Games people play: An overview of strategic decision-making theory in conflict situations

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Abstract — In this paper, a gentle introduction to Game Theory is presented in the form of basic concepts and examples. Minimax and Nash's theorem are introduced as the formal definitions for optimal strategies and equilibria in zero-sum and nonzero-sum games. Several elements of cooperaive gaming, coalitions, voting ensembles, voting power and collective efficiency are described in brief. Analytical (matrix) and extended (tree-graph) forms of game representation is illustrated as the basic tools for identifying optimal strategies and "solutions" in games of any kind. Next, a typology of four standard nonzero-sum games is investigated, analyzing the Nash equilibria and the optimal strategies in each case. Signaling, stance and third-party intermediates are described as very important properties when analyzing strategic moves, while credibility and reputation is described as crucial factors when signaling promises or threats. Utility is introduced as a generalization of typical cost/gain functions and it is used to explain the incentives of irrational players under the scope of "rational irrationality". Finally, a brief reference is presented for several other more advanced concepts of gaming, including emergence of cooperation, evolutionary stable strategies, two-level games, metagames, hypergames and the Harsanyi transformation.

Index Terms — Game Theory, Minimax theorem, Nash equilibrium, coalitional gaming, indices of power, voting ensembles, analytical form, extended form, Leader, Battle of the Sexes, Chicken, Prisoner's Dilemma, Hostage Situation, Kamikaze, signaling, bluff, credibility, promises, threats, utility, rational irrationality, two-level games, hypergames, evolutionary stable strategies, Harsanyi transformation, metagames.

GAME THEORY is a vast scientific and research area, based almost entirely on Mathematics and some experimental methods, with applications that vary from simple board games to Evolutionary Psychology and Sociology-Biology in group behavior of humans and animals. Conflict situations are presented everywhere in the real world, every day, for thousands of years - not only in human societies but also in animals. The seller and the buyer have to come up with a mutually acceptable price for the grocery. The employer and the employee have to bargain in order to reach a mutually satisfying value for the salary. A buyer in an auction has to continuously estimate the cost/gain value of making (or not) the next higher bid for some object. The primary adversaries in a wolf pack have to decide when it is beneficial to fight over the leadership and when to stop before they are severely wounded. A swarm of fish has to collectively "decide" what is the optimal number and distance of the piket members or "scouts" that serve as the early warning for the group, perhaps even self-sacrificing if required. All these cases are typical examples, simpler or more complex, of conflict situations that depend on bargaining, coordination and evolutionary optimization. Game Theory provides a unified framework with robust mathematical foundations for the proper formulation and analysis of such systems.

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The building blocks

In principle, the mathematical theory of games and gaming was first developed as a model for situations of conflict. *Game Theory* is the area of research that provides mathematical formulations and a proper framework for studying adversarial situations. Although E. Borel looked at similar problems in the 1920s, John Von Neumann and Oskar Morgenstern provided two breakthrough papers (1928, 1937) as a kick-start of the field. Since the early 1940's, with the end of World War II and stepping into the era of the Cold War that followed, the work of Von Neumann and Morgenstern has provided a solid foundation for the most simple types of games, as well as analytical forms for their solutions, with many applications to Economics, Operations Research and Logistics. However, there are several limitations that fail to explain various aspects of real-world conflicts (Luce & Raiffa, 1957), especially when the human factor is a major factor. The application of game-theoretic formulations in designing experiments in Psychology and Sociology is usually referred to as *gaming* (Thomas, 1984; Camerer, Ho, & Chong, 2002).

Games, strategies and solutions

The term *game* is the mathematical formulation of adversarial situations, where two or more *players* are involved in competitive or cooperative acts. The *zero-sum* games are able to model situations of conflict between two or more players, where one's gain is the other's loss and vice versa. Most military problems can be modeled as some form of two-player zero-sum game. When the structure of the game and the rationale of the players is known to all, then the game is one of *complete information*, while if some of these information is somehow hidden or unknown to some players, it is one of *incomplete information*. Furthermore, if all players are fully informed about their opponents' decisions, the game is one of *perfect information*. In contrast, if some of the information about the other players' moves, the game is one of *partial* or *imperfect information*. Such games of both complete and perfect information are all board games, like Chess, Go and Checkers, and they are all zero-sum by nature.

Von Neumann and Morgenstern (von Neumann & Morgenstern, 1947) proved that there is at least one optimal plan of decisions or *strategy* for each player in all zero-sum games, as well as a *solution* to the game that comes naturally as a result of all players following their optimal strategies. At the game's solution, each player can guarantee that the maximum gain an opponent can gain is kept under a specific minimal limit, defined only by this player's own strategy. This assertion was formulated as a theorem called *Minimax* and in the simple case of two opposing players with only two strategies each the Minimax solution of the game can be calculated analytically as a solution of a 2x2 set of linear equations, which determine the stable solution or *saddle-point*.

The consequences of the Minimax theorem have been thoroughly studied for many years after its proof. As an example, it mathematically proves the assertion that all board games, including the most complex ones like Chess, have at least one solution, i.e., an optimal (pure) strategy for both players that can be analytically calculated, at least in theory (Stahl, 1999; Thomas, 1984; Prywes, 1999). Of course, in the case of Chess the game space is so huge that it is still unfeasible today to calculate this theoretically optimal strategy, even with the help of parallel processing in supercomputers. In contrast, Checkers is a much smaller (3x3) and simpler game, making it possible to create the complete game space in any typical desktop computer[1] and calculate the exact optimal strategy - in fact, it is the same strategy that every child soon learns by trial-and-error, playing in a way that always leads to a win or a draw (never loose).

In general, if the chosen strategy of one player is known to its opponent, then an optimal counter-strategy is always available. Hence, in *simultaneous* games where the opposing moves are conducted at the same time, each player would normally try not to employ a deterministic way of choosing its strategy and conceal this choice until the very last moment. However, the Minimax theorem provides a mathematically solid way of nullifying any stochastic aspect in determining the opponent's choice and, in essence, make its exact choice irrelevant: no matter what the opponent does, the Minimax solution ensures the *minimum losses* to each player, given a specific game setup. In other words, it provides an analytic way to determine the best *defensive* strategy, instead of a preference to offensive strategies. In some zero-sum games this leads to one stable outcome or *equilibrium*, where each player would have no incentive not to choose its Minimax strategy; however, if this choice leads to a negative handicap for this player if it is known with complete certainty by the others, then this choice should not be manifested as certain. In practice this means that the Minimax solution would not be any single one of the player's *pure* strategies but rather a weighted combination of them in a *mixed* strategy scheme, where each weight corresponds to the probability of choosing one of the available pure strategies via a random mechanism. This notion of using mixtures of pure strategies for randomly choosing between them leads to a false sense of security in single-turn games, since the optimality of the expected outcome of the mixed strategy scheme refers to the asymptotic (long-term) and not the "spot" (one-shot) payoff. Moreover, a game may involve an infinite number of strategies for the players, in a discrete or continuous set; in this case the game is labeled as *continuous* or *infinite*, while a *finite* game is one with a limited number of (discrete) strategies (Dresher, 1961; Thomas, 1984).

When the game is inherently repetitive or *iterative*, i.e., includes multiple turns and not just one, even the pure strategy suggested by Minimax should not be chosen deterministically in every turn if according to the game setup this information might provide a handicap to the opponent. This is a topic of enthusiastic discussion about the optimality of the Minimax solution and its inherent defensive nature, as it is not clear in general when information about an opponent's *next* move is available and trustworthy enough to justify any deviation from this Minimax strategy.

[1] In Checkers, the board size is 3x3 and each position can be either empty or host the mark of of one of the two players, "X" or "O". Hence, if the two players are treated as interchangeable (i.e., who plays first) and no other symmetries are considered, the total number of all possible distinct board setups is: $9 \cdot 8 \cdot \ldots \cdot 2 \cdot 1 = 9! = 362,880$. After applying the game rules and pruning the game tree for early stops (with incomplete boards), the true number of game states is about 2/3 of that set. Using simple tree-node representation for each board setup, e.g. a 3-value 9-positions vector dictionary ($= 3^9 \simeq 2^{14.265} \leq 2^{15} < 2^{16} = 2$ bytes), such a program would only require about 484 KB or less than 0.5 MB. This is roughly the size of a small-sized photo taken by the camera of a low-end smartphone today, while in the '80s this was almost the total size of RAM in a typical PC.

Summary:
• In *zero-sum* games, one player's gains is another's losses (and vice versa).
• Information about the game structure and the opponents' moves may be *complete* or not, *perfect* or not.
• All board games are inherently zero-sum, of complete and perfect information.
• The *Minimax theorem* assures that all board games have at least one theoretically optimal way to play them, although its exact calculation may be unfeasible in practice for some games (e.g. Chess, Go).
• The *Minimax solution* of a game is the combination of players' strategies that lead to an *equilibrium* or *saddle-point*.

Nonzero-sum games and Nash equilibria

Although the Minimax theorem provided a solid base for solving many types of games, it is only applicable in practice for the zero-sum type of games. In reality, it is common that in a conflict not all players receive their opponents' looses as their own gain and vice versa. In other words, it is very common a specific combination of decisions between the players to result in a certain amount of "loss" to one and a corresponding "gain", not of equal magnitude, to another. In this case, the game is called *nonzero-sum* and it requires a new set of rules for estimating optimal strategies and solutions. As each player's gains and losses are not directly related to the opponents', the optimal solution is only based on the assertion that it should be the one that ensures that the player has "no regrets" when choosing between possible decision options. This essentially means that, since each player is now interested in his/her own gains and losses, the optimal solution should only focus on maximizing each player's own *expectations* (Owen, 1995; Montet & Serra, 2003; Dixit & Nalebuff, 2008). The Minimax property can still be applied in principle when the single most "secure" option must be identified, but now the solution of the game gains a new meaning.

