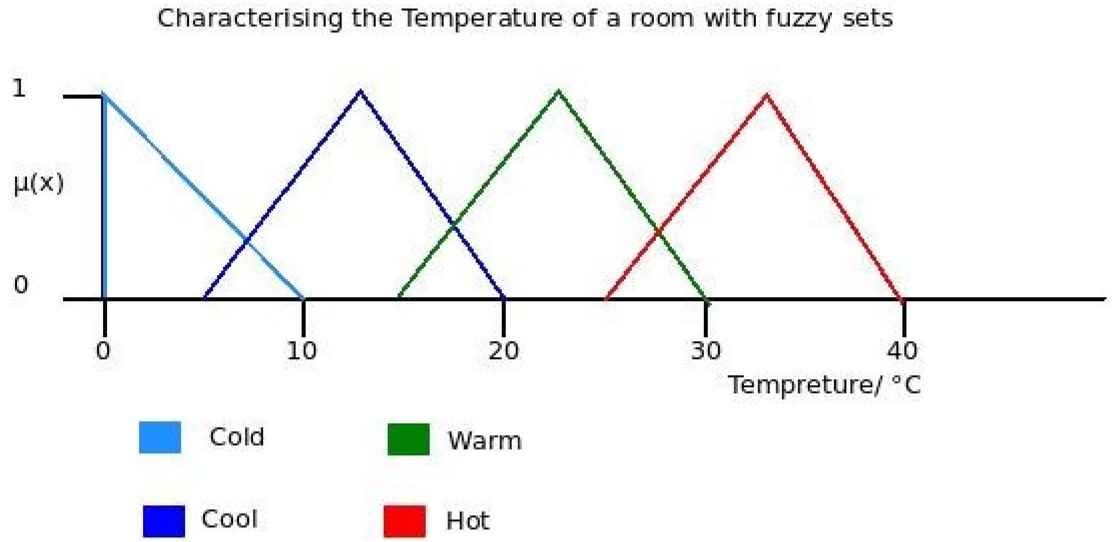
Some Preliminary notes on extending the interval of fuzzy logic

A.D Albert

Introduction

Fuzzy logic membership values are always specified in the unit interval of $[0,1]$. If the membership value is '0' then it represents falsity ; value of '1' represents truth. Values in between these two boundaries represent various degrees of truth {1} . But in certain circumstances it's advantageous to arbitrarily extend this interval to values greater than 1 or values less than 0 (i.e extended interval will be defined as $[\alpha,\beta],\ \alpha\geq 1,\ \beta\leq 0,\ \alpha,\beta\in\mathbb{R}$) . If membership value is greater than '1', it's meaning will be defined as “better truth” . If the membership value is less than 0 then it will be defined as “worse falsity” (or worse lies).

You may ask, where would I find these “better truths” and “worse falsity”. Example are abound, so I will take the well known example of characterising the room temperature in terms of fuzzy sets.

