A survey of the General Relativity Manifolds and their Variation

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Abstract

The paper investigates the possibility of continuous variation of a manifold starting from a given one. Theoretical investigation engenders the fact that such a continuous passage is not possible provided you start from a given manifold specified by its metric coefficients. You have to move in discrete steps starting from the given one, satisfying some equations discussed in the paper. A manifold surface can always be constructed using arbitrary continuous and differentiable functions as metric coefficients. The ensuing Ricci tensor and Ricci scalar will always satisfy the Bianchi Identity and hence the field equations. The functionals in the general Relativity use the Ricci scalar [ensuing from the metric coefficients] as arguments. Different surfaces[manifolds] are generated by varying the metric coefficients [in order to vary the Ricci scalar or such functions as dependent on it]. In each case the manifold satisfies the Bianchi identity and hence the Field Equations prior to the application of the stationary action principle. This perhaps induces a motivation for discretization. Discretization will modify all the principles involved in General Relativity making them of a suitable nature in the present context. This may open up the gates for including the General Relativity Lagrangians in a more rigorous manner.

Keywords: Metric Tensor, Transformations, Manifolds, Gauss Egregema, Christoffel Symbols

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1.Introduction

The paper aims to establish the following ideas: (1)From a given manifold we may move to other manifolds only in discrete steps but not in a continuous manner by changes in the metric coefficients. Any effort to access manifolds in a continuous manner would lead to contradictions. This idea is intimately tied to the idea of the General Relativity Lagrangians starting from the Einstein Hilbert Action to the f(R) gravity and other Lagrangians in the Extended gravity Theories. The common feature connected with the actions corresponding to the stated Lagrangians is making continuous changes in the metric coefficients to allow for the process of variation. But such continuous variation is not possible as we shall see in this paper.(2)If a surface(manifold) is constructed with arbitrary continuous differentiable functions as metric coefficients, the ensuing Ricci tensor together with the Ricci scalar and the metric coefficients will satisfy the Bianchi identity:$\nabla_\beta\left(R^{\alpha\beta} - g^{\alpha\beta}R\right) = 0$. The physical basis behind the field Equations is that energy density curves space and time and that the covariant derivative of the stress energy tensor with respect to time and spatial coordinates is zero. All surfaces [manifolds] constructed from arbitrary using continuous differentiable functions as metric coefficients will serve this purpose. The general Relativity like those of Einstein Hilbert and the f(R) gravities use manifold surfaces expressed through their metric coefficients as independent variables. But we cannot vary them in a continuous manner in applying the stationary action principle.

2.Global Transformations

Transformation of tensors in Curved Space Time

A second rank contravariant tensor[1] $T^{\alpha\beta}$ has been chosen to discuss some salient features related to the current investigation.

$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} T^{\alpha\beta}$ (1)

$\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ are elements from the transformationmatrix. You may consider transformation from spherical to rectangular system in Schwarzschild geometry. Manifold[2] remaining the same we consider these coordinate transformations from one system to another. Such transformations do always exist

Transformations described by (1) do not contain or involve the metric coefficients: they are independent of the nature of the manifold.The transformations may be of a local nature but the variable are global variables. It means $\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ may be a function of space time variables (t, x, y, z) but the variables t,x,y and z are global variables figuring in local transformations.

It may be noted that the metric coefficients are described as functions of the global coordinates[grid coordinates] but the transformations described by (1) do not involve the metric coefficients

Incidentally these metric coefficients themselves participate in global transformations

$$\bar{g}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}$$

$$\bar{g}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} g_{\alpha\beta}$$

Standard literature proof of the above relates to the invariance of the line element and the involvement of global coordinates and is provided below:

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

$$ds'^2 = g'_{\mu\nu} dx'^{\mu} dx'^{\nu}$$

[prime in the above denotes transformation and not derivative]

Due to invariance:

$$ds'^2 = ds^2$$

$$g'_{\mu\nu}dx'^{\mu}dx'^{\nu} = ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}dx'^{\mu}\frac{\partial x^{\beta}}{\partial x'^{\nu}}dx'^{\nu}$$

Therefore

$$g'_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}dx'^{\mu}dx'^{\nu}$$

$$\Rightarrow g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}g_{\alpha\beta}$$

The proof of the fact that $g_{\alpha\beta}$ is a second order covariant tensor is connected with global transformations

Now if you transformed from spherical to rectangular system for the Reissner Nordstrom[3] metric the elements $\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ in the transformation matrix will not change though the manifold has changed. **This is corroborated by the proof[4] that the difference of two connections is a tensor. It considers changes in the manifold[changes in $g_{\alpha\beta}$] with unchanged values of the transformation matrix elements[Supplementary Material ; Section 2: Difference of Two Connections]**

We have two types of transformations

1) Manifold remaining the same the system of coordinates change: the metric coefficients change according to the transformation given above
2) The manifold itself changes: in this case the metric coefficients may change in an arbitrary manner, remaining continuous differentiable functions. We may use arbitrary continuous differentiable functions to construct a manifold surface:thev following will hold (1) Covariant derivative of $g_{\mu\nu}$ with respect to time and spatial coordinates will be satisfied.(2) The Bianchi identities will remain valid
   During the process of variation information in point (2) above becomes active.
   Again while we are on some particular manifold before or after variation information in point (1) becomes relevant

In a particular type of geometry for example Schwarzschild Geometry[5] you may consider in relative motion of complicated type between the two global coordinate systems. In such situations we have the same type of relation as (1) except that the transformation elements will contain velocity components, angular speed acceleration etc.

$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}T^{\alpha\beta}$ (2)

The matrix elements for (1) and (2) will be different when we change the system of coordinates with out changing the manifold. But they will not change when we pass from one geometry to another[from one manifold to another] due to the absence of $g_{\mu\nu}$in provided we maintain the same type of relative motion between the global grid systems. The transformation elements will contain speed acceleration and other higher order derivatives in the global perspective

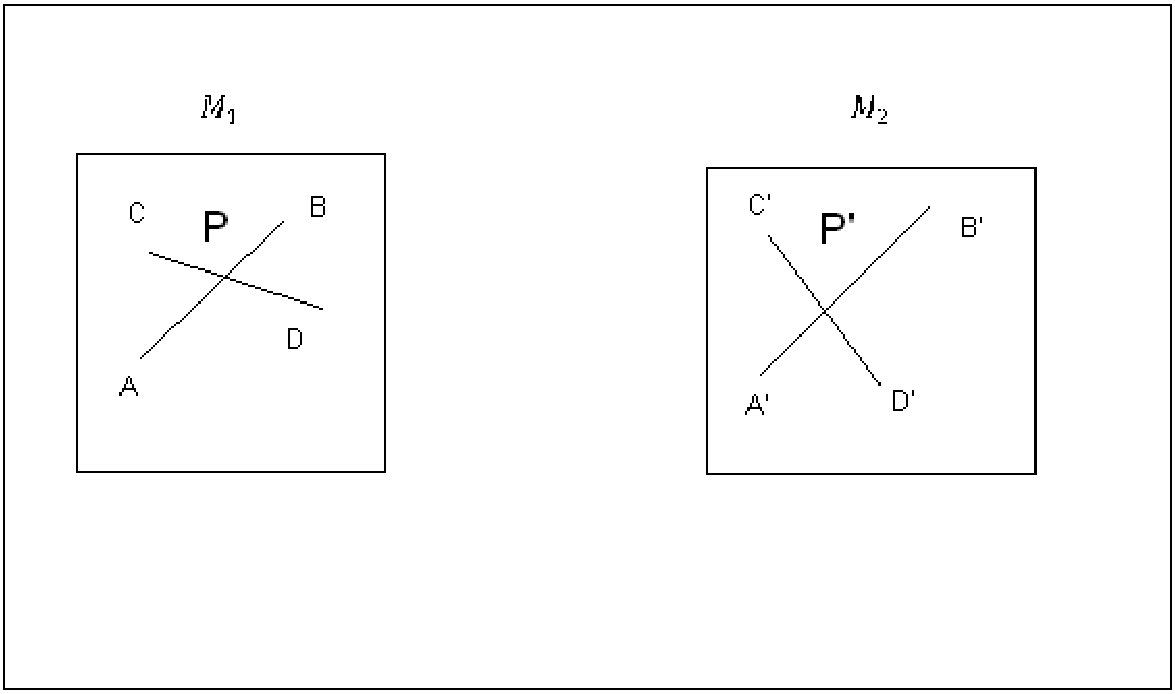
We may consider a manifold due to several masses in the fray. The global coordinate system will have world lines where the observer is spatially at rest but such observers may be in motion with respect to all the bodies in the fray.

In this paper we are concerned with the global transformations in relation o both the points (1) and (2). The tensors are being viewed from the perspective of global coordinates. For example if you are in Schwarzschild geometry you are viewing the tensors as a spatially stationary observer in the commonly used spherical system of coordinates or in some other system like the rectangular one which is less common.. This corresponds to some particular world line at any specified point on the manifold. Let us call it the stationary world line at the point P

Local Transformations

Though we are concerned with global transformations we have a brief discussion on the local transformations in this section.

P is a point on manifold M1 and P' is the image on the close manifold M2. At the same coordinate location((t, x, y, z), the tangent planes at P and P' have been considered before and after the infinitesimal change of the manifold.



First we consider the tangent plane at some point P in curved space time geometry. Observers passing along AB and CD have their own mutual local transformation laws, simplest being the Lorentz Transformations when relative motion is uniform.

If curvature changes as we change the manifold by some valid infinitesimal amount the inclination of the said tangent plane changes [with respect to the previous position: inclination between planes not shown in the figure]] and the angles between curves passing through P also change. Angle between AB and CD is different from the angle between A'B' and C'D'

At each point P and P' we will have a world line where the observer is at rest with respect to the global system of coordinates. Each point will have such a "stationary" world line passing through it. Metrics like Schwarzschild's are relative to spatially stationary observers at origin.

The transformation elements as depicted in equation (1) will not be different for the two manifolds for observers on the stationary world lines.

We may work out "global transformations" for observers along these stationary world lines when we pass from one manifold to another(point 1) and while on any particular manifold we consider global transformations between coordinate systems as depicted in point 2

Again we have transformations for observers moving along different world lines on same the tangent plane. The motion between such observers may not be uniform and will not necessarily fall into the category of the Lorentz Transformations though such relative motion pertains to the tangent plane. Lorentz transformations are the simplest type of transformations on the tangent plane when relative motion is uniform in nature. The tangent plane is not an exclusive uniform motion plenum: it is simply Minkowski space where acceleration and relative accelerations are permitted. A p[air of frames may have relative acceleration between them

We simply do not know the laws for frames that translate non uniformly between them but we may assume the existence of relevant laws.

In formal literature tensors and their transformations are defined in respect of the tangent plane[5] at some point P on a manifold. We can pass locally from curved space time at point P to the tangent plane[Euclidean plane: Minkowski space] by numerous local chartings and each local chart creates a basis for tensors in the tangent space .This basis incidentally has the same dimensionality as the manifold.. The bases thus created on the same tangent plane at point P may be transformed into one another and the corresponding tensors defined on them may also be transformed using relations like (1) where the coordinates involved are local coordinates , derived from the global ones [obviously by transgression of the Gauss Egregema: we are passing from curved space time to Minkowski space by such mapings/diffeomorphisms].