During the early 1950's, John Nash has focused primarily on the problem of finding a set of *equilibrium points* in nonzero-sum games, where the players eventually settle after a series of competitive rounds of the game (Nash, 1950a, 1950b). The failure of the Minimax approach to predict real-world outcomes in nonzero-sum games comes from the fact that the players are assumed to act independently and simultaneously, while in reality they usually are not. Experience shows that *possibly* better payoffs with what a player *might* choose, after observing the opponent's moves, is a very strong motivator when choosing its actual strategy (Mero, 1998). In strict mathematical terms, these equilibrium points would not be the same in essence with the Minimax solutions, as they would come as a result of the players' competitive behavior over several "turns" of moves and not as an algebraic solution of the mathematical formulation in a single-turn game.

In 1957 Nash has successfully proved that indeed such equilibrium points exist in all nonzero-sum games, in a way that is analogous to the Minimax theorem assertion. This new type of stable outcome is referred to as *Nash equilibrium* after his name and can be considered a generalization of the corresponding Minimax equilibrium in zero-sum games. In essence, they are the manifestation of the *no regrets* principle for all players, i.e., not regretting their final choice after observing their opponents' behavior (Stahl, 1999; Thomas, 1984). However, although the Nash theorem ensures that at least one such Nash equilibrium exists in all nonzero-sum games, there is no clear indication on how the game's solution can be analytically calculated at this point. In other words, although a solution is known to exist, there is no closed form for nonzero-sum games until today. Seminal works by C. Daskalakis & Ch. Papadimitriou in 2006-2007 and on have proved that, while Nash equilibria exist, they may be unattainable and/or practically impossible to calculate due to the inherent algorithmic complexity of this problem, e.g. see: (Daskalakis & Papadimitriou, 2006; Papadimitriou, 2011).

It should be noted that players participating in a nonzero-sum game may or may not have the same options available as alternative course of action, or the same set of options may lead to different gains or *payoffs* between the players. When players are fully interchangeable and their ordering in the game makes not difference to the game setup and its solutions, the game is called *symmetrical*. Otherwise, if exchanging players' position does not yield a proportional exchange of their payoffs, then the game is called *asymmetrical*. Naturally, symmetrical games lead to Nash equilibrium points that appear in pairs, as an exchange between players creates its symmetrical counterpart.

Summary:
• In *nonzero-sum* games, the payoffs of the players are separated (although may be correlated).
• If players are allowed to observe their opponents moves over several iterations, then the "no regrets" principle is a strong incentive to revise their own strategies, even though their payoffs are separated.
• The *Nash equilibrium theorem* ensures that, under these conditions, there are indeed stable solutions in nonzero-sum games, similarly to the Minimax theorem for zero-sum games.
• However, calculating the optimal strategies and the game solution for these Nash equilibria is a vastly more complex and generally unfeasible task.

Cooperation instead of competitiveness

The seminal work of Nash and others in nonzero-sum games was a breakthrough in understanding the outcome in real-world adversarial situations. However, the Nash equilibrium points are not always the globally optimal option for the players. In fact, the Nash equilibrium is optimal only when players are strictly competitive, i.e., when there is no chance for a mutually agreed solution that benefits them more. These strictly competitive forms of games are called *non-cooperative* games. The alternative option, the one that allows communication and prior arrangements between the players, is called a *cooperative* game and it is generally a much more complicated form of nonzero-sum gaming. Naturally, there is no option of having cooperative zero-sum games, since the game structure itself prohibits any other settlement between the players other than the Minimax solution.

The cooperative option

The problem of cooperative or possibly cooperative gaming is the most common form of conflict in real life situations. Since nonzero-sum games have at least one equilibrium point when studied under the strictly competitive form, Nash has extensively studied the cooperative option as an extension to it. However, the possibility of finding and mutually adopting a solution that is better for both players than the one suggested by the Nash equilibrium, essentially involves a set of behavioral rules regarding the players' stance and "mental" state, rather than strict optimality procedures (Mero, 1998). Nash named this process a *bargain* between the players, trying to mutually agree on one solution between multiple candidates within a *bargaining set* or *negotiation set*. In practice, each player should enter a bargaining procedure if and only if there is a chance that a cooperative solution exists and it provides at least the same gain as the best strictly competitive solution, i.e., the best Nash equilibrium. In this case, if such a solution is agreed between the players, it is called *bargaining solution* of the game (Montet & Serra, 2003; Owen, 1995).

As mentioned earlier, each player acts upon the property of no regrets, i.e., follow the decisions that maximize their own expectations. Nevertheless, the game setup itself provides means of improving the final gain in an agreed solution. In some cases, the bargaining process may involve the option of *threats*, that is a player may express the intention to follow a strategy that is particularly costly for the opponent. Of course, the opponent can do the same, focusing on a similar threat. This procedure is still a cooperative bargaining process, with the threshold of expectations raised for both players. The result of such a process may be a mutually *deterring* solution, which in this case is called a *threatening solution* or *threat equilibrium*. There is also evidence that, while cooperative strategies do exist, in some cases "cooperation" may be the result of *extortion* between players with unbalanced power and choices (Press & Dyson, 2012).

In his work, Nash has formulated a general and fairly logical set of six axioms, the *Nash's bargaining axioms*, regarding the behavior of *rational* players, in order to establish a non-empty bargaining set, i.e., to have at least one stable solution (equilibrium) (Montet & Serra, 2003; Owen, 1995; Nash, 1950a). In non-strict form, these axioms can be summarized in the following propositions:

• Any of the cooperative options under consideration must be feasible and yield at least the same payoff as the best strictly non-cooperative option for all players, i.e., cooperation must be mutually beneficial.

• Strict (mathematical) constraints: Pareto optimality, independence of irrelevant alternatives, invariance under linear transformations, symmetry (Thomas, 1984; Owen, 1995; Montet & Serra, 2003).

The first proposition essentially defines the term "rationality" for a player: he/she always acts with the goal of maximizing own gains and minimizing losses, regardless if this means strictly competitive or possibly cooperative behavior. The second proposition names a set of strict mathematical preconditions (not always satisfied in practice), in order for such a bargaining set to exist. Having settled on these axioms, Nash was able to prove the corresponding *bargaining theorem*: under these axioms, there exists such a *bargaining process*, it is unique and it leads to a bargaining solution, i.e., equilibrium. However, as in the general case of strictly competitive games, Nash's bargaining theorem does not provide analytical means of finding such solutions.

The notion of bargaining sets and threat equilibrium is often extended in special forms of games that include iterative or recursive steps in gaming, either in the form of multi-step analysis (*meta-games*) or focusing on the transitional aspects of the game (*differential games*). Modern research is focused on methods that introduce probabilistic models into games of multiple realizations and/or multiple stages (Owen, 1995).

> *Summary:*
> • In nonzero-sum games, there may be non-competitive (cooperative) options that are mutually beneficial to all players.
> • Under some general rationality principles, *Nash's bargaining theorem* ensures that these cooperative outcomes may indeed become the game solution, provided that strict competitiveness yields lower gains for all.
> • The procedure of structuring the "common ground" of cooperation between the players, normally conducted over several iterations, is the bargaining process.

Coalitions, stable sets, the Core

Nash's work on the Nash equilibrium and bargaining theorem provides the necessary means to study n-person non-cooperative and cooperative games under a unifying point of view. Specifically, a nonzero-sum game can be realized as a strictly competitive or a possibly cooperative form, according to the game's rules and restrictions. Therefore, the cooperative option can be viewed as a generalization to the strictly competitive mode of gaming.

When players are allowed to cooperate in order to agree on a mutually beneficial solution of game, they essentially choose one strategy over the others and bargain this option with all the others in order to come to an agreement. For symmetrical games, this is like each player chooses to join a group of other players with similar preference over their initial choice. Each of these groups is called a *coalition* and it constitutes the basic module in this new type of gaming: the members of each coalition act as cooperative players joined together and at the same time each coalition competes over the others in order to impose its own position and become the *winning coalition*. This setup is very common when modeling voting schemes, where the group that captures the relative majority of the votes becomes the winner.

Coalition Theory is closely related to the classical Game Theory, especially the cooperating gaming (Owen, 1995; Montet & Serra, 2003). In essence, each player still tries to maximize its own expectations, not individually any more but instead as part of a greater opposing term. Therefore, the individual gains and capabilities of each player is now considered in close relation to the coalition this player belongs, as well as how its individual decision to join or leave a coalition affects this coalition's winning position. As in classic nonzero-sum games, the notion of equilibrium points and solutions is considered under the scope of domination or not in the game at hand. Furthermore, the theoretical implications of having competing coalitions of cooperative players is purely combinatorial in nature, thus making its analysis very complex and cumbersome. There are also special cases of collective decision schemes where a single player is allowed to *abstain* completely from the voting procedure, or prohibit a contrary outcome of the group via a *veto* option.

In order to study the properties of a single player participating in a game of coalitions, it is necessary to analyze the wining conditions of each coalition. Usually each player is assigned a fixed value of "importance" or "weight" when participating in this type of games and each coalition's power is measured as a sum over the individual weights of all players participating in this coalition. The coalition that ends up with the highest cumulative value of power is the winning coalition. Therefore, it is clear that, while each player's power is related to its individual weight, this relation is *not* directly mapped on how the participation in any arbitrary coalition may affect this coalition's winning or losing position. As this process stands true for all possible coalitions that can be formed, this competitive type of "claiming" over the available pool of players/voters by each coalition suggests that there are indeed configurations that marginally favor the one or the other coalition, i.e., a set of "solutions".

