A Plain Proof of Beal's Conjecture[1]

ABSTRACT. This paper offers a plain proof of Beal's conjecture using the cosine rule.

1 Introduction

Beal's conjecture states that no pairwise coprimes X,Y,Z satisfy $X^a + Y^b = Z^c$ for positive integers $a,b,c > 2$. This paper will offer a plain proof of Beal's conjecture using the cosine rule.

2 Proof

$$X^a + Y^b = Z^c; 2 < a,b,c \in \mathbb{Z}^+; X,Y,Z : \text{pairwise coprime}; \mathbb{Z}^+\text{: positive integer} \tag{1}$$

2.1 For the case at least one of a,b,c : odd prime

If there exist X,Y,Z satisfying (1), and let at least one of a,b,c be odd prime p, then X,Y,Z satisfy

$$(X^{a/p})^p + (Y^{b/p})^p = (Z^{c/p})^p. \tag{2}$$

Now then, let $X^{a/p} = x, Y^{b/p} = y, Z^{c/p} = z$, then (2) can be written as

$$x^p + y^p = z^p. \tag{3}$$

From (3) it follows that $(x+y)^p > z^p$, $x+y > z$, $z+x > y, z+y > x$. Accordingly, x,y,z always form a triangle. Thus, x,y,z satisfy

$$x^2 + y^2 - 2xy\cos\zeta = z^2; \ \angle\zeta\text{: opposite of } z. \tag{4}$$

From (3),(4) it follows that

$$(x^p + y^p)^2 = (x^2 + y^2 - 2xy\cos\zeta)^p. \tag{5}$$

Then, let z be a constant, the graphs of (3) and (4) must meet each other at least at one point (x,y). Thus, there is no need for (5) to be an identity. However, if we treat as if x,y of the point (x,y) were integers, x,y must satisfy $x+y \mid x^p + y^p = z^p$, i.e. $(x+y)^2 \mid (x^p+y^p)^2$, hence x,y of the point (x,y) must satisfy

$$(x+y)^2 \mid (x^2 + y^2 - 2xy\cos\zeta)^p. \tag{6}$$

Then, $x^2 + y^2 - 2xy\cos\zeta = (x+y)^2 - 2xy(1+\cos\zeta)$, and $(x+y)^2 > 2xy(1+\cos\zeta)$ because $(x-y)^2 < (x+y)^2 - 2xy(1+\cos\zeta) < (x+y)^2$. Hence, $(x+y)^2 \mid x^2+y^2-2xy\cos\zeta$ is possible only when $1+\cos\zeta = 0$, i.e. $p = 1$.[2]. Hence, (5) cannot be satisfied when $(x+y)^2 \mid x^2+y^2-2xy\cos\zeta$.
Moreover, (6) cannot be satisfied, when $x^2+y^2-2xy\cos\zeta$ is divisible not by $(x+y)^2$ but only by $x+y$. It is because $2 \nmid p$. This means that in this case (5) cannot be satisfied.
Accordingly, no pairwise coprimes X,Y,Z satisfy (1) when at least one of a,b,c: odd prime. This means that according to the laws of exponents no pairwise coprimes X,Y,Z satisfy (1), even when $p \mid a,b,c$.
Hence, no pairwise coprimes X,Y,Z satisfy (1) for $2 < a,b,c \in \mathbb{Z}^+$, unless $a = 2^{m_1}, b = 2^{m_2}, c = 2^{m_3}$, where $2 \le m_1,m_2,m_3 \in \mathbb{Z}^+$.

2.2 For the case $a = 2^{m_1}, b = 2^{m_2}, c = 2^{m_3}$

$$X^4 + Y^4 = Z^4 \tag{7}$$

That no positive integers X,Y,Z satisfy (7) was proven by Fermat.([1]) Hence, according to the laws of exponents no positive integers X,Y,Z satisfy (1) for $a = 2^{m_1}, b = 2^{m_2}, c = 2^{m_3}$.

3 Conclusion

No pairwise coprimes X,Y,Z satisfy $X^a + Y^b = Z^c$ for positive integers $a,b,c > 2$. QED.

References

[1] Freeman, L., Fermat's One Proof, http://fermatslasttheorem.blogspot.kr/, Retrieved 2015-04-18.

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[1] Yun, J., Daegu Univ., 712-714, South Korea; jmyun@daegu.ac.kr

[2] Hence, X,Y,Z satisfying $X^a + Y^b = Z^c$ can exist, e.g. when at least any one of a,b,c is 1. Then, for reference, the cases e.g. $3^3 + 6^3 = 3^5, 27^4 + 162^3 = 9^7, 3^{3n} + [2(3^n)]^3 = 3^{3n+2} (1 \le n \in \mathbb{Z}^+), 7^3 + 7^4 = 14^3, 2^n + 2^n = 2^{n+1}$ cannot come under (5) because they can be written as $1 + 2^3 = 3^2, 1 + 2^3 = 3^2, 1 + 2^3 = 3^2, 1 + 7 = 2^3, 1 + 1 = 2$ respectively, if divided by their common integer factors.