Electromagnetic theory and transformations between reference frames: a new proposal

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Abstract

This paper proposes a Galilean-invariant theory of electromagnetism, applicable at first order in v/c, describing both instantaneous and propagative interactions.
To extend this theory to higher orders, definitions of universal time and proper time are introduced. New transformations between reference frames are suggested.

Abbreviations:
STR: Special Theory of Relativity
GTR: General Theory of Relativity
IRF/s: Inertial Reference Frame/s
GT: Galilean Transformations
LT: Lorentz Transformations

Introduction

The Special Theory of Relativity (STR) plays a key role in the physical description of the world.
It implies that space and time cannot be conceived as separate structures, but must be taken together as a single structure: the space-time continuum.
Its points – the so-called space-time events – are the natural elements of reality, namely the fundamental elements around which a convenient description of the world can be formulated.

According to STR, physical laws are described in the same way in all inertial reference frames (IRFs), and therefore each IRF possesses equal dignity.
The only kinematic elements relevant to a phenomenological description are positions and velocities (of particles or field disturbances) relative to an arbitrarily chosen inertial observer.
The concept of “absolute motion” is excluded from the physical description of reality.
According to STR, the Lorentz transformations (LT) are the relationships between measurements, in different IRFs, of spatial distances or time intervals between events.
The LT imply, changing the IRF, invariance of the so-called space-time interval between a generic pair of events.
The invariant interval between events can be geometrically interpreted as a distance between points in a four-dimensional pseudo-Euclidean space, the so-called Minkowski space-time.
Unlike space-time intervals, spatial distances and time intervals lose their property of invariance.
One is compelled to abandon the concept of absolute simultaneity of spatially separated events.
In this context, to express physical theories, the most appropriate mathematical entities are four-tensors.
Unsurprisingly, Maxwell’s electromagnetism can be expressed by this formalism.
STR has reified the LT as the transformation laws between IRFs and those transformations leave the form of Maxwell’s equations unchanged (it is said that Maxwell’s equations are covariant under LT).
The adoption of the STR and therefore of the LT starts from the following observation: if we assume the contemporary validity of Galilean transformations (GT) and of Maxwell’s equations into a privileged reference frame (the postulated medium for the propagation of light, called

"ether"), this produces conclusions that contradict the results of the Michelson-Morley interferometer experiments.

However, the resolution of the incompatibility between experiments and theory can be theoretically pursued (and historically it has been) via different approaches.
They can be classified into the following options:

1. Accept Maxwell's equations as the correct description of electromagnetic phenomena, or at least as a sufficiently correct description to capture the "true" symmetry of space-time, missed by the equations of classical mechanics.
   This implies adopting LT in place of GT as the correct laws of transformation between IRFs.

2. Leave both GT and Maxwell's equations unchanged, but modify the model of the propagation medium.

3. Maintain the GT and change the description of electromagnetic phenomena.

4. Change both the transformation laws between IRFs and the electromagnetic theory.

The STR is the full development of the first option.

Complete or partial ether dragging theories are examples of the second option.

An example of a theory that pursues the third possibility was formulated by T.E. Phipps[1].
He proposed an electromagnetic theory that was formally equal to Hertz's original theory, but reinterpreted it in the meaning of the terms.
Old mathematics and a new symbolic interpretation produced a new theory – which I shall refer to it as Hertz-Phipps electromagnetism – displaying invariance under GT.
Phipps proposed this theory as a valid approximation to the first order in v/c.
The extension of the theory, which was described by Phipps as "Neo-Hertzian" and claimed to overcome the limitations of the first order, represents a transition to the fourth option.
In fact, the extended theory involves the introduction of a new type of time (the proper time) and a consequent change of the GT.
The result is a strictly relativistic theory, since it implies no privileged reference frame and the kinematic quantities are expressed relatively to a generic inertial observer.
However, the Neo-Hertzian version of the theory produces paradoxical predictions, as I will show later in this paper.

The third option was also pursued by F. Selleri[2] through a theory called "Weak Relativity".
In Selleri's theory the existence of an absolute reference frame – $\boldsymbol{S_0}$ – is assumed.
Assuming the validity of Maxwell's electromagnetism in this preferred reference frame, the speed of light would only be isotropic in respect to $\boldsymbol{S_0}$.
In IRFs different from $\boldsymbol{S_0}$ (which therefore have absolute speeds), the speed of light would be anisotropic.
However, for any closed path, the average speed of light should remain constant in all IRFs.
In Selleri's Weak Relativity, the LT are replaced by so-called Inertial Transformations.
The transformations of the electromagnetic fields in accordance with Inertial Transformations are described in a paper by G.D. Puccini[3] and in another paper by B. Buonaura[4].
The theory is called weak because, unlike the STR, it requires the existence of an absolute reference frame, but maintains the name of relativity because the implied state of absolute motion is locally unmeasurable.

It is a kind of relativity in which the slowing down of moving clocks has an asymmetrical description. In contrast to the predictions made by the STR, two different inertial observers would agree on which of their identical clocks is beating a slower pace.
Equally asymmetric is the length contraction.
Weak Relativity, even if it provides different rates for clocks at rest in different IRFs, restores the absolute simultaneity of spatially separated events.

In this theory the absolute reference frame can be interpreted as a Lorentz-type ether that justifies the effects (shortening of the bodies' lengths and slowing of the clocks as absolute effects) produced by motion.

In his book "Weak Relativity" and other publications[5,6], Franco Selleri is particularly effective at showing the weaknesses of the TRR interpretation capability in different areas (the Sagnac effect, non-inertial systems or stellar aberration descriptions, for example).
Among the objections that Selleri poses to the STR, the most philosophical one can be summarized as follows.
The LT, which constitute the relational structure between different IRFs, impose a form of ontological confusion (since they pertain to the fundamental categories of being), giving the same status of reality to the past, present and future.
According to the LT, different inertial observers "cut" space-time in different constant time slices, meaning that each of the observers attributes different collections of events to their present.
This means that events placed in my future (namely events not yet manifested in my co-moving IRF present) may belong to the past of a different inertial observer.
The equal dignity of all inertial observers also requires the equal dignity of all these possible different "nows".
Therefore, what I call "my future", in its having to be the past of another "present" with equal rights of reality, is also reified.
This implies an absolute determinism.
Such a description is therefore philosophically (though not mathematically) irreconcilable with quantum mechanics, which admits an inherently probabilistic description of the future.
If one accepts the description of space and time provided by LT as correct, one must also accept that uncertainty (which is necessary to a quantum description) is merely a kind of mirage or illusion.

These considerations motivate the search for theoretical alternatives that are capable of breaking the symmetry of the LT and reintroducing the absolute simultaneity concept.

Selleri's theory is interesting in this sense, and it would be even more interesting to test it experimentally through measurements of the speed of light on one-way paths in different directions.
Measurements of the flight time of an electromagnetic pulse could be carried out by means of distant clocks, not synchronized at a distance via the Einstein method, but synchronized in contiguity conditions and transported in quasistatic conditions (low speed) to their final positions.
Clocks synchronized in this way make it possible to reveal any differences in the flight time of electromagnetic pulses which propagate in opposite directions along the straight line joining two clocks.
However, since Selleri's theory is only alternative to STR, it is only applicable to the physical descriptions made by inertial observers.
"Inertial observers" must be understood in the narrower sense of observers moving at a constant speed in regions far from significant masses; freefalling observers in regions with a gravity gradient are therefore excluded.
Consequently these tests should ideally be conducted far away from gravitational sources.

By adopting Inertial Transformations instead of LT, Weak Relativity is not compatible with that generalization of STR, the General Theory of Relativity (GTR).
Thus, if one regards the dependence of the beating of the clocks on the gravitational potential as an empirical truth, in absence of an alternative gravitational theory to GTR that is able to justify this dependence, it is not obvious how to correct the Inertial Transformations to make them applicable in extended regions where a gravitational field is present.

Although I agree with Selleri's criticism of STR and I consider his theory plausible, I believe that a revision of the classical electromagnetic description as indicated by Phipps in his book can suggest further interesting alternatives.

Phipps's proposed modification of the classical electromagnetic theory is persuasive in the necessity to use the total time derivative in order to fully represent the experimental Faraday results regarding induction.
His theory is invariant (not covariant) in respect to GT.

I will show that his theory must be modified, since it predicts results that disagree with experience.
In fact, using Phipps's theory to calculate forces between stationary current elements, one finds results that are incompatible with empirical evidence.
I also believe that his equations should be rendered compatible with the experimental results, documented in:

- L. Kholmetskii, O. V. Missevitch, R. Smirnov-Rueda: Measurement of propagation velocity of bound electromagnetic fields in near zone - JOURNAL OF APPLIED PHYSICS 102, 013529 (2007).
- A. Calcaterra, R. de Sangro, G. Finocchiaro, P. Patteri, M. Piccolo, G. Pizzella: Measuring Propagation Speed of Coulomb Fields – arXiv:1211.2913v1 [gr-qc] 13 Nov 2012.

Both these experimental approaches show evidence of instantaneous interactions.

Instantaneous interactions are not representable within Maxwell's electromagnetism, since the latter must obey the space-time symmetries described by LT.
Instantaneous interactions imply that the principle of energy conservation takes a non-local form.
Instead, as pointed out by Feynman[7], the conservation principle imposed on energy by the STR has a strictly local nature.

In short: energy conservation in local form firstly means that energy is a quantity placeable in space, describable by a density function.
It also means that, if the energy in a region changes, this may only occur through a flow of the same energy crossing the boundaries of that region.
A non-local principle of conservation – according to which a certain physical quantity decreases in a place and simultaneously increases in another place distant from the former, so that the sum remains constant at every instant – is in contradiction with the STR since the simultaneity of spatially separated events is not shared by different inertial observers.
The appearance-disappearance of energy at distant points, evaluated as occurring simultaneously by one inertial observer, would be evaluated by another inertial observer as the disappearance of energy at a certain instant and the appearance of the same amount of energy in another instant. Thus there would be a time interval with a shortage or excess of energy.

Instantaneous interactions are, instead, representable in an electromagnetism with properties of invariance or covariance under GT.

Not having problems with the contemporaneity of distant events, this electromagnetism can be formulated to admit instantaneous interactions, without falling into contradiction.
Put another way, it is expressible in a manner that is consistent with these recent experimental observations, because the various electrical quantities are not obliged to be components of four vectors or four tensors.

Premises

<u>*Lorentz Transformations*</u> (more extensive notes in Appendix - A)

Given two IRFs: ***S*** and ***S'***, let $(\vec{r}, t)$ and $(\vec{r}\,', t')$ be the spatial and temporal coordinates of the same event in the two reference frames. Let $\vec{v}$ be the velocity of ***S'*** with respect to ***S***.
Expressed in vector form, the Lorentz Transformations are:

$$\vec{r}\,' = \vec{r} + (\gamma - 1)\frac{\vec{v}\cdot\vec{r}}{v^2}\,\vec{v} - \gamma\vec{v}\,t \qquad t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right) \qquad \text{where: } \gamma = 1\Big/\sqrt{1 - v^2/c^2}$$

<u>*Galilean Transformations and operators*</u>

Given two IRFs: ***S*** and ***S'***, in vector form, the Galilean transformations are:

$$\vec{r}\,' = \vec{r} - \vec{v}\,t \qquad t' = t$$

From these transformations, relations between operators are derived (details in Appendix - B):

$$\nabla' = \nabla \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$$

The operator ∇ is therefore invariant, while the operator $\partial/\partial t$ is not.

<u>*Maxwell's*</u> <u>*equations in vacuum*</u> (according to International System of Units)

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (1)$$

$$\nabla\cdot\vec{B} = 0 \qquad (2)$$

$$\nabla\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (3)$$

$$\nabla\wedge\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad (4)$$

These four field equations, with appropriate initial and boundary conditions, determine the electric field $\vec{E}(x, y, z, t)$ and the magnetic field $\vec{B}(x, y, z, t)$ at a generic point (x, y, z) in space at a generic instant t.
The equations are not invariant under GT due to the non-invariance of the operator $\partial/\partial t$ which appears in equations (3) and (4).

Phipps' criticisms of Maxwell's electromagnetism

Under-parameterization

Maxwell's equations lack of reciprocity in considering motions of charges.
In an observer's inertial reference frame, the movements of the source charges are described by the $\vec{J}$ current density field.
The movements of the field detector (or absorber), which is as conceptually essential as the source, is absent from the description.
Maxwell's equations are under-parameterized with respect to the detector's state of motion because they do not contemplate it.
The connection between fields and detector is introduced through the definition of force: the Lorentz force.
It is reasonable to consider a change of Maxwell's equations, introducing a role for the state of motion of the charge detector-absorber.

