_0|, \mathbf{b}_{\mathbf{N}}, |\mathbf{b}_0|). \quad (5.9)$$

2.Using Robinson transfer principle from Eq.(5.6) and Eq.(4.19) one obtains directly

$$\forall r(r \in \mathbb{N}) :$$

$$^{*}\Xi(\mathbf{N},\mathbf{p},r) = \sum_{l=1}^{r}(^{*}a_l)\sum_{k=1}^{k_l}(^{*}M_{k,l}(\mathbf{N},\mathbf{p})) = \mathbf{p}\mathbf{C}_r \in {}^{*}\mathbb{Z}_{\infty}. \tag{5.10}$$

and therefore

$$\begin{gathered}\forall r(r \in \mathbb{N}) : \\ \mathbf{p}|^{*}\Xi(\mathbf{N},\mathbf{p},r).\end{gathered} \tag{5.11}$$

3.Using Robinson transfer principle from Eq.(5.7) and Eq.(4.21) one obtains directly

$$\begin{cases} |^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})| = \left|(^{*}e^{^{*}\beta_{k,l}})\displaystyle\int_{0}^{^{*}\beta_{k,l}}\frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1}z^{\mathbf{p}-1}f^{\mathbf{p}}(z)[^{*}e^{-z}]dz}{(\mathbf{p}-1)!}\right| \leq \\ \dfrac{1}{(\mathbf{p}-1)!}\displaystyle\int_{0}^{^{*}\beta_{k,l}}|b_{\mathbf{N}}^{-1}f(z)(^{*}e^{-z+(^{*}\beta_{k,l})})|[|b_{N_r}^{-1}zf(z)|]^{\mathbf{p}-1}dz \leq \dfrac{[^{*}g_0(\mathbf{r})][^{*}g^{\mathbf{p}-1}(\mathbf{r})]|^{*}\beta_{k,l}|}{(\mathbf{p}-1)!} \\ \leq \dfrac{[^{*}g_0(\mathbf{r})][^{*}g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!}, \end{cases} \tag{5.12}$$

where $k = 1,\ldots,{}^{*}k_l, l = 1,\ldots,\mathbf{r}$.Note that $\forall\epsilon(\epsilon \in^{*} \mathbb{R})[\epsilon \approx 0]$, there exists $\mathbf{p} = \mathbf{p}(\epsilon)$

$$\frac{[^{*}g_0(\mathbf{r})][^{*}g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!} \leq \epsilon. \tag{5.13}$$

4. From (5.13) follows that for any $\epsilon \in [0,\delta]$ there exists an infinite prime $\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$ such that

$$\forall r(r \in \mathbb{N}) :$$

$$\sum_{l=1}^{r}(^{*}a_l)\sum_{k=1}^{k_l}(^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})) = \epsilon(\mathbf{p}) < 1 \tag{5.14}$$

where $k = 1,\ldots,{}^{*}k_l, l = 1,\ldots,\mathbf{r}.$.

5. From Eq.(5.5)-Eq.(5.7) we obtain

$$\left[{}^{*}e^{^{*}\beta_{k,l}} = \frac{^{*}M_{k,l}(\mathbf{N},\mathbf{p}) + (^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))}{^{*}M_0(\mathbf{N},\mathbf{p})}\right], \tag{5.15}$$

where $k = 1,\ldots,{}^{*}k_l, l = 1,\ldots,\mathbf{r}$.

Part II.The Wattenberg imbedding ${}^{*}e^{^{*}\beta_{k,l}}$ into ${}^{*}\mathbb{R}_{\mathbf{d}}$

1.By using Wattenberg imbedding ${}^{*}\mathbb{R} \overset{\#}{\hookrightarrow} {}^{*}\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer (see subsection 2.8 Theorem 2.17) from Eq.(5.8) one obtains

$$\begin{cases} ({}^*M_0(\mathbf{N},\mathbf{p}))^\# = \left(\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1}\mathbf{b}_0^{\mathbf{p}}\right)^\# + \mathbf{p}^\#\mathbf{C}^\# = \\ = (\mathbf{b}_{\mathbf{N}}^\#)^{(\mathbf{N}^\#-1)\mathbf{p}^\#-1}(\mathbf{b}_0^\#)^{\mathbf{p}^\#} + \mathbf{p}^\#\mathbf{C}^\# \end{cases} \quad (5.16)$$

where $\mathbf{b}_{\mathbf{N}}^\#\mathbf{b}_0^\# \neq 0^\#, \mathbf{C}^\# \in {}^*\mathbb{Z}_{\mathbf{d}}$. We choose now an infinite prime $\mathbf{p} \in {}^*\mathbb{N}$ such that

$$\left\{ \mathbf{p}^\# > \max(|\mathbf{a}_0^\#|, \mathbf{b}_{\mathbf{N}}^\#, |\mathbf{b}_0^\#|). \right. \quad (5.17)$$