The notion of solution in coalition games is somewhat different from the one suggested for typical nonzero-sum games, as it identifies minimal settings for coalitions that dominate all the others. In other words, they do not identify points of maximal gain for a player or even a coalition, but equilibrium "points" that determine which of the forming coalitions is the winning one. This type of "solutions" in coalition games is defined in close relation to *domination* and *stability* of such points and they are often referred to as *the Core*. Von Neumann and Morgenstern have defined a somewhat more relaxed definition of such conditions and the corresponding solutions are called *stable sets* (Owen, 1995; Montet & Serra, 2003). It should be noted that, in contrast to Nash's theorems and the Minimax assertion of solutions, there is generally no guarantee that solutions in the context of the Core and stable sets need to exist in an arbitrary coalition game.

Summary:
- Players of similar preferences and mutual benefits may join in groups or *coalitions*; these coalitions may be competing with each other, similarly to competitive games between single players.
- The study of games between coalitions is inherently more complex than with single players, as in this case every player contributes to the collective "power" and enjoys a share of the wins.
- In general, coalitions are formed and structured under the scope of *voting ensembles*, where the voting weight of each individual player contributes to the combined weight of the coalition.

Indices of power in committees

The notion of the Core and stable sets in coalition gaming is of vital importance when trying to identify the winning conditions and the relative power of each individual player in affecting the outcome of the game. The observation that a player's weight in a weighted system may not intuitively correspond to its voting "power" goes back at least to Shapley and Shubik (1954). For example, a specific weight distribution to the players may make them relatively equivalent in terms of voting power, while only a slight variation of the weights may render some of them completely irrelevant on determining the winning coalition (Taylor & Zwicker, 1993).

Shapley and Shubik (1954) and later Banzhaf and Coleman (1965, 1971) suggested a set of well-defined equations for calculating the relative power of each player, as well as each forming coalitions as a whole (Owen, 1995; Montet & Serra, 2003). The *Shapley-Shubik index of power* is based on the calculation of the actual contribution of each player entering a coalition, in terms of improving a coalition's gain and winning position. Similarly, the *Banzhaf-Coleman index of power* calculates how an individual player's decision to join or leave a coalition ("swing vote") results in a winning or loosing position for this coalition, accordingly. Both indexes are basically means of translating each player's individual importance or weight within the coalition game into a quantitative measure of power in terms of determining the winner. While both indices include combinatorial realizations, the Banzhaf index is usually easier to calculate, as it is based on the sum of "shifts" on the winning condition a player can incur (Berg, 1997). Furthermore, its importance in coalition games is made clearer when the Banzhaf index is viewed as the direct result of calculating the derivatives of a *weighted majority game* (WMG).

Seminal work by L. S. Penrose (Penrose, 1952), as well as more recent studies with computer simulations (Chang, Chua, & Machover, 2006), have shown that this discrepancy between voting weights and actual voting power is clearly evident when there is large variance in the weighting profile and/or when the voting group has less than 12-15 members.

Even in large voting pools, the task of designing optimal voting mechanisms and protocols with regard to some *collective efficiency* criterion is one of the most challenging topics in Decision Theory.

Summary:
• *Weighted majority games* (WMG) are the typical theoretical structures of the process of formulating the collective decision within a coalition.
• In voting ensembles, each player's voting weight is *not* directly proportional to his/her true *voting power* within the group, i.e., the level of steering the collective decision towards its own choices.

Voting ensembles and majority winners

In most cases, majority functions that are employed in practice very simplistic when it comes to weighting distribution profile or they imply a completely uniform weight distribution. However, a specific weighting profile usually produces better results, provided that is simple enough to be applied in practice and attain a consensus in accepting it as "fair" by the voters. Taylor and Zwicker (Taylor & Zwicker, 1993) have defined a voting system as *trade robust* if an arbitrary series of trades among several winning coalitions can never simultaneously render them losing. Furthermore, they proved that a voting system is trade robust if and only if it is weighted. This means that, if appropriate weights are applied, at least one winning coalition can benefit from this procedure.

As an example, institutional policies usually apply a non-uniform voting scheme when it comes to collective board decisions. This is often referred to as the "inner cabinet rule". In a hospital, senior staff members may attain increased voting power or the chairman may hold the right of a tie-breaking vote. It has been proven both in theory and in practice that such schemes are more efficient than simple majority rules or any restricted versions of them like trimmed means. Nitzan and Paroush (Nitzan & Paroush, 1982) have studied the problem of optimal *weighted majority rules* (WMR) extensively and they have proved that they are indeed the optimal decision rules for a group of decision makers in dichotomous choice situations. This proof was later extended by Ben-Yashar and Paroush, from dichotomous to polychotomous choice situations (Ben-Yashar & Nitzan, 2001); hence, the optimality of the WMR formulation has been proven theoretically for any n-label voting task.

The weight optimization procedure has been applied experimentally in trained or other types of combination rules, but analytical solutions for the weights is not commonly used. However, Shapley and Grofman (Shapley & Grofman, 1984) have established that an analytical solution for the weighting profile exists and it is indeed related to the individual player skill levels or *competencies* (Karotkin, 1998). Specifically, if decision independence is assumed for the participating players, the optimal weights in a WMR scheme can be calculated as the log-odds of their respective skill probabilities, i.e.:

$$w_k = \log(O_k) = \log\left(\frac{p_k}{1 - p_k}\right) \tag{1}$$

where p_k is the competency of player k and w_k is its corresponding voting weight. Interestingly enough, this is exactly the solution found by analytical Bayesian-based approaches in the context of decision fusion of independent experts in Machine Learning (Kuncheva, 2004). The optimality assertion regarding the WMR, together with an analytical solution for the optimal weighting profile, provides an extremely powerful tool for designing theoretically optimal collective decision rules. Even when the independence assumption is only partially satisfied in practice, studies have proved that WMR-based models employing log-odds weighting profiles for combining pattern classifiers confirm these theoretical results (Georgiou, Mavroforakis, & Theodoridis, 2006; Georgiou & Mavroforakis, 2013).

Summary:
• *Weighted majority rules* (WMR) have been proven theoretically as the optimal decision-making structures in weighted majority games.
• The *log-odds* model has been proven both as the theoretically optimal way to weight the individual player's votes, provided that they decide independently.
• The optimality of the log-odds weighting method has also been proven experimentally, even when the independence assumption is only partially satisfied.

Collective efficiency

Condorcet (1785) (Condorcet, 1989) was the first to address the problem of how to design and evaluate an efficient voting system, in terms of fairness among the people that participating in the voting process, as well as the optimal outcome for the winner(s). This first attempt to create a probabilistic model of a voting body is known today as the *Condorcet Jury Theorem* (Young, 1988). In essence, this theorem says that if each of the voting individuals is somewhat more likely than not to make the "better" choice from a set of alternative options; and if each individual makes its own choice *independently* from all the others, then the probability that the group majority is "correct" is greater than the individual probabilities of the voters. Moreover, this probability of correct choice by the group increases as the number of independent voters increases. In practice, this means that if each voter decides independently and performs marginally higher than 50%, then a group of such voters is *guaranteed* to perform better than each of the participating individuals. This assertion has been used in Social sciences for decades as a proof that decentralized decision making, like in a group of juries in a court, performs better than centralized expertise, i.e., a sole judge. The Condorcet Jury Theorem and its implications have been used as one guideline for estimating the efficiency of any voting system and decision making in general

(Young, 1988). Under this context, the coalition games are studied by applying quantitative measures on *collective competence* and *optimal distribution of power* in the ensemble, e.g. tools like the Banzhaf or Shapley indices of power. The degree of consistency of such a voting scheme on establishing the pair-wise winner(s), as the Condorcet Jury Theorem indicates, is often referred to as the *Condorcet criterion*.

Shapley-Shubik and Banzhaf-Coleman are only two of several formulations for the indices of power in voting ensembles, each defining different payoff distributions or realizations among the members of winning coalitions. In general, these formulations are collectively referred to as *semivalue functions* or *semivalues* and they are considered more or less equivalent in principle, although may be different in exact values. Almost all of them are based on combinatorial functions (inclusion-exclusion operations in subsets) and, as a result, there is no easy way to formulate proper inverse functions that can be calculated in polynomial time. Therefore, the design of exact voting profiles with weights based on semivalues, instead of competencies as described above (log-odds), is generally impractical even for ensembles of small sizes.

For further insight on weighted majority games, weighted majority voting, collective decision efficiency and Condorcet efficiency, as well as applications to Machine Learning for designing pattern classifiers, see (Georgiou, 2015; Georgiou et al., 2006; Georgiou & Mavroforakis, 2013).

Summary:
- Under the assumption of independent voters and that each decides "correctly" marginally higher than 50% of the time, then their collective decision as a group is theoretically proven to be asymptotically better any single member of the ensemble.
- Furthermore, as the size of the ensemble increases, its collective competency is guaranteed to increase too.
- In the other hand, the problem of formulating an analytical solution for the optimal distribution of voting power within such a group, i.e., the design of theoretically optimal *voting mechanisms*, is still an open research topic.

Game Analysis & Solution Concepts

One of the most important factors in understanding and analyzing games correctly is the way they are represented. Games can be represented and analyzed in two generic formulations: (a) the *analytical* or *normal* form, where each player is manifested as one dimension and its available choices (strategies) as offsets on it, and (b) the *extensive* or *tree-graph* form, where each player's "move" correspond to a node split in a tree representation. Each one of them has its own advantages and disadvantages, but theoretically they are equivalent.

Table 1
Generic 2x2 zero-sum game in analytical form.