[The transformations on the tangent described above are transformations in Minkowski space]

The same tensor on the tangent plane can have different representations in the distinct bases on the said plane. Any tensor on the local tangent plane may be transported to some other tangent plane on the manifold another by the method of parallel transport.

In the case of variations, the manifold itself changes though by an infinitesimal amount at each coordinate location (t,x,y,z). We have close tangent planes at P before and after variation. We may call the point P , P', after variation but it is actually the same coordinate location (t,x,y,z)

The Tangent planes representing Minkowski space are inclined with respect to each other and the angles between curves passing through them have changed. Our aim would be to locate a world line L1 at P and L2 at P' so that the values of elements of the local transformation matrix $\frac{\partial\bar{\xi}^{\mu}}{\partial\xi^{\alpha}}$ have not changed though the local coordinates may have changed due to subtle changes in the local charting due and due to variation of the manifold.[ξ^{α} and $\bar{\xi}^{\mu}$ are local coordinates on the same tangent plane pertaining to distinct basis formed by different local charts:$x^{\mu} \leftrightarrow \xi^{\nu}\ and\ x^{\mu} \leftrightarrow \bar{\xi}^{\nu}$ at P

Here x^{μ} is the global coordinate at P while ξ^{ν} and $\bar{\xi}^{\nu}$ are the local coordinates on the same tangent plane at P due to different local chartings

On the new tangent plane formed due to variation we could as before think of an infinite number of local charts from x^{μ} to ξ'^{ν} and $\bar{\xi}'^{\nu}$. We can choose a pair so that $\frac{\partial\bar{\xi}^{\mu}}{\partial\xi^{\alpha}} = \frac{\partial\bar{\xi}'^{\mu}}{\partial\xi'^{\alpha}}$

This technique can help us in adding and subtracting tensors on different tangent planes using relations like (1). The value of the metric tensor coefficients are independent of local charts and they conform to global transformations of the type described by relation (1)

These metric coefficients are expressed in terms of global coordinates in the known metrics like Schwarzschild's metric. There is always a necessity of transforming them in respect of the global coordinates

Considering the fact that $g_{\mu\nu}$ *is a tensor [metric tensor of rank 2 we have in the global perspective*

$$\bar{g}^{\mu\nu} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}g^{\alpha\beta}$$

$$\bar{g}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}g_{\alpha\beta}$$

Any denial of global transformation would render the tensor attribute of $g_{\mu\nu}$ *as invalid*

We may inter relate global and local tensors by suitable processes /mappings though such procedure is not necessary for our article. The local transformations on any particular tangent plane are not of concern to us. We are concerned in global transformations of two types: in one type the manifold itself does not change: only the coordinate system described on it changes. In the other type of transformation we are concerned with the manifold changes globally without the coordinate system on it changing.

Let us take the stationary world line on each tangent plane at the points P and P'. Observers on such lines are at rest with the global coordinate system. Due to the process of infinitesimal change the in the manifold the metric coefficients have changed but events continue to be labeled by the same global coordinates for example (t,x,y,z) unless we change the system of coordinates like from rectangular to spherical.

We can always link the local tensors on the tangent plane to some global tensor[many to one mapping], the global tensor obeying relation (1) between the two manifolds as considered in the process of variation. The local tensor follows similar rules with the local coordinates

For the two tangent planes considered ion the variation we may take such local charts in each case maintaining that $\frac{\partial \bar{\xi}^{\mu}}{\partial \xi^{\alpha}}$ remain the same in value in each case though the local coordinate ξ may change. This is just for our convenience. But local transformations are not so much of concern to us: we are interested in the global changes. Nevertheless we have in this section we have alluded to a small discussion on local transformations on the tangent plane

Example of the GPS

Time changes due to General Relativity effects predominate over Special Relativity time dilation for GPS calculations. The two effects have to be calculated separately. This is done in the case of the Global Positioning System[6] Schwarzschild's metric does not provide us with the time dilation of Special relativity. Relation (1) relates only to global transformations[for example from rectangular to spherical and vice versa]. Such transformations are not necessary for GPS in relation to what we are discussing .

We require time changes at different heights due to potential difference and we come to know of it from Schwarzschild's metric. This time difference predominates over Special Relativity time dilation

3.Variation of the Action

The Einstein Hilbert Action[7] and the f(R) gravity[8] Action

$S = \int d^4x \left[\frac{1}{2\kappa} R\sqrt{-g} + L_m\sqrt{-g}\right]$ (3)

Action for f(R) gravity

$S = \int d^4x \left[\frac{1}{2\kappa} f(R)\sqrt{-g} + L_m\sqrt{-g}\right]$ (4)

We consider different functions R to work out the variation of S due to change in the metric coefficients $\delta g_{\mu\nu}$

[$\nabla_\alpha g_{\mu\nu} = 0$ as a relevant feature of metric compatibility]

S is a set of real numbers corresponding to the integration of the Ricci scalar R or different functions R , f(R

Let the initial manifold be represented by M1[R_1being the function representing Ricci scalar on it. We have $\nabla_\alpha g_{\mu\nu} = 0$ on M1

We choose an infinitesimal $\delta g_{\mu\nu}$. Then we go on to define another surface M2

having as metric coefficients, $\left(g_{\mu\nu} + \delta g_{\mu\nu}\right)$. Now $\left(g_{\mu\nu} + \delta g_{\mu\nu}\right)$. Automatically we have $\nabla_p\left(g_{\mu\nu} + \delta g_{\mu\nu}\right) = 0$ with respect to M2 The result $\nabla_p\left(g_{\mu\nu} + \delta g_{\mu\nu}\right) = 0$ will follow automatically when the surface M2 is constructed from the functions $g_{\mu\nu} + \delta g_{\mu\nu}$ as metric coefficients. This has already been discussed in the earlier section.

With respect to M2:

$$g'_{\mu\nu} + \delta g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\left(g_{\alpha\beta} + \delta g_{\alpha\beta}\right)$$

With respect to M1

$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$ (5)

[Prime above denotes transformed values and not differentiation]

The quantities $\frac{\partial x^\alpha}{\partial x'^\mu}$ are identical for M1 and M2 since they involve the transformation of grid system

By subtraction we obtain:

$\delta g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \delta g_{\alpha\beta}$ (6)

Therefore $\delta g_{\alpha\beta}$ is a tensor with respect to each surface M1 and M2[the grid transformations do not change]

[$g^{\alpha i} + \delta g^{\alpha i}$]is a symmetric tensor being a metric coefficient on M2: $\delta g^{\alpha i}$ will also be a symmetric tensor on M1 and on M2];

We have from tensor properties[$\delta g_{\mu\nu}$ being a tensor]

$\delta g^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$ (7)

On M1: $\nabla_p g_{\mu\nu} = 0$ but it is not necessary to have $\nabla_p \delta g_{\mu\nu} = 0$. We do have $\delta g_{\mu\nu}$ as a tensor on M1 but not as a metric tensor

On M2: $\nabla_p \left(g_{\mu\nu} + \delta g_{\mu\nu}\right) = 0$

With respect to M2, $\nabla_p \left(g_{\mu\nu} + \delta g_{\mu\nu}\right) = 0$

But on M2 it is not necessary to have $\nabla_p g_{\mu\nu} = 0$ and $\nabla_p \delta g_{\mu\nu} = 0$ individually/separately

$g_{i\beta} + \delta g_{i\beta}$ are metric coefficients with respect to M2

Therefore $\left(g^{\alpha i} + \delta g^{\alpha i}\right)\left(g_{i\beta} + \delta g_{i\beta}\right) = \delta^{\alpha}{}_{\beta}$ ((8)

We could have used the above to define $\delta g^{\alpha\beta}$from $\delta g_{\alpha\beta}$. But we know that $\delta g_{\alpha\beta}$ is a tensor with respect to both M1 and M2:

$$\delta g^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$$

So we do not have the advantage of defining $\delta g^{\alpha\beta}$from $\delta g_{\alpha\beta} using$ (3)without using extra conditions as given by

$$\delta g^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$$

Now,

$$g^{\alpha i} g_{i\beta} + g^{\alpha i} \delta g_{i\beta} + g_{i\beta} \delta g^{\alpha i} + \delta g^{\alpha i} \delta g_{i\beta} = \delta^{\alpha}{}_{\beta}$$

Referring to M1:$g^{\alpha i} g_{i\beta} = \delta^{\alpha}{}_{\beta}$:Kroneker delta transforms to the Kronetker delta in all systems:

Therefore,

$$\delta^{\alpha}{}_{\beta} + g^{\alpha i} \delta g_{i\beta} + g_{i\beta} \delta g^{\alpha i} + \delta g^{\alpha i} \delta g_{i\beta} = \delta^{\alpha}{}_{\beta}$$

Or,

$g^{\alpha i} \delta g_{i\beta} + g_{i\beta} \delta g^{\alpha i} + \delta g^{\alpha i} \delta g_{i\beta} = 0$ (9)

$g^{\alpha i} \delta g_{i\beta} + g_{k\beta} \delta g^{\alpha k} + \delta g^{\alpha l} \delta g_{l\beta} = 0$ (10)

Referring to M1 and an orthogonal system on it we have for fixed indices $\alpha$and :

$g^{\alpha\alpha}\delta g_{\alpha\beta} + g_{\beta\beta}\delta g^{\alpha\beta} + \delta g^{\alpha i}\delta g_{i\beta} = 0$ (11)

[Referring to (4.1) $g^{\alpha i} = 0$ if $i \neq \alpha$ and $g_{k\beta} = 0$ if if $k \neq \beta$]

Now

$$\delta g^{\alpha i} = g^{\alpha m} g^{in} \delta g_{mn}$$

And

$$\delta g^{\alpha\beta} = g^{\alpha p} g^{\beta q} \delta g_{pq}$$

$$g^{\alpha\alpha}\delta g_{\alpha\beta} + g_{\beta\beta} g^{\alpha p} g^{\beta q} \delta g_{pq} + g^{\alpha m} g^{in} \delta g_{mn} \delta g_{i\beta} = 0$$

But the system is orthogonal

Therefore for non trivial metric coefficients:

$m = \alpha; n = i$and$p = \alpha; q = \beta$

$g^{\alpha\alpha}\delta g_{\alpha\beta} + g_{\beta\beta} g^{\alpha\alpha} g^{\beta\beta} \delta g_{\alpha\beta} + g^{\alpha\alpha} g^{ii} \delta g_{\alpha i} \delta g_{i\beta} = 0$[summation on$i$ only]

In the orthogonal systems:

$g^{\beta\beta} = \frac{1}{g^{\beta\beta}}$[summation on β not implied in this formula]

Or,

$g^{\alpha\alpha}\delta g_{\alpha\beta} + g^{\alpha\alpha}\delta g_{\alpha\beta} + g^{\alpha\alpha} g^{ii} \delta g_{\alpha i} \delta g_{i\beta} = 0$ [summation holds on i: alpha and beta are fixed indices]