2.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$, and Gonshor transfer from Eq.(5.10) one obtains directly

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ ({}^*\Xi(\mathbf{N},\mathbf{p},r))^\# = \sum_{l=1}^{r}\left(({}^*a_l)^\#\right)\sum_{k=1}^{k_l}({}^*M_{k,l}(\mathbf{N},\mathbf{p}))^\# = \mathbf{p}^\#\mathbf{C}_r^\# \in {}^*\mathbb{Z}_{\mathbf{d}} \end{cases} \quad (5.18)$$

and therefore

$$\forall r(r \in \mathbb{N})\left[\mathbf{p}^\# | ({}^*\Xi(\mathbf{N},\mathbf{p},r))^\#\right]. \quad (5.19)$$

3.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$,and Gonshor transfer from Eq.(5.14) one obtains directly

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ \sum_{l=1}^{r}\left(({}^*a_l)^\#\right)\sum_{k=1}^{k_l}({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))^\# = \epsilon^\#(\mathbf{p}^\#) < 1. \end{cases} \quad (5.20)$$

4.By using Wattenberg imbedding ${}^*\mathbb{R} \overset{\#}{\hookrightarrow} {}^*\mathbb{R}_{\mathbf{d}}$,and Gonshor transfer from Eq.(5.15) one obtains directly

$$\left\{ e^{\beta_{k,l}^\#} \triangleq ({}^*e^{{}^*\beta_{k,l}})^\# = \frac{({}^*M_{k,l}(\mathbf{N},\mathbf{p}))^\# + ({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))^\#}{({}^*M_0(\mathbf{N},\mathbf{p}))^\#}, \right. \quad (5.21)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,\mathbf{r} \in {}^*\mathbb{N}$.

Part III.Main equality

Remark 5.1 Note that in this subsection we often write for a short $a^\#$ instead $({}^*a)^\#, a \in \mathbb{R}$. For example we write

$$\begin{cases} \forall r(r \in \mathbb{N}) : \\ e^{\beta_{k,l}^\#} = \dfrac{M_{k,l}^\#(\mathbf{N},\mathbf{p})^\# + \varepsilon_{k,l}^\#(\mathbf{N},\mathbf{p})}{M_0^\#(\mathbf{N},\mathbf{p})} \end{cases}$$

instead Eq.(5.21).

Assumption 5.1. Let $f_l(z), l = 1,2,\ldots,$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q} : \beta_{1,l},\ldots,\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots,r$ form a

complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots \quad (5.22)$$

$l = 1,2,\ldots,a_0 \in \mathbb{Q}, a_0 \neq 0, r = 1,2,\ldots$.

Note that from Assumption 5.1 by Robinson transfer follows that algebraic numbers over

$^{*}\mathbb{Q}$: $^{*}\beta_{1,l},\ldots,{}^{*}\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots,$ for any $l = 1,2,\ldots,$ form a complete set of the roots

of $^{*}f_l(z)$ such that

$$^{*}f_l(z) \in {}^{*}\mathbb{Z}[z], \deg({}^{*}f_l(z)) = k_l, l = 1,2,\ldots\ . \tag{5.23}$$

Assumption 5.2. We assume now that there exists a sequence

$$\breve{a}_l = \frac{q_l}{m_l} \in \mathbb{Q}, l = 1,2,\ldots; r = 1,2,\ldots \tag{5.24}$$

and rational number

$$\breve{a}_0 = \frac{q_0}{m_0} \in \mathbb{Q}, \tag{5.25}$$

such that

$$\sum_{l=1}^{\infty} |\breve{a}_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{5.26}$$

and

$$\breve{a}_0 + \sum_{l=1}^{\infty} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} = 0. \tag{5.27}$$

Assumption 5.3. We assume now for a short that the all roots $^{*}\beta_{1,l},\ldots,{}^{*}\beta_{k_l,l},$ $k_l \geq 1, l = 1,2,\ldots$ of $^{*}f_l(z)$ are real.