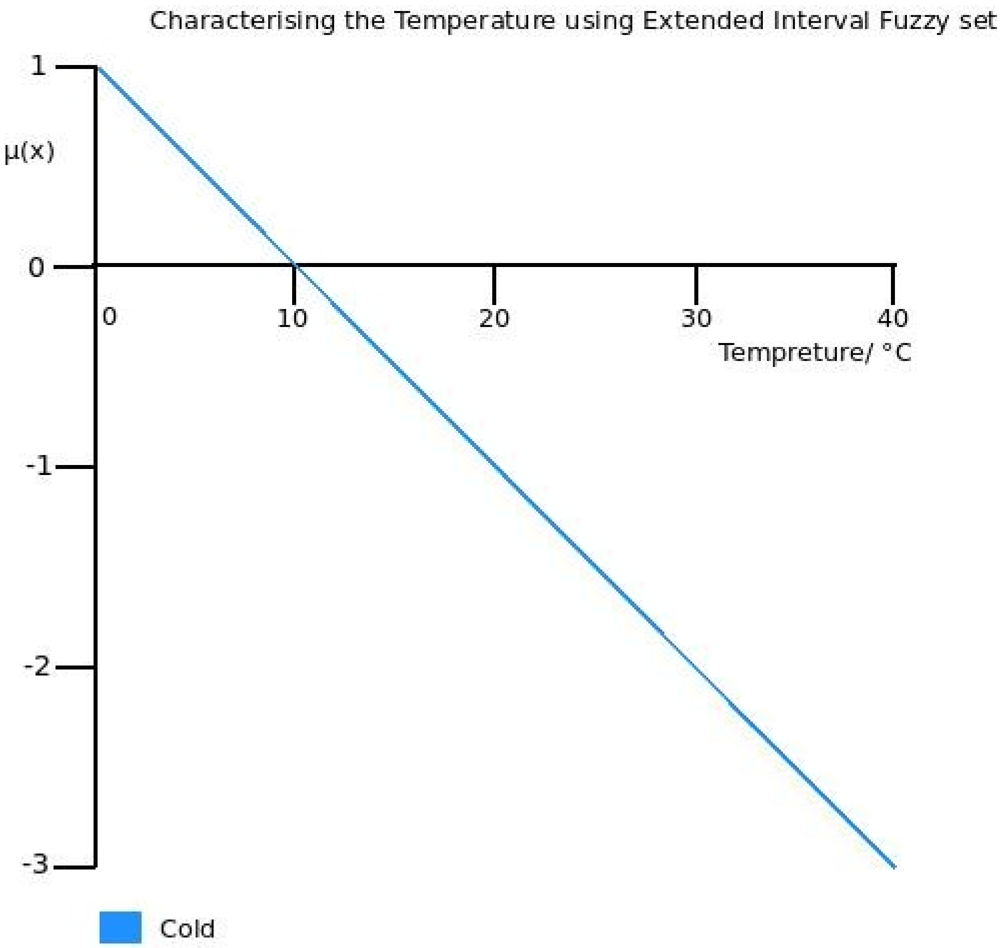


Now according to these fuzzy sets $\mu_{Cold}(10^o C)=0$, $\mu_{Cold}(20^o C)=0$ and $\mu_{Cold}(30^o C)=0$. But, what if we could, without assigning the same falsity('0') to all of these values we could assign various grades of falseness to to these values. $"20^0 C\ \ is\ \ cold"$ is indeed a false statement (it has no truth to it) , but it's a worse falsehood than $"10^0 C\ \ is\ \ cold"$ (which is also false, and has no truth to it).

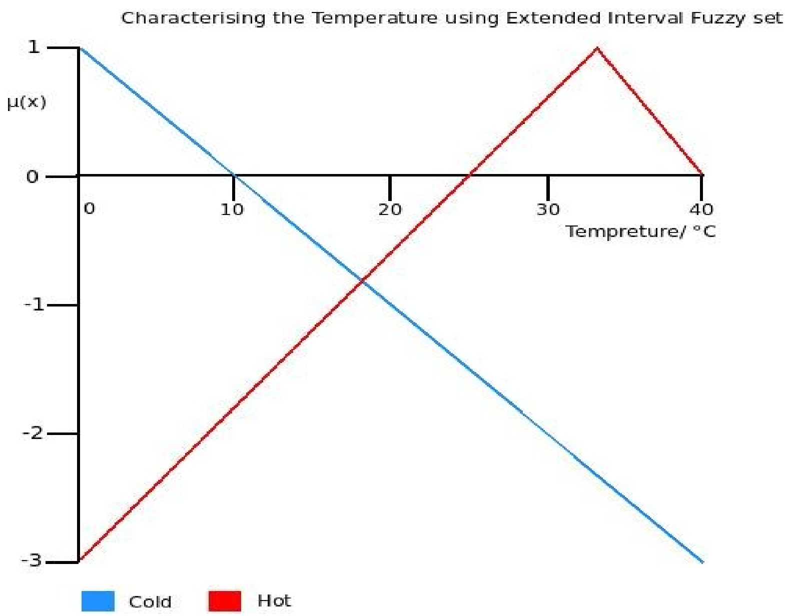
We have already assigned a truth value of '0' to $"10^0 C\ \ is\ \ cold"$, then what kind of a numeric value should be assigned to $"20^0 C\ \ is\ \ cold"$ which is also completely false ? (since it has no truth, this numeric value cannot be in the region of $0<\mu(x)\leq 1$) . Since $"20^0 C\ \ is\ \ cold"$ is a worse lie than saying $"10^0 C\ \ is\ \ cold"$, truth value for $"20^0 C\ \ is\ \ cold"$ should be less than $"10^0 C\ \ is\ \ cold"$ (i.e $\mu_{Cold}(20^o C)<\mu_{Cold}(10^o C)$ or $\mu_{Cold}(20^o C)<0$) .

In another point you cannot normalize this truth value, so that it will always be inside the unit interval of $[0,1]$. For example if you were to normalize then, $\mu_{Cold}(20^o C)=0$ and $\mu_{Cold}(10^o C)>0$, but definitely know $\mu_{Cold}(10^o C)=0$ so normalization will violate our definition of “Cold” fuzzy set. Straightforward way around this problem is to expand our interval to some arbitrary real values such as $[\alpha,\beta]$, $\alpha,\beta\in\mathbb{R}$ and $\alpha\leq 0,\beta\geq 1$. This arbitrary boundary is problem specific. So for this example I'll use an extended interval of $[-3,1]$.

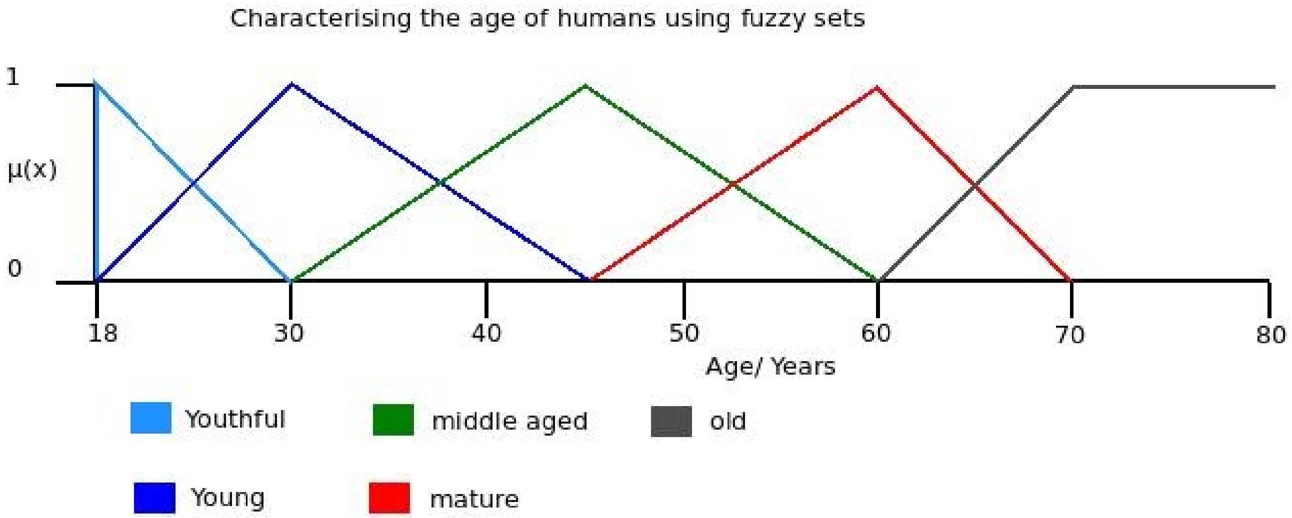
Now in this extended interval we can specify fuzzy set "Cold" in which $\mu_{Cold}(10^o C)=0$ but $\mu_{Cold}(40^o C)<\mu_{Cold}(30^o C)<\mu_{Cold}(20^o C)<\mu_{Cold}(10^o C)$. You can specify this as a linear extension or a non linear extension (fuzzy set even in a extended interval is still subjective).