Faraday's observations and the use of the time derivative operator

Although based on Faraday's observations, Maxwell's equation (3) appears lacking as a translation of those same observations.
The results of Faraday's experiments can be summarized in the integral form:

$$\oint_L \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot \vec{n} \; dS$$

where Φ is the $\vec{B}$-field flux through a surface S bounded by a closed conductive circuit L.
In Faraday's experiments, variations of the magnetic flux passing through a closed electrical circuit were realized not only by acting on the field source, but also through a change of the circuit's shape.
Because the path of the line integral may be time-variant – $L = L(t)$ – the use of the total derivative in place of the partial derivative is mandatory.

These considerations also compel the use of the total derivative operator in the differential formulation.

Equation (3) $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ should be replaced by: $\nabla \wedge \vec{E} = -\frac{d\vec{B}}{dt}$

The total derivative

The above summarized considerations suggest an approach similar to the Lagrangian one, used in fluid dynamics, in which a fluid is described by the movement of its various parts.
The individual fluid particles are individually labeled and followed in their motions.
The alternative approach, called Eulerian, describes the fluid through functions of position and time, i.e. through scalar or vector fields (speed, pressure, density, etc.).
In the Eulerian description, the position that appears as an argument of a function is the position of a geometrical point, regardless of the presence of a specific fluid particle.
In this context it is natural to define the "partial time derivative", denoted by the symbol $\partial/\partial t$, as the limit of the ratio between the variation of a quantity in a fixed point of space and the time interval of this variation, when this interval tends towards zero.

In the Lagrangian description it is rather useful to define the "total time derivative", denoted by the symbol d/dt, as the limit of the ratio between the variation of a quantity to follow the motion of a particle and the time interval of this variation, when this interval tends towards zero.

If $x_p(t),\ y_p(t),\ z_p(t)$ are the coordinates of a moving particle, the time derivative of the property f evaluated on that particle is

$$\frac{df}{dt}=\frac{d\,f(t,x,y,z)}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{dx_p}{dt}+\frac{\partial f}{\partial y}\frac{dy_p}{dt}+\frac{\partial f}{\partial z}\frac{dz_p}{dt}=\frac{\partial f}{\partial t}+v_{px}\frac{\partial f}{\partial x}+v_{py}\frac{\partial f}{\partial y}+v_{pz}\frac{\partial f}{\partial z}$$

In vector notation:

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\left(\vec{v}_p\cdot\nabla\right)f$$

In the electromagnetic framework the use of the d/dt operator assigns a role to the motion of the field detector.

Given an inertial reference frame ***S***, to which the coordinates of each relevant entity are referred, the use of the operator d/dt is intended to mean that the temporal variations of quantities are not those "seen" from the fixed point instantaneously occupied by the detector, but are those "seen" in the same point by the detector in motion.

More generally, the use of the total time derivative operator in differential equations of fields means that temporal variations of variables are not measured on fixed points in ***S***, but on points in motion, which share the same instantaneous speed of the detector.
So temporal variations of all quantities are measured in the non-inertial (but not rotating) reference frame, $\boldsymbol{S_d}$, moving with speed $\vec{v}_d$ in respect to ***S***.

Considering a point particle field detector in arbitrary motion with velocity $\vec{v}_d=\vec{v}_d(t)$ relative to an arbitrarily chosen inertial observer, from the chain rule it follows that:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{v}_d\cdot\nabla$$

Invariance of d/dt under GT

Applying the Galilean law of composition of velocities to the detector motion:

$$\vec{v}_d' = \vec{v}_d - \vec{v}$$

$\vec{v}_d$ is the speed of the detector evaluated in ***S***.
$\vec{v}_d'$ is the speed of the detector evaluated in ***S'***.
$\vec{v}$ is the speed of ***S'*** evaluated in ***S***.

Having already established:

$$\nabla' = \nabla \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}_d\cdot\nabla$$

it follows that:

$$\left(\frac{d}{dt}\right)' = \left(\frac{\partial}{\partial t} + \vec{v}_d\cdot\nabla\right)' = \frac{\partial}{\partial t'} + \vec{v}_d'\cdot\nabla' = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right) + \left(\vec{v}_d - \vec{v}\right)\cdot\nabla = \frac{\partial}{\partial t} + \vec{v}_d\cdot\nabla = \frac{d}{dt}$$

which verifies the first-order invariance of d/dt.

$\vec{v}$ is necessarily constant, given the hypothesis of inertiality.

$\vec{v}_d$ is not necessarily constant, as an attribute of the non-inertial motion of a Lagrangian particle.

Hertz-Phipps Electromagnetism

Referring to a generic IRF ***S***, in empty space the equations for the electromagnetic field valid to first order of v/c are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(1H)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(2H)}$$

$$\nabla \wedge \vec{E} = -\frac{d\vec{B}}{d\,t} \qquad \text{(3H)}$$

$$\nabla \wedge \vec{B} = \mu_0 \vec{J}_m + \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t} \qquad \text{(4H)}$$

They are complemented by a definition of force:

$$\vec{F} = q\,\vec{E} \qquad \text{(5H)}$$

In contrast to Maxwell's equations, $d/d\,t$ replaces $\partial/\partial t$. Also, the density $\vec{J}_m$ is interpreted as the current density "seen" by a detector in motion, with velocity $\vec{v}_d(t)$ in ***S***.
$\vec{J}_m$ is related to the Maxwellian current density $\vec{J}$, the current density "seen" by a detector fixed at that point in ***S***, by the following expression:

$$\vec{J}_m = \vec{J} - \rho\,\vec{v}_d \qquad \text{(6H)}$$

$-\rho\,\vec{v}_d$ is an equivalent current due to the detector motion at velocity $\vec{v}_d$.

A detector stationary in ***S*** represents the special case: $\vec{v}_d = 0$.

Given that: $$\frac{d}{d\,t} = \frac{\partial}{\partial t} + \vec{v}_d \cdot \nabla\,,$$

for $\vec{v}_d = 0$, total and partial derivatives become equal.
In this case, the field equations become indistinguishable from Maxwell's.
Furthermore, the force expressions coincide in the two theories.

Therefore, in the case of a non-accelerated detector, supposing equal descriptions of the sources (ρ and $\vec{J}$) in the IRF in which the field detector is at rest ($\vec{v}_d = 0$), all the predictions of Maxwell's theory are reproduced by the Hertz-Phipps theory.

This does not mean, however, as Phipps stated, that the new theory constitutes a "covering theory" of Maxwell's electromagnetism, since the description of the sources differs, in general, in the two theories.

In Maxwell's theory, ρ and $\vec{J}$ densities are transformed when the reference system changes, whereas in the Hertz-Phipps theory ρ and $\vec{J}_m$ densities and are invariant (as will be shown later).

In the case of a detector moving in ***S***, a comparison between the two theories cannot be accomplished by only considering the field equations, but by also considering the differences in the definitions of the electrodynamic force adopted in the two theories.

In Maxwell's theory the Lorentz force is assumed: $\vec{F} = q\,\vec{E}_{Maxwell} + q\,\vec{v}_d \wedge \vec{B}_{Maxwell}$

In the Hertz-Phipps theory the force law is: $\vec{F} = q\,\vec{E}_{Hertz}$

It is therefore postulated that the electric field vector is sufficient to evaluate the force acting on a charge. The magnetic field plays an indispensable role in determining the electric field in dynamic situations, but it does not appear explicitly in the law of force.

Galilean source transformation equations

Consider two generic IRFs: ***S*** and ***S'***.
Let ***S'*** in motion be at constant velocity $\vec{v}$ with respect to ***S***.
The charge density transformation equation is:

$$\rho'(\vec{r}\,', t') = \rho(\vec{r}\;, t)$$

This assumption can be understood by considering that:

$\vec{r}\,'$ and $\vec{r}$ refer to the same point **P** in space viewed in ***S'*** and ***S***, respectively.

$t' = t$ for the assumption of GT validity.

$\rho(\vec{r}\;, t)$ must be intended as the ratio between the amount of net charge (the algebraic sum of a finite number of point-like charges whose conservation is postulated) contained in a given infinitesimal volume dV, centered in **P**, stationary in ***S***, and the volume itself.

$\rho'(\vec{r}\,', t')$ must be intended as the ratio between the amount of net charge contained in a given infinitesimal volume dV', with the same linear size of dV, centered in **P**, stationary in ***S'***, namely in motion at speed $\vec{v}$ in ***S***, and the volume itself.

dV' is instantaneously coincident with dV because of the invariance of the lengths implied by GT.

The coincidence of reference volumes in different IRFs implies the same amount of net charge and therefore the same evaluation for the density.

To evaluate the transformation of the current density, it is appropriate to begin with the definitions of $\vec{J}$ and $\vec{J}_m$ in a given IRF ***S***.
Consider a limited region of space where there is a positive electric charge distribution with density $\rho_+(\vec{r},t)$ and a negative electrical charge distribution with density $\rho_-(\vec{r},t)$.
ρ_+ is therefore a positive real number, while ρ_- is a negative real number.
The net charge density will be:

$$\rho = \rho_+ + \rho_-$$

The volume element dV identified by $\vec{r}$, at time t, will have a total electric charge:

$$\rho(\vec{r},t)\,dV = [\rho_+(\vec{r},t) + \rho_-(\vec{r},t)]\,dV$$

In ***S***, if all positive charges contained in dV share the $\vec{v}_+$ velocity, while all negative charges share the $\vec{v}_-$ velocity, the Maxwellian current density is defined as:

$$\vec{J} = \rho_+\vec{v}_+ + \rho_-\vec{v}_-$$

In ***S***, the Hertzian current density is defined as:

$$\vec{J}_m = \vec{J} - \rho\ \vec{v}_d = \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho\ \vec{v}_d$$

In the IRF ***S'*** (moving at speed $\vec{v}$ with respect to ***S***), adopting the GT, the Maxwellian current density becomes:

$$\begin{aligned}
\vec{J}' &= \rho_+\vec{v}_+' + \rho_-\vec{v}_-' = \\
&= \rho_+(\vec{v}_+ - \vec{v}) + \rho_-(\vec{v}_- - \vec{v}) = \\
&= \rho_+\vec{v}_+ - \rho_+\vec{v} + \rho_-\vec{v}_- - \rho_-\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho_+\vec{v} - \rho_-\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - (\rho_+ + \rho_-)\,\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho\vec{v} = \\
&= \vec{J} - \rho\vec{v}
\end{aligned}$$

Since the expression of $\vec{J}$ contains the speed of ***S'*** with respect to ***S***, it shows the non-invariance under Galilean transformations (unless the source is neutral).
In contrast, the Hertzian current density remains unchanged (invariant) passing from ***S*** to ***S'***:

$$\begin{aligned}
\vec{J}_m' &= \rho_+\vec{v}_+' + \rho_-\vec{v}_-' - \rho\ \vec{v}_d' = \\
&= \rho_+(\vec{v}_+ - \vec{v}) + \rho_-(\vec{v}_- - \vec{v}) - \rho\ (\vec{v}_d - \vec{v}) = \\
&= \rho_+\vec{v}_+ - \rho_+\vec{v} + \rho_-\vec{v}_- - \rho_-\vec{v} - \rho\vec{v}_d + \rho\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho_+\vec{v} - \rho_-\vec{v} - \rho\vec{v}_d + \rho\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - (\rho_+ + \rho_-)\vec{v} - \rho\vec{v}_d + \rho\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho\vec{v} - \rho\vec{v}_d + \rho\vec{v} = \\
&= \rho_+\vec{v}_+ + \rho_-\vec{v}_- - \rho\vec{v}_d = \vec{J}_m
\end{aligned}$$

The Hertzian current density $\vec{J}_m$, described as "measured" by Phipps (although it is not clear how its measurability deviates from the Maxwellian one), is not invariant because it is measured but because it is suitably defined.
In fact, its definition only makes relative velocities relevant between source charges and detector.
$\vec{J}_m$ can be expressed as:

$$\vec{J}_m = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho\ \vec{v}_d = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - (\rho_+ + \rho_-)\ \vec{v}_d =$$
$$= \rho_+ (\vec{v}_+ - \vec{v}_d) + \rho_- (\vec{v}_- - \vec{v}_d) = \rho_+ \vec{v}_{rel+d} + \rho_- \vec{v}_{rel-d}$$

or:

$$\vec{J}_m = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho\ \vec{v}_d = \rho_+ \vec{v}_+ + (\rho - \rho_+) \vec{v}_- - \rho\ \vec{v}_d = \rho_+ \vec{v}_+ - \rho_+ \vec{v}_- + \rho \vec{v}_- - \rho\ \vec{v}_d =$$
$$= \rho_+ (\vec{v}_+ - \vec{v}_-) + \rho (\vec{v}_+ - \vec{v}_d) = \rho_+ \vec{v}_{rel+-} + \rho \vec{v}_{rel+d}$$

or:

$$\vec{J}_m = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho\ \vec{v}_d = (\rho - \rho_-) \vec{v}_+ + \rho_- \vec{v}_- - \rho\ \vec{v}_d = \rho_- (\vec{v}_- - \vec{v}_+) + \rho (\vec{v}_+ - \vec{v}_d) =$$
$$= \rho_- \vec{v}_{rel-+} + \rho \vec{v}_{rel+d}$$

$\vec{v}_{rel+d}$ is the velocity of positive charges contained in dV with respect to the detector
$\vec{v}_{rel-d}$ is the velocity of negative charges contained in dV with respect to the detector
$\vec{v}_{rel+-}$ is the velocity of positive charges in dV with respect to negative charges in dV
$\vec{v}_{rel-+}$ is the velocity of negative charges in dV with respect to positive charges in dV

In any case, $\vec{J}_m$ is only expressible through the use of the relative speeds between charges.
And since the relative speeds under the GT are the same in any inertial reference frame, the invariance of $\vec{J}_m$ follows.