Game example		Player-2	
		y	$1-y$
Player-1	x	a	b
	$1-x$	c	d

Games in analytical (matrix) form

In Table 1, an example of a zero-sum game in analytical form is presented. Player-1 is usually referred to as the "max" player and Player-2 is referred to as the "min" player, while rows and columns correspond to each player's available strategies, respectively. Since this is a zero-sum game and one player's gains is the other player's losses, the "max" player tries to maximize the game value (outcome) while the "min" player tries to minimize it. In the context of the Minimax theorem, Player-2 chooses the *maximum-of-minimums*, while Player-2 chooses the *minimum-of-maximums*. The x and y correspond to the weight or probability of choosing the first strategy and, since this is a 2x2 game, the other strategies are attributed with the complementary probabilities, 1-x and 1-y.

The exact Minimax solution for x and y depends solely on the values of the individual payoffs for each of the four outcomes. Here, it is assumed that there is no *domination* in strategies, i.e., there is no row/column that is strictly "better" than another row/column (column-wise/row-wise, respectively, all payoffs). For example, Player-1 would have a dominating strategy in the first row if and only if $a \geq c$ and $b \geq d$. Based on this generic setup, this is a typical 2x2 system of linear equations and, if no domination is present, its solution can be determined analytically as (Stahl, 1999; Dresher, 1961; Maschler, Solan, & Zamir, 2013):

$$[x, 1-x] = \left[\frac{d-c}{a-b-c+d}, \frac{a-b}{a-b-c+d} \right] \tag{2}$$

$$[y, 1-y] = \left[\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d} \right] \tag{3}$$

$$u = \frac{ad-bc}{a-b-c+d} \tag{4}$$

The Minimax solution $[x, y]$ determines the saddle-point, i.e., the equilibrium that is reached when both opponents play optimally in the Minimax sense, when the game has no pure (non-mixed) solution. In this case, the expected payoff or *value* of the game for both players is calculated by u (remember, this is a zero-sum game). If the game has a pure solution, then it is determined as either 0 or 1 for each probability x and y. Table 2 illustrates a zero-sum game and the corresponding pure Minimax solution, by selecting the appropriate strategies for each player. In this case, "max" Player-1 chooses the the maximum {1} between the two minimum values {-3,1} from its own two possible worst-case outcomes, while "min" Player-2 chooses the the minimum

Table 2
Example 2x2 zero-sum game in analytical form.

Game example		Player-2 (0)	Player-2 (1)
Player-1	(0)	0	-3
	(1)	4	**1**

Table 3
Example of a 2x2 nonzero-sum game in analytical form.

Game example		Player-2 y	Player-2 $1-y$
Player-1	x	(a_1,a_2)	(b_1,b_2)
	$1-x$	(c_1,c_2)	(d_1,d_2)

{1} between the two maximum values {4,1} from its own two possible worst-case outcomes. Hence, the pure solution [1,1] is the Minimax outcome.

In nonzero-sum games, the analytical form is still a matrix, but now the payoffs for each player are separate, as illustrated in Table 3. Here, since the payoffs are separated, both players are treated as "max" and the Minimax solution for each one is calculated by selecting the maximum-of-minimums as described before for zero-sum games, focused solely on its own payoffs from each value pair.

Although a (pure) Minimax solution can always be calculated for nonzero-sum games, the exact Nash equilibrium solution is a non-trivial task that cannot be solved analytically in the general case. However, pure Nash equilibrium outcomes can be identified by locating any payoff pairs (z, w) such that z is the maximum of its column and w is the maximum of its row. In other words, every row for Player-1 is scanned and every entry in it is compared to the values in the same column, marking it if it is the maximum among them; the same process is conducted for every column for Player-2, scanning each value row-wise for its maximum; any payoff pair that has both values marked as maximums is a Nash equilibrium in the game. Table 4 illustrates such an example, where asterisk (*) marks the identified max-values and the single Nash equilibrium for [A,B] at (2,4). Here, although the strategies are the same for both players, their (separated) payoffs are not, hence the game is referred to as *asymmetric*. According to the *oddness theorem* by Wilson (1971), the Nash equilibria almost always appear in odd numbers (Stahl, 1999; Owen, 1995), at least for *non-degenerate* games, where mixed strategies are calculated upon k linearly independent pure strategies.

Table 4
*Example of a 2x2 nonzero-sum game with one Nash equilibrium at [*A,B*]:(2,4).*

Game example		Player-2 A	Player-2 B
Player-1	A	(3,3)	(2*,4*)
	B	(4*,1)	(1,2*)

Summary:
- Game representation in *analytical* form introduces a game matrix, with row and column positions associated to the strategies available to the players and contents associated to the corresponding payoffs.
- Analytical-form representation introduces very convenient ways to identify Minimax solutions and Nash equilibria in games.
- However, they are appropriate mostly for 2-player simultaneous games, since any other configuration cannot be fully illustrated.

Games in extensive (tree-graph) form

In the extensive form the game is represented as a tree-graph, where each node is a state labeled by a player's number and each (directed) edge is a player's choice or "move". Strictly speaking, this is a form of state-transition diagram that illustrates how the game evolves as the players choose their strategies. Figure 1 shows such a 2x2 nonzero-sum game of perfect information, while Figure 2 shows a similar 2x2 game of imperfect information (Thomas, 1984; Montet & Serra, 2003; Wikipedia.org, n.d.-a; Schalk, 2003; Fudenberg & Tirole, 1991; Dresher, 1961). Nodes with numbers indicate players, edges with letters indicate chosen strategies (here, symmetric) and separated payoffs (in parentheses) indicate the game outcome after one full round. The dashed line between the two nodes for Player-2 indicate that its current true state is not clearly defined due to imperfect information regarding Player-1's move. In practice, these two states form an *information set* for Player-2, which has no additional information to differentiate between them. This is also valid in the case of simultaneous moves, where Player-2 cannot observe Player-1's move in advance of its own, and vice versa. In extensive form, an information set is indicated by a dotted line or by a loop, connecting all nodes in that set.

The extensive form of game is usually the preferred way to represent the tree-graph of simple 2-player board games, where each node is clearly a state and each edge is a player's move. Even in single-player games, where a puzzle has to be solved through a series of moves (e.g. Rubik's cube)[2], the

[2] The combinatorial analysis of the classic 3x3x6 Rubik's cube should take into account tile permutations that can only be reached by the available shifts and turns of the slices of the device. Therefore, a totally "free" permutation scheme would produce: $8! \cdot 3^8 \cdot$

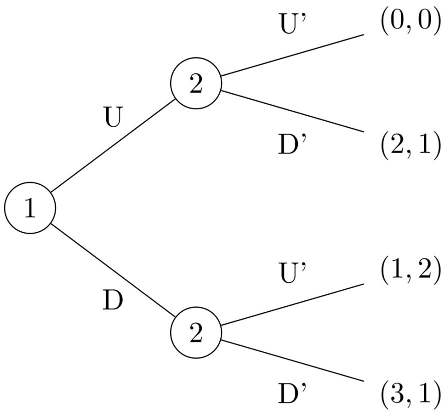

Figure 1. Example of a 2x2 nonzero-sum game of *perfect* information.

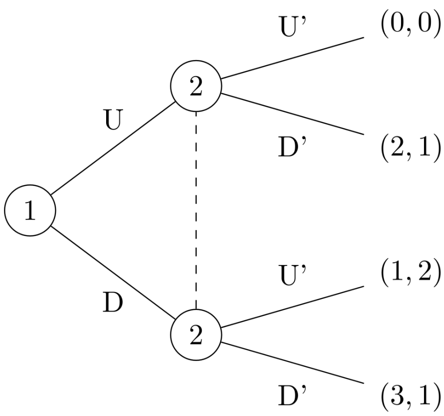

Figure 2. Example of a 2x2 nonzero-sum game of *imperfect* information.

tree-graph is a very effective way to organize the game under an algorithmic perspective, in order to program a "solver" in a computer. In practice, the problem is structured as sequences of states and transitions in a tree-graph manner and the "game" is explored as it is evolving, move after move, expanding the tree-graph from every terminal node. The tree-graph can be expanded either by full a level ("breadth-first"), or from a branch all the way down to non-expandable terminal nodes ("depth-first"), or some hybrid scheme between these two alternatives.

As described above, small games like Checkers can be structured and expanded fully, with their tree-graph having only internal (already expanded) and terminal nodes; however, in larger games like Chess or Go this is practically unfeasible even with super-computers. In such cases, the algorithm should assess the "optimality" of each expandable terminal node with regard to relevance towards the predefined goal ("win" or "solution"), sort all these nodes according to their ranking and choose the "best" ones for expansion in the next iteration. This way, the search is sub-optimal but totally feasible with almost any memory constraints - this is exactly how most computer players are programmed for playing board games or solving complex puzzle games. In Artificial Intelligence, algorithms like $A*$ and AB solve this type of problems as a path-finding optimization procedure towards a specified goal (Russell & Norvig, 2009; Nilsson, 1998).

Figure 3 illustrates the way a path-finding algorithm like A* would work in expanding a tree-graph as described above. The "root" node is the starting state in a puzzle game (single-player) and each node represents a new state after a valid move. The numbers indicate the sequence in which the nodes are expanded, according to some optimality-ranking function (not relevant here). For example, node "4" in the 3rd level is expanded before node "5" in the 2nd level, node "21" in the 5th level is expanded before node "22" in the 3rd level, etc. Here, node "30" in the 5th level is the last and most relevant terminal node (still expandable) towards the goal, hence the optimal path from the "root" state is currently the: "5"→"7"→"11"→"30" and the next "best" single-step move is the one towards "5". The tree-graph can be expanded in an arbitrary number of levels according to the current memory constraints for the program, but the same path-finding procedure has to be reset and re-applied after the realization of each step when two or more players are involved, since every response from the opponent effectively nullifies every other branch of the tree-graph.