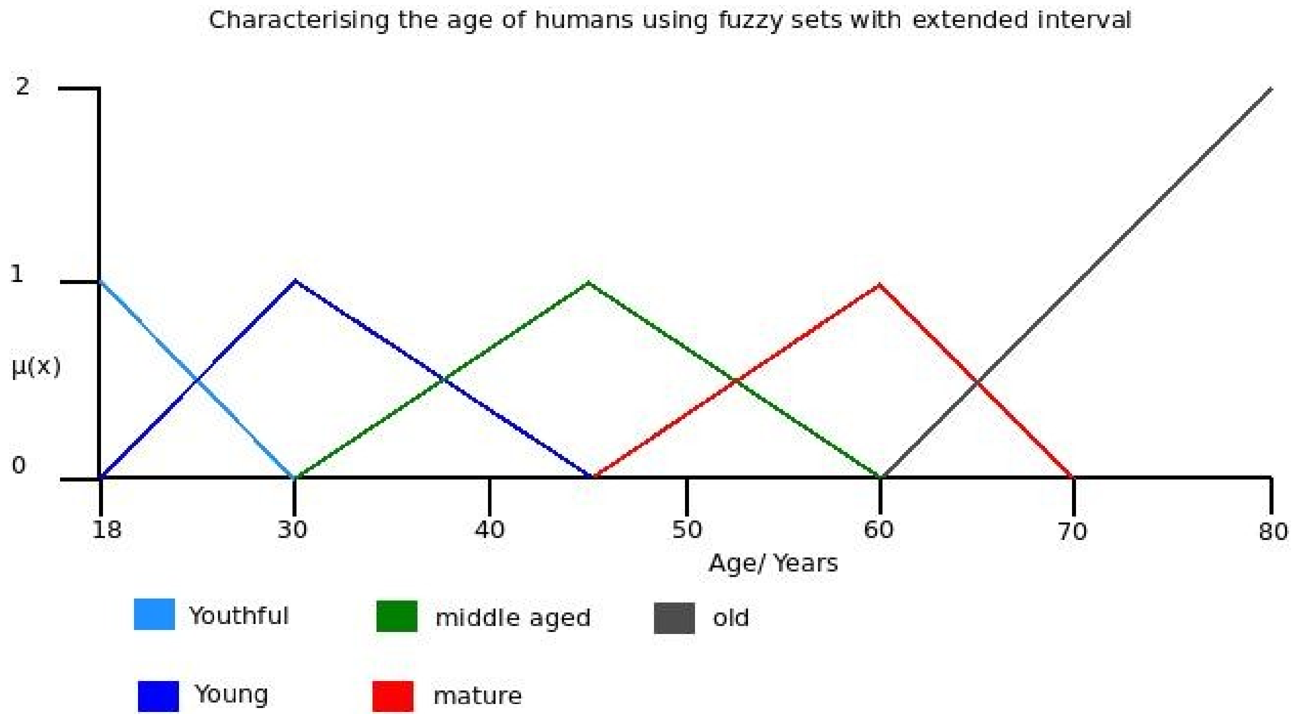
In fact you can take any fuzzy set and extend it over the range of the independent variable (in this case variable is temperature) . The fuzzy set "Hot" has the same value of falsity for all these values $\mu_{HOT}(25^o C)=\mu_{HOT}(20^o C)=\mu_{HOT}(10^o C)=0$. But we can set "$10^o C$ *is Hot*" to have less truth value than "$25^o C$ *is Hot*" (because "$10^o C$ *is Hot*" is a worse lie than "$25^o C$ *is Hot*").
we can extend the "Hot" fuzzy set so that they will obey this truth value relationship $\mu_{HOT}(25^o C)>\mu_{HOT}(20^o C)>\mu_{HOT}(10^o C)>\mu_{HOT}(0^o C)$.