Galilean transformations of fields

Since the operators ∇ and d/dt appearing in the Hertzian field equations are Galileo-invariant, and since ρ and $\vec{J}_m$ are quantities on which all observers agree, the field equations are Galileo-invariant too.

The field transformation laws, in passing from ***S*** to ***S'***, are therefore:

$$\vec{E}' = \vec{E} \qquad \vec{B}' = \vec{B}$$

Continuity equation

Using the total derivative, one also arrives at a Galilean invariant continuity equation:

$$\nabla \cdot \vec{J}_m + \frac{d\rho}{d\,t} = 0 \qquad (7H)$$

This equation (like the corresponding Maxwellian one) is implicitly contained in the filed equations, as can be verified by differentiating the equation (1H) with respect to time:

$$\nabla \cdot \frac{d\vec{E}}{d\,t} = \frac{1}{\varepsilon_0}\frac{d\rho}{d\,t} \qquad (\square)$$

calculating the divergence of equation (4H):

$$0 = \mu_0 \nabla \cdot \vec{J}_m + \mu_0 \varepsilon_0 \nabla \cdot \frac{d\vec{E}}{d\,t} \qquad (\blacksquare)$$

introducing the equation (□) into (■):

$$0 = \mu_0 \nabla \cdot \vec{J}_m + \mu_0 \varepsilon_0 \frac{1}{\varepsilon_0}\frac{d\rho}{d\,t}$$

from which the equation (7H) follows.

Wave equations

Taking the rotor of (4H) and using the vector identity $\nabla \wedge (\nabla \wedge \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$:

$$\nabla \wedge (\nabla \wedge \vec{B}) = \nabla \wedge (\mu_0 \vec{J}_m) + \nabla \wedge \left(\mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t}\right)$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \nabla \wedge \vec{J}_m + \mu_0 \varepsilon_0 \frac{d}{d\,t}(\nabla \wedge \vec{E})$$

$$-\nabla^2 \vec{B} = \mu_0 \nabla \wedge \vec{J}_m + \mu_0 \varepsilon_0 \frac{d}{d\,t}\left(-\frac{d\vec{B}}{d\,t}\right)$$ – considering (2H) and (3H) –

it follows that:

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{d^2\vec{B}}{dt^2} = -\mu_0 \nabla \wedge \vec{J}_m \qquad (8H)$$

which expresses a local constraint obeyed by the $\vec{B}$ field, described in ***S***, at any point in space.
The points are identified in ***S***, but the temporal variations of the field at these points are evaluated as they move with instantaneous velocity $\vec{v}_d$ with respect to ***S***.
In a similar way, starting from (3H), it is possible to obtain an equation for the $\vec{E}$ field:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{d^2\vec{E}}{dt^2} = \mu_0 \frac{d\vec{J}_m}{d\,t} + \frac{1}{\varepsilon_0}\nabla\rho \qquad (9H)$$

(8H) and (9H) are propagation equations with local forcing terms (functions of charge density and current density).
They express local constraints, but also depend in a non-local way on the speed of the detector $\vec{v}_d$ through the interpretation of the total time derivative and the definition of $\vec{J}_m$.

Comparison between Maxwell's theory and the Hertz-Phipps theory

Let ***S*** be the observer's IRF, in respect to which the source positions and the field points are described.

In the case of a stationary detector in ***S***, if the descriptions of the sources coincide, the two theories would make the same predictions about the electric field and the force acting on the detector.
The expected magnetic field would also have the same value, but with different operational meaning.
According to both theories, the magnetic field does not produce effects on a stationary detector.
However, while in Maxwell's theory the magnetic field calculated at the position of the detector could be used to evaluate the force experienced by another detector with the same instantaneous position but not null speed, in Hertz's theory this is not true, because the field depends on the detector's motion.
The different description of the sources in the two theories is due to the fact that, according to Maxwell, charge and current densities depend on the adopted IRF, while according to Hertz-Phipps the densities are invariant.
The question can be better understood using an example.
Consider a neutral magnetostatic source consisting of an electric circuit in which a constant current flows.
Assume that the circuit is realized by means of an ideal conductor at rest in the inertial system of the laboratory $\boldsymbol{S_L}$.
In $\boldsymbol{S_L}$ the circuit identifies a region of space characterized by $\rho=0$ and $\vec{J}\neq 0$.
According to Maxwell, a detector moving at a certain speed in the laboratory frame undergoes a force totally justified by the magnetic component of the Lorentz force, the electric field being zero.
Instead, in $\boldsymbol{S_{d\text{-}I}}$, i.e. the IRF instantaneously co-moving with the detector, the force must have a purely electric justification and thus the source cannot show local neutrality.
In $\boldsymbol{S_{d\text{-}I}}$ the circuit identifies a (moving) region of space characterized by $\rho\neq 0$.
As is well known, different evaluations of ρ made by different inertial observers are possible, assuming the conservation of charge, by means of the non-invariance of lengths provided by LT.
According to Hertz-Phipps's theory, if the source is neutral in $\boldsymbol{S_L}$ then it is also neutral in any other IRF.
Therefore, the description of the field sources made by an observer co-moving with the detector is different in the two theories.

In the case of a moving detector in ***S***, the comparison between Hertzian and Maxwellian predictions is more complex. One must consider the differences of the operators (partial and total time derivative) as well as differences in the definitions of force.
According to Maxwell, in a generic IRF ***S***, at every instant and in every point of the space, the magnetic field must satisfy the equation:

$$\nabla^2 \vec{B}_M - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}_M}{\partial t^2} = -\mu_0 \nabla \wedge \vec{J}$$

$\vec{J}$ is the Maxwellian current density in ***S***.
The field is described as independent from the moving charge intended to detect it.

According to Hertz-Phipps, the magnetic field must satisfy the equation:

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2} = -\mu_0 \nabla \wedge \vec{J}_m$$

The spatial and temporal coordinates that are the arguments of the vector function $\vec{B}$ are the coordinates of ***S***. However, the variations of $\vec{B}$ are valued by an observer co-moving with the detector, i.e. as if the evaluation points were moving with velocity $\vec{v}_d$ in ***S***.
$\vec{J}_m$ is the Hertzian current density in ***S*** or, by reason of its invariance, in any other IRF.

The previous two differential equations also produce different solutions in the case of a neutral source (when $\vec{J} = \vec{J}_m$), as becomes evident expanding the total derivative in ***S*** (details in Appendix - C):

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ \frac{\partial}{\partial t} \left[(\vec{v}_d \cdot \nabla) \vec{B} \right] + (\vec{v}_d \cdot \nabla) \frac{\partial \vec{B}}{\partial t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla) \vec{B} \right\} = -\mu_0 \nabla \wedge \left(\vec{J} - \rho\, \vec{v}_d \right)$$

Similarly, it can be said that, according to Maxwell, in ***S*** the electric field should satisfy:

$$\nabla^2 \vec{E}_M - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_M}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon_0} \nabla \rho$$

According to Hertz-Phipps:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2} = \mu_0 \frac{d \vec{J}_m}{d\,t} + \frac{1}{\varepsilon_0} \nabla \rho$$

ρ is the charge density field in ***S***.
According to Maxwell ρ describes the charge density exclusively in ***S***.
According to Hertz-Phipps ρ describes the charge density both in ***S*** and in any other IRF.

This Hertzian wave equation can also be expressed in ***S*** in the form (details in Appendix - C):

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ \frac{\partial}{\partial t} \left[(\vec{v}_d \cdot \nabla) \vec{E} \right] + (\vec{v}_d \cdot \nabla) \frac{\partial \vec{E}}{\partial t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla) \vec{E} \right\} = \mu_0 \frac{d \vec{J}_m}{d\,t} + \frac{1}{\varepsilon_0} \nabla \rho$$

Which highlights the role of the detector's instantaneous velocity on the wave equation of $\vec{E}$.

In the Hertz-Phipps theory, for the calculation of fields, it is convenient to use their invariance property and choose $\boldsymbol{S_{d\text{-}I}}$, i.e. the IRF instantaneously co-moves with the detector, as a reference frame.
The variables evaluated in $\boldsymbol{S_{d\text{-}I}}$ will be marked by an asterisk as superscript.
The relations between $\boldsymbol{S_{d\text{-}I}}$ and $\boldsymbol{S}$ are:

$$\vec{v}_d{}^* = 0 \qquad \vec{r}^* = \vec{r} - \vec{v}_{d\,(t)}\, t \qquad t^* = t \qquad \rho = \rho^*$$

$$\vec{J}_m = \vec{J}^*_m \qquad \nabla^* = \nabla \qquad \left(\frac{d}{d\,t}\right)^* = \frac{d}{d\,t} \qquad \frac{\partial}{\partial\, t^*} = \frac{\partial}{\partial\, t} + \vec{v}_d \cdot \nabla$$

In $\boldsymbol{S_{d\text{-}I}}$, the equations (8H) and (9H) become:

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial\, t^{*2}} = -\mu_0 \nabla \wedge \vec{J}_m \tag{10H}$$

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial\, t^{*2}} = \mu_0 \frac{\partial\, \vec{J}_m}{\partial t^*} + \frac{1}{\varepsilon_0} \nabla \rho \tag{11H}$$

They are formally identical to the Maxwellian ones in $\boldsymbol{S_{d\text{-}I}}$, but differing in the description of sources.
They only provide a consistent description at each instant in the case of the detector's uniform motion (because in the case of an accelerated detector $\boldsymbol{S_{d\text{-}I}}$ changes continuously) and imply the retarded solutions:

$$\vec{B}\left(\vec{r}^*, t\right) = \frac{\mu_0}{4\pi} \int \frac{\nabla \wedge \vec{J}_m\left(\vec{r}^*, t - \left|\, \vec{r}^* - \vec{r}^{*}{}'(t_R)\, \right| / c\right)}{\left|\, \vec{r}^*(t) - \vec{r}^{*}{}'(t_R)\, \right|} dV'$$

$$\vec{E}\left(\vec{r}^*, t\right) = -\frac{\mu_0}{4\pi} \int \frac{\dfrac{\partial\, \vec{J}_m\left(\vec{r}^*, t - \left|\, \vec{r}^* - \vec{r}^{*}{}'(t_R)\, \right| / c\right)}{\partial t}}{\left|\, \vec{r}^*(t) - \vec{r}^{*}{}'(t_R)\, \right|} dV' - \frac{1}{4\pi\varepsilon_0} \int \frac{\nabla \rho\left(\vec{r}^*, t - \left|\, \vec{r}^* - \vec{r}^{*}{}'(t_R)\, \right| / c\right)}{\left|\, \vec{r}^*(t) - \vec{r}^{*}{}'(t_R)\, \right|} dV'$$

Where the "retarded time" t_R is implicitly defined by the equation: $t_R = t - \left|\, \vec{r}^* - \vec{r}^{*}{}'(t_R)\, \right| / c$.

The perturbations of the fields are "seen" propagating with speed $\boldsymbol{c}$ in $\boldsymbol{S_{d\text{-}I}}$.
That such spatial and temporal field distributions are "seen" is obviously metaphorical and should be understood as: "inferred in order to justify the observed forces, in a manner consistent with the theory".
The totally inferred nature of the fields is particularly evident in this theory. The fields, whose perturbations propagate according to the wave equations (10H) and (11H), are not testable, even conceptually.
At any instant and point in space not belonging to the world line of the detector, the electric field may be intended as the force per unit charge perceived by a virtual detector sharing the instantaneous speed of the real detector and placed in that given point at that instant.
However, where a real charge is not present to operate as a detector, there is no possibility of introducing a charge later in order to test a perturbation "intended" for another detector.