Or,

$2g^{\alpha\alpha}\delta g_{\alpha\beta} + g^{\alpha\alpha} g^{ii} \delta g_{\alpha i} \delta g_{i\beta} = 0$ (12)[summation on i only]

If the product term $\delta g_{\alpha i}\delta g_{i\beta}$ ignored with respect to $\delta g_{\alpha\beta}$,

$g^{\alpha\alpha}\delta g_{\alpha\beta} = 0$[not summed]

Implies: $\delta g_{\alpha\beta} = 0$

If we do not ignore $g^{\alpha\alpha} g^{ii} \delta g_{\alpha i} \delta g_{i\beta}$[$\alpha, \beta$ fixed indices: summation on i]

We have from (12)

$g^{\alpha\alpha}\delta g_{\alpha\beta} = -\frac{1}{2} g^{\alpha\alpha} g^{ii} \delta g_{\alpha i} \delta g_{i\beta}$[summation only on i]

Multiplying both sides of the above by $g_{\alpha\alpha}$

$g_{\alpha\alpha}g^{\alpha\alpha}\delta g_{\alpha\beta} = -\frac{1}{2}g_{\alpha\alpha}g^{\alpha\alpha}g^{ii}\delta g_{\alpha i}\delta g_{i\beta}$ [we have summed on i:alpha,beta fixed]

$g_{\alpha\alpha} = \frac{1}{g^{\alpha\alpha}}$[orthogonalsystems;alpha not in summation]

$\delta g_{\alpha\beta} = -\frac{1}{2}g^{ii}\delta g_{\alpha i}\delta g_{i\beta}$ (13)[summation on i: α, β fixed indices]

$$\delta g_{\alpha\beta} = -\frac{1}{2}g^{00}\delta g_{\alpha 0}\delta g_{0\beta} - g^{11}\delta g_{\alpha 1}\delta g_{1\beta} - g^{22}\delta g_{\alpha 2}\delta g_{2\beta} - g^{33}\delta g_{\alpha 3}\delta g_{3\beta}$$

The 10 quadratic equations have been listed below[$\delta g_{\alpha\beta} = \delta g_{\beta\alpha}$]

$$\delta g_{00} = -\frac{1}{2}g^{00}\delta g_{00}\delta g_{00} - g^{11}\delta g_{01}\delta g_{10} - g^{22}\delta g_{02}\delta g_{20} - g^{33}\delta g_{03}\delta g_{30}$$

$$\delta g_{11} = -\frac{1}{2}g^{00}\delta g_{10}\delta g_{01} - g^{11}\delta g_{11}\delta g_{11} - g^{22}\delta g_{12}\delta g_{21} - g^{33}\delta g_{13}\delta g_{31}$$

$$\delta g_{22} = -\frac{1}{2}g^{00}\delta g_{20}\delta g_{02} - g^{11}\delta g_{21}\delta g_{12} - g^{22}\delta g_{22}\delta g_{22} - g^{33}\delta g_{23}\delta g_{32}$$

$$\delta g_{33} = -\frac{1}{2}g^{00}\delta g_{30}\delta g_{03} - g^{11}\delta g_{31}\delta g_{13} - g^{22}\delta g_{32}\delta g_{21} - g^{33}\delta g_{33}\delta g_{33}$$

$$\delta g_{10} = -\frac{1}{2}g^{00}\delta g_{10}\delta g_{00} - g^{11}\delta g_{11}\delta g_{10} - g^{22}\delta g_{12}\delta g_{20} - g^{33}\delta g_{13}\delta g_{01}$$

$$\delta g_{20} = -\frac{1}{2}g^{00}\delta g_{20}\delta g_{00} - g^{11}\delta g_{21}\delta g_{10} - g^{22}\delta g_{22}\delta g_{20} - g^{33}\delta g_{\alpha 2}\delta g_{30}$$

$$\delta g_{30} = -\frac{1}{2}g^{00}\delta g_{30}\delta g_{00} - g^{11}\delta g_{31}\delta g_{10} - g^{22}\delta g_{32}\delta g_{20} - g^{33}\delta g_{33}\delta g_{30}$$

$$\delta g_{12} = -\frac{1}{2}g^{00}\delta g_{10}\delta g_{02} - g^{11}\delta g_{11}\delta g_{12} - g^{22}\delta g_{12}\delta g_{22} - g^{33}\delta g_{13}\delta g_{32}$$

$$\delta g_{23} = -\frac{1}{2}g^{00}\delta g_{20}\delta g_{03} - g^{11}\delta g_{21}\delta g_{13} - g^{22}\delta g_{22}\delta g_{23} - g^{33}\delta g_{23}\delta g_{32}$$

$$\delta g_{31} = -\frac{1}{2}g^{00}\delta g_{30}\delta g_{01} - g^{11}\delta g_{31}\delta g_{11} - g^{22}\delta g_{32}\delta g_{21} - g^{33}\delta g_{33}\delta g_{31}$$

We have 10 quadratic equations for 10 unknowns $\delta g_{\alpha\beta}$ in (13), g^{ij} being known to us

In fact $\delta g_{\alpha\beta}$ is a tensor as shown by relation ….and. $\delta g_{\alpha\beta}$ and $\delta g^{\alpha\beta}$ related by the usual index raising and index lowering rule of the tensors.

Thus we have a discrete set of values for $\delta g_{\alpha\beta}$ or $\delta g^{\alpha\beta}$ because of the ten quadratic equations. The action cannot be varied by varying the metric coefficients in a continuous manner

For a particular manifold M1 we can move to some specific ones by solutions of (13) but we cannot move to any arbitrary close manifold . There are $2^{10} = 1024$specific manifolds M2 according to solutions of (13) and from any one of them we may move to another specific manifold which is again the solution of (7). Thus we may move through distinct manifolds in discrete steps but not in a continuous manner.

Incidentally we also have,

$\delta g^{\alpha\beta} = (g^{\alpha\mu} + \delta g^{\alpha\mu})(g^{\beta\nu} + \delta g^{\beta\nu)})\delta g_{\mu\nu}$ (14)

[Since $\delta g_{\mu\nu}$ is a tensor wrt M2].

$$\delta g^{\alpha\beta} = \left(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\mu}\delta g^{\beta\nu} + g^{\beta\nu}\delta g^{\alpha\mu} + \delta g^{\alpha\mu}\delta g^{\beta\nu}\right)\delta g_{\mu\nu}$$

$$\delta g^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} + \left(g^{\alpha\mu}\delta g^{\beta\nu} + g^{\beta\nu}\delta g^{\alpha\mu} + \delta g^{\alpha\mu}\delta g^{\beta\nu}\right)\delta g_{\mu\nu}$$

$$\delta g^{\alpha\beta} = \delta g^{\alpha\beta} + \left(g^{\alpha\mu}\delta g^{\beta\nu} + g^{\beta\nu}\delta g^{\alpha\mu} + \delta g^{\alpha\mu}\delta g^{\beta\nu}\right)\delta g_{\mu\nu}$$

$$\left(g^{\alpha\mu}\delta g^{\beta\nu} + g^{\beta\nu}\delta g^{\alpha\mu} + \delta g^{\alpha\mu}\delta g^{\beta\nu}\right)\delta g_{\mu\nu} = 0$$

$\left(g^{\alpha i}\delta g^{\beta j} + g^{\beta j}\delta g^{\alpha i} + \delta g^{\alpha i}\delta g^{\beta j}\right)\delta g_{ij} = 0$

Again

$\left(g^{\alpha i} + \delta g^{\alpha i}\right)\left(g_{i\beta} + \delta g_{i\beta}\right) = \delta^{\alpha}{}_{\beta} \Rightarrow g^{\alpha i}\delta g_{i\beta} + g_{i\beta}\delta g^{\alpha i} + \delta g^{\alpha i}\delta g_{i\beta} = 0$(A)(15)[rank 2 mixed tensor terms]

Or,

$$g^{\alpha i}g_{ni}g_{\beta k}\delta g^{nk} + g_{ni}g_{\beta k}g^{nk}\delta g^{\alpha i} + g_{ni}g_{\beta k}\delta g^{\alpha i}\delta g^{nk} = 0$$

Or,

$g_{ni}g_{\beta k}\left(g^{\alpha i}\delta g^{nk} + g^{nk}\delta g^{\alpha i} + g^{\alpha i}\delta g^{nk}\right) = 0$(16)

$$\delta g^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}$$

$\delta g^{\alpha\beta} = (g^{\alpha\mu} + \delta g^{\alpha\mu})(g^{\beta\nu} + \delta g^{\beta\nu)})\delta g_{\mu\nu} \Rightarrow \left(g^{\alpha\mu}\delta g^{\beta\nu} + g^{\beta\nu}\delta g^{\alpha\mu} + \delta g^{\alpha\mu}\delta g^{\beta\nu}\right)\delta g_{\mu\nu} = 0 \Rightarrow \left(g^{\alpha i}\delta g^{\beta j} + g^{\beta j}\delta g^{\alpha i} + \delta g^{\alpha i}\delta g^{\beta j}\right)\delta g_{ij} = 0$ (17)

A’ and B are identical. Therefore movement of manifold in discrete steps is possible.

We have to consider these discrete process

We refer to familiar relation[9] stated below

$$\frac{\delta g_{\alpha\beta}}{\delta g^{ik}} = -g_{\alpha\mu} g_{\beta\nu} \frac{\delta g^{\mu\nu}}{\delta g^{ik}}$$

To investigate the above relation we start with;

$g_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} g^{\mu\nu}$relation

$$\delta g_{\alpha\beta} = (\delta g_{\alpha\mu}) g_{\beta\nu} g^{\mu\nu} + (\delta g_{\beta\nu}) g_{\alpha\mu} g^{\mu\nu} + g_{\alpha\mu} g_{\beta\nu} (\delta g^{\mu\nu})$$

$$\delta g_{\alpha\beta} = (\delta g_{\alpha\mu}) \delta_\beta{}^\mu + (\delta g_{\beta\nu}) \delta_\alpha{}^\nu + g_{\alpha\mu} g_{\beta\nu} (\delta g^{\mu\nu})$$

$$\delta g_{\alpha\beta} = \delta g_{\alpha\beta} + \delta g_{\beta\alpha} + g_{\alpha\mu} g_{\beta\nu} (\delta g^{\mu\nu})$$

Only assuming symmetricity of $\delta g_{\alpha\beta}$ in terms of interchange of alpha and beta , we may write :

$\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} (\delta g^{\mu\nu})$

Dividing both sides of the above by δg^{ik} where I and k are fixed indices we have relation (17) of the

$$\frac{\delta g^{\mu\nu}}{\delta g_{ik}} = -g^{\mu\alpha} g^{\nu\beta} \frac{\delta g_{\alpha\beta}}{\delta g_{ik}}$$

The process of differentiation has been applied ;incidentally conventional type of differentiation is not possible due to discreteness indicated in the earlier section.Refreshment pf mathematical proceedures is necessary.