It should be mentioned that, although the extensive form of game representation is often inefficient for large games like Chess, it can be used as a tool in the proof of the existence of an optimal solution (Ferguson, 2014; Thomas, 1984). Specifically, in every such game of complete and perfect information (all board games), each player knows its exact position in the graph-tree prior to choosing the next move. In other words, each player is not only aware of the complete structure of the game but also knows all the past moves of the game, including the ones of random choice. Hence, since there is no uncertainty in the moves, each player can remove the dominated strategies and subsequently identify the optimal choice, which is always a pure strategy, i.e., the one that corresponds to the saddle-point of the game. This proof actually ensures the existence of a (pure) optimal strategy in every typical board game, no matter how large or complex it is. Examples include Tic-Tac-Toe, Chess, Backgammon, etc.

$12! \cdot 2^{12} = 519,024,039,293,878,272,000$ cube instances, while in practice the possible permutations are only: $8! \cdot 3^7 \cdot (12!/2) \cdot 2^{11} = 43,252,003,274,489,856,000$ cube instances (about 12 times fewer) (Wikipedia.org, n.d.-b).

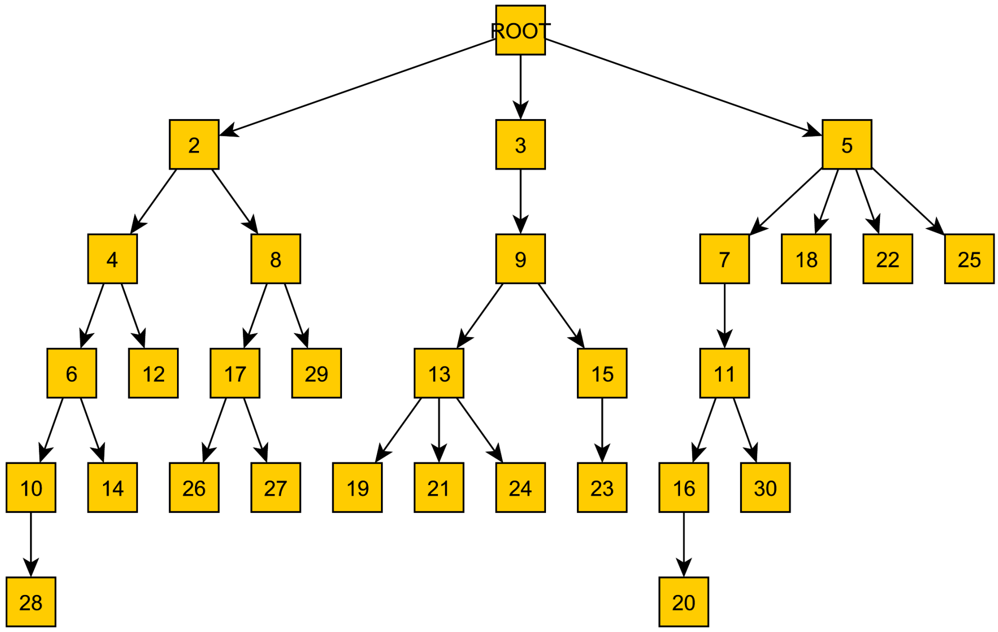

Figure 3. Example of the way a path-finding algorithm like A* would work in expanding the tree-graph of a single-player "puzzle" game like Rubik's cube.

Summary:
- Game representation in *extended* form introduces a tree-graph, with nodes associated to individual players and (directed) edges associated to selected strategies ("moves").
- Extended-form representation introduces very convenient ways to identify chains of moves and solution paths.
- However, the calculation of Minimax solutions and Nash equilibria is not straight-forward.

The four interesting cases

In the real world, games may be either zero-sum or nonzero-sum by nature. As described previously, the case of zero-sum games can be considered simpler and much easier to solve analytically, since it can be formulated as a typical algebraic set of linear equations that define the Minimax solution, regardless if it contains pure or mixed strategies (Stahl, 1999; Dresher, 1961). However, nonzero-sum games are inherently much more complex and require non-trivial solution approaches, usually via some Linear Programming (constraint) optimization procedure, e.g. see: (Gu, 2008; Sierksma, 2001). In fact, it has been proven that the general task of finding the Nash equilibria is algorithmically intractable[3] (Daskalakis & Papadimitriou, 2006; Daskalakis, Goldberg, & Papadimitriou, 2009a, 2009b; Papadimitriou, 2011) - something that puts into a "philosophical" question the very nature and practical usefulness of having proof of game solutions (i.e., stable outcomes) that we may not be able to calculate.

Table 5
The general analytical (matrix) form of a 2x2 nonzero-sum symmetric game.

Game template		Player-2 *C*	*D*
Player-1	*C*	(R,R)	(S,T)
	D	(T,S)	(P,P)

Some cases of nonzero-sum games are particularly interesting, especially when they involve symmetric configurations. The players can switch places, the actual payoff values are usually of much less importance than their relative ordering as a simple preference list, the Minimax and Nash equilibria can be easily identified, yet these simple games seem to capture the very essence of bargaining and strategic play in a vast set of real-world conflict situations with no trivial outcomes.

Table 5 shows a generic template for such very simple symmetric nonzero-sum games, employing only two strategies and four payoff values to completely define such games in analytical (matrix) form. Here, the game is symmetric

[3] In their seminal works, Daskalakis, Goldberg and Papadimitriou have shown that the task of finding a Nash equilibrium is PPAD-complete; informally, PPAD is the class of all search problems which always have a solution and whose proof is based on the parity argument for directed graphs. Due to the proof of intractability, the existence of Nash equilibrium in all nonzero-sum games somewhat loses its credibility as a predictor of behavior.

because the players can switch roles without any effect in their corresponding payoff pairs. Furthermore, they share two common strategies C and D, named typically after the choices of "cooperate" or "defect", while constants P, R, S and T are the real-valued payoffs in each case (Casti, 1997).

In practice, a player's preference of strategies (and hence, the equilibria) depends only on the relative ordering of the corresponding payoffs and not their exact values, which become of real importance only when the actual payoff value of the game solution is to be calculated for each player. There is a finite number of rank combinations, i.e., permutations, of these four constants, which produce all the possible unique game matrices of this type. Specifically, there are $4! = 24$ different ways to order these four numbers, 12 of which can be discarded as qualitatively equivalent to other game configurations. Out of the 12 remaining games, eight of them possess optimal pure strategies for both players, therefore they can be considered trivial in terms of calculating their solution. The four remaining configurations are the most interesting ones, as they do not possess any optimal pure strategy. These are the following:

- *Leader*: $T > S > R > P$.
- *Battle of the Sexes*: $S > T > R > P$.
- *Chicken*: $T > R > S > P$.
- *Prisoner's Dilemma*: $T > R > P > S$.

These four qualitatively unique games seem to capture the essence of most of the majority real-world conflict situations historically. Although they have been studied extensively in the past, there are still many open research topics regarding the feasibility, tractability and stability of the theoretical solutions.

Leader

The *Leader* or *Coordination* game (Montet & Serra, 2003; Owen, 1995; Thomas, 1984; Stahl, 1999; Casti, 1997; Fudenberg & Tirole, 1991) is named after the typical problem of two drivers attempting to enter a stream of increased traffic from opposite sides of an intersection. When the road is clear, each driver has to decide whether to move in immediately or concede and wait for the other driver to move first. If both drivers move in (i.e., choose D), they risk crashing onto each other, while if they both wait (i.e., choose C), they will waste time and possibly the opportunity to enter the traffic. The former case is the worst, hence the payoff of (1,1), while the later case is slightly more preferable with a payoff of (2,2). The best outcome is for one driver to become the "leader" and move first, while the other becomes the "follower" and move second. There is still some difference in their absolute gains, but now the deadlock is resolved in the best possible way, no matter who is actually the leader and who is the follower.

Table 6 illustrates the analytical form of this game setup, where numbers indicate relative preferences rather than absolute gain values. There are two pure Nash equilibria, (3,4) and (4,3), which correspond to the proper assignment of roles to the players, explicitly or implicitly, such that coordination is achieved. Since the game is symmetric the two players can switch roles, with only marginal increase/decrease to their payoffs. In terms of Minimax strategies, each player is free to choose the strategy that guarantees the maximum-of-the-minimums without any concern about the opponent's payoffs, since this is a nonzero-sum game. Hence, the Minimax solution is [C,C] at (2,2) marked in bold.

Table 6
The typical setup of the Leader *game with two players. Nash equilibria are marked with paired asterisks and the Minimax solution with bold numbers.*

Leader game		Player-2 C	Player-2 D
Player-1	C	(**2**,**2**)	(3*,4*)
	D	(4*,3*)	(1,1)

In the real world, the assignment of roles as leader/follower is more effective when applied explicitly, typically by some external mechanism or a predefined set of rules. Street signs, traffic policemen and highway code for driving properly are all such mechanisms for explicit resolution of deadlocks via priority assignment in traffic.

Battle of the Sexes

In the *Battle of the Sexes* game (Montet & Serra, 2003; Owen, 1995; Thomas, 1984; Casti, 1997; Fudenberg & Tirole, 1991), a married couple has to decide between entertainment options for the evening. The husband prefers one choice, while the wife prefers another. The problem is that they would both prefer to concede to the same choice together even if it is not their own, rather than follow their own choices alone. For example, of he wants to watch a sports match on TV and she wants to go out for dinner, they both prefer either watching TV or going out for dinner as long as they are together.

Table 7 illustrates the analytical form of the game, where strategy C is for conceding to the other's preference and D is for defecting to his/her own choice. If they both concede the payoff (1,1) is the worst outcome, since they both end up miserable and bored. If they both defect the payoff (2,2) is marginally better for both, but they end up being alone. The two other cases of someone following the other yields the best payoffs for both, since the game is symmetric and they can switch places. The outcomes (3,4) and (4,3) are actually the two Nash equilibria, similarly to the *Leader* game; however, the Minimax solution (2,2) here corresponds to both players choosing D (not C as in *Leader*) as their best Minimax strategy.