In this subsection we obtain an reduction of the equality given by Eq.(5.27) in $\mathbb{R}$ to some equivalent equality given by Eq.(3.) in $^{*}\mathbb{R}_{\mathbf{d}}$. The main tool of such reduction that external countable sum defined in subsection 2.8.

Lemma 5.1.Let $\Delta_{\leq}(r)$ and $\Delta_{>}(r)$ be the sum correspondingly

$$\begin{cases} \Delta_{\leq}(r) = \breve{a}_0 + \sum_{l=1}^{r\geq 1} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}}, \\ \Delta_{>}(r) = \sum_{l=r+1}^{\infty} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}}. \end{cases} \tag{5.28}$$

Then $\Delta_{>}(r) \neq 0, r = 1,2,\ldots$

Proof. Suppose there exist r such that $\Delta_{>}(r) = 0$.Then from Eq.(5.27) follows $\Delta_{\leq}(r) = 0$. Therefore by Theorem 4.1 one obtains the contradiction.

Remark 5.2. Note that from Eq.(5.27) follows that in generel case there is a sequence $\{m_i\}_{i=1}^{\infty}$ such that

$$\begin{gathered} \lim_{i\to\infty} m_i = \infty, \\ \forall (i \in \mathbb{N}) \left[\breve{a}_0 + \sum_{l=1}^{m_i} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} < 0 \right], \\ \breve{a}_0 + \lim_{i\to\infty} \left(\sum_{l=1}^{m_i} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \right) = 0, \end{gathered} \tag{5.29}$$

or there is a sequence $\{m_j\}_{j=1}^{\infty}$ such that

$$\begin{cases} \lim\limits_{i\to\infty} m_j = \infty, \\ \forall(j \in \mathbb{N})\left[\breve{a}_0 + \sum\limits_{l=1}^{m_j} \breve{a}_l \sum\limits_{k=1}^{k_l} e^{\beta_{k,l}} > 0 \right], \\ \breve{a}_0 + \lim\limits_{j\to\infty} \left(\sum\limits_{l=1}^{m_j} \breve{a}_l \sum\limits_{k=1}^{k_l} e^{\beta_{k,l}} \right) = 0, \end{cases} \quad (5.30)$$

or both sequences $\{m_i\}_{i=0}^{\infty}$ and $\{m_j\}_{j=0}^{\infty}$ with a property that is specified above exist.

Remark 5.3. We assume now for short but without loss of generelity that (5.29) is satisfied. Then from (5.29) by using Definition 2.17 and Theorem 2.14 (see subsection 2.8) one obtains the equality [4]

$$({}^*\breve{a}_0)^{\#} + \left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge} ({}^*\breve{a}_l)^{\#} \sum_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#} \right] = -\varepsilon_{\mathbf{d}}. \quad (5.31)$$

Remark 5.4.Let $\Delta^{\#}_{\leq}(r)$ and $\Delta^{\#}_{>}(r)$ be the upper external sum defined by

$$\begin{cases} \Delta^{\#}_{\leq}(r) = \breve{a}_0 + \sum\limits_{l=1}^{r\geq 1} ({}^*\breve{a}_l)^{\#} \sum\limits_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#}, \\ \Delta^{\#}_{>}(r) = \#Ext\text{-}\sum\limits_{\substack{n\in\mathbb{N} \\ l=r+1}}^{\wedge} ({}^*\breve{a}_l)^{\#} \sum\limits_{k=1}^{k_l} ({}^*e^{{}^*\beta_{k,l}})^{\#}. \end{cases} \quad (5.32)$$

Note that from Eq.(5.31)-Eq.(5.32) follows that

$$\Delta^{\#}_{\leq}(r) + \Delta^{\#}_{>}(r) = -\varepsilon_{\mathbf{d}}. \quad (5.33)$$

Remark 5.5. Assume that $\alpha, \beta \in {}^*\mathbb{R}_{\mathbf{d}}$ and $\beta \notin {}^*\mathbb{R}$. In this subsection we will write for a short $\mathbf{ab}[\alpha|\beta]$ iff β absorbs α, i.e. $\beta + \alpha = \beta$.