The Gauss Egregema context:

Gauss Egregema is concerned with the transformation of coordinates. In so far as the problem we are investigating, the coordinate grid remains unchanged: only the metric coefficients are changing on the same grid. For the rectangular system will remain rectangular while the metric coefficients will change producing another metric compatible manifold; the spherical system of coordinates will remain spherical there while the expressions [and consequently the values] of the metric coefficients between any pair of coordinate labels will change.

We are not considering grid changes: rectangular to spherical or from spherical to elliptic. Our formulations are not in violation of the Gauss Egregema

Tensors incidentally are defined by transformation rules: these transformations relate to changes in the grid system keeping the manifold constant/fixed. Changes may be from spherical to cylindrical, rectangular to elliptic etc. …

Example:

$$\bar{T}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} T^{\alpha\beta}$$

These transformations involve coordinate transformations on the same type of manifold(differentiation allowed: discreteness issue crops up when we are trying to move into an adjacent manifold]. The coordinate system will change for example from rectangular to spherical etc. The manifold will not change: flat space time will remain flat space time or Schwarzschild geometry will remain Schwarzschild geometry.

But in our case the metric coefficients will change on the same coordinate grid.

Discussion:

Between two surfaces M1 and M2 we may have millions of metric compatible surfaces but all the 10 equations given by (7) will not be simultaneously satisfied by them. Starting from a given manifold M1 only 2^{10} specific surfaces will adhere to the required equation s as listed earlier. **We are considering metric compatible surfaces only in the process of variation because of the Ricci scalar worked by changes in the metric coefficients.**

Each metric compatible surface satisfies the Bianchi Identity and consequently is a solution of the Field Equation[Einstein's Field equations: incidentally they may be deduced without using the Einstein Hilbert Action] So the procedure indicated provides us the opportunity to locate different solutions to the Field equations starting from a known one.

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Supplementary Material

1.On the Covariant Derivative of the Metric Tensors

We may choose arbitrary differentiable functions $f_{\mu\nu}(t,x,y,z)$: four of them for orthogonal systems and sixteen of them for on orthogonal systems. If a surface[manifold] is considered where the above functions are chosen to be metric coefficients that is $g_{\mu\nu} \equiv f_{\mu\nu}$, then have

$\nabla_\alpha g_{\mu\nu} = 0$ or $\nabla_\alpha f_{\mu\nu} = 0$

Proof[1]:

Step 1:

We will first show:

1. $\frac{\partial g_{\mu\nu}}{\partial x^m} = [\mu m.\nu] + [\nu m.\mu]$; where $[ab,c] = \frac{1}{2}\left(\frac{\partial g_{ac}}{\partial x^b} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^c}\right)$: FChristoffel symbols of type1]
2. $\frac{\partial g^{\mu\nu}}{\partial x^m} = -g^{\mu n}\Gamma_{mn}{}^{\nu} - g^{\nu n}\Gamma_{mn}{}^{\mu}$

[where Christoffel symbols of the second type: $\Gamma_{\beta\gamma}{}^{\alpha} = \frac{1}{2} g^{\alpha k}\left(\frac{\partial g_{\beta k}}{\partial x^\gamma} + \frac{\partial g_{k\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^k}\right)$]

Proof of 1.

$$[\mu m.\nu] = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^m} + \frac{\partial g_{\nu m}}{\partial x^\mu} - \frac{\partial g_{\mu m}}{\partial x^\nu}\right)$$
$$[\nu m.\mu] = \frac{1}{2}\left(\frac{\partial g_{\nu\mu}}{\partial x^m} + \frac{\partial g_{m\mu}}{\partial x^\nu} - \frac{\partial g_{\nu m}}{\partial x^\mu}\right)$$

Therefore[by direct addition],

$$[\mu m.\nu] + [\nu m.\mu] = \frac{\partial g_{\mu\nu}}{\partial x^m}$$

Proof of 2.

$$g^{ik} g_{kj} = \delta_j{}^i$$

Partial Differentiating with respect to x^m we obtain:

$$g^{ik}\frac{\partial g_{kj}}{\partial x^m}+\frac{\partial g^{ik}}{\partial x^m}g_{kj}=0$$
$$g^{ik}\frac{\partial g_{kj}}{\partial x^m}=-\frac{\partial g^{ik}}{\partial x^m}g_{kj}$$

Multiplying both sides of the above by g^{ir} we obtain

$g^{jr}g_{kj}\frac{\partial g^{ik}}{\partial x^m}=-g^{jr}g^{ik}\frac{\partial g_{kj}}{\partial x^m}$

$Or,$

$$\delta_k{}^r\frac{\partial g^{ik}}{\partial x^m}=-g^{rj}g^{ik}[km,j]-g^{jr}g^{ik}[jm,k]$$

$$\frac{\partial g^{ir}}{\partial x^m}=-g^{ik}\Gamma_{km}{}^r-g^{jr}\Gamma_{jm}{}^i$$

Same as

$$\frac{\partial g^{\mu\nu}}{\partial x^m}=-g^{\mu n}\Gamma_{mn}{}^{\nu}-g^{\nu n}\Gamma_{mn}{}^{\mu}$$

Step 2:

Proof of $\nabla_m g^{\mu\nu}=0$

$$\nabla_m g^{\mu\nu}=\frac{\partial g^{\mu\nu}}{\partial x^m}+\Gamma_{km}{}^{\mu}g^{k\nu}+\Gamma_{km}{}^{\nu}g^{\mu k}$$

But

$$\frac{\partial g^{\mu\nu}}{\partial x^m}=-g^{k\nu}\Gamma_{km}{}^{\mu}-g^{\mu k}\Gamma_{km}{}^{\nu}$$

[from 2 of step1]

Therefore $\nabla_m g^{\mu\nu}=0$

Proof of $\nabla_m g_{\mu\nu}=0$

$$\nabla_m g_{\mu\nu}=\frac{\partial g_{\mu\nu}}{\partial x^m}-\Gamma_{\mu m}{}^k g_{k\nu}-\Gamma_{\nu m}{}^k g_{\mu k}$$

$$\nabla_m g_{\mu\nu}=[\mu m.\nu]+[\nu m.\mu]-[\mu m.\nu]-[\nu m.\mu]=0$$

Results used:

1. $\frac{\partial g_{\mu\nu}}{\partial x^m}=[\mu m.\nu]+[\nu m.\mu]$

2. $g_{kv}\Gamma_{\mu m}{}^{k} = \delta_{v}{}^{s}[\mu m, s] = [\mu m, v]$ and $\Gamma_{vm}{}^{k} g_{\mu k} = [vm.\mu]$

[Discussion: $\Gamma_{\mu m}{}^{k} g_{kv} = [\mu m.v]: Proof: \Gamma_{\mu m}{}^{k} = \frac{1}{2} g^{ks}\left(\frac{\partial g_{\mu s}}{\partial x^{m}} + \frac{\partial g_{sm}}{\partial x^{\mu}} - \frac{\partial g_{\mu m}}{\partial x^{s}}\right) = g^{ks}[\mu m, s]\ or, g_{kv}\Gamma_{\mu m}{}^{k} = g_{kv} g^{ks}[\mu m, s]$

Or,

$$g_{kv}\Gamma_{\mu m}{}^{k} = \delta_{v}{}^{s}[\mu m, s] = [\mu m, v]$$

Therefore

$\nabla_{\alpha}(g_{\mu v}) = 0$ and $\nabla_{\alpha} g^{\mu v} = 0$ follow as necessary conditions for covariant differentiation. We cannot part with this so long as we do not change the fundamental premises relating to the space being considered in the Lagrangian.

Moreover the said conditions are tied to the fact that parallel transport of a pair of vectors along a curve lying on the manifold does not change the dot product/inner product between them.

2 Difference of two Connections

We may analyze it the following way.

We are given metric compatible functions $g_{\mu v}(t, x, y, z)$

We consider an incremented $g_{\mu v}(t, x, y, z) + \delta g(t, x, y, z)$. The new functions are metric compatible in respect of the manifold constructed by using them as metric coefficients . Then we have constructed a new transformation of passing from one manifold to another[both are metric compatible satisfying $\nabla_{\alpha}\left[g_{\mu v}(t, x, y, z)\right] = 0$ and $\nabla_{\alpha}[g_{\mu v}(t, x, y, z) + \delta g(t, x, y, z)] = 0$]

Suppose M2 is not metric compatible in the sense $\nabla_{\alpha} g_{\mu v}(t, x, y, z) \neq 0$

The final metric figuring in the field equations will be of the type M1, that is metric compatible, and not of the type M2 which may be metric incompatible for arbitrary increments . Possibly that may allow inclusion of fictitious metric incompatible surfaces in the variation. Differentiation will always take place on the metric compatible surfaces like M1 and not on surfaces of the type M2. . But this idea cannot be entertained in view of the fact that we consider the difference of two connections as a tensor

Standard Proof from Literature[Sean Carroll]

Transformation of Christoffel Symbols

$$\Gamma_{\mu' v'}{}^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{v}}{\partial x^{v'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma_{\mu v'}{}^{\lambda} - -\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{v}}{\partial x^{v'}}\frac{\partial^{2} x^{\lambda'}}{\partial x^{\mu}\partial x^{v}}$$

In the above coordinate system has changed on the same manifold. Now the manifold changes and we have coordinate transformation on the new manifold.

$$\bar{\Gamma}_{\mu'\nu'}{}^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\bar{\Gamma}_{\mu\nu'}{}^{\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu}\partial x^{\nu}}$$

The transformation elements $\frac{\partial x^{\mu}}{\partial x^{\mu'}}$ remaining unchanged with change of the manifold

Difference between two connections:

$$\Gamma_{\mu'\nu'}{}^{\lambda'} - \bar{\Gamma}_{\mu'\nu'}{}^{\lambda'}$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma_{\mu\nu'}{}^{\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\lambda'}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\bar{\Gamma}_{\mu\nu'}{}^{\lambda} \quad (1)$$

Therefore,

$$\Gamma_{\mu'\nu'}{}^{\lambda'} - \bar{\Gamma}_{\mu'\nu'}{}^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\left[\Gamma_{\mu\nu}{}^{\lambda} - \bar{\Gamma}_{\mu\nu}{}^{\lambda}\right] \quad (2)$$

We have been able to factor out the common quantity $\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}$ which has not changed due to change in the manifold at the concerned point. This is a vital point for us

The quantity $\Gamma_{\mu\nu'}{}^{\lambda} - \bar{\Gamma}_{\mu\nu}{}^{\lambda}$ transforms like a tensor.

$\Gamma_{\mu\nu'}{}^{\lambda}and\bar{\Gamma}_{\mu\nu'}{}^{\lambda}$ stand on the same coordinate grid. $\Gamma_{\mu'\nu'}{}^{\lambda'}and\bar{\Gamma}_{\mu'\nu'}{}^{\lambda'}$ stand on the same grid but different from the previous one. Variation takes place on the same grid. This variation is then considered on other grids through coordinate transformation.