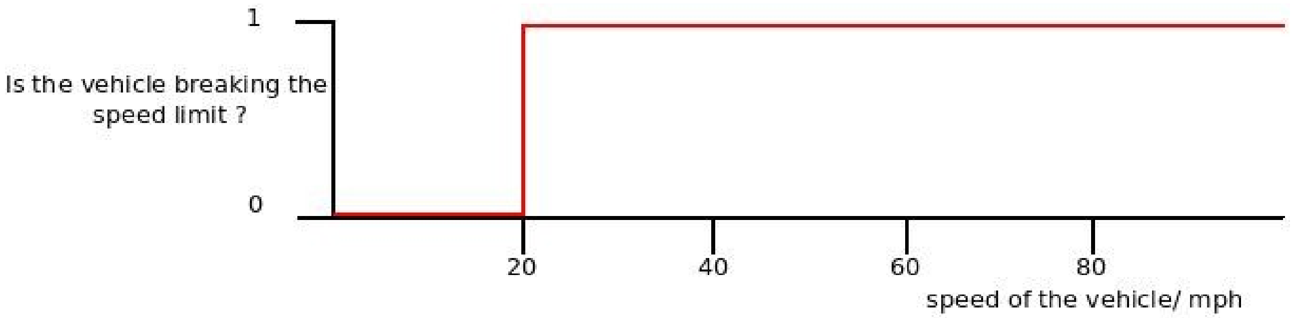
In another example, let's characterise age of humans (over 18 only) using fuzzy sets.



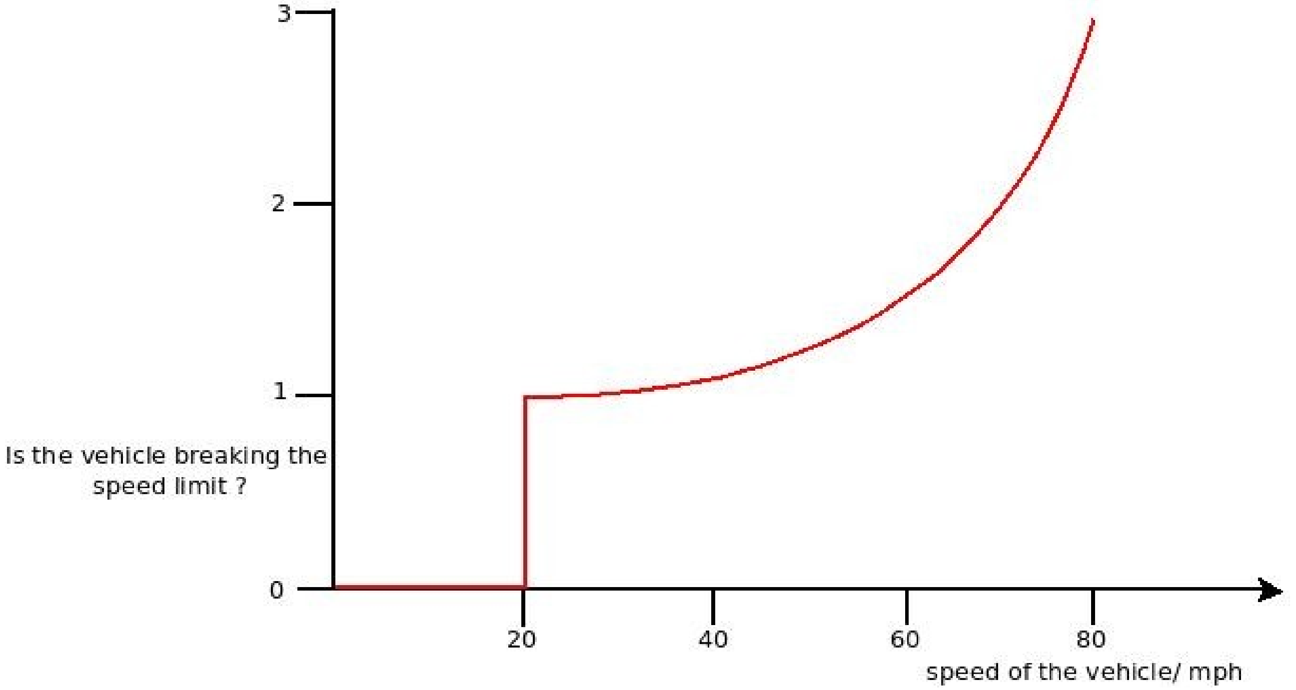
In this example $\mu_{OLD}(70)=1$ and $\mu_{OLD}(80)=1$. But this is not satisfactory, because if you are 80 years old, then you are older than someone who is 70 years old. In other words, “someone is old if they are 80 years of age” has more truth value to it than “someone is old if they are 70 years of age”. So we can say $\mu_{OLD}(80)>\mu_{OLD}(70)$; but $\mu_{OLD}(70)=1$. If we are to normalize the fuzzy set “old” then $\mu_{OLD}(70)$ will become less than 1 , which in turn will violate the definition of the this fuzzy set. So $\mu_{OLD}(70)=1$ is completely true, but it just happens to be that $\mu_{OLD}(80)$ is more truthful, thereby has a higher truth value . If we are to extrapolate linearly, fuzzy set “old” will look like this (see below).



I another example Let's think of a public road with a speed limit of 20 Miles per hour. Anyone driving their vehicle above this limit is committing an offence.



But above crisp set does not reflect reality. Drivers travelling at 30Mph may get a financial penalty while, Drivers caught travelling at 60 Mph will have their licence suspended. Severity of the punishment is strongly dependent on speed of the vehicle, Not everybody breaking the speed limit will get the same punishment. Below graph models this scenario though a non-linear function in an extended interval.



Above graph shows that; as the speed of the vehicle increases, it breaks the law to a higher degree.

So there seems to be 3 distinct regions of truth values. If a truth value is in the interval of [0,1] then it represents partial truth (fuzzy logic interpretation) .