The presence of the detector must be conceptually contemplated a priori in the description of the electromagnetic system in question, since its state of motion influences the propagation.

Applying the GT, one can deduce that, in ***S***, the field perturbations are "seen" propagating with speed $c+\vec{u}_K \cdot \vec{v}_d$, where $\vec{u}_K$ is the unit vector in the propagation direction.
This means that, given a source contained in a limited and motionless spatial region in ***S***, the perturbations of the fields propagating from the source travel at speeds higher (or lower) than ***c*** if associated with a detector moving away from (or approaching) the region.

This wave dragging mechanism, which makes the speed of field perturbations equal to ***c*** with respect to the detector, is compatible with the null results of the Michelson-Morley interferometer experiments.

Irrespective of the source-detector distance, the radiated electromagnetic field, behaves as if it knew its destination, changing as a function of the detector-absorber motion.
Every charge is connected with each other regardless of the distance (since all the charges must share information about their mutual state of motion).
Therefore, the total time derivative as used in Hertz-Phipps's theory introduces a link between fields and detectors of a non-local nature, a kind of entanglement.

It should be noted that, despite the non-local constraint that acts on the propagation speed, the electromagnetism formulated by means of the equations (1H), (2H), (3H) and (4H) only implies propagative solutions, i.e. delayed.

Solutions of the wave equation

Consider a homogeneous wave equation (of $\vec{E}$, but the procedure also applies to $\vec{B}$):

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{d t^2} = 0$$

and a linearly polarized monochromatic plane-wave solution:

$$\vec{E} = \vec{E}_o \sin\left(\vec{k} \cdot \vec{r} - \omega t + \phi\right) = \vec{E}(p)$$

Since a constant phase can be set to zero by a suitable choice of the zero value for time, for the phase ***p*** one may consider the simplified expression:

$$p = \vec{k} \cdot \vec{r} - \omega t = x k_x + y k_y + z k_z - \omega t$$

which corresponds to a phase velocity:

$$v_{ph} = \omega / k \, .$$

Introducing the solution in the wave equation (details in Appendix - D), considering only non-accelerated detectors, one obtains:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{d t^2} = \left\{ k^2 - \frac{1}{c^2} \left[\omega - \left(\vec{v}_d \cdot \vec{k} \right) \right]^2 \right\} \frac{\partial^2 \vec{E}}{\partial p^2} = 0$$

It follows that:

$$k^2 - \frac{1}{c^2} \left[\omega - \left(\vec{v}_d \cdot \vec{k} \right) \right]^2 = 0$$

And so:

$$\omega = \pm c k + \vec{k} \cdot \vec{v}_d \qquad \text{(eo1)}$$

In IRF ***S***, the wave phase propagation speed of a plane wave "hooked" on a given non-accelerated detector, with speed $\vec{v}_d$ in ***S***, is:

$$v_{ph} = \frac{\omega}{k} = \pm c + \frac{\vec{k}}{k} \cdot \vec{v}_d \qquad \text{(eo2)}$$

Considering a different IRF ***S'***, moving with speed $\vec{v}$ in ***S***, the invariance of fields implies:

$$\vec{E}(p) = \vec{E}'(p'),$$

so:

$$p = p',$$

which means:

$$\vec{k} \cdot \vec{r} - \omega t = \vec{k}' \cdot \vec{r}' - \omega' t'$$

This describes a constant phase value of the wave in the two IRFs.

Applying the GT: $\vec{r}\,' = \vec{r} - \vec{v}\,t\,,\quad t' = t$

the previous relation becomes: $\vec{k}\cdot\vec{r} - \omega t = \vec{k}\,'\cdot\left(\vec{r} - \vec{v}\,t\right) - \omega' t$

so: $\left(\vec{k} - \vec{k}\,'\right)\cdot\vec{r} = \left(\omega - \omega' - \vec{k}\,'\cdot\vec{v}\right)t$

Since $\vec{r}$ and t are arbitrary and independent variables, the validity of equality implies that the coefficients that multiply them are zero. It follows that:

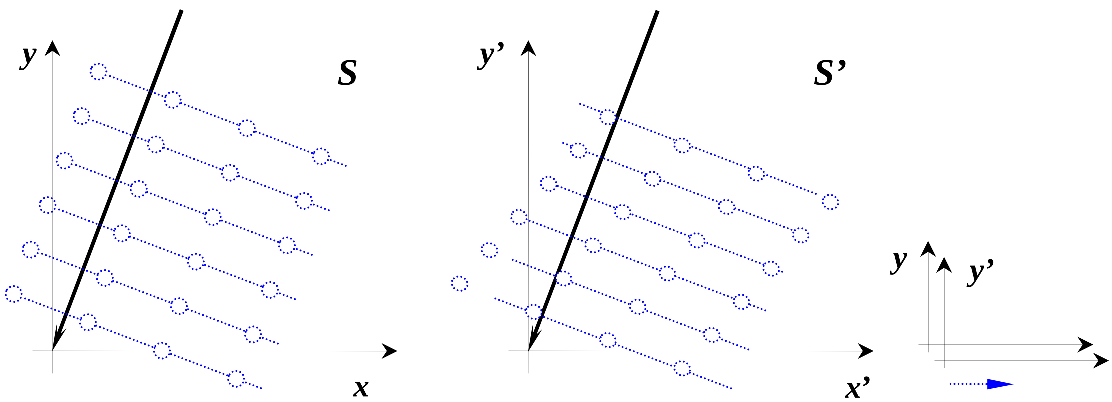
$$\vec{k}\,' = \vec{k} \qquad\qquad \omega' = \omega - \vec{k}\cdot\vec{v} \qquad\qquad \text{(eo3)}$$

The first result shows that, in the description of a plane wave associated with that given detector, the wave propagation direction does not change in the transition from ***S*** to ***S'***.
The second result describes a variation of the frequency ascribed to the plane wave in the change of IRF. This frequency variation is not a measured quantity, since the measurements only pertain to the detector.

Assuming that ***S'*** coincides with $\boldsymbol{S_{d\text{-}I}}$, i.e. the IRF instantaneously co-moving with the inertial detector ($\vec{v}_d = \vec{v}$), it follows that:

$$\vec{k}\,' = \vec{k} \qquad\qquad \omega' = \omega - \vec{k}\cdot\vec{v}_d \qquad\qquad \text{(eo4)}$$

Now the second result binds the angular frequency measured by the detector with the angular frequency of the wave described in ***S***, in which the detector has velocity $\vec{v}_d$.



By applying the GT to the IRF variation, the figure shows how the direction of any given point of the wave front changes, while it does not change the direction of the front as a whole.

Phipps interprets the result $\vec{k}\,' = \vec{k}$ as if he were describing the aberration of light, which, therefore, would be zero in the first order version of the theory.

The conclusion is wrong.
The result only says that the wave vector $\vec{k}$ of a plane wave associated with a given detector is invariant when we change the IRF in which the wave is described.
To evaluate the Hertz-Phipps theory about light aberration, the change of the IRF in which the motion of the inertial detector is described is not a relevant question.

Instead, having chosen the source, it is important to understand if and how the wave vector varies when the speed of a detector in the same IRF changes.
Alternatively, we must understand how the wave vectors of electromagnetic waves, generated by the same source but associated with two detectors with different speeds in the same IRF, are linked.
Plane waves associated with different instantaneously coincident detectors but with different speeds are not necessarily subject to the constraint of equality for their phases in a generic point of the space at a given instant.
The same can be said for the waves associated with the same detector, having different uniform speeds at different times.
Therefore, the previously used procedure cannot be followed to infer the invariance of $\vec{k}$ to the change of the detector or its state of motion.

For this purpose it is, instead, useful to consider the role of the initial conditions for determining the orientation of a planar travelling wave solution.
Far from the source, the wave equation is:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ \frac{\partial}{\partial t}\left[\left(\vec{v}_d \cdot \nabla \right) \vec{E} \right] + \left(\vec{v}_d \cdot \nabla \right) \frac{\partial \vec{E}}{\partial t} + \left(\vec{v}_d \cdot \nabla \right)\left(\vec{v}_d \cdot \nabla \right) \vec{E} \right\} = 0$$

It can be rewritten in the form:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ \left(\vec{a}_d \cdot \nabla \right) \vec{E} + 2\left(\vec{v}_d \cdot \nabla \right) \frac{\partial \vec{E}}{\partial t} + \left(\vec{v}_d \cdot \nabla \right)^2 \vec{E} \right\} = 0$$

Consider a monochromatic source of angular frequency ω_{source}, placed at a great distance from the detector, so that a plane solution constitutes an accurate local approximation of the spherical surface of the front at the detector's position. It is a simplified model of a stellar source.

Let ***S*** be the IRF in which the source is at rest.

Let ***d1*** and ***d2*** be two different detectors: ***d1,*** stationary in ***S***, and ***d2***, with constant velocity $\vec{v}_d$ in ***S***.

Suppose there is an instant, defined as $t = 0$, in which the two detectors ***d1*** and ***d2*** are coincident, at a (large) distance ***L*** from the source.

Let the x-axis be oriented as the straight line joining the source and ***d1***.

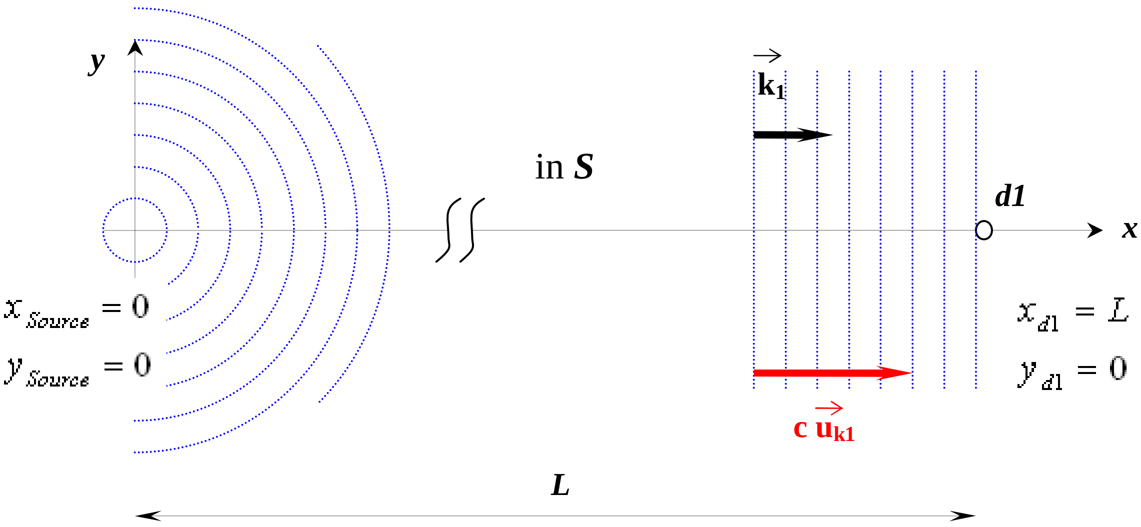
Let the origin of the axes be placed into the source.

Let ***S'*** be the IRF in which the detector ***d1*** is stationary.

Let the origins of the axes of the two IRFs be coincident at $t = 0$.