The Christoffel symbols have been considered on the same coordinate grid that allows the cancellation of two identical middle terms on the right side of (1). That leads to the tensor transformation criterion in (2). We have different sets of $g_{\mu\nu}$ hanging on the same grid

$\Gamma_{\mu'\nu'}{}^{\lambda'}and\bar{\Gamma}_{\mu'\nu'}{}^{\lambda'}$ correspond to different manifolds but they are on the same coordinate grid.

$\Gamma_{\mu\nu}{}^{\lambda}and\bar{\Gamma}_{\mu\nu}{}^{\lambda}$ correspond to different manifolds but they are on the same coordinate grid.

$\Gamma_{\mu'\nu'}{}^{\lambda'}and\Gamma_{\mu\nu}{}^{\lambda}$enjoy the same manifold but different coordinate grids

$\bar{\Gamma}_{\mu'\nu'}{}^{\lambda'}and\bar{\Gamma}_{\mu'\nu'}{}^{\lambda'}$ are on the same manifold but their coordinate grids are different

[the above derivation does not consider $\delta g_{\alpha\beta}$ as a tensor: this goes in favor of the formula as we shall see soon.

$(g_{\mu\nu} + \delta g_{\mu\nu})$is a tensor and also a metric coefficient in respect of M2

3.Tensors

Coordinate and Physical Values Infinitesimal Separations:

We start with the metric

$c^2 d\tau^2 = g_{tt} d(ct)^2 - g_{xx} dx^2 - g_{yy} dy^2 - g_{zz} dz^2$ (1)

$c^2 d\tau^2 = d(cT)^2 - dL^2$ (2)

Physical Separations

Physical Time interval: $\boldsymbol{dT = \sqrt{g_{tt}} dt}$ (3)

Physical Length:$dL = \sqrt{g_{xx} dx^2 + g_{yy} dt^2 + g_{zz} dz^2}$ (4)

Physical separations along the x , y and the z direction:

$\boldsymbol{dx_{ph} = \sqrt{g_{xx}} dx}$ **(5.1)**

$\boldsymbol{dy_{ph} = \sqrt{g_{yy}} dy}$ **(5.2)**

$\boldsymbol{dz_{ph} = \sqrt{g_{zz}} dz}$ **(5.3)**

[Sufficx: "ph" stands for physical]

For the Null Geodesic: $ds^2 = c^2 d\tau^2 = 0$

$$d(cT)^2 - dL^2 = 0$$

Or $c^2 dT^2 = dL^2$

$\Longrightarrow \frac{dL}{dT} = c$ (6)

In this physical separation formulation locally we always have $\frac{dL}{dT} = c$ for the null geodesic.

Now we take a light ray travelling along the x axis: rather we orient the x axis[an infinitesimal part of it] al.ong the direction of propagation of light ray in curved space

$$ds^2 = g_{tt} c^2 dt^2 - g_{xx} dx^2$$

$$ds^2 = 0$$

Therefore, $0 = g_{tt} c^2 dt^2 - g_{xx} dx^2$

$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{g_{tt}}{g_{xx}}} c$ (7)

Coordinate speed of light in curved space time

$\frac{dx}{dt} \neq c$ (8)

But physical speed of light [Local physical Speed], $\frac{dL}{dT} = c$

It is interesting to note that relation (7)

Reduces to $\frac{dx}{dt} = c$ when we are in Minkowski space that is when $g_{tt} = 1$ and $g_{xx} = 1$

wE do have an allied concept of proper speed

Coordinate and Physical Values of Tensors:

Proper speed in General Relativity is defined by:

$\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$. Now proper time interval $d\tau$ is not expected to be the same for flat space time and curved space time. So we have different values for proper speed components in flat space time and in curved space time: due to difference in the value of proper time interval

$$\left(\frac{cdt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \equiv \frac{cdt}{\sqrt{g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2}}, \frac{dx}{\sqrt{g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2}}$$

$$\frac{dy}{\sqrt{g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2}}, \frac{dz}{\sqrt{g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2}}$$

In the flat space time context $g_{tt} = g_{xx} = g_{yy} = g_{zz} = 1$ and we have the proper time interval for Minkowski space. In a thought experiment you may turn on gravity starting from Minkowski space : the proper time interval will change gradually as the manifold changes.

Replacing proper time interval of curved space time by proper time interval; of Minkowski space is theoretically incorrect. But we can always do the following:

Locally we can replace

$$c^2 d\tau^2 = g_{tt}d(ct)^2 - g_{xx}dx^2 - g_{yy}dy^2 - g_{zz}dz^2$$

By

$$c^2 d\tau^2 = c^2 dT^2 - {dx_{ph}}^2 - {dy_{ph}}^2 - {dz_{ph}}^2 \text{ (9)}$$

Relation (9) locally has the " form " of the Minkowski metric. We have to be care ful that this metric contains gravity. But the advantage with it is that we have the form of the flat space time metric

Four speed as Tensor[Rank 1]$\equiv \left(\frac{cdT}{d\tau}, \frac{dx_{ph}}{,d\tau}, \frac{dy_{ph}}{,d\tau}, \frac{dz_{ph}}{,d\tau}\right)$

The above is a tensor considering the Minkowskian form of the metric (9). The proper time in (9) is not the proper time of Minkowski space but we consider it to be locally invariant in its own manifold.

The justification of considering physical values becomes cl;ear.

In a transformation relation like

$$\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} A^{\alpha}$$

$\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ does not contain the metric coefficients$g_{\mu\nu}$. As a mater of fact A^{α} and $\bar{A}^{\mu}$ are Euclidean quantities. But the physical values (orthogonal systems being considered):$A^{\alpha}{}_{ph} = \sqrt{g_{\alpha\alpha}} A^{\alpha}$ and

$\bar{A}^{\mu}{}_{ph} = \sqrt{\bar{g}_{\mu\mu}} A^{\alpha}$(summation over α or or μ not implied]These physical values are cvharacterizedby the metric properties of the manifold. These physical quantities exhibit local Lorentz covariance in view of metric (9)

The variables in $\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}$ are those of (1) but he physical variables pertain to (9).

Important to note that if you look at relation (1) the quantities t, x, y and z do not contain the metric coefficients $g_{\mu\nu}$

The metric cvoefficients are responsible o=]for curving space time . The global variables standing independent of them do not curve space.

Now we are write the following two equations

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$c^2 d\tau'^2 = g_{tt} c^2 dt^2 - g_{xx} dx^2 - g_{yy} dy^2 - g_{zz} dz^2$$

Absence of metric coefficients in **t, x, y and z pints to un structured space, But lookingat the above two equations the interval ,** dx, dy and dz are Euclidean **if the variables t,x,y and z are identical in the two equations.**

We may justify this physically by a thought experiment. Starting from Minkowski space you turn =on gravity gradually the quantities dt, dx, dy and dz will change to **$\sqrt{g_{tt}}dt$, $\sqrt{g_{xx}}dx$, $\sqrt{g_{yy}}dy$ and $\sqrt{g_{zz}}dz$** for the same pair of space time coordinate locations (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$. Thus the first equation defines the Euclidean background while the second onerepresents curved space time with the labels t x y and z remaining Euclidean

IN the curved spasce time context the measurable quantities having the same dimensions of length are to c**$\sqrt{g_{tt}}dt$, $\sqrt{g_{xx}}dx$, $\sqrt{g_{yy}}dy$ and $\sqrt{g_{zz}}dz$**

Let us try to comprehend the situation with Schwarzschild metric:

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2Gm}{c^2 r}\right) dt^2 - \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 - r^2 (d\theta^2 + Sin^2\theta d\varphi^2)$$

The radial distance between two infinitesimally close points is not dr but it is : $\left(1 - \frac{2Gm}{c^2 r}\right)^{-1/2} dr = \sqrt{g_{rr}} dr$

Suppose In try transfortming Schwarzschild metric from Spherical to Cartesian system for the stationary observer in the global system of coordinates.

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$

$$x = rCos\varphi Sin\theta$$

$$y = rSin\varphi Sin\theta$$

$$z = rCos\theta$$

$t' = t$ [relatively stationary observers in the global system of coordinates]

Cartesian coordinates related to Schwarzschild coordinates by familiar relations[2]

The above transformations are use $\frac{\partial x^{\alpha}}{\partial x'^{\mu}}$ in consideration of the Euclidean grid

Special Example [3]:

$$f^{\alpha} = m_0 \left[\frac{d^2 x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}{}_{\beta\gamma} \frac{dx^{\beta}}{d\tau} \frac{dx^{\gamma}}{d\tau}\right]$$

We consider a spatially stationary observer near a Schwarzcchild mass so that we have non geodesic motion and $f^{\alpha} \neq 0$

Four speed≡ $(u_t, 0,0,0)$

Norm of four speed=$c^2 = 1$

$$\Rightarrow \left(1 - \frac{2m}{r}\right) {u_t}^2 = 1$$

$$u_t = \left(1 - \frac{2m}{r}\right)^{-1/2}$$

We are in the (r, θ, φ) $system$ .For(spatially) stationary observer $u_r = \frac{dr}{d\tau} = 0; u_\theta = \frac{d\theta}{d\tau} = 0; u_\varphi = \frac{d\varphi}{d\tau} = 0$

Radial force: $f^r = m_0 \left[\frac{d^2 x^r}{d\tau^2} + \Gamma^r{}_{tt} \left(\frac{dx^t}{d\tau}\right)^2\right] = m_0 \Gamma^r{}_{tt} \left(\frac{dx^t}{d\tau}\right)^2 = \frac{m}{r^2}(1 - 2m/r)\left(1 - \frac{2m}{r}\right)^{-1} = \frac{m}{r^2}$=

$f^r = \frac{m}{r^2}$ gives us the **coordinate value** of the radial component of the force four vector[Minkowski force]**. It is the same as Newtonian thrust**

The physical component is given by :$\sqrt{g_{rr}} f^{\alpha} = \left(1 - \frac{2m}{r}\right)^{-1/2} \frac{m}{r^2}$. The physical value is much greater than Newtonian thrust and infinitely larger a r approaches 2m.

Additional Point

We start with the definition of four acceleration:

$$a^{\alpha} = \left[\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau}\right]$$

For motion along a geodesic each component of four acceleration is zero. In particular for radial motion under gravity the radial component of four acceleration=0. Though four acceleration and four force are elegant mathematical formulations they do not conform to what we understand by acceleration in the physical sense. An apple falling from a tree will accelerate at the rate3 9.8 m/s^2. If gravity were a million times stronger it would have accelerated at a much faster rate. But four acceleration would always remain zero for geodesics.