When truth value is between $[\alpha, 0)$ (when $\alpha < 0, a \in \mathbb{R}$) then it represents degrees of complete falsity (or worse lies). If $x, y \in \mathbb{R}$ and in the interval of $[\alpha, 0)$, let's say $x < y$ then x and y are both complete lies, but x is a worse lie than y.

When truth value is between $(1, \beta]$ (when $\beta < 0, \beta \in \mathbb{R}$) then it represents degrees of complete truth (or better truths). If $p, q \in \mathbb{R}$ and in the interval of $[1, \beta)$, let's say $p < q$ then p and q are both complete truths, but q is a better truth than p.

Limits α and β are arbitrary real numbers, which can be constrained according the problem at hand. They can even be very large positive and negative real numbers (i.e. tend towards negative and positive infinity) . This abstraction can be useful when these limits are unknown.

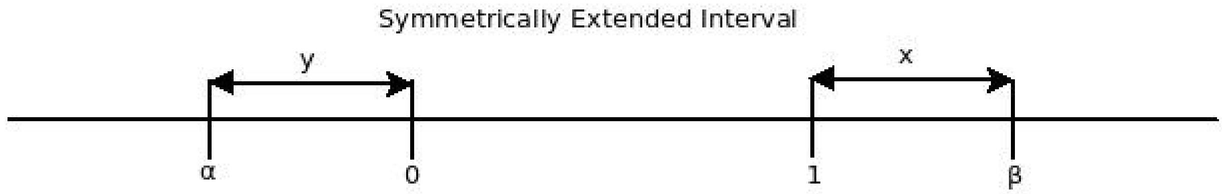
Now, we have to know what kind of logical operators can be applied in this extended interval. To find out these operators let's just take the axioms used in fuzzy logic and change the boundary conditions. Then we can use these modified axioms to find the logical operators in the extended interval.

Extending the interval

Say that the logic is defined in the interval of $[\alpha,\beta]$ where $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}$. α,β will also have additional boundary conditions, such as $\alpha\leq 0$ and $\beta\geq 1$ .If $\alpha=0$ and $\beta=1$ this logic will become fuzzy logic (i.e. multivalued logic defined in the interval of $[0,1]$) .If $\alpha<0$ or $\beta>1$ then we can say interval has been extended .

Symmetrical extension of the interval

It's possible to extend the interval symmetrically. Refer to the illustration below.



Here length $y=0-\alpha$ is equal to, length $x=\beta-1$. So by equating these two expressions (i.e. $x=y$) result will be $0-\alpha=\beta-1$. Simplifying these equations we can get conditions for symmetrical extension.

- $\alpha=1-\beta$ [when $\alpha<0,\beta>1$ **],** then interval has been symmetrically extended.
- $\alpha\neq 1-\beta$ [when $\alpha<0,\beta>1$ **],** then interval has been asymmetrically extended.

If the interval was extended symmetrically, then mid point is still $\frac{1}{2}$. Treat the interval as a line segment. Then mid point of the interval $[\alpha,\beta]$ is = $\frac{\alpha+\beta}{2}$ but Interval was symmetrically extended so $\alpha=1-\beta$ substitute this into previous expression mid point of symmetrically extended interval is = $\frac{(1-\beta)+\beta}{2}=\frac{1-\beta+\beta}{2}=\frac{1}{2}$. If the interval was asymmetrically extended then mid point is $\frac{\alpha+\beta}{2}$.

Extended Interval Fuzzy Sets

Suppose space X contains n objects $x_i : X=\{x_0,x_1,x_2,x_3,....,x_n\}$. Additionally we require that all of these elements require to have membership value which is the maximum in the interval of $[\alpha,\beta]$. So $\mu_X(x_i)=\beta$ since $max([\alpha,\beta])=\beta$. Now we can write down the Universal set when interval has been extended.

$$X=\{\frac{\beta}{x_0}+\frac{\beta}{x_1}+\frac{\beta}{x_2}+....+\frac{\beta}{x_n}\}=\{\sum_{i=0}^{n}\frac{\beta}{x_i}\}$$

Horizontal bar is a delimiter. Summation symbol is aggregation of each element {1}. Suppose that we have an extended interval fuzzy set in this space called $\underset{\sim ei}{A}$ this can be written down as

$$\underset{\sim ei}{A}=\{\frac{\mu_{\underset{\sim ei}{A}}(x_0)}{x_0}+\frac{\mu_{\underset{\sim ei}{A}}(x_1)}{x_1}+\frac{\mu_{\underset{\sim ei}{A}}(x_2)}{x_2}+....+\frac{\mu_{\underset{\sim ei}{A}}(x_n)}{x_n}\}=\{\sum_{i=0}^{n}\frac{\mu_{\underset{\sim ei}{A}}(x_i)}{x_i}\}$$

Note that truth value $\mu_{\underset{\sim ei}{A}}()$ for any element is defined in the interval of $[\alpha,\beta]$.