Lemma 5.2. $\neg\mathbf{ab}[\Delta^{\#}_{\leq}(r)|\Delta^{\#}_{>}(r)], k = 1, 2, \ldots$

Proof.Suppose there exists $r \in \mathbb{N}$ such that $\mathbf{ab}[\Delta^{\#}_{\leq}(r)|\Delta^{\#}_{>}(r)]$. Then from Eq.(5.33) one obtains

$$\Delta^{\#}_{>}(r) = -\varepsilon_{\mathbf{d}}. \quad (5.34)$$

From Eq.(5.34) by Theorem 2.11 follows that $\Delta_{>}(r) = 0$ and therefore by Lemma 5.1 one obtains the contradiction.

Theorem 5.2.[4] The equality (5.31) is inconsistent.

Proof.Let us considered hypernatural number $\mathfrak{I} \in {}^*\mathbb{N}_{\infty}$ defined by countable sequence

$$\mathfrak{I} = (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) \quad (5.35)$$

From Eq.(5.31) and Eq.(5.35) one obtains

$$\Im^{\#}\times({}^{*}\breve{a}_{0})^{\#}+\Im^{\#}\times\left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}({}^{*}\breve{a}_{l})^{\#}\sum_{k=1}^{k_{l}}({}^{*}e^{{}^{*}\beta_{k,l}})^{\#}\right]=$$
$$=\Im_{0}^{\#}+\left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\Im_{l}^{\#}\times\sum_{k=1}^{k_{l}}({}^{*}e^{{}^{*}\beta_{k,l}})^{\#}\right]=-\Im^{\#}\times\varepsilon_{\mathbf{d}} \tag{5.36}$$

where

$$\Im_{0}^{\#}=\Im^{\#}\breve{a}_{0}=\frac{\Im^{\#}q_{0}^{\#}}{m_{0}^{\#}},$$
$$\Im_{l}^{\#}=\Im^{\#}\breve{a}_{l}^{\#}=\frac{\Im_{0}^{\#}q_{l}^{\#}}{m_{l}^{\#}}. \tag{5.37}$$

Remark 5.6.Note that from inequality (5.12) by Gonshor transfer one obtains

$$\left|{}^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})^{\#}\right|\leq\frac{[{}^{*}g_{0}(\mathbf{r})]^{\#}[{}^{*}g^{\mathbf{p}-1}(\mathbf{r})]^{\#}|{}^{*}\beta_{k,l}^{\#}|}{(\mathbf{p}^{\#}-1)!^{\#}}$$
$$\mathbf{N},\mathbf{p}\in{}^{*}\mathbb{N}_{\infty}. \tag{5.38}$$

Substitution Eq.(5.21) into Eq.(5.36) gives

$$\Im_{0}^{\#}+\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\Im_{l}^{\#}\times\sum_{k=1}^{k_{l}}\frac{M_{k,l}^{\#}(\mathbf{N},\mathbf{p})^{\#}+\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})}{M_{0}^{\#}(\mathbf{N},\mathbf{p})}=-\Im^{\#}\times\varepsilon_{\mathbf{d}}. \tag{5.39}$$

Multiplying Eq.(5.39) by Wattenberg hyperinteger $[{}^{*}M_{0}(\mathbf{N},\mathbf{p})]^{\#}\in{}^{*}\mathbb{Z}_{\mathbf{d}}$ by Theorem 2.13 (see subsection 2.8) we obtain

$$\Im_{0}^{\#}\times M_{0}^{\#}(N_{r},\mathbf{p})+\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_{l}}\Im_{l}^{\#}\times\sum_{k=1}^{k_{l}}\left[M_{k,l}^{\#}(\mathbf{N},\mathbf{p})+\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})\right]=$$
$$=-\Im^{\#}\times[{}^{*}M_{0}(\mathbf{N},\mathbf{p})]^{\#}\times\varepsilon_{\mathbf{d}}. \tag{5.40}$$