We can always formulate relations closer to what we mean by acceleration in the day to day physical sense. We do the following

The Geodesic Equation:

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau} = 0$$

For radial motion under gravity, Schwarzschild metric being considered we have,

$$\frac{d^2r}{d\tau^2} = -\Gamma^{r}{}_{tt}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^{r}{}_{rr}\left(\frac{dr}{d\tau}\right)^2$$

$$\frac{d^2r}{d\tau^2} = -\frac{m}{r^2}(1-2m/r)\left(\frac{dt}{d\tau}\right)^2 + \frac{m}{r^2}(1-2m/r)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

Or,

$$\frac{d^2r}{d\tau^2} = -\frac{m}{r^2}\left((1-2m/r)\left(\frac{dt}{d\tau}\right)^2 - (1-2m/r)^{-1}\left(\frac{dr}{d\tau}\right)^2\right)$$

[Christoffel symbols in the above have been taken in the (-,+,+,+) signature]

Now we write the Schwarzschild metric:

$$d\tau^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 + r^2(d\theta^2 + Sin^2\theta d\varphi^2)$$

For radial motion

$$d\tau^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2$$

Dividing both sides of the above by the proper time interval squared $d\tau^2$ we have:

$$1 = -(1 - 2m/r)\left(\frac{dt}{d\tau}\right)^2 + (1 - 2m/r)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

$$\frac{d^2r}{d\tau^2} = +\frac{m}{r^2}$$

$m \to \frac{Gm}{c^2}$and $\tau \to c\tau; t \to ct$

Explanation for the positive sign: In our chosen signature(-,+,+,+),$d\tau$ is imaginary for time like interval separations. $d\tau$ contains the imaginary "I" as a factor. When we double differtentiate to evaluate $\frac{d^2r}{d\tau^2}$ on the left side of the formula we evidently have ,$i \times i = -1$

In effect we are having

$$\frac{d^2r}{d\tau^2} = -\frac{m}{r^2}$$

Which is equivalent to

$$\frac{d^2r}{d\tau^2} = -\frac{Gm}{r^2}$$

Notable issue: How does the relation stand in view of transformations considering the fact that $\frac{d^2r}{d\tau^2}$ is not a four vector component like $\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$, given that transformations comprise a core aspect in physical considerations?

The tangent to a geodesic curved gets parallel transported along it. In presence of gravity alone only geodesics are available for physical motion . But parallel transport may be considered for all types of world lines , even the imaginary ones

We are concerned with radial motion under gravity[geodesic in terms of spatial and temporal coordinates]

For parallel transport along non geodesics the norm of the vector is preserved. But in these situations we have agents other than gravity in operation.

We may start with the velocity vector for [geodesic] motion in the radial direction and move it along the same radius. The four speed and the corresponding four acceleration will remain unchanged. If we consider a Cartesian system with origin on the same radial line and x –axis along the radial direction then,

we will have $\frac{d^2r}{d\tau^2} = \frac{d^2x}{d\tau^2}$

This Cartesian system may have the same origin as the global Schwarzschild coordinate system. But the x axis has to coincide with the outward radial direction

In fact we have without any type of mathematical fanfare, $dr = dx$ and $d\tau$ unchanged when we align the x axis of the Cartesian frame in the direction with out changing the origin]

On the Curving of Space time Geometry

In general relativity coordinate distances and physical distances are identical only for flat spacetime. But in curved spacetime the coordinate separations and the physical separations are different. They may become radically different if the curvature is strong enough. Lets consider the physical curving of 3D space in view of the above fact. We consider two flat surfaces parallel to the x-y plane at two levels, z=a and z=b in the flat spacetime context. Several points are considered on the two mentioned planes. A gravitational change is now considered. The metric coefficients change and the physical distances of the points lying on each plane change . The points may be considered pair wise on each plane and also pair wise on the two separate planes. Their mutual distances change with changes in the gravitational field and change may occur differently for the different pairs. The planes become undulating surfaces—space gets curved!

Let's consider a spherical planet like the earth. A dense mass approaches it in our thought experiment. The value of the metric coefficients change at each point in the concerned field changes.. Due to gravitational effects, even in our classical interpretation, the shape of the earth's surface might change due to an interaction between the changes in the space time curvature and non-gravitational factors like the resistance of the earth's crust etc In our "experiment" in the first paragraph we may consider the coordinates as labels---stickers of different colors at different points on the two planes. Initially they were on a flat surface. After the gravitational change they lie on a pair of undulating surfaces. A straight line on some plane becomes a curved line – the path of a light ray bends and the straight line path of a test particle the Minkowski space picks up the curved path of a planet!

We may start with a spherical coordinate system describing Schwarzschild geometry. Due to manifold changes the coordinate planes become curved surfaces. The axes become curved lines instead of straight lines. Prior to Schwarzchild geometry there could have been some other type of geometry with global Cartesian or spherical coordinates. The straight line x,yz axis became curved lines in Schwarzschlild geometry. Old curved lines became strait in the physical sense. The Euclidean background remains maintained.

Events are Labeled by Coordinate Values

Events are labeled by Coordinate values and not by physical values. The light cone is created by such coordinate labels. If there is a change in the nature of the manifold the surface of the light cone will get distorted

Dot product and its invariance

$a.b = g_{\mu\nu}a^{\nu}b^{\mu} = a_{\mu}b^{\mu}$: Invariant (1)

In the orthogonal system: $a.b = g_{\mu\mu}a^{\mu}b^{\mu}$

[*since* $g_{\mu\nu} = 0$ *if* $\mu \neq \nu$]

a^{ν} and b^{μ} are coordinate values of the vectors: vectors/tensors are denoted by coordinate values and not by physical values." $g_{\mu\nu}$" pertain to the nature of the space where the objects are situated.

Physical value of a^{ν} in orthogonal system :$\sqrt{g_{\nu\nu}}a^{\nu} = a^{\nu}{}_{ph}$ [$\sqrt{|g_{\mu\nu}|}$ has been implied by $\sqrt{g_{\mu\nu}}$]

Physical value of b^{μ} in orthogonal system :$\sqrt{g_{\mu\mu}}b^{\mu} = b^{\mu}{}_{ph}$

$a.b = g_{\mu\mu}a^{\mu}b^{\mu} = g_{\mu\mu}\frac{a^{\mu}{}_{ph}}{\sqrt{g_{\mu\mu}}}\frac{b^{\mu}{}_{ph}}{\sqrt{g_{\mu\mu}}} = (appropriate\ sign)a^{\mu}{}_{ph}b^{\mu}{}_{ph} = a_{\mu}b^{\mu}$ (2)

Example for clarifying the sign aspect:For flat space time(With the $(+,-,-,-)$ signature

] $g_{\mu\mu}\frac{a^{\mu}{}_{ph}}{\sqrt{g_{\mu\mu}}}\frac{b^{\mu}{}_{ph}}{\sqrt{g_{\mu\mu}}} = g_{00}\frac{a^{0}{}_{ph}}{\sqrt{g_{00}}}\frac{b^{0}{}_{ph}}{\sqrt{g_{00}}} + g_{11}\frac{a^{1}{}_{ph}}{\sqrt{g_{11}}}\frac{b^{1}{}_{ph}}{\sqrt{g_{11}}} + g_{22}\frac{a^{2}{}_{ph}}{\sqrt{g_{22}}}\frac{b^{2}{}_{ph}}{\sqrt{g_{22}}} + g_{33}\frac{a^{3}{}_{ph}}{\sqrt{g_{33}}}\frac{b^{3}{}_{ph}}{\sqrt{g_{33}}} = +1\frac{a^{0}{}_{ph}}{\sqrt{+1}}\frac{b^{0}{}_{ph}}{\sqrt{+1}} + (-1)\frac{a^{1}{}_{ph}}{\sqrt{|-1|}}\frac{b^{1}{}_{ph}}{\sqrt{|-1|}} + (-1)\frac{a^{2}{}_{ph}}{\sqrt{|-1|}}\frac{b^{2}{}_{ph}}{\sqrt{|-1|}} + (-1)\frac{a^{3}{}_{ph}}{\sqrt{|-1|}}\frac{b^{3}{}_{ph}}{\sqrt{|-1|}} = a^{0}{}_{ph}b^{0}{}_{ph} - a^{1}{}_{ph}b^{1}{}_{ph} - a^{2}{}_{ph}b^{2}{}_{ph} - a^{3}{}_{ph}b^{3}{}_{ph}$
(3)

Similar extensions

[Appropriate sign has to be taken since $\sqrt{g_{\mu\mu}}$ is always positive]

The above signifies the MInkowskian nature of the physical quantities a and b.

They agree with the definition of dot product

Now

$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ *(4)*

The above is in relation to coordinate values

In relation to physical values we may write[orthogonal systems]

$$F^{\mu\nu} = \partial^{\mu}\left(\frac{A^{\nu}{}_{physical}}{\sqrt{g_{\nu\nu}}}\right) - \partial^{\nu}\left(\frac{A^{\mu}{}_{physical}}{\sqrt{g_{\mu\mu}}}\right)$$

But A(coordinate value) is gauge dependent…gauging on A should not alter the physical nature of $F^{\mu\nu}$*(which is not violated in the above.)*

Definition of physical value in the General type [Literature based]: orthogonal or non orthogonal:

$F^{ik}{}_{ph} = \sqrt{g_{i\mu}g_{k\nu}}F^{\mu\nu}$ *(5)[following summation convention]*

Identical with

$$F^{\mu\nu}{}_{ph} = \sqrt{g_{\mu\alpha}g_{\nu\beta}}F^{\alpha\beta}$$

Multiplying both sides of (5)by $g_{i\mu}g_{k\nu}$

$g_{i\mu}g_{k\nu}F^{ik}{}_{ph} = \sqrt{g_{i\mu}g_{k\nu}}g_{i\mu}g_{k\nu}F^{\mu\nu}$*Implies that*

$g_{i\mu}g_{k\nu}F^{ik}{}_{ph} = \sqrt{g_{i\mu}g_{k\nu}}F_{ik}$ (6)

Define $F_{\mu\nu:ph} = \sqrt{g^{i\mu}g^{k\nu}}F_{ik}$ (7)[allowing summation convention]

$F_{\mu\nu:ph}F^{\mu\nu}{}_{ph} = \sqrt{g^{i\mu}g^{k\nu}}F_{ik}\sqrt{g_{\mu\alpha}g_{\nu\beta}}F^{\alpha\beta}$(6.1)

You have in the orthogonal system, $i = \mu = \alpha$:and $k = \nu = \beta$ and

$F_{\mu\nu:ph}F^{\mu\nu}{}_{ph} = F_{\mu\nu}F^{\mu\nu}$ (8)[summation considered]

[On the left side we have curved space time metric coefficients on the right we don't have.

[In the orthogonal system:$g^{jj} = \frac{1}{g_{jj}}$ [summation not implied]: this simplifies the rhs of (6.1) causing cancellation][]

Thus in the orthogonal system the dot product of the physical values is invariant

You may compare equation (8) with the relation $a^{\nu}{}_{ph}b^{\mu}{}_{ph} = a_{\mu}b^{\mu}$

Further investigation of the tensorial nature of $F^{\mu\nu}{}_{ph}$ and $F_{\mu\nu:ph}$

Recalling (6)

$$g_{i\mu}g_{k\nu}F^{ik}{}_{ph} = \sqrt{g_{i\mu}g_{k\nu}}F_{ik}$$

And treating $F^{ik}{}_{ph}$ as a tensor let us lower its indices :$F_{\mu\nu:ph} = \sqrt{g_{i\mu}g_{k\nu}}F_{ik}$. This agrees with our definition of $F_{\mu\nu:ph}$.