Wave equation associated to detector ***d1***, in ***S***:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$



This equation admits, in ***S***, a solution with a propagation speed equal to ***c***, of form:

$$\vec{E} = \vec{E}_1 \sin\left(\vec{k}_1 \cdot \vec{r} - \omega_1 \, t\right)$$

With:

Angular frequency $\quad \omega_1 = \omega_{1 \ in \ S} = \omega_{source}$

Wavelength $\quad \lambda_{1 \ in \ S} = \dfrac{v_{ph1 \ in \ S}}{f_{1 \ in \ S}} = \dfrac{2\pi \, v_{ph1 \ in \ S}}{\omega_{1 \ in \ S}} = \dfrac{2\pi \, c}{\omega_{source}}$

Wavenumber $\quad k_1 = \dfrac{\omega_{1 \ in \ S}}{v_{ph1 \ in \ S}} = \dfrac{\omega_{source}}{c}$

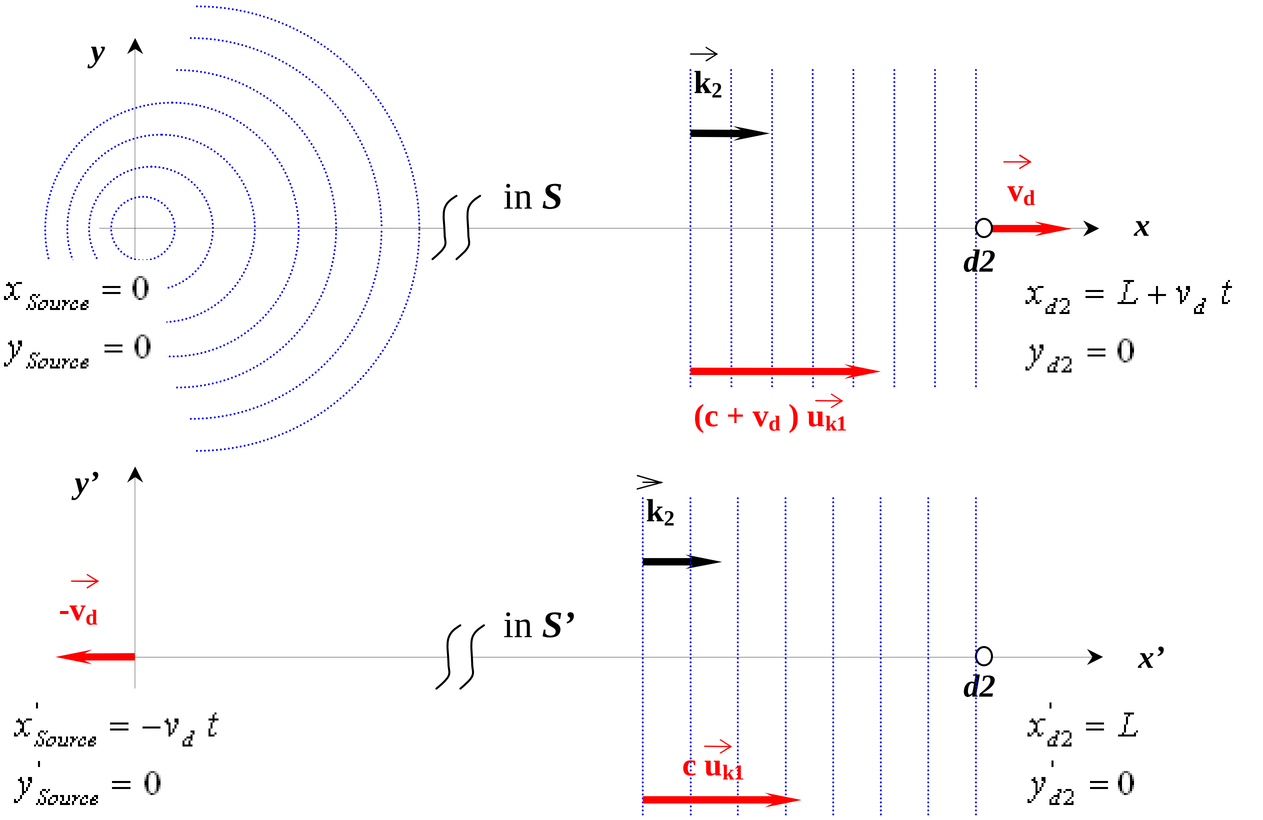
The propagation direction is coincident with the x axis:

$$\vec{u}_{k1} = \vec{u}_x$$

Wave equation associated to detector ***d2***, with ***d2*** moving away from the source in the direction of the positive x-axis $\left(\vec{v}_d \ // \ \vec{u}_x \ \right)$:

in ***S***: $$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ 2\left(\vec{v}_d \cdot \nabla\right) \frac{\partial \vec{E}}{\partial t} + \left(\vec{v}_d \cdot \nabla\right)^2 \vec{E} \right\} = 0$$

in ***S'***: $$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$



The second equation admits, in ***S'***, a solution with a propagation speed equal to ***c***, of form:

$$\vec{E} = \vec{E}_2 \sin\left(\vec{k}_2 ' \cdot \vec{r}\,' - \omega_2 ' \, t\right)$$

Due to the symmetry of the system, the propagation direction is coincident with the x-axis: $\dfrac{\vec{k}_2 '}{k_2 '} = \vec{u}_x$

This means that the first equation (describing the same entity in ***S***) admits a solution, with propagation velocity in ***S***, evaluated by (eo2), equal to $c + \vec{u}_{k2} \cdot \vec{v}_d = c + v_d$.
This solution can be expressed as:

$$\vec{E} = \vec{E}_2 \sin\left(\vec{k}_2 \cdot \vec{r} - \omega_2 \, t\right)$$

Equating the phases of the two descriptions and introducing the coordinate transformations $\left(\vec{r}\,' = \vec{r} - \vec{v}_d \ t \ \right)$, the expressions (eo4) are obtained.

Therefore: $\vec{k}_2 ' = \vec{k}_2$ $\qquad$ $\vec{u}_{k2} = \vec{u}_x$.

In ***S***:

$$\omega_2 = \omega_{2\ \ in\ S} = \omega_{source}$$

$$\lambda_{2\ \ in\ S} = \frac{v_{ph2\ \ in\ S}}{f_{2\ \ in\ S}} = \frac{2\pi\ v_{ph2\ \ in\ S}}{\omega_{2\ \ in\ S}} = \frac{2\pi\left(c+\vec{u}_{k2}\cdot\vec{v}_d\right)}{\omega_{source}} = \frac{2\pi\left(c+v_d\right)}{\omega_{source}}$$

$$k_2 = k_{2\ \ in\ S} = \frac{\omega_{2\ \ in\ S}}{v_{ph2\ \ in\ S}} = \frac{\omega_{2\ \ in\ S}}{c+\vec{u}_{k2}\cdot\vec{v}_d} = \frac{\omega_{source}}{c+v_d} \qquad \text{(eo5)}$$

In ***S'***, considering (eo4):

$$\omega_2' = \omega_{2\ \ in\ S'} = \omega_{2\ \ in\ S} - \vec{k}_2\cdot\vec{v}_d$$

$$\omega_2' = \omega_{2\ \ in\ S} - \left(\frac{\omega_{2\ \ in\ S}}{c+\vec{u}_{k2}\cdot\vec{v}_d}\right)\vec{u}_{k2}\cdot\vec{v}_d = \omega_{source} - \frac{\omega_{source}\ v_d}{c+v_d} = \omega_{source}\left(1-\frac{v_d}{c+v_d}\right)$$

$$\lambda_2' = \lambda_{2\ \ in\ S'} = \frac{v_{ph2\ \ in\ S'}}{f_{2\ \ in\ S'}} = \frac{2\pi\ v_{ph2\ \ in\ S}}{\omega_{2\ \ in\ S'}} = \frac{2\pi\ c}{\omega_{source}\frac{c}{c+v_d}} = \frac{c+v_d}{f_{source}}$$

Therefore:

$$\omega_2' = \omega_{source}\frac{c}{c+v_d} \qquad \text{(eo6)}$$

$$\lambda_2' = \frac{c+v_d}{f_{source}} \qquad \text{(eo7)}$$

(eo6) and (eo7) describe the Doppler effect, in terms of angular frequency and wavelength, associated with a detector moving away from the source, according to the first order theory.

In case of ***d2*** approaching the source along the x-axis:

$$\omega_2' = \omega_{source}\frac{c}{c-v_d} \qquad \text{(eo8)}$$

$$\lambda_2' = \frac{c-v_d}{f_{source}} \qquad \text{(eo9)}$$

Wave equation associated with detector ***d2***, with ***d2*** moving in respect to the source in the direction of the positive y-axis $\left(\vec{v}_d \perp \vec{u}_x \right)$:

in ***S***: $$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{ 2\left(\vec{v}_d \cdot \nabla\right) \frac{\partial \vec{E}}{\partial t} + \left(\vec{v}_d \cdot \nabla\right)^2 \vec{E} \right\} = 0$$

in ***S'***: $$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The second equation admits, in ***S'***, a solution with a propagation speed equal to ***c***, of form:

$$\vec{E} = \vec{E}_2 \sin\left(\vec{k}_2' \cdot \vec{r}' - \omega_2' t \right)$$

So the first equation (describing the same entity in ***S***) admits a solution with propagation velocity in ***S*** equal to $c + \vec{u}_{k2} \cdot \vec{v}_d$:

$$\vec{E} = \vec{E}_2 \sin\left(\vec{k}_2 \cdot \vec{r} - \omega_2 t \right)$$

For the equality of the phases in the two descriptions, one still obtains: $\vec{k}_2' = \vec{k}_2$.

However, in this case: $\vec{u}_{k2} \neq \vec{u}_x$.

In fact, the propagation direction must be consistent with the description of the source in the IRF in which the solution is expressed.

In ***S'***, where ***d2*** is stationary, the source is described in movement with velocity $-\vec{v}_d$.

Let t_{prop} be the propagation time, i.e. the time taken by a wave front to traverse the space between the source and the detector.
It follows that the distance traversed by the wave front (the hypotenuse of the right triangle with cathets ***D*** and ***L*** in the figure below) is: $c\, t_{prop}$

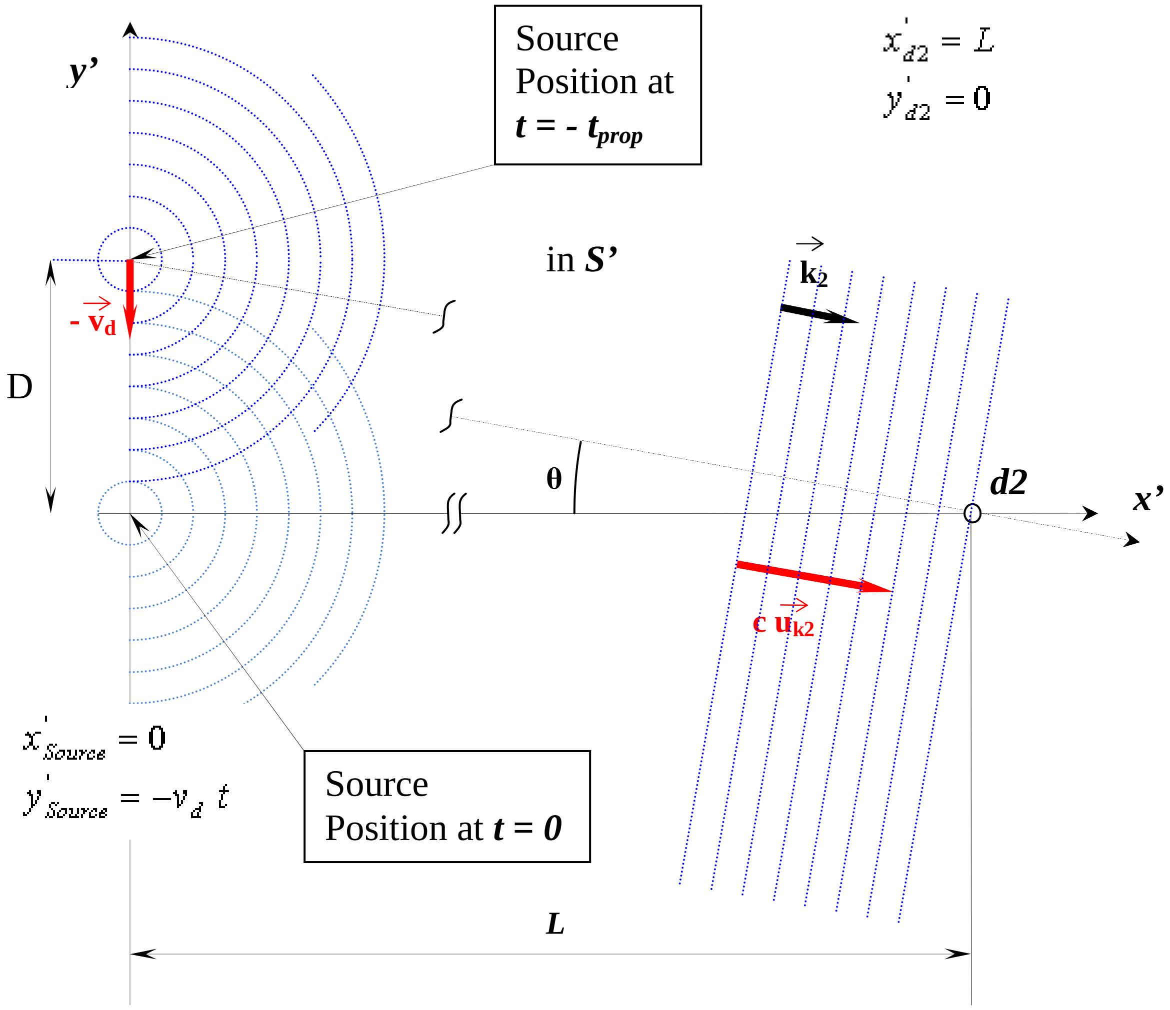
D has a length equal to: $v_d\ t_{prop}$.