By using inequality (5.38) for a given $\delta\in{}^{*}\mathbb{R},\ \delta\approx 0$ we will choose infinite prime integer $\mathbf{p}\in{}^{*}\mathbb{N}_{\infty},\mathbf{p}=\mathbf{p}(\delta)$ such that:

$$\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_{l}}\Im_{l}^{\#}\sum_{k=1}^{k_{l}}\ \varepsilon_{k,l}^{\#}(\mathbf{N},\mathbf{p})\subseteq-\delta^{\#}\times\varepsilon_{\mathbf{d}}. \tag{5.41}$$

Therefore from Eq.(5.40) and (5.41) by using definition (2.15) of the function $\mathbf{Int.p}(\alpha)$ given by Eq.(2.20)-Eq.(2.21) and corresponding basic property **I** (see subsection 2.7) of the function $\mathbf{Int.p}(\alpha)$ we obtain

$$\mathbf{Int.p}\left(\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l}\left[M_{k,l}^\#(\mathbf{N},\mathbf{p}) + \varepsilon_{k,l}^\#(\mathbf{N},\mathbf{p})\right]\right) =$$
$$\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}) = \quad (5.42)$$
$$-\mathbf{Int.p}\left(\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}\right) = -\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}.$$

From Eq.(5.42) finally we obtain the main equality

$$\Im_0^\# \times M_0^\#(\mathbf{N},\mathbf{p}) + \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}) = -\Im^\# \times [{}^*M_0(\mathbf{N},\mathbf{p})]^\# \times \varepsilon_{\mathbf{d}}. \quad (5.43)$$

We will choose now infinite prime integer $\mathbf{p}$ in Eq.(3.56) $\mathbf{p} = \widehat{\mathbf{p}} \in {}^*\mathbb{N}_\infty$ such that

$$\left\{ \widehat{\mathbf{p}}^\# > \max(|\mathbf{a}_0^\#|, \mathbf{b}_{\mathbf{N}}^\#, |\mathbf{b}_0^\#|, \Im_0^\#). \right. \quad (5.44)$$

Hence from Eq.(5.16) follows

$$\widehat{\mathbf{p}}^\# \nmid M_0^\#(\mathbf{N},\widehat{\mathbf{p}}). \quad (5.45)$$

Note that $[{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]^\# \neq 0^\#$. Using (5.44) and (5.45) one obtains:

$$\widehat{\mathbf{p}}^\# \nmid M_{k,l}^\#(\mathbf{N},\widehat{\mathbf{p}},r) \times \Im_0^\#. \quad (5.46)$$

Using Eq.(5.11) one obtains

$$\widehat{\mathbf{p}}^\# \mid M_{k,l}^\#(\mathbf{N},\widehat{\mathbf{p}}), k,l = 1,2,\ldots. \quad (5.47)$$

Part IV.The proof of the inconsistency of the main equality (5.43)

In this subsection we wil prove that main equality (5.43) is inconsistent. This prooff is based on the Theorem 2.10 (**v**), see subsection 2.6.

Lemma 5.3.The equality (5.43) under conditions (5.46)-(5.47) is inconsistent.

Proof. (**I**) Let us rewrite Eq.(5.43) in the short form

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}) = -\Lambda^\#(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}, \quad (5.48)$$

where

$$\begin{cases} \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}) = \#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}\sum_{k=1}^{k_l}\Im_l^\# \times \sum_{k=1}^{k_l} M_{k,l}^\#(\mathbf{N},\mathbf{p}), \\ \Gamma(\mathbf{n},\widehat{\mathbf{p}}) = \Im_0^\# \times [{}^*M_0(\mathbf{N},\widehat{\mathbf{p}})]^\#, \Lambda^\#(\widehat{\mathbf{p}}) = \Im^\# \times [{}^*M_0(\mathbf{N},\widehat{\mathbf{p}})]^\#. \end{cases} \quad (5.49)$$

From (5.46)-(5.47) follows that

$$\begin{cases} \widehat{\mathbf{p}}^\# \nmid \Gamma(\mathbf{N},\widehat{\mathbf{p}}), \\ \widehat{\mathbf{p}}^\# \mid \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}). \end{cases} \quad (5.50)$$

Remark 5.7.Note that $\Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}}) \notin {}^*\mathbb{R}$. Otherwise we obtain that

$$\mathbf{ab.p}(\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \Sigma^\wedge(\mathbf{N},\widehat{\mathbf{p}})) = \{\varnothing\}. \quad (5.51)$$

But the other hand from Eq.(5.48) follows that

$$\mathbf{ab}.\,\mathbf{p}(\Gamma(\mathbf{N},\widehat{\mathbf{p}})+\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})) = -\Lambda^{\#}(\widehat{\mathbf{p}})\times\varepsilon_{\mathbf{d}}. \tag{5.52}$$

But this is a contradiction. This contradiction completed the proof of the statement (**I**).