Containment

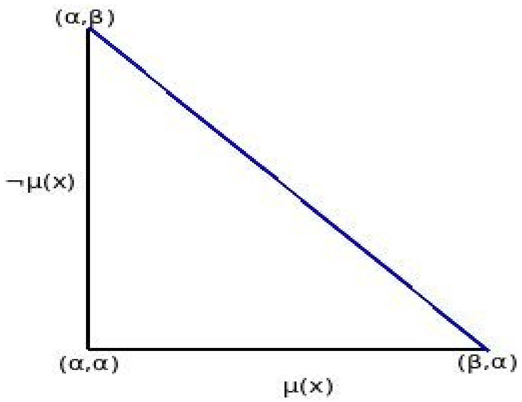
$\underset{\sim ei}{A}\subseteq X$ If and only if $\mu_{\underset{\sim ei}{A}}(x)\leq\beta$ for all elements $x\in X$. But $\beta=\mu_X(x)$ so we can say, A is subset of universal set X , or $\underset{\sim ei}{A}\subseteq X$ if $\mu_{\underset{\sim ei}{A}}(x)\leq\mu_X(x)$. We can further extend this notion and say, $\underset{\sim ei}{A}$ is a subset of $\underset{\sim ei}{B}$, *or* $\underset{\sim ei}{A}\subseteq\underset{\sim ei}{B}$ when $\mu_{\underset{\sim ei}{A}}(x)\leq\mu_{\underset{\sim ei}{B}}(x)$ for all elements $x\in X$. Just as in fuzzy logic defined in unit interval {1}.

Complement Operator

When Interval is extended , Negation operator should obey these axioms. These are based on axioms required by the complement operator for fuzzy logic in the unit interval {2}, but the boundary conditions were changed.

- $C(\alpha)=\beta$ and $C(\beta)=\alpha$ [boundary condition]
- For all $p,q\in[\alpha,\beta]$, if $p\leq q$ then $C(p)\geq C(q)$ [monotonicity condition]
- $C()$ should be continuous
- $C(C(p))=p$ for each $p\in[\alpha,\beta]$ [involution condition]

I can use the boundary conditions to draw a linear function between membership value and it's complement. To find the complementary operator I just need to find out the equation of this function.



Let's say $\underset{\sim ei}{A}$ is an extended interval fuzzy set on the universe X . Modelling the above function as a linear equation. $(\bar{\mu}_{\underset{\sim ei}{A}}-\beta)=(\frac{\beta-\alpha}{\alpha-\beta})\cdot(\mu_{\underset{\sim ei}{A}}-\alpha)$ since $\beta-\alpha=-1\cdot(\alpha-\beta)$ so $(\bar{\mu}_{\underset{\sim ei}{A}}-\beta)=(-1)\cdot(\mu_{\underset{\sim ei}{A}}-\alpha)$. Can be further simplified to $\bar{\mu}_{\underset{\sim ei}{A}}=\alpha+\beta-\mu_{\underset{\sim ei}{A}}$; This is the general complement operator, when the interval has been extended. If the interval was extended symmetrically then $\alpha=1-\beta$. substituting this into previous expression , we get $\bar{\mu}_{\underset{\sim ei}{A}}=(1-\beta)+\beta-\mu_{\underset{\sim ei}{A}}=1-\mu_{\underset{\sim ei}{A}}$ which is the same as fuzzy complement.

Summarising these results :

$$C(p)=\begin{cases}1-p & ,p\in[a,b]\text{ and }\alpha=1-\beta\\ \alpha+\beta-p & ,p\in[a,b]\text{ and }\alpha\neq1-\beta\end{cases}$$

Let's go ahead and see whether the general complement operator obeys all the axioms

Boundary Conditions
$C(\alpha)=\alpha+\beta-\alpha=\beta$, $C(\beta)=\alpha+\beta-\beta=\beta$ so it obeys the boundary conditions.

Monotonicity Conditions
Let's say $p<q$ and $p,q\in[\alpha,\beta]$
Then multiply both sides by -1 and you will get $-p>-q$
Then add $\alpha+\beta$ to both sides $\alpha+\beta-p>\alpha+\beta-q$
From this we can see $C(p)>C(q)$ so it obeys monotonicity conditions.

Continuity Conditions
$C()$ Is continuous because we modelled it as a unbroken line between two points

Involution Conditions
$C(C(p))=(\alpha+\beta-(\alpha+\beta-p))=(\alpha+\beta-\alpha-\beta+p)=p$ so it obeys involution condition.

$C(p)=\alpha+\beta-p$ Obeys all the relevant conditions so it's one of the complementary operators in the interval $[\alpha,\beta]$ (there may be other operators which obey these conditions)

When $\alpha=0$ and $\beta=1$ complement operator will become $C(p)=\alpha+\beta-p=0+1-p=1-p$.

Intersection Operator
It's possible to slightly modify T-norm axioms to fit into extended interval. These are based on axioms required by the intersection operator for fuzzy logic in the unit interval {2}, but the boundary conditions were changed.

For all $p,q,r\in[\alpha,\beta]$

- $I(p,\beta)=p$ [boundary condition]
- $I(p,q)=I(q,p)$ [commutativity condition]
- If $q\leq r$ then $I(p,q)\leq I(p,r)$ [monotonicity condition]
- $I(p,I(q,r))=I(I(p,q),r)$ [associativity condition]

Actually $min()$ function fulfils all these conditions (similar to fuzzy logic in the unit interval). Let's see whether this is correct .

Boundary Condition
Since β is the maximum value in the interval of $[\alpha,\beta]$. It must be $p\leq\beta$ then $min(p,\beta)=p$. So $min()$ obeys the boundary condition.

Commutativity Condition
If $p>q$ then $min(p,q)=q$ and $min(q,p)=q$
If $p<q$ then $min(p,q)=p$ and $min(q,p)=p$
If $p=q$ then $min(p,q)=q=p$ and $min(q,p)=q=p$
So $min()$ obeys the commutativity condition.