Chicken

One of the most well-known strategic games is *Chicken* (Ferguson, 2014; Maschler et al., 2013; Montet & Serra, 2003; Owen, 1995; Thomas, 1984; Casti, 1997), dating back at least as far as the Homeric era. Two or more adversaries engage in a very dangerous or even lethal confrontation, each having a set of choices at his/her disposal and each of these

Table 7
The typical setup of the Battle of the Sexes *game with two players. Nash equilibria are marked with paired asterisks and the Minimax solution with bold numbers.*

Battle of the Sexes		Player-2 *C*	Player-2 *D*
Player-1	*C*	(1,1)	(3*,4*)
Player-1	*D*	(4*,3*)	(**2**,**2**)

Table 8
The typical setup of the Chicken *game with two players. Nash equilibria are marked with paired asterisks and the Minimax solution with bold numbers.*

Chicken game		Player-2 *C*	Player-2 *D*
Player-1	*C*	(**3**,**3**)	(2*,4*)
Player-1	*D*	(4*,2*)	(1,1)

choices producing more or less damage to all players if their choice is the same. Typically, this translates to the Hollywood's favorite version of two cars speeding towards each other, the drivers can choose to turn and avoid collision or keep the course and risk death if the other driver do not turn either. The game seems simple enough, but there are several theoretical implications that make it one of the most challenging situations, appearing in many real-world conflicts throughout History.

Table 8 illustrates the typical *Chicken* game setup with two players and two strategic choices. Option *C* corresponds to turning away ("swerve") and losing the game, while option *D* corresponds to keeping the course and risk death. The worst possible outcome is at (1,1) when players persist in keeping course and eventually crashing against each other. The mutually beneficial outcome or "draw" is at (3,3) when both players decide to play safe and turn away; this is actually the Minimax solution of the game, i.e., the most conservative and "rational" outcome if the game is a one-off round. On the other hand, there are two Nash equilibria for the two outcomes when only one player turns away and one persists.

One particularly interesting feature of the *Chicken* game is that it is impossible to avoid playing it with some insistent adversary, since refusing to play is effectively equivalent to choosing *C* (swerve). Furthermore, the player who succeeds in making his/her commitment to *D* adequately convincing is always the one that can win at the expense of the other player, assuming that the other player is rational and would inevitably decide to avoid disaster. In other words, the player that is somehow bounded to avoid losing *at any cost* and makes this commitment very clear to the opponent, is the one that will always win against any rational player.

This aspect of credible commitment is closely related to the notion of *reputation*, as well as the strange conclusion that in this game the most effectively "rational" strategy is the manifestation of "irrational" commitment to lethal risk. This becomes especially relevant in cases where the game is played a number of times repeatedly and previous behaviors directly affect the players' strategic choices in the future: once the risky player starts winning he/she may maintain or even improve this advantage, as confidence and prior "risky" behavior makes it more and more difficult for future opponents to decide and deviate from their cautious Minimax choice of swerving. The *Chicken* game is perhaps the most descriptive and simple case where players' previous behavior (i.e., *reputation*) is of such importance for predicting the actual outcome.

Table 9
The typical setup of the Prisoner's Dilemma *game with two players. Nash equilibria are marked with paired asterisks and the Minimax solution with bold numbers.*

Prisoner's Dilemma		Player-2 *C*	Player-2 *D*
Player-1	*C*	(3,3)	(1,4*)
Player-1	*D*	(4*,1)	(**2***,**2***)

Prisoner's Dilemma

This forth basic type of non-trivial, nonzero-sum game is by far the most interesting one. The *Prisoner's Dilemma* game (Ferguson, 2014; Maschler et al., 2013; Montet & Serra, 2003; Owen, 1995; Thomas, 1984; Stahl, 1999; Casti, 1997; Fudenberg & Tirole, 1991) typically involves two prisoners who are accused of a crime. Each of them has the option of remaining silent and withholding any information or confessing to the police and accusing the other by disclosing details about the crime. The first choice *C* is effectively the cooperative option, while the second choice *D* corresponds to purely competitive behavior in order to reduce he/her own damages.

Table 9 illustrates the typical *Prisoner's Dilemma* game setup with two players and two strategic choices. The payoffs here correspond simply to preferences and not real gain/cost values, but the essence and the strategic properties of the game remain intact. In practice, what the game matrix says is that if the two prisoner's remain silent, i.e., mutually cooperate, they will not be freed but they will share an equal, relatively mild conviction. If they both talk and accuse each other, i.e., mutually defect, they will share and equal but more severe conviction. If only one of them talks to the police and the other remains silent, the one that talked is freed and the other serves a full-time conviction for both. It is of course imperative that the two prisoners are immediately separated upon capture and no communication between them is allowed; this does not nullifies any prior arrangements they may have, but isolation after being captured means that neither of them can confirm they loyalty of the other. This is one of the main reasons why police always isolates suspects prior and during any similar investigation.

The real beauty and singularity of the *Prisoner's Dilemma* is that it implies a paradox. A quick analysis of the payoffs in Table 9 yields two extremes at (1,4) and (4,1), corresponding to the two interchangeable cases one player cooperating (C) and one not (D), but in contrast to the three previous games these are *not* Nash equilibria. There is only one Nash equilibrium at (2,2), which is in fact the Minimax solution too. This means that under the solution concepts of both Minimax strategy and Nash equilibrium, theory suggests that the two prisoner's will probably choose to betray one another, despite any previous arrangements. It is clearly evident that the outcome (3,3) is *mutually beneficial* and at the same time unattainable due to lack of communication. However, in therms of strict personal gain, defecting (D) is the dominant strategy for both and neither of them has any incentive to deviate from it. In other words, it appears that defecting is always the optimal choice *regardless* of what the other prisoner does - but if both adopt the same rationale, they will end up at (2,2) which is clearly worse than the (3,3) that they could have gotten if they had chosen mutual cooperation.

The essence of the paradox of *Prisoner's Dilemma* lies in the inherent conflict between *individual* and *collective rationality*. While individual rationality is well-understood, collective rationality deals with the scope of optimizing the *mutual* gain of the players. This is not a default behavior in strictly competitive situations, as in zero-sum games, or nonzero-sum games that do not imply cooperation. However, nonzero-sum games permit the idea of *mutually* optimal gains as a combination of *simultaneously* optimal separate payoffs. Under this broader scope, even (4,1) and (1,4) are worse than (3,3) since they yield a sum of 5 in gain value rather than 6, respectively.

It should also be noted that the single Nash equilibrium in *Prisoner's Dilemma* is stable, while the corresponding pairs of Nash equilibria in the three previous games are inherently unstable, since the players are not in agreement as to which of the two equilibria is preferable. Furthermore, in the three previous games the worst possible outcome comes when both players choose their non-Minimax strategy; in *Prisoner's Dilemma* this is not so. In fact, *Prisoner's Dilemma* has produced lengthy academic debates and hundreds of studies in a wide range of disciplines, from Game Theory and Mathematics to Sociology and Evolutionary Biology. The paradox of this game (as described above) has been illustrated as a notorious example where theory often fails to predict the true "gaming" outcomes in the real world: cooperation can emerge spontaneously, even though theory says it should not (Axelrod, 1984; Axelrod & Dion, 1988; Mero, 1998; Casti, 1997).

Summary:
- There are four basic nonzero-sum game types of particular interest namely: *Leader* (or *Coordination*), *Battle of the Sexes*, *Chicken* and *Prisoner's Dilemma*.
- Three of these games (except *Prisoner's Dilemma*) have two "mirrored" pure Nash equilibria and players receive the worst possible payoff when they choose to deviate from their optimal Minimax strategy.
- *Prisoner's Dilemma* is a very unique type of game, since neither Minimax solution or Nash equilibrium (single one in this case) point to the best *mutually* beneficial outcome; this is informally labeled as the *paradox* of this game.

Signals, Mechanisms & Rationality

Game formulation and representation in analytical or extensive form are imperative for proper analysis and identification of equilibria. However, they fail to capture many elements of *gaming* as a multi-aspect process, especially in relation to *strategic moves*; these are actions performed by the players at different places and times, even before the realization of the current game, with the goal of enhancing strategic advantages and increasing the effectiveness of chosen strategies. Sometimes the "moves" are no more than message exchanges between the players, explicit or implicit, or simply tracking the history of previous choices in iterated games. Formulating these factors into a proper mathematical model can be very difficult, but nevertheless they are matters of great importance in real-world conflict situations.

Signals, carriers & bluffs

The exchange of messages between the players is a very useful option when a player is trying to model or even predict the behavior of its opponent(s). A message or *signal* from one player to another may be voluntary or involuntary, direct or indirect, explicit or implicit (Thomas, 1984; Stahl, 1999). In any case, it carries some sort of strategic information, which is always valuable to the other player if it can be asserted as credible with a high degree of confidence. On the other hand, if this credibility can be manipulated and falsely asserted as such, the source player may gain some strategic advantage by means of deceiving its opponent.

Strategic *signaling* is the process of information exchange between two or more players in a game, using any means or intermediate third-parties as carriers. If the source player does this deliberately, the purpose is to project some strategic preference or *stance* ("posturing") in the game without making any actual "move", in order to intimidate or coordinate with the opponent(s). This is particularly useful in situations where mutually beneficial equilibria are achievable but lack of preference ranking can lead to disastrous lack of coordination. The *Leader* and *Battle of the Sexes* games are such examples (see Tables 6 and 7). On the other hand, if the source player signals its opponent unintentionally, this strategic information could be a "leak" of such importance that may determine the actual outcome of the game.