(**II**) Let $\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{N},\widehat{\mathbf{p}}),\widetilde{\Delta}^{\#}_{>}(k,\mathbf{N},\widehat{\mathbf{p}}),\widetilde{\Delta}^{\#}_{\leq}(k_1,k_2,\mathbf{N},\widehat{\mathbf{p}})$ and $\widetilde{\Delta}^{\#}_{\leq}(k,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{n}),\widetilde{\Delta}^{\#}_{>}(k,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{n})$, be the external sum correspondingly

$$\begin{cases}
\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}}) = \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \sum_{l=1}^{r\geq 1}\sum_{k=1}^{k_l}\mathfrak{I}^{\#}_{l}\times\sum_{k=1}^{k_l}M^{\#}_{k,l}(\mathbf{N},\mathbf{p}),\\
\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}}) = \sum_{l\geq r+1}^{\wedge}\sum_{k=1}^{k_l}\mathfrak{I}^{\#}_{l}\times\sum_{k=1}^{k_l}M^{\#}_{k,l}(\mathbf{N},\mathbf{p}),\\
\widetilde{\Delta}^{\#}_{\leq}(r_1,r_2,\mathbf{N},\widehat{\mathbf{p}}) = \sum_{l=r_1}^{r_2}\sum_{k=1}^{k_l}\mathfrak{I}^{\#}_{l}\times\sum_{k=1}^{k_l}M^{\#}_{k,l}(\mathbf{N},\mathbf{p}),\\
\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = \Gamma(\mathbf{n},\widehat{\mathbf{p}}) + \sum_{l=1}^{r\geq 1}\sum_{k=1}^{k_l}\mathfrak{I}^{\#}_{l}\times\sum_{k=1}^{k_l}\{M^{\#}_{k,l}(\mathbf{N},\mathbf{p})+\varepsilon^{\#}_{k,l}(\mathbf{N},\mathbf{p})\},\\
\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = \#Ext\text{-}\sum_{l\geq r+1}^{\wedge}\sum_{k=1}^{k_l}\mathfrak{I}^{\#}_{l}\times\sum_{k=1}^{k_l}\{M^{\#}_{k,l}(\mathbf{N},\mathbf{p})+\varepsilon^{\#}_{k,l}(\mathbf{N},\mathbf{p})\}.
\end{cases} \tag{5.53}$$

Note that from Eq.(5.43) and Eq.(5.53) follows that

$$\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}})+\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}}) = -\Lambda^{\#}(\widehat{\mathbf{p}})\times\varepsilon_{\mathbf{d}},\quad r=1,2,\ldots \tag{5.54}$$

Lemma 5.4. (**i**) Under conditions (5.46)-(5.47)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l})\middle|\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l})\right], r=1,2,\ldots\ . \tag{5.55}$$

And (**ii**) Under conditions (5.46)-(5.47)

$$\neg\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}})\middle|\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}})\right], r=1,2,\ldots\ . \tag{5.56}$$

Proof. (**i**) First note that under conditions (5.46)-(5.47) one obtains

$$\left[\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l})\neq 0\right], r=1,2,\ldots \tag{5.57}$$

Suppose that there exists $r\geq 0$ such that $\mathbf{ab}\left[\widetilde{\Delta}^{\#}_{\leq}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l})\middle|\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l})\right]$. Then from Eq.(5.54) one obtains

$$\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = -\Lambda^{\#}(\widehat{\mathbf{p}})\times\varepsilon_{\mathbf{d}}. \tag{5.58}$$

From Eq.(5.58) by Theorem 2.17 one obtains

$$-\varepsilon_{\mathbf{d}} = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1}\times\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = [\Lambda^{\#}(\widehat{\mathbf{p}})]^{-1}\times\widetilde{\Delta}^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}) = \Delta^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}). \tag{5.59}$$