Recalling (7) let is raise the indices of $F_{\mu\nu:ph} = \sqrt{g^{i\mu}g^{k\nu}}F_{ik}$. Multiplying both sides by $g^{i\mu}g^{k\nu}$ we have

$$g^{i\mu}g^{k\nu}F_{\mu\nu:ph} = \sqrt{g^{i\mu}g^{k\nu}}\ g^{i\mu}g^{k\nu}F_{ik}$$

Treating $F_{\mu\nu:ph}$ as a tensor and raising indices ,

$F^{ik}{}_{ph} = \sqrt{g^{i\mu}g^{k\nu}}F^{\mu\nu}$: agrees with our definition

From relations (5) and (7) the quantities $F^{\mu\nu}{}_{ph}$ and $F_{\mu\nu:ph}$ cannot be directly recognized as tensors

From the theoretical point of view you may take $F^{ik} = P^{i}Q^{k}$

We have the following differential equation: $\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = P^{\mu}Q^{\nu}$: There two quantities P and Q and so we can exert choice. But P^{μ} and Q^{μ}each must transform like a vector

If A transforms like a 4 vector and if P and Q are 4 vectors preserving $\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = P^{\mu}Q^{\nu}$we have to write.

$$\partial^{\mu}(\Lambda^{\nu}{}_{\alpha}A'^{\alpha}) - \partial^{\nu}(\Lambda^{\mu}{}_{\beta}A'^{\beta}) = (\Lambda^{\nu}{}_{\alpha}P'^{\mu})(\Lambda^{\nu}{}_{\alpha}Q'^{\mu})$$

The above equation will ensure the appropriate transformation properties

[the background system is Euclidean and Lorentz transformation holds for uniform relative motion between the frames].

F^{ik} *is the coordinate value of the vector.*

But

$$F^{ik} = \partial^{i}\left(\frac{A^{k}{}_{physical}}{\sqrt{g_{\nu\nu}}}\right) - \partial^{k}\left(\frac{A^{i}{}_{physical}}{\sqrt{g_{\mu\mu}}}\right)$$

Therefore:

$$F^{\mu\nu}{}_{ph} = \sqrt{g^{i\mu}g^{k\nu}}\left(\partial^{i}\left(\frac{A^{k}{}_{physical}}{\sqrt{g_{\nu\nu}}}\right) - \partial^{k}\left(\frac{A^{i}{}_{physical}}{\sqrt{g_{\mu\mu}}}\right)\right)$$

The above is identical with

$$F^{\mu\nu}{}_{ph} = \sqrt{g^{i\mu}g^{k\nu}}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})$$

Transformation:

$$F'^{ik} = \Lambda^{i}{}_{\mu}\Lambda^{k}{}_{\nu}F^{\mu\nu}$$

$$\sqrt{g^{i\mu}g^{k\nu}}F'^{ik} = \sqrt{g^{i\mu}g^{k\nu}}\Lambda^{i}{}_{\mu}\Lambda^{k}{}_{\nu}F^{\mu\nu}$$

$$\sqrt{g'^{i\mu}g'^{k\nu}}\sqrt{g^{i\mu}g^{k\nu}}F'^{ik} = \sqrt{g'^{i\mu}g'^{k\nu}}\sqrt{g^{i\mu}g^{k\nu}}\Lambda^{i}{}_{\mu}\Lambda^{k}{}_{\nu}F^{\mu\nu}$$

$$\sqrt{g^{i\mu}g^{k\nu}}F'^{\mu\nu}{}_{ph} = \sqrt{g'_{i\mu}g'_{k\mu}}\Lambda^{i}{}_{\mu}\Lambda^{k}{}_{\nu}F^{\mu\nu}{}_{ph}$$

Again:

We have the curved space time transformations for the physical values of $F^{\mu\nu}$

$g'_{i\mu}$ and $g'_{k\mu}$ are transformed values of the metric coefficients.

When we consider transformation of tensors we involve only the coordinate values and not the physical values of the tensor for example we take A^{μ} instead of $\sqrt{g_{\mu\mu}}A^{\mu}$

References

[1]Spiegel M R, Vector Analysis and an Introduction to Tensor Analysis ;Schaum Series,McGraw-Hill Company,Capter 8; Solved problems

[2] Hartle J B,, Gravity: An Introduction to Einstein's General Relativity: Chapter 14 Section 14.5 page 304; link https://drive.google.com/file/d/0BymT8iD6LY1nQ1VURFNhWDdtVkE/view?usp=sharing

[3] Hartle J B,, Gravity: An Introduction to Einstein's General Relativity: Chapter 12;Box12.2: A Myth about Black Holes] Section 14.5 page 261;

Link https://drive.google.com/file/d/0BymT8iD6LY1nQ1VURFNhWDdtVkE/view?usp=sharing

Extra Supplementary Material

Certain calculations that were provided to stand out against my work :

$g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}$ *(A)*

$\delta g^{\mu\nu} = (\delta g^{\mu\alpha}) g^{\nu\beta} g_{\alpha\beta} + g^{\mu\alpha} (\delta g^{\nu\beta}) g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} (\delta g_{\alpha\beta})$

$\delta g^{\mu\nu} = (\delta g^{\mu\alpha}) g^{\nu\beta} g_{\alpha\beta} + g^{\mu\alpha} (\delta g^{\nu\beta}) g_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$

$\delta g^{\mu\nu} = (\delta g^{\mu\alpha}) \delta_\alpha{}^\nu + (\delta g^{\nu\beta}) \delta_\beta{}^\mu + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$

$\delta g^{\mu\nu} = \delta g^{\mu\nu} + \delta g^{\nu\mu} + g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$ *(B)*

There fore,

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

But I have in my paper

$$\delta g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

Solution/answer offered by me:

$\delta g^{\mu\nu} = \delta g^{\mu\nu} + \delta g^{\nu\mu} + \delta g^{\mu\nu}$ *[if $\delta g_{\alpha\beta}$ is treated as a tensor] (C)*

We have: $\delta g^{\mu\nu} + \delta g^{\nu\mu} = 0$ *ensuring the validity of (C)*

$\delta g^{\mu\nu} = -\delta g^{\nu\mu}$

$\delta g^{\mu\nu}$ is an antisyymmetric tensor

Relation (B) reads $\delta g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$ *(D)*

Again in relation B if you cancel $\delta g^{\mu\nu}$ *from either side you have*

$\delta g^{\nu\mu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$ *(E)*

Adding (D) and (E)

$$\delta g^{\nu\mu} + \delta g^{\nu\mu} = \left(-g^{\mu\alpha} g^{\nu\beta} + g^{\nu\alpha} g^{\mu\gamma}\right)\delta g_{\alpha\beta}$$

[μ, ν are fixed indices]

The left side is symmetric with respect to interchange of μ and ν while the right side is antisymmetric with respect to the interchange of μ and ν. Therefore we have $\delta g^{\nu\mu} + \delta g^{\nu\mu} = 0 \Longrightarrow \delta g_{\alpha\beta} = 0$ for general type of non orthogonal systems

[Since generally speaking $-g^{\mu\alpha} g^{\nu\beta} + g^{\nu\alpha} g^{\mu\gamma} \neq 0$]

In the orthogonal system(C) and (D)

$\delta g^{\nu\mu} = -g^{\mu\mu} g^{\nu\nu} \delta g_{\mu\nu}$ *[[μ, ν are fixede indices: summation on them is not implied here](C')*

$\delta g^{\nu\mu} = g^{\nu\nu} g^{\mu\mu} \delta g_{\mu\nu}$ *[[μ, ν are fixede indices: summation on them is not implied here (D)*

$$\delta g^{\nu\mu} + \delta g^{\nu\mu} = -g^{\mu\mu} g^{\nu\nu} \delta g_{\mu\nu} + g^{\nu\nu} g^{\mu\mu} \delta g_{\mu\nu} = 0$$

Therefore $2\delta g^{\nu\mu} = 0$ or, $\delta g^{\mu\nu} = 0$. Impossibility of continuous variation is evident

Simultaneous symmetricity and asymmetricity suggests zero value. But asymmetricity has been deduced using differentiation. In basic calculationsinn my paper "survey…" I have used$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$ and

$(g^{\alpha\mu} + \delta g^{\alpha\mu})(g_{\alpha\nu} + \delta g_{\alpha\nu}) = \delta^{\mu}{}_{\nu}$

Covaraiant differentiation of $\delta g_{\alpha\beta}$ is extensively used in the extended gravity theories only because $\delta g_{\alpha\beta}$ taken to be is a tensor. But the 10 equations I th paper "Survey"show that $\delta g_{\alpha\beta}$ can move through discrete steps only

We have $\delta g^{\nu\mu}$ as anti symmetric tensor instead of symmetric

Now $g^{\alpha\beta}$ is symmetric and $\delta g^{\alpha\beta}$ is antisymmetric. Therfore $g^{\alpha\beta} + \delta g^{\alpha\beta}$ is neither symmetric or anti symmetric

Salient Point:

If continuous changes in $g_{\mu\nu}$ are impossible due to discrete values of $\delta g_{\mu\nu}$, differentiation of becomes impossible and we do not have relation s C or D. In we do not have anti symmetric $\delta g_{\mu\nu}$

Ancillary Aspect

Even then we may assert the following though it is not so much necessary if discreteness renders usual type of differentiation impossible.

We consider the quantityr:

$$\phi = A_{ij}a^i a^j$$

$$\phi = A_{ij}a^i a^j = A_{ji}a^j a^i$$

$$2\phi = A_{ij}a^i a^j + A_{ji}a^j a^i$$

Or,

$$\phi = \frac{1}{2}(A_{ij} + A_{ji})a^i a^j$$

Or,

$$\phi = B_{ij}a^i a^j$$

Therefore in the context of a dot product like, $\phi = A_{ij}a^i a^j$ *[or* $ds^2 = (g^{\alpha\beta} + \delta g^{\alpha\beta})dx^i dx^j$*]*

I can always replace A_{ij} *by a symmetric tensor* B_{ij}*: this is a context dependent replacement valid for dot products . But may not be valid in an arbitrary situation*

Symmetricity for the metric tensor is not essential for our work or otherwise:

Definition:$g^{ij} = \frac{G(i,j)}{g}$

Where $G(i,j)$ is the cofactor of g_{ij} and g is the determinant of g_{ij}

Again $g = g_{ij}G(i,j)$; [summation over j only]

Also: $g_{kj}G(i,j) = 0$ if $k \neq i$

Now, $g^{ij}g_{kj} = \frac{G(i,j)}{g}g_{kj} = \delta^i{}_k$

Symmetricity of g_{kj} has not been assumed here.

Incidentally $G(i,j)$ is not a tensor in the usual sense . It is a relative tensor of weight two

But the quantity $\frac{G(i,j)}{g}$ is a tensor by division rule since g_{kj} and $\delta^i{}_k$ are tensors.

The result $g^{ij}g_{kj} = \delta^i{}_k$ has been used in derivation of my equations.