So the following must apply: $v_d\ t_{prop} = c\, t_{prop} \sin(\theta)$.

Finally, it follows that:

$$\sin(\theta) = \frac{v_d}{c} \qquad \text{(eo10)}$$

The expression (eo10) describes aberration according to the first order theory for a detector moving with respect to the source in a direction perpendicular to the straight line joining source and detector.



Therefore:

In ***S***:

$$\omega_2 = \omega_{2\ \ in\ S} = \omega_{source}$$

$$\lambda_{2\ \ in\ S} = \frac{2\pi\, v_{ph2\ \ in\ S}}{\omega_{2\ \ in\ S}} = \frac{2\pi\left(c + \vec{u}_{k2}\cdot\vec{v}_d\right)}{\omega_{source}} = \frac{2\pi\left(c + v_d \sin(\theta)\right)}{\omega_{source}} = \frac{2\pi\left(c + \frac{v_d^{\,2}}{c}\right)}{\omega_{source}}$$

$$k_2 = k_{2\ \ in\ S} = \frac{\omega_{2\ \ in\ S}}{v_{ph2\ \ in\ S}} = \frac{\omega_{2\ \ in\ S}}{c + \vec{u}_{k2}\cdot\vec{v}_d} = \frac{\omega_{source}}{c + \frac{v_d^{\,2}}{c}} \qquad \text{(eo11)}$$

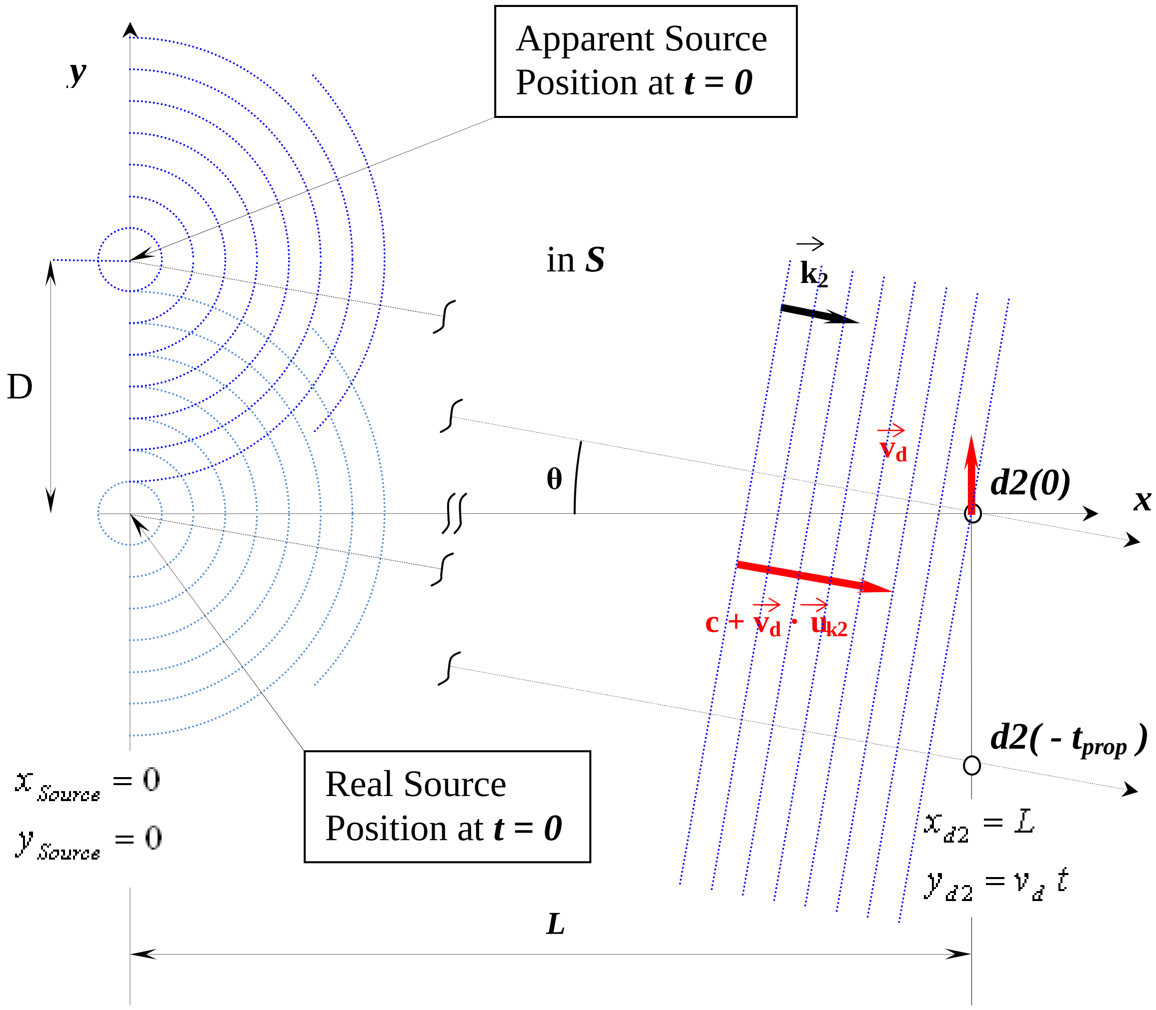
In ***S'***, considering (eo4):

$$\omega_2' = \omega_{2\ in\ S'} = \omega_{2\ in\ S} - \vec{k}_2 \cdot \vec{v}_d$$

$$\omega_2' = \omega_{2\ in\ S} - k_2\, \vec{u}_{k2} \cdot \vec{v}_d = \omega_{source} - \frac{\omega_{source}}{c + \frac{v_d^{\ 2}}{c}} \frac{v_d^{\ 2}}{c} = \omega_{source}\left(1 - \frac{v_d^{\ 2}}{c^2 + v_d^{\ 2}}\right)$$

Therefore:

$$\omega_2' = \omega_{source}\left(1 - \frac{v_d^{\ 2}}{c^2 + v_d^{\ 2}}\right) \qquad \text{(eo12)}$$



Hertzian potentials

A zero divergence vector field can be expressed as the curl of another vector field.
Therefore:

$$\vec{B} = \nabla \wedge \vec{A} \qquad (12H)$$

A vector potential is determined up to the gradient of an arbitrary scalar field; on the basis of (12H) the vector fields $\vec{A}$ and $\vec{A}' = \vec{A} + \nabla\psi$ are equivalent.
Introducing (12H) into equation (3H):

$$\nabla \wedge \vec{E} = -\frac{d\vec{B}}{dt} = -\frac{d}{dt}\left(\nabla \wedge \vec{A}\right) = -\nabla \wedge \frac{d\vec{A}}{dt}$$

If the permutability between the total time derivative and curl operator is not evident, see the proof in Appendix - E.

$$\nabla \wedge \left(\vec{E} + \frac{d\vec{A}}{dt}\right) = 0$$

so $\vec{E} + \frac{d\vec{A}}{dt}$ is conservative and therefore may be expressed as a gradient of a scalar potential.

$$\vec{E} + \frac{d\vec{A}}{dt} = -\nabla\varphi$$

$$\vec{E} = -\nabla\varphi - \frac{d\vec{A}}{dt} \qquad (13H)$$

Gauge transformations

The constraint (12H) between $\vec{B}$ and $\vec{A}$ leaves the divergence of $\vec{A}$ undefined.
This implies the possibility to add the gradient of any scalar function, the so-called gauge, to $\vec{A}$ without altering the magnetic field.
By applying a variation in the gauge of $\vec{A}$:

$$\vec{A}^{\wedge} = \vec{A} + \nabla\psi \qquad (14H)$$

introducing (14H) into (3H):

$$\nabla \wedge \vec{E} = -\frac{d\vec{B}}{dt} = -\frac{d}{dt}\left(\nabla \wedge \vec{A}^{\wedge}\right) = -\nabla \wedge \frac{d\vec{A}^{\wedge}}{dt}$$

$$\nabla \wedge \left(\vec{E} + \frac{d\vec{A}^{\wedge}}{dt}\right) = 0$$

Hence: $$\vec{E}+\frac{d\vec{A}^{\wedge}}{dt}=-\nabla\varphi^{\wedge}$$

Arranging the terms and substituting $\vec{A}^{\wedge}$ with its expression, it follows that:

$$\vec{E}=-\nabla\varphi^{\wedge}-\frac{d\vec{A}^{\wedge}}{dt}=-\nabla\varphi^{\wedge}-\frac{d\vec{A}}{dt}-\frac{d}{dt}\nabla\psi=-\nabla\left(\varphi^{\wedge}+\frac{d\psi}{dt}\right)-\frac{d\vec{A}}{dt}$$

From the comparison of the previous expression with (13H), it follows that:

$$\nabla\left(\varphi^{\wedge}+\frac{d\psi}{dt}\right)=\nabla\varphi$$

$$\varphi^{\wedge}=\varphi-\frac{d\psi}{dt} \tag{15H}$$

which shows how the scalar potential is affected by the choice of the vector potential's gauge.

Gauge invariance

It is known that Maxwell's electromagnetism is gauge-invariant.
This property is due to the structure of Maxwell's equations and the Lorentz force, which exclude the influence of the divergence of $\vec{A}$ in any measurable physical manifestation.
The gauge invariance also applies to electromagnetism described by the Hertz-Phipps equations.
To verify this assertion it is sufficient to search the expression of the force (or of the electric field) acting on a charge-detector according to Hertz-Phipps's equations, expressed in terms of potential, and apply a variation of the gauge.

$$\vec{E}=-\nabla\varphi-\frac{d\vec{A}}{dt}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}-(\vec{v}_d\cdot\nabla)\vec{A} \tag{16H}$$

Since: $$\nabla\left(\vec{a}\cdot\vec{b}\right)=(\vec{a}\cdot\nabla)\vec{b}+\left(\vec{b}\cdot\nabla\right)\vec{a}+\vec{a}\wedge\left(\nabla\wedge\vec{b}\right)+\vec{b}\wedge(\nabla\wedge\vec{a})$$

by placing: $\vec{a}=\vec{v}_d \qquad \vec{b}=\vec{A}$

it follows that:
$$\nabla\left(\vec{v}_d\cdot\vec{A}\right)=(\vec{v}_d\cdot\nabla)\vec{A}+\left(\vec{A}\cdot\nabla\right)\vec{v}_d+\vec{v}_d\wedge\left(\nabla\wedge\vec{A}\right)+\vec{A}\wedge(\nabla\wedge\vec{v}_d)$$
$$\nabla\left(\vec{v}_d\cdot\vec{A}\right)=(\vec{v}_d\cdot\nabla)\vec{A}+\quad 0 \quad+\vec{v}_d\wedge\left(\nabla\wedge\vec{A}\right)+\quad 0$$
$$(\vec{v}_d\cdot\nabla)\vec{A}=\nabla\left(\vec{v}_d\cdot\vec{A}\right)-\vec{v}_d\wedge\left(\nabla\wedge\vec{A}\right)$$

$$\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}-\nabla\left(\vec{v}_d\cdot\vec{A}\right)+\vec{v}_d\wedge\left(\nabla\wedge\vec{A}\right) \tag{17H}$$

Since: $$\vec{F}=q\vec{E}$$

it follows that: $$\vec{F} = q\left(-\nabla\varphi - \frac{d\vec{A}}{dt} \right) \tag{18H}$$

$$\vec{F} = q\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t} \right) - q\,(\vec{v}_d \cdot \nabla)\vec{A} \tag{19H}$$

$$\vec{F} = q\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t} \right) + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right) \tag{20H}$$

$$\vec{F} = \vec{F}_{Lorentz} - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right) \tag{21H}$$