Monotonicity Condition
If $q\leq r$ then there are 3 possible values for p in the interval of $[\alpha,\beta]$.

Case 1 :

If $p \leq q \leq r$ then $min(p,q) \leq min(p,r)$.

Case 2 :
If $q \leq p \leq r$ then $min(p,q) \leq min(p,r)$.

Case 3 :
If $q \leq r \leq p$ then $min(p,q) \leq min(p,r)$.

So monotonicity condition is obeyed by minimum function.

Associativity Condition
This proof is trivial.
$min(p, min(q,r)) = min(p,q,r)$
Which is just the minimum of all three variables.
$min(min(p,q),r) = min(p,q,r)$
So these two terms must be equal to each other.
$min(min(p,q),r) = min(p, min(q,r))$
Associativity condition is obeyed .

Union Operator

It's possible to slightly modify S-norm axioms to fit into extended interval. These are based on axioms required by the union operator for fuzzy logic in the unit interval {2}, but the boundary conditions were changed.

For all $p,q,r \in [\alpha,\beta]$

- $U(p,\alpha) = p$ [boundary condition]
- $U(p,q) = U(q,p)$ [commutativity condition]
- If $q \leq r$ then $U(p,q) \leq U(p,r)$ [monotonicity condition]
- $U(p, U(q,r)) = U(U(p,q),r)$ [associativity condition]

Actually $max()$ function fulfils all these conditions (similar to fuzzy logic in the interval of $[0,1]$). Let's see whether this is correct .

Boundary Condition
Since α is the minimum value in the interval of $[\alpha,\beta]$. it must be $p \geq \alpha$. So $max(p,\alpha) = p$. So $max()$ function obeys the boundary condition.

Commutativity Condition
If $p > q$ then $max(p,q) = p$ and $max(q,p) = p$
If $p < q$ then $max(p,q) = q$ and $max(q,p) = q$
If $p = q$ then $max(p,q) = q = p$ and $max(q,p) = q = p$
So $max()$ obeys the commutativity condition.

Monotonicity Condition
If $q \leq r$ then there are 3 possible values for p in the interval of $[\alpha,\beta]$.

Case 1 :

if $p \le q \le r$ then $max(p,q) \le max(p,r)$.

Case 2 :
If $q \le p \le r$ then $max(p,q) \le max(p,r)$.

Case 3 :
If $q \le r \le p$ then $max(p,q) \le max(p,r)$.

So monotonicity condition is obeyed by maximum function.

Associativity Condition

$max(p, max(q,r)) = max(p,q,r)$

because it's the maximum of all three elements.

$max(max(p,q),r) = max(p,q,r)$

So these two terms must be equal

$max(max(p,q),r) = max(p, max(q,r))$

so associativity condition is obeyed by maximum function.

Aggregation Operator

These are based on axioms required by the aggregation operator for fuzzy logic in the unit interval {2}, but the boundary conditions were changed.

- $A(\alpha,\alpha,\alpha,\alpha....) = \alpha$ and $A(\beta,\beta,\beta,\beta....) = \beta$ [boundary condition]
- For any $(a_0, a_1, a_2, ..., a_n)$ and $(b_0, b_1, b_2, ..., b_n)$ such that $a_i, b_i \in [\alpha, \beta]$ and $a_i \le b_i$ for $i = 0,1,2...n$.
  $A(a_0, a_1, a_2 ... a_n) \le A(b_0, b_1, b_2 ... b_n)$ [monotonicity condition]
- $A()$ is continuous.
- For all $a \in [\alpha, \beta]$ aggregation function should be $A(a, a, a, ..., a) = a$ [idempotent condition] .
- For any permutation p on $\{1,2,...,n\}$
  $A(a_1, a_2, ..., a_n) = A(a_{p(1)}, a_{p(2)}, ..., a_{p(n)})$ [commutativity or symmetric condition]

Arithmetic mean (just like in fuzzy logic) full fills all the criteria.

Boundary condition

$$A(\alpha,\alpha,\alpha,...,\alpha) = \frac{1}{n}\sum_{i=0}^{n}\alpha = \frac{(\alpha+\alpha+\alpha+\alpha+\alpha+....+\alpha)}{n} = \frac{n\cdot\alpha}{n} = a$$

$$A(\beta,\beta,\beta,...,\beta) = \frac{1}{n}\sum_{i=0}^{n}\beta = \frac{(\beta+\beta+\beta+\beta+\beta+....+\beta)}{n} = \frac{n\cdot\beta}{n} = \beta$$

So arithmetic mean obeys the boundary conditions.

Monotonicity Condition

$a_i \le b_i$ then say $a_i + c_i = b_i$ for $i = 0,1,2...n$ and $a, b, c \in [\alpha, \beta]$

$$A(a_0, a_1, a_2, \ldots, a_n) = \frac{1}{n}\sum_{i=0}^{n} a_i$$

$$A(b_0, b_1, b_2, \ldots, b_n) = \frac{1}{n}\sum_{i=0}^{n} b_i = \frac{1}{n}\sum_{i=0}^{n} a_i + c_i$$

$$\frac{1}{n}\sum_{i=0}^{n} a_i \leq \frac{1}{n}\sum_{i=0}^{n} a_i + c_i$$

So arithmetic mean obeys the monotonicity condition.