Thus

$$-\varepsilon_{\mathbf{d}} = \Delta^{\#}_{>}(r,\mathbf{N},\widehat{\mathbf{p}},\varepsilon^{\#}_{k,l}). \tag{5.60}$$

From Eq.(5.60) by Theorem 2.11 follows that $\Delta_{>}(r)=0$ and therefore by Lemma 5.2 one obtains the contradiction. This contradiction finalized the proof of the Lemma 5.4 (**i**)

Proof. (**ii**) This is immediate from the Definition 2.14 (**Property I**), see subsection 2.7. (**III**)

Remark 5.8.(**i**) Note that from Eq.(5.49) by Theorem 2.10 (**v**) follws that $\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})$ has the form

$$\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) = \mathbf{q}^{\#} + \mathbf{ab}.\mathbf{p}(\Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}})) = \\ = \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) \qquad (5.61)$$

where

$$\mathbf{q}^{\#} \in \Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) = \widetilde{\Delta}^{\#}_{>}(1,\mathbf{N},\widehat{\mathbf{p}}), \\ \mathbf{q} \in {}^{*}\mathbb{Z}_{\infty} \text{ and} \widehat{\mathbf{p}}|\mathbf{q}. \qquad (5.62)$$

(**ii**) Substitution by Eq.(5.61) into Eq.(5.48) gives

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \Sigma^{\wedge}(\mathbf{N},\widehat{\mathbf{p}}) = \Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \qquad (5.63)$$

Remark 5.9. Note that from (5.63) by definitions follows that

$$\mathbf{ab}[(\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#})|(-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}})]. \qquad (5.64)$$

From Eq.(5.63) follows that

$$\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#} + (-\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}) = -\Lambda^{\#}(\widehat{\mathbf{p}}) \times \varepsilon_{\mathbf{d}}. \qquad (5.68)$$

Therefore

$$(\Lambda^{\#}(\widehat{\mathbf{p}}))^{-1}[\Gamma(\mathbf{N},\widehat{\mathbf{p}}) + \mathbf{q}^{\#}] + (-\varepsilon_{\mathbf{d}}) = -\varepsilon_{\mathbf{d}}. \qquad (5.69)$$

Now we dealing with semiring ${}^{*}\mathbb{Q}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})$,(see subsection 2.16.2).By consideration similarly

as above we obtain

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{N},,\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#^{\approx}_{\varepsilon}}] \approx 0^{\#^{\approx}_{\varepsilon}}, \\ (\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{N},,\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#}]\ {}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)1} \check{\gtrsim} (\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \qquad (5.70)$$

and

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \bullet_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} [\Gamma(\mathbf{N},,\widehat{\mathbf{p}}) \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} \mathbf{q}^{\#^{\approx}_{\varepsilon}}] \check{+}_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon)} (-_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon;{}^{*}\mathbb{R}^{\approx}(\varepsilon))) = \\ = -_{{}^{*}\mathbb{R}^{\approx}_{\mathbf{d}}(\varepsilon,\mathbf{p})} \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon;{}^{*}\mathbb{R}^{\approx}(\varepsilon)). \qquad (5.71)$$

From inequality (5.38) follows that we willin to choose $\widehat{\mathbf{p}}$ and $\varepsilon \approx 0$ such that

$$(\Lambda^{\#^{\approx}_{\varepsilon}}(\widehat{\mathbf{p}}))^{-1} \notin \breve{\boldsymbol{\varepsilon}}^{\approx}_{\mathbf{d}}(\varepsilon;{}^{*}\mathbb{R}^{\approx}(\varepsilon)). \qquad (5.72)$$

But this is a contradiction. This contradiction completed the proof of the Lemma 5.3.

Remark 5.11. Note that by Definitions 2.19-2.20 and Theorem 2.18 from Assumption 5.1 and Assumption 5.2 follows

$$\left|({}^{*}\breve{a}_0)^{\#} + \left[\#Ext\text{-}\sum_{l\in\mathbb{N}}^{\wedge}({}^{*}\breve{a}_l)^{\#}\sum_{k=1}^{k_l}({}^{*}e^{{}^{*}\beta_{k,l}})^{\#}\right]\right|^2 = |-\varepsilon_{\mathbf{d}}|^2 = \varepsilon_{\mathbf{d}}. \qquad (5.73)$$

Theorem 5.3.The equality (5.73) is inconsistent.