For the force on the detector, Hertz-Phipps's equations imply an expression equal to that of Lorentz, corrected with an additional term.
Applying a gauge transformation to potentials, i.e. introducing (14H) and (15H) into (20H):

$$\vec{F}^{\wedge} = q\left(-\nabla\varphi^{\wedge} - \frac{\partial\vec{A}^{\wedge}}{\partial t} \right) + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}^{\wedge}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}^{\wedge}\right)$$

$$\vec{F}^{\wedge} = q\left[-\nabla\left(\varphi - \frac{d\psi}{dt} \right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla\psi\right) \right] + q\,\vec{v}_d \wedge \left[\nabla \wedge \left(\vec{A} + \nabla\psi\right)\right] - q\,\nabla\left[\vec{v}_d \cdot \left(\vec{A} + \nabla\psi\right)\right]$$

$$\vec{F}^{\wedge} = q\left[-\nabla\varphi + \frac{d\nabla\psi}{dt} - \frac{\partial\vec{A}}{\partial t} - \frac{\partial\nabla\psi}{\partial t} \right] + q\,\vec{v}_d \wedge \left[\nabla \wedge \vec{A}\right] - q\,\nabla\left[\vec{v}_d \cdot \vec{A}\right] - q\,\nabla\left[\vec{v}_d \cdot \nabla\psi\right]$$

$$\vec{F}^{\wedge} = q\left[-\nabla\varphi + \frac{\partial\nabla\psi}{\partial t} + \vec{v}_d \cdot \nabla(\nabla\psi) - \frac{\partial\vec{A}}{\partial t} - \frac{\partial\nabla\psi}{\partial t} \right] + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right) - q\,\nabla(\vec{v}_d \cdot \nabla\psi)$$

$$\vec{F}^{\wedge} = q\left[-\nabla\varphi + (\vec{v}_d \cdot \nabla)\nabla\psi - \frac{\partial\vec{A}}{\partial t} \right] + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right) - q\,\nabla(\vec{v}_d \cdot \nabla\psi)$$

$$\vec{F}^{\wedge} = -q\nabla\varphi - q\frac{\partial\vec{A}}{\partial t} + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right) - q\,\nabla(\vec{v}_d \cdot \nabla\psi) + q\,(\vec{v}_d \cdot \nabla)\nabla\psi$$

Using the vector identity: $\nabla\left(\vec{a} \cdot \vec{b}\right) \equiv (\vec{a} \cdot \nabla)\vec{b} + \left(\vec{b} \cdot \nabla\right)\vec{a} + \vec{a} \wedge \left(\nabla \wedge \vec{b}\right) + \vec{b} \wedge (\nabla \wedge \vec{a})$

setting: $\vec{a} = \vec{v}_d \qquad \vec{b} = \nabla\psi$

it follows that: $\nabla(\vec{v}_d \cdot \nabla\psi) = (\vec{v}_d \cdot \nabla)(\nabla\psi) + (\nabla\psi \cdot \nabla)\,\vec{v}_d + \vec{v}_d \wedge (\nabla \wedge \nabla\psi) + \nabla\psi \wedge (\nabla \wedge \vec{v}_d)$

Because the curl of a gradient is null and $\vec{v}_d$ behaves like a constant with respect to any spatial differential operator, we can say:

$$\nabla(\vec{v}_d \cdot \nabla\psi) = (\vec{v}_d \cdot \nabla)(\nabla\psi)$$

Therefore:

$$\vec{F}^{\wedge} = q\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t} \right) + q\,\vec{v}_d \wedge \left(\nabla \wedge \vec{A}\right) - q\,\nabla\left(\vec{v}_d \cdot \vec{A}\right)$$ which means $\vec{F}^{\wedge} = \vec{F}$.

So gauge invariance applies.

<u>Hertzian equations expressed through potentials</u>

Introducing (12H) into (1H):

$$\nabla \cdot \left(-\nabla \varphi - \frac{d\vec{A}}{dt} \right) = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2 \varphi + \frac{d}{dt} \nabla \cdot \vec{A} = -\frac{\rho}{\varepsilon_0}$$

Introducing (12H) and (13H) into (4H):

$$\nabla \wedge \left(\nabla \wedge \vec{A} \right) = \mu_0 \vec{J}_m + \mu_0 \varepsilon_0 \frac{d}{dt} \left(-\nabla \varphi - \frac{d\vec{A}}{dt} \right)$$

Using the vector identity: $\nabla \wedge \left(\nabla \wedge \vec{A} \right) = \nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A}$

$$\nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = \mu_0 \vec{J}_m - \mu_0 \varepsilon_0 \frac{d\,\nabla \varphi}{dt} - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}}{dt^2}$$

$$\nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = \mu_0 \vec{J}_m - \mu_0 \varepsilon_0 \nabla \frac{d\varphi}{dt} - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}}{dt^2}$$

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}}{dt^2} - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{d\varphi}{dt} \right) = -\mu_0 \vec{J}_m$$

Thus the new system of differential equations, which replaces (1H), (2H), (3H) and (4H), is:

$$\begin{cases} \nabla^2 \varphi + \dfrac{d}{dt} \nabla \cdot \vec{A} = -\dfrac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}}{dt^2} - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \dfrac{d\varphi}{dt} \right) = -\mu_0 \vec{J}_m \end{cases} \qquad \text{(22H)}$$

which must be completed by the force law expressed in terms of potential, i.e. (18H).
The decoupling of the equations of system (22H) can be obtained by using the properties of gauge invariance.

Lorenz-like gauge: $\nabla \cdot \vec{A}_L = -\mu_0 \varepsilon_0 \dfrac{d\varphi_L}{dt}$ (۞)

With such a gauge, system (22H) takes the form:

$$\begin{cases} \nabla^2 \varphi_L - \mu_0 \varepsilon_0 \dfrac{d^2 \varphi_L}{dt^2} = -\dfrac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A}_L - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}_L}{dt^2} = -\mu_0 \vec{J}_m \end{cases} \qquad \text{(23H) - Lorenz}$$

with: $\vec{E} = -\nabla \varphi_L - d\vec{A}_L / dt$

The equations have propagative solutions for both potentials.

It should be noted that $\vec{A}_L$ is not uniquely defined. The gauge (۞) acts on the divergence but does not make it unique.

If $\vec{A}_L$ and φ_L are a pair of potentials satisfying (۞), then this same condition (۞) is also satisfied by $\vec{A}_L'$ and φ_L', defined by

$$\vec{A}_L' = \vec{A}_L + \nabla\lambda \qquad\qquad \varphi_L' = \varphi_L - \frac{d\lambda}{dt},$$

where λ is the solution of $\quad \nabla^2\lambda - \mu_0\varepsilon_0\dfrac{d^2\lambda}{dt^2} = 0$

The potentials, as well as the fields, are attributed to points in space at rest in the IRF, but evaluated in points that are moving with the detector's velocity.

In the IRF instantaneously co-moving with the detector, the potentials – calculated using the gauge (۞) – are described as propagating at speed ***c***.

Instead, the potentials' propagation speed is described as different from ***c*** in other IRFs.

Using a different gauge, for example:

Coulomb-like gauge: $\qquad \nabla\cdot\vec{A}_C = 0 \qquad\qquad (\infty)$

system (22H) takes the form:

$$\begin{cases} \nabla^2\varphi_C = -\dfrac{\rho}{\varepsilon_0} \\ \nabla^2\vec{A}_C - \mu_0\varepsilon_0\dfrac{d^2\vec{A}_C}{dt^2} = -\mu_0\vec{J}_m + \mu_0\varepsilon_0\nabla\dfrac{d\varphi_C}{dt} \end{cases} \qquad \text{(24H) - Coulomb}$$

with: $\quad \vec{E} = -\nabla\varphi_C - d\vec{A}_C / dt$

The system's first equation has an instantaneous Coulomb potential as a solution.

The second equation provides propagative solutions.

The forcing term has a local component, which is proportional to the current density, and a non-local component, which is proportional to the gradient of the temporal derivative of the instantaneous scalar potential.

It is clear that, just like for Maxwell's electromagnetism, gauge invariance allows the use of different pairs of potentials, with different propagation speeds, obtaining the same physical predictions.

Calculation of force between steady current elements, according to the Hertz-Phipps theory

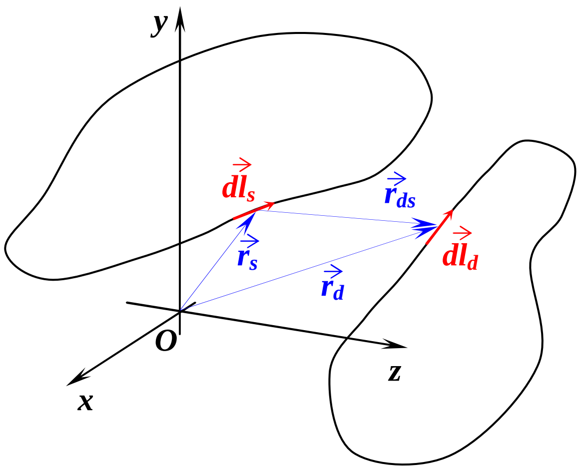
Consider a magnetostatic situation with two stationary circuits in which continuous currents flow. Evaluate the forces acting between the circuits' elements.

Let the closed circuit Γ_S, traversed by a constant current with magnitude I_S, be the source.
Let Γ_d be the second closed circuit, traversed by a constant current I_d.
Let the conductor of the circuits be thin but with negligible electrical resistance.
Let $\boldsymbol{S_L}$ be the IRF of the laboratory in which the conductors are fixed.
Let $d\vec{l}_S$ be an infinitesimal portion of Γ_S, placed in position $\vec{r}_S$ and oriented as the positive direction of the current.

As a detector, choose the charge element dq_d (conduction electrons), moving at a constant speed v_d and associated with an infinitesimal line element $d\vec{l}_d$ of Γ_d, oriented in the positive direction of the current. Let $\vec{r}_d$ be the position of $d\vec{l}_d$.

The link between these quantities is obviously: $\qquad dq_d = -I_d\ dt = -I_d\ \dfrac{dl_d}{v_d}$.

Treating the wire sections as infinitesimal, the circuits are described by lines, one-dimensional entities immersed in $\mathbb{R}^3$.



Let:

$\vec{r}_{dS} = \vec{r}_d - \vec{r}_S$

$\vec{u}_{dS} = \vec{r}_d - \vec{r}_S / r_{dS}$

$\vec{u}_{Sd} = \vec{r}_S - \vec{r}_d / r_{dS} = -\vec{u}_{dS}$

$d^2\vec{F}_d$ force exerted on $I_d d\vec{l}_d$ by $I_S d\vec{l}_S$

$d^2\vec{F}_S$ force exerted on $I_S d\vec{l}_S$ by $I_d d\vec{l}_d$

(See Appendix - F for a note on alternative historically proposed expressions of the force between current elements).

In $\boldsymbol{S_L}$, in which the conductors are stationary, the wires are described by charge densities: the positive $\rho_{S+}(\vec{r})$, $\rho_{d+}(\vec{r})$ – representing fixed charges – and the negative $\rho_{S-}(\vec{r})$, $\rho_{d-}(\vec{r})$, representing conduction electrons in motion.
Adopting the hypothesis of neutrality for conductors implies:

$$\rho_{S+}(\vec{r}) + \rho_{S-}(\vec{r}) = 0; \qquad \rho_{d+}(\vec{r}) + \rho_{d-}(\vec{r}) = 0.$$

Because GT preserve length invariance, this neutrality remains true in all reference frames.
Furthermore, due to the neutrality, Maxwellian and Hertzian current densities associated with the sources are equal:

$$\vec{J}_m = \vec{J}$$

Consider the potential equations with the constraint imposed by the Lorenz-like gauge, system (23H). Adopting the hypothesis of neutrality of the source, the system is reduced to the single equation (the subscript "L" is omitted for brevity):

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}}{dt^2} = -\mu_0 \vec{J}_m$$

In $\boldsymbol{S_{d-I}}$, i.e. the IRF instantaneously co-moving with the detector, the equation takes the form:

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\left(\partial t^*\right)^2} = -\mu_0 \vec{J}_m$$

In $\boldsymbol{S_{d-I}}$, the conductors are not stationary, but moving with velocity $-\vec{v}_d$.
The vector potential produced by the single current element $\vec{J}_m dV_S = I_S d\vec{l}_S$, in the detector position, is:

$$d\vec{A}\left(\vec{r}_d^{\,*}, t\right) = \frac{\mu_0 I_S}{4\pi} \frac{d\vec{l}_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|}$$

Under the condition:

$$t_R = \quad t - \frac{\left|\vec{r}^{\,*} - \vec{r}^{\,*}{}'(t_R)\right|}{c} \quad \approx \quad t$$

Therefore neglecting the propagation delay:

$$d\vec{A}\left(\vec{r}_d^{\,*}, t\right) \approx \frac{\mu_0 I_S}{4\pi} \frac{d\vec{l}_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t)\right|}$$

Remembering that the force on a charge, in any IRF, can be expressed as:

$$\vec{F} = q\vec{E} = q\left(-\nabla\varphi - \frac{d\vec{A}}{dt}\right)$$

that, in $\boldsymbol{S_{d-I}}$, becomes:

$$\vec{F} = q\vec{E} = q\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t^*}\right)$$