Explicit signaling means that the source player sends out a clear message with undeniable association and content. An explicit signal may be voluntary or involuntary; in the later case, the message is simply a "leak" with very clear origin and content. *Implicit* signaling happens when the origin or (most commonly) the content of the message is somehow inconclusive or "plausibly deniable" as to the intentions of the source player. A signal exchange may occur directly between the players or via a third-party that performs the role of a carrier. A number of combinations of these attributes are possible in practice, employing direct/indirect messaging, voluntary/involuntary information exchange, with explicit/implicit messages. For example, a third-party carrier may share an implicit signal or "leaked" (involuntary) information about a player's stance with another player, participating in the game only as a mediator, coordinator or "referee", rather than an actively involved player.

A very special type of signaling is when the message exchange involves false information, i.e., a *bluff*. This kind of signals is a very common practice in games of imperfect and/or incomplete information, where the players do not have a complete view of the game structure itself and/or the opponents' choices, respectively. In this case, *false signaling* or *bluffing* is usually a strategic option by itself, exploiting this uncertainty regarding the true status of the game to enhance advantages or mitigate disadvantages. A very common example of such games is Poker, where a player with weaker deck of cards can project a *false stance* to its opponents, in order to avoid defeat or even secure a victory against players with better decks of cards (Thomas, 1984; Stahl, 1999). Bluffing can be realized directly between players or indirectly via a third-party carrier. In the later case, especially when the signaling is implicit and assumed involuntary, the credibility of the assertion is strongly associated with the credibility of the carrier itself. In other words, even if the source player could not project a successful bluff on its own, a credible third-party carrier might be the necessary intermediate to achieve such a move. The role of third-party mediators in signaling is a special topic in the study of strategic moves and how they affect the final outcome in games.

Summary:
- A *signal* between players is a voluntary or involuntary, direct or indirect, explicit or implicit exchange of a message; it is usually a declaration of *stance* ("posture") in the game, i.e., intent to include or exclude a strategy from a set of open options.
- *Strategic moves*, e.g. signaling, project some strategic preference without making any actual "move", in order to intimidate or coordinate with the other player(s).
- A *bluff* is a projection of false information, i.e., exploiting the incomplete/imperfect information structure of a game to gain some strategic advantage that could not be achievable if the game was of complete/perfect information.

Credibility, reputation, promises & threats

The effectiveness of projecting a strategic stance via signaling, regardless if it is true or bluff, depends heavily on the *credibility* of that signal, as well as the credibility of the player itself (Thomas, 1984; Stahl, 1999). When it comes to a single signal or stance, the credibility is closely linked to the level of compatibility of that signal or stance with the rationality of the player. Although rationality per se may be only an assumption with regard to one's opponent, in general terms it is fairly easy to examine the matrix or the tree-graph representation of a game and establish whether a declared stance is beneficial or not to the associated player. In other words, if that player is assumed to behave rationally, Minimax strategies and Nash equilibria can be used to filter out choices that are clearly excluded, at least with a high probability.

The set of previous stances and/or moves, as well as their associated credibility values, can be used as the history or *reputation* of that player, which is in fact the a priori probability for any future stance and/or move of being consistent with its previous behavior (Mero, 1998). Since games of complete and perfect information, e.g. Chess, are not compatible with false signaling and bluffs, the true theoretical aspect of credibility and reputation is relevant only in games of incomplete and/or imperfect information. Hence, Poker players are indeed characterized as being cautious or risk-takers according to their reputation on using bluffs in lower or higher frequency, respectively.

A player with a specific reputation can signal a specific stance to the others, projecting either a *promise* or a *threat*. A promise is a signal that usually declares the intent to cooperate, i.e., choose the less aggressive approach. This is particularly useful when the players need to coordinate in order to avoid much worse outcomes, as in the games *Leader* and *Battle of the Sexes* (see Tables 6 and 7). On the other hand, a threat is a signal that usually declares the intent to compete, i.e., choose the more aggressive approach. This is still useful as the means to enforce some kind of coordination, now in the form of extortion rather than willful cooperation. The *Chicken* game is such any example (see Table 8), where one player must force the other to swerve, in order to naturally end up in one of the two Nash equilibria and avoid the worst outcome of crash.

As it was mentioned earlier, *Prisoner's Dilemma* is a very special type of game, since neither Minimax solution or Nash equilibrium points to the mutually beneficial option of cooperation; however, if signaling between the prisoners is possible, i.e., if they are allowed to communicate with each other, cooperation becomes much more plausible: all they have to do is to promise each other to remain silent and threat to accuse the other as a retaliation if they see the other doing such thing. One of the most interesting topics in modern Game Theory is the study and analytical formulation of the conditions, the constraints and the exact processes of the *evolution of cooperation* in games like *Prisoner's Dilemma*, where typical theory fails to predict optimal strategies, although such strategies seem to exist, usually in accordance to some *Tit-for-Tat* variation (Axelrod, 1984; Axelrod & Dion, 1988; Mero, 1998; Casti, 1997).

In any case, whether it is a promise or a threat, the signal or stance is labeled as credible or not. Hence, a *credible promise* is one that comes from a player with a reputation of being consistently reliable in fulfilling that promise, i.e., actually choosing less aggressive strategies when signaling intent to cooperate. Similarly, a *credible threat* is one that comes from a player with a reputation of being consistently reliable in fulfilling that threat, i.e., actually choosing more aggressive strategies when signaling intent to compete (Montet & Serra, 2003; Owen, 1995).

Summary:
- *Promise* is a signal that usually declares the intent to cooperate, i.e., choose the less aggressive approach; it is useful when players need to coordinate in order to avoid much worse outcomes.
- *Threat* is a signal that usually declares the intent to compete, i.e., choose the more aggressive approach; it is useful a player wants to enforce some kind of coordination, in the form of extortion.
- *Credibility* is closely linked to the level of compatibility of a signal or stance with the rationality of the player; in practice, it is a measure (probability) of whether the player will fulfill a promise or a threat, if necessary.
- *Reputation* of a player is the a priori probability for any future stance and/or move of being consistent with its previous behavior.
- *Credible promises* and *credible threats* are associated with the reputation and credibility of each player, as well as the actual payoffs in the corresponding game matrix.

Utility, incentives & "rational irrationality"

As it was mentioned earlier, if that player is assumed to behave rationally, i.e., trying to minimize losses and maximize gains in terms of actual payoffs in each outcome, the credibility of a promise or a threat can be easily established with a high probability. Nevertheless, the fact that this is just a probability and not a perfect forecast comes from the fact that, in turn, the level of rationality of that player cannot be evaluated perfectly and in exact terms.

Rationality and *incentives* of a player emerge naturally from the exact formulation of its own *utility function*, which is nothing more than a generalization of the loss/gain function that is described by the matrix or the tree-graph of the game (Owen, 1995; Fudenberg & Tirole, 1991; Montet & Serra, 2003). If the formulation of the game's payoff matrix is perfect, then it is clear when a strategy is optimal for a player and when it is not. However, the truth is that these payoff values may not reflect the exact *utility*, i.e., overall loss/gain value for that player, usually due to some "hidden" outcomes or side-effects. For example, a game matrix may describe the payoffs for each outcome and each player correctly, but with the assumption that these players are rational in the same way: winning over their opponent; this may not be true, e.g. when one player cares more about securing that their opponent does not win, rather than securing their own win. In other words, when the players' rationality is not symmetrically the same, then they do not share the same utility function and the true payoffs in the game matrix may actually be quite different.

A very classic example of such games, assumed to be symmetric when they are actually asymmetric by nature, is the *Hostage Situation*, described in analytical form by Table 10. If the two opponents are treated as similarly rational, i.e., symmetric in terms of incentives and behavior, then the game is not much different than the classic *Chicken*, where one must convince the other to swerve first, in order to avoid the crash. This translates to either the authorities give in to the assaulter's demands or the assaulter eventually surrenders to the authorities, both outcomes assumed to be equally rational, correspondingly, to each player. However, if for some reason the assaulter is more determined than initially presumed, preferring to fight to the death rather than surrendering and ending up in jail, then the game is inherently asymmetric and the payoff matrix is quite different, as illustrated in Table 10. What the matrix shows is that now Player-1, i.e., the assaulter, has a dominant strategy of always choosing the most aggressive stance, no matter what the authorities choose to do. There is no pure Minimax solution here, since there is no pure saddle-point (see payoffs "3" and "2" in bold); however, there is now a single Nash equilibrium at (4,2), i.e., aggressive assaulter and passive authorities - this is in fact the standard approach internationally in all hostage situations: the authorities start with trying to establish a communication link and negotiate with the assaulter, rather than choosing a rescue operation by direct action that could put the hostages in danger.

As it is evident from the *Hostage Situation* game of Table 10, the authorities are normally guided to a more passive and cooperative approach of negotiating rather than using force, because the incentive is to protect the hostages at all costs. This effectively translates to employing a utility function that includes a high priority on the hostages' lives, higher than the immediate capture or incapacitation of the assaulter. Hence, the rationality of Player-2 dictates a more passive, cooper-

Table 10
The typical setup of the Hostage Situation *game with two players. Player-1 is the assaulter and Player-2 is the rescuer-protector.*

Hostage Situation		Player-2 *C*	Player-2 *D*
Player-1	*C*	(2,3)	(1,4*)
	D	(4*,**2***)	(**3***,1)

Table 11
The typical setup of the Kamikaze *game with two players. Player-1 is the "kamikaze" and Player-2 is the defender.*

Kamikaze		Player-2 *C*	Player-2 *D*
Player-1	*C*	(2,3)	(1,4*)
	D	(4*,*1*)	(**3***,**2***)

ative stance. This changes drastically if, during this evolution, the lives of hostages are put in severe danger, e.g. when the assaulter poses a very credible threat or actually harms a hostage (assuming there are more). In this case, the authorities should change stance and employ the more aggressive option, because this is now the optimal response.