Proof. The proof of the Theorem 5.3 copies in main details the proof of the Theorem 5.2.

Theorem 5.3 completed the proof of the main Theorem 1.6.

References

[1] Nesterenko,Y.V., Philippon.Introduction to Algebraic Independence Theory. Series: Lecture Notes in Mathematics,Vol.1752 Patrice (Eds.) 2001, XIII, 256 pp.,Softcover ISBN: 3-540-41496-7

[2] Waldschmidt M., Algebraic values of analytic functions.Journal of Computational and Applied Mathematics 160 (2003) 323–333.

[3] Foukzon J., 2006 Spring Central Sectional Meeting Notre Dame,IN, April 8-9,2006 Meeting #1016 The solution of one very old problem in transcendental numbers theory. Preliminary report. http://www.ams.org/meetings/sectional/1016-11-8.pdf

[4] Foukzon J., Non-archimedean analysis on the extended hyperreal line ${}^*\mathbb{R}_d$ and some transcendence conjectures over field $\mathbb{Q}$ and ${}^*\mathbb{Q}_\omega$. http://arxiv.org/abs/0907.0467v9

[5] Goldblatt,R., Lectures on the Hyperreals. Springer-Verlag, New York, NY,1998.

[6] Wattenberg,F., $[0,\infty]$-valued, translation invariant measures on $\mathbb{N}$ and the Dedekind completion of ${}^*\mathbb{R}$. Pacific J. Math. Volume 90, Number 1 (1980), 223-247.

[7] Gonshor,H., Remarks on the Dedekind completion of a nonstandard model of the reals.Pacific J. Math. Volume 118, Number1 (1985), 117-132.

[8] Shidlovsky, A.B., "Diophantine Approximations and Transcendental Numbers", Moscov, Univ.Press,1982 (in Russian). http://en.bookfi.org/book/506517 http://bookre.org/reader?file=506517&pg=129

[9] I.Moerdijk A model for intuitionistic nonstandard arithmetic. Ann. Pure Appl. Logic 73(1995),297-325.

[10] I.Moerdijk and G. E. Reyes, Models of Smooth In nitesimal Analysis Springer 1991.

[11] P. Martin Löf, Mathematics of infinity. In P. Martin Lof and G. E. Mints eds COLOG Computer Logic Lecture Notes in Computer Science vol. 417 Springer Berlin 1990,146-197.

[12] E. Palmgren Sheaf theoretic foundation for nonstandard analysis. Ann. Pure Appl. Logic 85(1997) 69-86.

[13] E. Palmgren Developments in constructive nonstandard analysis. Bulletin of Symbolic Logic 4(1998) 233-272.

[14] E. Palmgren, Unifying Constructive and Nonstandard Analysis. Vol. 306 of the series Synthese Library, pp.167-183.Springer 2001.

[15] S.G.Simpson, Subsystems of second order arithmetic, Perspectives in Logic (2nd ed.),Cambridge University Press,(2009), ISBN 978-0-521-88439-6, MR 2517689

[16] E. Bishop, Foundations of Constructive Analysis (McGraw-Hill, New York, 1967).

[17] E.Bishop and D.Bridges, Constructive Analysis (Springer, Berlin, 1985).
[18] Foukzon J.,There Is No Standard Model of ZFC and ZFC_2 with Henkin Semantics
Advances in Pure Mathematics Vol.9 No.9, September 2019
https://doi.org/10.4236/apm.2019.99034
[19] Foukzon J.,Inconsistent Countable Set in Second Order ZFC and Nonexistence of the Strongly Inaccessible Cardinals, British Journal of Mathematics & Computer Science (ISSN: 2231-0851) BJMCS 2015,Volume 9,Issue 5,Page 380-393
DOI : 10.9734/BJMCS/2015/16849
[20] Foukzon J.,Men'kova E., There Is No Standard Model of ZFC and ZFC2 with Henkin Semantics, Advances in Pure Mathematics,Vol.9 No.9(2019), Paper ID 95029, 60 pages DOI:10.4236/apm.2019.99034
[21] Foukzon J.,Men'kova E.,There is No Standard Model of ZFC and ZFC2, Book, Advances in Mathematics and Computer Science Vol.1,chapt.3,pp.26-75
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