<u>Commutativity Condition</u>

$$A(a_{1,} a_2, \ldots, a_n) = \frac{1}{n}\sum_{i=0}^{n} a_i$$

Order of summation does not have an effect on the result of the summation.

$$A(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}) = \frac{1}{n}\sum_{i=0}^{n} a_i$$

So :

$$A(a_{1,} a_2, \ldots, a_n) = A(a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)})$$

So arithmetic mean obeys the monotonicity condition.

<u>Implication Operator</u>

An implication operator based on t-conorm S() and strong negation N() is called a s-implication{3}.

$$I_{S,N}(x, y) = S(N(x), y)$$

so in an extended interval one of the s-implication operator will be $max(\alpha+\beta-x,\ y)$ (based on Zadeh's implication operator {1}). This implication operator should obey these conditions.

1) if $x \leq z$ then $I(x, y) \geq I(z, y)$ $for\ all$ $x, y, z \in [\alpha, \beta]$

2) if $y \leq t$ then $I(x, y) \leq I(x, t)$ $for\ all$ $x, y, t \in [\alpha, \beta]$

3) $I(\alpha, y) = \beta$

4) $I(x, \beta) = \beta$

5) $I(\beta, \alpha) = \alpha$

6) $I(\beta, x) = x$

7) $I(x,\ I(y, z)) = I(y,\ I(x, z))$

8) $I(x, y) = I(N(y), N(x))$ when N( ) is a strong negation.

Condition 1 can be proved as :

say $\neg x = \alpha+\beta-x$ and $\neg z = \alpha+\beta-z$ then $\neg x \geq \neg z$:

case 1 : $y \leq \neg z \leq \neg x$

$max(\neg x, y) = \neg x$ and $max(\neg z, y) = \neg z$ so $I(x, y) \geq I(z, y)$

case 2: $\neg z \leq y \leq \neg x$
$max(\neg x, y) = \neg x$ and $max(\neg z, y) = y$ so $I(x, y) \geq I(z, y)$

case 3 : $\neg z \leq \neg x \leq y$
$max(\neg x, y) = y$ and $max(\neg z, y) = y$ so $I(x, y) \geq I(z, y)$

Condition 2 can be proved as :
say $\neg x = \alpha + \beta - x$ and $y \leq t$ then :

case 1 : $\neg x \leq y \leq t$
$max(\neg x, y) = y$ and $max(\neg x, t) = t$ so $I(x, y) \leq I(x, t)$

case 2 : $y \leq (\neg x) \leq t$
$max(\neg x, y) = \neg x$ and $max(\neg x, t) = t$ so $I(x, y) \leq I(x, t)$

case 3 : $t \leq y \leq (\neg x)$
$max(\neg x, y) = \neg x$ and $max(\neg x, t) = \neg x$ so $I(x, y) \leq I(x, t)$

Condition 3 can be proved as :
$max(\alpha + \beta - \alpha, y) = max(\beta, y)$ then β is the highest possible value in the interval of $[\alpha, \beta]$ so
$I(\alpha, y) = max(\beta, y) = \beta$

Condition 4 can be proved as :
$I(x, \beta) = max(\alpha + \beta - x, \beta) = \beta$ because β is the highest possible value in the interval of $[\alpha, \beta]$.

Condition 5 can be proved as :
$I(\beta, \alpha) = max(\alpha + \beta - \beta, \ \alpha) = max(\alpha, \alpha) = \alpha$

Condition 6 can be proved as :
$I(\beta, x) = max(\alpha + \beta - \beta, \ x) = max(\alpha, x) = x$ because α is the lowest value in the interval of $[\alpha, \beta]$.

Condition 7 can be proved as :
$max(\alpha + \beta - x, \ max(\alpha + \beta - y, \ z)) = max(\alpha + \beta - x, \ \alpha + \beta - y, z)$

$max(\alpha + \beta - y, \ max(\alpha + \beta - x, \ z)) = max(\alpha + \beta - x, \ \alpha + \beta - y, z)$

So :
$max(\alpha + \beta - x, \ max(\alpha + \beta - y, \ z)) = max(\alpha + \beta - y, \ max(\alpha + \beta - x, \ z))$

Condition 8 can be proved as

$max(\alpha + \beta - (\alpha + \beta - y), \ \alpha + \beta - x) = max(y, \alpha + \beta - x) = max(\alpha + \beta - x, \ y)$

Possible Applications

Extended interval maybe useful in fuzzy inference systems. They can be utilised reduce dimensional explosion. This is mainly because you can use smaller number of “extended interval fuzzy sets” to specify input(s) or output(s) [see temperature example]. This will lead to reduced number of rules, thereby reducing dimensional explosion.

Conclusion

It's possible to extend the interval of fuzzy logic to arbitrary limit of $[\alpha,\beta]$, where $\alpha,\beta\in\mathbb{R}$ and $\alpha\leq 0$, $\beta\geq 1$. If the truth value is between $[0,1]$ then it represents degrees partial truth (i.e fuzzy logic interpretation) . If truth value is between $[\alpha,0)$ then it represents degrees of complete falsity. If the truth value is between $(1,\beta]$ then it represents degrees of complete truth. Various logical operators for this extended interval can be found through first principles. Also various other operators can be found through using the same first principles.

operator	In interval [0,1]	In interval [α,β]
complement	1-x	α+β-x
intersection	min(x,y)	min(x,y)
union	max(x,y)	max(x,y)
aggregation	(x+y)/2	(x+y)/2
implication	max(1-x,y)	max(α+β-x,y)

References

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{2}Natasa Sladoje, Fuzzy Sets and Fuzzy Techniques
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