Table 11 illustrates the *Kamikaze* game, which is actually a slightly modified *Hostage Situation* game in terms of payoff matrix. The game is still asymmetric and the only variation is the swapping of payoff values {2} and {1} for Player-2 (marked in italics), which illustrates the new fact that at this point it is more harmful for the hostages to remain idle rather than using direct force to rescue them, even if this too poses some danger to them - again, this is exactly the standard approach internationally in all hostage situations: the authorities follow strict rules-of-engagement which state that, once it is established that the lives of hostages is in clear and severe danger, direct action is to be employed immediately. The same setup emerges when the *Kamikaze* game is studied according to its name: when one player (assaulter) is more concerned about damaging the opponent (defender) rather than protecting itself, then there is indeed a dominant strategy of always choosing the most aggressive stance, no matter what the defender chooses to do. Likewise, the defender is now forced to choose between its two worst outcomes and naturally chooses the less damaging one, i.e., direct counteraction rather than swerve. Here, the passive stance is established as more damaging than all-out-conflict, exactly as in *Hostage Situation* with a very aggressive assaulter. In terms of game analysis, now there is indeed a pure Minimax solution at (3,2), which is also the single Nash equilibrium of the game. This explains why there is practically no other rational (strategically optimal) way to defend against a murderous hostage-taker or a desperate kamikaze than employing equally aggressive response.

The concepts described along the strategic analysis and "rationalization" of the players in games like *Hostage Situation* and *Kamikaze* illustrate how a seemingly irrational course of actions can be easily explained and even classified as rational behavior, if the proper utility functions are employed. In other words, if the utility of each and every player is defined correctly, then all players in any game can be labeled as "rational" ones. This proposition is often referred to as "rational irrationality" (valid/explainable behavior), rather than "irrational rationality" (incomprehensible behavior) (Mero, 1998).

Summary:
- *Utility* is the generalized cost/gain function of a player in a specific game, depending on the outcomes but including any "hidden" regards and side-effects.
- Given a specific *utility function*, a player's *incentives* emerge naturally as the rational behavior of the underlying payoff-optimization process.
- A player's behavior may seem "irrational" if its utility function is incomplete; given a properly defined utility function, a player's behavior can always be labeled as rational per se.
- *Hostage Situation* and *Kamikaze* are two examples of (asymmetric) stand-off games where the notion of "rational irrationality" is fully explained via proper definition of the corresponding utility functions for the assaulter.

The Frontier

This paper included only some of the most basic concepts of Game Theory, including solution methods and representations of typical games of special interest, like *Chicken* and *Prisoner's Dilemma*. However, these are only a scratch on the surface of what lies beneath, the rigorous mathematical theory and the complex, some still unsolved, problems in this extremely interesting and useful scientific area.

All the games and setups presented thus far was somewhat "too perfect", too simple compared to real-world situations of conflict. There are few cases where only two players are involved, their moves are full observable and their incentives clear and consistent. In most conflicts, groups of players are spiraling in alternating rounds competing and cooperating, each knowing its own utility function and very little about the others', while signaling, third-party credibility assertions and continuous bargaining are common things. Is there really a way Game Theory can address all these aspects in the same clarity, mathematical robustness and universality as is does with simple cases of zero-sum and nonzero-sum games like the ones presented previously?

The short answer is "No". Game Theory is the mathematical way to approach some of the most complex problems the human mind has ever encountered. For example, what are the prerequisites, the dynamics and the survivability of the evolution of cooperation as a strategy, in human or animal societies? What is the asymptotic behavior of such "cooperative" groups? Can they survive in an environment of pure competition? These issues are addressed in other aspects of the theory, namely the *Evolutionary Stable Strategies* (ESS), not analyzed in this study. In short, ESS are patterns of behavior in games of pure competition and/or possible cooperation, such as the *Prisoner's Dilemma*, that not only may emerge spontaneously but also survive as optimal strategies in iterative games. *Tit-for-Tat* (Axelrod, 1984; Axelrod & Dion, 1988) is such an example of ESS in iterated *Prisoner's Dilemma*: cooperation can emerge spontaneously given a set of conditions, primarily (a) players "start nicely", (b) continue with reciprocity, (c) don't know when the game finishes. Although it seems simple enough, spontaneous cooperation in conflict situations is one of the most intriguing and theoretically complex problems in Game Theory today.

In a slightly simpler scenario, a player may be involved in a game with another player, while *at the same time* its strategic choices are relevant to a second game, with some other player. For example, a politician may be in a "bargain" with voters, trying to gain their support by promising specific actions if elected, while at the same time a second "bargain" may be taking place in parallel with the party's main policies and governmental plan if it comes to power. If some of that politician's promises are on conflict with the party's main lines, then as a player is involved in what is called a *two-level game* (Putnam & Henning, 1986; Putnam, 1988). This form of gaming was first proposed by Putnam in the late '70s and it models two-level or multi-level conflict situations in general, where the strategic choices of a player affect two or more simultaneous games. The solution concepts and equilibria are not much different than those of simple games, but now a strategy is optimal and produces a stable outcome only if it is such *simultaneously* in all these games.

Another very interesting aspect of gaming in general is the evolution of strategies and each player's behavior as each observes the others' moves. In single-step games, the Minimax solution (pure or mixed) is the one that dictates the optimal strategy for each player. The concept of iterative gaming is much more general, since it includes cases where the same players may face one another in the same single-step games many times in the future. In this case, Nash equilibria predict the most probable outcomes with much better accuracy. But the knowledge that there will be a "next round", especially when players alternate moves and one can observe the other before making its own (e.g. in Chess), then the game analysis can expand to two or more steps ahead. In practice, the player does not only take into account the strategic choices available to the opponent(s) but also the "what if" combinations of moves-countermoves. Hence, the corresponding game matrix includes these combinations of composite states on the opponent(s) side and the payoffs are estimated accordingly. This type of composite multi-step setup is often referred to as a *metagame* (Thomas, 1984). The extended-form representation of metagames is more natural than the analytical (matrix) form, but the identification of equilibria and solutions is somewhat less straight-forward.

Some games involve elements of chance regarding the game's state or partial information regarding the observability of each player's moves. In such games of imperfect information, modeling via a game matrix or a tree-graph can be problematic, since many of the paths may be mutually exclusive and not just alternative choices. In the '60s, very early on in the history of Game Theory, Harsanyi introduced the so-called *Harsanyi transformation* (Harsanyi, 1962, 1967; Montet & Serra, 2003) for transforming a game of incomplete information to an equivalent game of complete but imperfect information. This may not seem much, but in reality there is a very distinct and important difference between them. If a random event dictates the exact structure and payoffs of the games, perhaps even the strategic behavior of the players, then the analysis of such a game is inherently a very difficult task. On the other hand, the Harsanyi transformation models this random event as a deterministic one, removing the element of chance and introducing the notion of "hidden" information about it. In practice, this results in creating multiple variations of the game, one for each possible configuration, and treating them separately. After they are individually analyzed, solutions and equilibria are combined together within a probabilistic framework, introducing the more generalized concept of Bayesian Nash equilibria (Montet & Serra, 2003).

In real-world conflict situations it is not uncommon that one or some of the players have a different knowledge or "view" of the game structure, its payoffs and the other players' preferences. This means that each player acts upon its own payoff matrix, possibly very different in structure and values than the one used by the other players. Of course, all players are involved in the same, single game and the payoffs on each outcome is effectively a single one, despite each player's unique view of the game. This is extremely important if some of the players have a more complete view of the game, i.e., when they address the game as one of (almost) complete information, while some opponents address it as one of incomplete information. These special types of conflict are often referred to as *hypergames* (Vane, 2006; Bennett & Dando, 1979). Introduced by Bennett and Dando in late '70s and later revised in the '00s by Vane and others, hypergames is a very efficient way to describe games of asymmetric information between players by employing different variations of the game matrix or tree-graph, according to each player's view. In practice, hypergames are treated the same way as simple games, with each player deciding its strategic choices according to its own view and, subsequently, combining the (partial) outcomes together.

Game Theory is a vast scientific and research area, based almost entirely on Mathematics and some experimental methods, with applications that vary from simple board games and auctions to Evolutionary Psychology and Sociology-Biology in group behavior of humans and animals. Although real-world situations reveal that sometimes its predictive value is limited, the robust theoretical framework and solution concepts provide an extremely valuable set of tools that clarifies the inner workings and dynamics of conflict situations.

Summary:
• In accordance to Nash's bargaining theorem, cooperation can *emerge spontaneously*, even in competitive games, when a specific set of pre-requisites are satisfied.
• *Evolutionary stable strategies* (ESS) are patterns of behavior in games of pure competition and/or possible cooperation that survive as optimal strategies in iterative games.
• In *two-level games*, a player may be involved in a game with another player, while *at the same time* its strategic choices are relevant to a second game, with some other player.
• *Metagames* are multi-step game setups where the corresponding game matrix includes combinations of "what if" composite states, regarding the future strategic choices of the opponent(s).
• The *Harsanyi transformation* is used in games of incomplete information, e.g. when the game structure and payoffs depend on some random event, to transform it to an equivalent game of complete but imperfect information.
• *Hypergames* is a very efficient way to describe games of asymmetric information between players by employing different variations of the game matrix or tree-graph, according to each player's view.
• In general, Game Theory is a vast scientific and research area with robust theoretical foundation that can be used as a predictive tool, as well as (mostly) an extremely valuable approach to analyze conflict situations.

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Acknowledgments. This work is dedicated to John F. Nash, pioneer and mathematical genius, who was killed earlier this month on May 23th 2015 in a car accident along with his wife Alicia. His inspirational work and breakthrough ideas has changed Game Theory and Economics forever.

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