On fR) Gravity

Force four Orthogonal to 4 velocity as suggested by f(R) gravity papers[1] is not necessary .There are logicalinconsistencies withsuch formulations

Part 1

$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu{}_{\nu\lambda}u^\nu u^\lambda=f^\mu$ (68:f®Gravity paper) ---(1)

Where,

$f^\mu=8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi+f_T(R,T)]}(g^{\mu\nu}-u^\mu u^\nu)$ (69:paper) ---------- (2)

We multiply both sides of (68) by $\frac{1}{i^2}$ and use the substitution $s'=is$ [prime on s does not relate to differentiation];$i=\sqrt{-1}$

Relation (68) reduces to

$\frac{d^2x^\mu}{ds'^2}+\Gamma^\mu{}_{\nu\lambda}u'^\nu u'^\lambda=-f^\mu$ (3)

Where,

$u'^\nu=\frac{dx^\nu}{ds'}=\frac{dx^\nu}{d(is)}=\frac{1}{i}\frac{dx^\nu}{ds}$ (4.1)

$u'^\lambda=\frac{dx^\lambda}{ds'}=\frac{dx^\lambda}{d(is)}=\frac{1}{i}\frac{dx^\lambda}{ds}$ (4.2)

We re write (3) as:\

$\frac{d^2x^\mu}{ds'^2}+\Gamma^\mu{}_{\nu\lambda}u'^\nu u'^\lambda=-8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi+f_T(R,T)]}(g^{\mu\nu}-u^\mu u^\nu)$ (5)

It is important to note that on the right side of (5) we do not have prime on u^μ and on u^ν

In relations (64) 65 ,66, 67 we could have used s'=is instead of s in getting the same form of result that is in obtaining (68).

$$\frac{d^2x^\mu}{ds'^2}+\Gamma^\mu{}_{\nu\lambda}u'^\nu u'^\lambda=f'^\mu(6)$$

With $f'^\mu=8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi+f_T(R,T)]}(g^{\mu\nu}-u'^{\mu\prime}u^\nu)$ (7)

On the right we now have u'^μ and u'^ν since we have used is instead of s

This is mathematical artifice and it is not mandatory to connect it in the physical sense. The manifold itself contains space like and mixed curves[worlds lines[] though the possibility of movement along such curves in the physical world come under restrictions.

From (3) and (6)

$$f'^\mu=-f^\mu$$

From (6) and (7) we have

$\frac{d^2x^\mu}{ds'^2}+\Gamma^\mu{}_{\nu\lambda}u'^\nu u'^\lambda=8\pi\frac{\nabla_\nu p}{(\rho+p)[8\pi+f_T(R,T)]}(g^{\mu\nu}-u'^{\mu\prime}u^\nu)$ (8)

From (5) and (8)

$$-8\pi \frac{\nabla_\nu p}{(\rho+p)[8\pi + f_T(R,T)]}(g^{\mu\nu} - u^\mu u^\nu) = 8\pi \frac{\nabla_\nu p}{(\rho+p)[8\pi + f_T(R,T)]}(g^{\mu\nu} - u'^{\mu\prime} u^\nu)$$

[important $\nabla_\nu p, p\ and \rho$ are identical on both sides: they are point functions on the manifold while proper velocity components depend on directions chosen at a given point on the manifold]

$$-(g^{\mu\nu} - u^\mu u^\nu) = (g^{\mu\nu} - u'^{\mu\prime} u^\nu)$$

Or

$$2g^{\mu\nu} = u^\mu u^\nu + u'^{\mu\prime} u^\nu = 0$$

Since $u'^{\mu\prime} u^\nu = \frac{1}{i^2} u^\mu u^\nu$ from (4.1) and (4.2)

Implies

$$g^{\mu\nu} = 0$$

This can be avoided only if

$f'^\mu = -f^\mu = 0$ (9)

You have nothing other than what you have from Einstein's theory.

Part 11

Separation between to particles/bodies moving along the same geodesic may increase

Consider two stones falling towards the earth along the same radius. The one at a greater height has lesser acceleration than the one closer to the earth. Their separation should increase with time as they move along the geodesic. Gravity should not be misunderstood to have the sole option of bring bodies closer together. If we pelt a stone upwards with a speed greater than 11 or 12 km/s[escape velocity] it will not return to the earth. In a many body scenario we can think of suitable initial condition like a big bang[classical big bang] where the bodies move outwards never to come back. In fact Laplace's equation does not allow stable equilibrium.

This type of classical expansion may explain the expansion of the universe [less satisfactorily]

But will not explain the accelerated pace of the expansion at least in the two body scenario. And the classical conservation of energy principle.

But in Einstein's theory gravity is not a force. where do you get work from?

If Power is the time component of force and this happens to be zero in geodesic motion. [but in principle it admits force not relating to gravity: in presence of gravity alone it is zero[geodesic motion)]

Is it possible to speculate accelerated pace of the universe with Einstein's theories?

First we have to consider the difference between four acceleration and three acceleration. A body falling freely under gravity accelerates at the rate of 9.8 m/s. But four acceleration[even the radial component] is exactly zero. If the earth were a dense object the three acceleration could have been 1000 m/s radially downwards, the four acceleration remaining zero.

The relation

$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\lambda}u^\nu u^\lambda = f^\mu$ represents four acceleration per unit rest mass in the general case

For geodesic motion

$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\lambda}u^\nu u^\lambda = 0$ (10)

The rate of fall [three acceleration] of the apple falling from a tree is governed by the value

$\frac{d^2x^\mu}{ds^2}$ which is physically palpable to us.

[Both the concepts four acceleration and three acceleration are consistent/valid formulations; four acceleration being more suitable for writing the equations].

We write relation (10) as

$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu{}_{\nu\lambda}u^\nu u^\lambda$ if the right side positive there will be positive there is a possibility of fours speeds in the radial direction continuously increasing in a time varying metric. This formula will obviously do better than classical results

[Our idea here is to explore such possibilities]

[1] Harko T,Lobo F S N,Nojiri S,Odinstov S D, f(R) GravityPhysical Review D,DOI:10.1103/PhysRev D. 84.024020 (2011)

Short note on Dual Spaces and how they may be applied to $g^{\alpha\beta}$ and $g_{\alpha\beta}$

[To justify the fact that $g_{\alpha\beta}$ is the dual of $g^{\alpha\beta}$]

Given any vector space *V* over a field *F*, the **dual space** *V** is defined as the set of all linear maps *φ*: *V* → *F* (linear functionals). The dual space *V** itself becomes a vector space over *F* when equipped with an addition and scalar multiplication satisfying:

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a\left(\varphi(x)\right)$$

A mapping may assign a scalar to every vector in some vector space . These mappings ***may be defined in such a manner*** that they form a vector space themselves: the scalar mentioned is just instrumental in the process of defining the mapping so that these mappings form a new [distinct]vector space.

Linear maps are most suitable

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a(\varphi(x))$$

[x:element from the original vector space V; φ and ψ are elements in the dual space V^*]

For linear maps we also have:

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

$$\varphi(cx) = c\varphi(x)$$

Example:

$$(g_{\alpha\beta})(g^{\alpha\beta}) = 4$$

The mapping $g_{\alpha\beta}$ has to be defined as a linear map . With any other element in V

$$(\varphi+\psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a(\varphi(x))$$

Have to be satisfied: point is that we have to create a vector space with thefunctions /mappings

$g_{\alpha\beta}$ along with other elements in the dual space .

The mapping $g_{\alpha\beta}$ should also satisfy:

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$

$$\varphi(cx) = c\varphi(x)$$

Usual definitions are good enough.

$g^{\alpha\beta}$ and $g_{\alpha\beta}$ belong to different spaces: the original vector spa e and its dual

Quantities like $g^{\alpha\beta} + g_{\alpha\beta}$ belong to neither of the spaces, original vector space or the dual spaces.

But we have mappings/operations connecting vectors from the two spaces V and its dual V*

$(g_{\alpha\beta})(g^{\alpha\beta}) = 4$ has the appearance of a dot product but it is not a dot product since the two elements are from different spaces.

For an arbitrary tensor:

$T_{\alpha\beta}T^{\alpha\beta} = s$[sacalar]

That is if you choose $T^{\alpha\beta}$ from V the mapping has to be defined by

$T_{\alpha\beta}\left(T^{\alpha\beta}\right) = S$ scalar

Usual tensor transformation rules[for example the contravariant tensor transformations:$T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T^{\alpha\beta}$] are good enough. We have the formula $T_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}T^{\mu\nu}$, which is consistent with

The transformation $T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T^{\alpha\beta}$ and also with the mapping $(g_{\alpha\beta})(g^{\alpha\beta}) = 4$

Could we get something more general than $T_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}T^{\mu\nu}$ from $(g_{\alpha\beta})(g^{\alpha\beta}) = 4$?

We have

$T^{\alpha\beta}T_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}T^{\alpha\beta}T^{\mu\nu} = T_{\mu\nu}T^{\mu\nu}$[a consistent picture in the same frame of reference]

We may choose any consistent definition for example the usual ones…

If the components of a tensor are zero in one frame , they are also zero in the dual space.

Any tensor equation may be written in terms of tensors in the original or the dual space or in mixed form expressing the same event.

Thus tensors I the original and the dual space express the same physical quantities in different forms

We consider two vectors T^1, T^2 belong to

Mappings $T_1$and T_2 are contained in the dual space V^* using the scalar concept:

$$T_1(T^1) = s_{11}$$

$$T_2(T^1) = s_{21}$$

$$T_1(T^2) = s_{12}$$

$$T_2(T^1) = s_{21}$$

$$T_1(T^1) \pm T_2(T^1) = s_{11} \pm s_{21}$$

Or,

$g_{\alpha\beta}(T^{\alpha\beta}) + \delta g_{\alpha\beta}(T^{\alpha\beta})$ = sum / difference of two scalars[sum and difference are operations defined consistently by linearity[linear mappings]

Given

$g^{\mu\nu}g_{\mu\nu} = 4$

$g^{\alpha\beta}g_{\alpha\beta} = 4$

We have to find a consistent formula for $g^{\mu\nu}g_{\alpha\beta}$

$g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}$ is a valid fomula which already exists . it leads to consistent formation of the dual space

When we take variations on both sides of $g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}$,the variation in one space is getting consistently linked with variation in the other [dual space]. Two different spaces are not a problem when they get connected by mappings especially if the mappings are linear functions. Dual space speaks of such connected or inter related vectors.

We denote

$$A^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}$$

Therefore $\delta g^{\mu\nu} = A^{\mu\nu} - g^{\mu\nu}$

If $A^{\mu\nu}$ is a tensor [as required by manifold variation to produce Ricci scalar R or some function of it at any stage of variation] $\delta g^{\mu\nu}$ us also a tensor being the difference of two tensor

$A^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}$ is a tensor with respect to both the manifolds M1 and M2 bu t it is a metric tensor with respect to M2 only. We $g^{\mu\nu}$ as metric tensor for M1. In have In the product

$A^{\mu\nu}A_{\mu\nu} = s$[scalar]

$$g^{\mu\nu}g_{\mu\nu} = 4$$

[To be continued/improved with further improvements/elaborations]