Considering the neutrality of the source and the constraint between the partial time derivatives expressed in $\boldsymbol{S_L}$ and $\boldsymbol{S_{d-I}}$, the force exerted on the current $I_d d\vec{l}_d$ by the element $I_S d\vec{l}_S$ will be:

$$d^2\vec{F}_d = dq_d \; d\vec{E} = -I_d \frac{dl_d}{v_d}\left[-\frac{\partial\left(d\vec{A}\right)}{\partial t^*}\right] = I_d \frac{dl_d}{v_d}\left[\frac{\partial\left(d\vec{A}\right)}{\partial t^{SL}} + \left(\vec{v}_d \cdot \nabla\right)\left(d\vec{A}\right)\right]$$

For the assumed stationarity of I_S in $\boldsymbol{S_L}$, it follows that: $\frac{\partial\left(d\vec{A}\right)}{\partial t^{SL}} = 0$

So:

$$d^2\vec{F}_d = I_d \frac{dl_d}{v_d}(\vec{v}_d \cdot \nabla)\left(d\vec{A}\right)$$

$$d^2\vec{F}_d = I_d \frac{dl_d}{v_d}(\vec{v}_d \cdot \nabla)\left(\frac{\mu_0 I_S}{4\pi}\frac{d\vec{l}_S}{|\vec{r}_d - \vec{r}_S|}\right) = \frac{\mu_0 I_d I_S}{4\pi}\frac{dl_d}{v_d}(\vec{v}_d \cdot \nabla)\left(\frac{d\vec{l}_S}{|\vec{r}_d - \vec{r}^*|}\right)$$

Using the identity $(\vec{v}_d \cdot \nabla)\vec{C} = \nabla\left(\vec{v}_d \cdot \vec{C}\right) - \vec{v}_d \wedge \left(\nabla \wedge \vec{C}\right)$:

$$d^2\vec{F}_d = \frac{\mu_0 I_d I_S}{4\pi}\frac{dl_d}{v_d}\left\{\nabla\left(\frac{\vec{v}_d \cdot d\vec{l}_S}{|\vec{r}_d - \vec{r}_S|}\right) - \vec{v}_d \wedge \left[\nabla \wedge \left(\frac{d\vec{l}_S}{|\vec{r}_d - \vec{r}_S|}\right)\right]\right\}$$

$$d^2\vec{F}_d = \frac{\mu_0 I_d I_S}{4\pi}\frac{dl_d}{v_d}\left\{\vec{v}_d \cdot d\vec{l}_S\,\nabla\left(\frac{1}{|\vec{r}_d - \vec{r}_S|}\right) - \vec{v}_d \wedge \left[\nabla \wedge \left(\frac{d\vec{l}_S}{|\vec{r}_d - \vec{r}_S|}\right)\right]\right\}$$

Mathematical note:

$$\nabla\frac{1}{|\vec{r} - \vec{r}\,'|} = -\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = -\frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{|\vec{r} - \vec{r}\,'|^2}$$

$$\nabla \wedge \left(\frac{d\vec{l}\,'}{|\vec{r} - \vec{r}\,'|}\right) = \frac{-1}{|\vec{r} - \vec{r}\,'|^3}(\vec{r} - \vec{r}\,') \wedge d\vec{l}\,' = -\frac{\vec{u}_{\vec{r}-\vec{r}\,'}}{|\vec{r} - \vec{r}\,'|^2} \wedge d\vec{l}$$ (Proof in Appendix - G).

Using the previous results:

$$d^2\vec{F}_d = \frac{\mu_0 I_d I_S}{4\pi}\frac{dl_d}{v_d}\left\{\vec{v}_d \cdot d\vec{l}_S\left(-\frac{\vec{u}_{dS}}{|\vec{r}_{dS}|^2}\right) - \vec{v}_d \wedge \left[-\frac{\vec{u}_{dS}}{|\vec{r}_{dS}|^2} \wedge d\vec{l}_S\right]\right\}$$

$$d^2\vec{F}_d = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\frac{dl_d}{v_d}\left\{-\left(\vec{v}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} + \vec{v}_d \wedge \left(\vec{u}_{dS} \wedge d\vec{l}_S\right)\right\}$$

Since: $\frac{dl_d}{v_d}\vec{v}_d = -d\vec{l}_d$ (the moving charge is negative, therefore $\vec{v}_d$ is opposite to the positive direction of the current, which is the direction of $d\vec{l}_d$), it follows that:

$$d^2\vec{F}_d = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\left[\left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} - d\vec{l}_d \wedge \left(\vec{u}_{dS} \wedge d\vec{l}_S\right)\right]$$

$$d^2\vec{F}_d = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\left[\left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} + d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right)\right]$$

The second of the two terms coincides with the expression of Grassmann's force, the force expected by Maxwell's electromagnetism.

$$d^2\vec{F}_d = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} + d^2\vec{F}_d^{\,G}$$

Using the identity: $\vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right) = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$, so:

$$d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right) = \left(d\vec{l}_d \cdot \vec{u}_{dS}\right) d\vec{l}_S - \left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS}$$

it is possible to express $d^2\vec{F}_d$ in another form:

$$d^2\vec{F}_d = \frac{\mu_0 \, I_S \, I_d}{4\pi \, r_{dS}^{\;2}} \left[\left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} + \left(d\vec{l}_d \cdot \vec{u}_{dS}\right) d\vec{l}_S - \left(d\vec{l}_d \cdot d\vec{l}_S\right)\vec{u}_{dS} \right]$$

$$d^2\vec{F}_d = \frac{\mu_0 \, I_S \, I_d}{4\pi \, r_{dS}^{\;2}} \left[\left(d\vec{l}_d \cdot \vec{u}_{dS}\right) d\vec{l}_S \right]$$

So the force is oriented like $d\vec{l}_S$.

By reversing the roles of source and detector, the following expression is obtained:

$$d^2\vec{F}_S = \frac{\mu_0 \, I_S \, I_d}{4\pi \, r_{dS}^{\;2}} \left[\left(d\vec{l}_S \cdot \vec{u}_{Sd}\right) d\vec{l}_d \right]$$

These expressions are not compatible with reality.

Considering, for example, the case of linear parallel conductors placed at distance ***d*** from each other, the integration of the elemental forces along the wires' path according to Maxwell's theory (but also according to the Ampere's original law) implies a force per unit length, attractive in the case of currents in the same direction, with amplitude:

$$\frac{\left|\vec{F}^{\,A}\right|}{L} = \frac{\left|\vec{F}^{\,G}\right|}{L} = \frac{\mu_0 \, I_S \, I_d}{2\pi \, d}$$

By repeating the same integration, using the expression of the force according to Hertz-Phipps's theory, since the elemental forces lie on the same line of the wires, no attraction is predicted.

In its present wording, the theory is therefore to be abandoned due to incompatibility with experimental observations.

Some considerations on predictive failure and an initial proposal for amendment

The previous expressions of the forces between current elements show that the relation between force and magnetic field, created only implicitly by the use of the total time derivative in the field equations, fails to represent the force component directed perpendicular to the motion of the detector.
It is not possible to adjust the law of force by simply adding a term, thus adopting Lorentz force, due to the invariant nature of the fields.
Since the fields $\vec{E}$ and $\vec{B}$ are invariant, the force $q\vec{E}$ is also invariant on account of the invariance of *q*, while the force $q\,\vec{v}_d \wedge \vec{B}$ is not, due to the dependence of $\vec{v}_d$ on the IRF.
And it does not seem possible to conceive a definition of force with an invariant and explicit role for $\vec{B}$ that is compatible with observations.
It does not even seem possible to modify the theory to make it covariant with respect to the change between IRFs, since covariant fields require covariant source terms. Furthermore, it is impossible to have a covariant charge density if one assumes the validity of the GT and conservation of charge.

I propose an initial amendment to the theory, which preserves its invariant character but changes the description of the sources with respect to the detector's motion.
Since the effect of this motion must be invariant, the detector's state of motion cannot be described using $\vec{v}_d$, which is the speed evaluated by an inertial observer, due to the arbitrariness of the observer. So the relative velocities between detector and sources must be considered.

Therefore, I assume that the divergence of the electric field (from the detector's "point of view") does not depend solely on the source charge density (understood as the number of charges in the unit volume), but on a scalar function, an equivalent density ρ_m, function of $\vec{v}_{dS}$, which is reduced to ρ for $\vec{v}_{dS} = 0$.

$$\nabla \cdot \vec{E} = \rho_m(\vec{v}_{dS})/\varepsilon_0$$

It should be noted that the current density in the Hertzian theory already plays a similar role.
In fact, in the general case of a non-neutral source, $\vec{J}_m$ introduces an effect of the relative velocity of the detector and source on the curl of the magnetic field produced by the source.

To clarify this proposal, consider a limited region of space with the presence of a positive charge with density $\rho_+(\vec{r},t)$ and a negative charge with density $\rho_-(\vec{r},t)$.
The net charge density is: $\rho = \rho_+ + \rho_-$
The volume element dV, located in $\vec{r}$, at the instant t, has a total electric charge:

$$\rho(\vec{r},t)\,dV = [\rho_+(\vec{r},t) + \rho_-(\vec{r},t)]\,dV$$

If, in ***S***, all positive charges contained in dV share the same average velocity $\vec{v}_+$, while all the negative charges share the same average velocity $\vec{v}_-$, it is possible to define the following current densities:

$$\vec{J}_m^{\,+} = \rho_+(\vec{v}_+ - \vec{v}_d) \qquad (25)$$

$$\vec{J}_m^{\,-} = \rho_-(\vec{v}_- - \vec{v}_d) \qquad (26)$$

The Hertzian current density in ***S*** is then expressible as:

$$\vec{J}_m = \vec{J}_m^{+} + \vec{J}_m^{-} \tag{27}$$

Indeed:

$$\vec{J}_m = \rho_+ (\vec{v}_+ - \vec{v}_d) + \rho_- (\vec{v}_- - \vec{v}_d) = \rho_+ \vec{v}_+ - \rho_+ \vec{v}_d + \rho_- \vec{v}_- - \rho_- \vec{v}_d = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho \vec{v}_d$$

which coincides with the original definition.

It has previously been shown that, by adopting GT, the Hertzian current density is invariant to changes of IRF. It has also been shown that its invariance corresponds to the possibility of being expressed by means of the only relative speeds between charges:

$$\vec{J}_m = \rho_+ \vec{v}_{rel+d} + \rho_- \vec{v}_{rel-d} = \rho_+ \vec{v}_{rel+-} + \rho \vec{v}_{rel+d} = \rho_- \vec{v}_{rel-+} + \rho \vec{v}_{rel+d}$$

So I define:

$$\rho_m^{+} = \rho_+ + \frac{1}{2c^2} \vec{J}_m^{+} \cdot (\vec{v}_+ - \vec{v}_d) = \rho_+ + \frac{1}{2c^2} \rho_+ (\vec{v}_+ - \vec{v}_d) \cdot (\vec{v}_+ - \vec{v}_d) = \rho_+ \left(1 + \frac{|\,\vec{v}_+ - \vec{v}_d|^2}{2c^2} \right) \tag{28}$$

$$\rho_m^{-} = \rho_- + \frac{1}{2c^2} \vec{J}_m^{-} \cdot (\vec{v}_- - \vec{v}_d) = \rho_- + \frac{1}{2c^2} \rho_- (\vec{v}_- - \vec{v}_d) \cdot (\vec{v}_- - \vec{v}_d) = \rho_- \left(1 + \frac{|\,\vec{v}_- - \vec{v}_d|^2}{2c^2} \right) \tag{29}$$

$$\rho_m \mathrel{\hat{=}} \rho_m^{+} + \rho_m^{-} \tag{30}$$

The scalar fields $\rho_m^{+}(\vec{v}_{dS+})$, $\rho_m^{-}(\vec{v}_{dS-})$ and $\rho_m(\vec{v}_{dS+}, \vec{v}_{dS-})$ are invariant – meaning they have values on which all inertial observers agree – because they depend on the only relative speeds. It follows that:

$$\rho_m^{+} = \rho_+ \left(1 + \frac{\vec{v}_+ \cdot \vec{v}_+ - 2\vec{v}_+ \cdot \vec{v}_d + \vec{v}_d \cdot \vec{v}_d}{2c^2} \right) = \rho_+ \left(1 + \frac{v_+^{\,2} + v_d^{\,2} - 2\vec{v}_+ \cdot \vec{v}_d}{2c^2} \right)$$

$$\rho_m^{-} = \rho_- \left(1 + \frac{\vec{v}_- \cdot \vec{v}_- - 2\vec{v}_- \cdot \vec{v}_d + \vec{v}_d \cdot \vec{v}_d}{2c^2} \right) = \rho_- \left(1 + \frac{v_-^{\,2} + v_d^{\,2} - 2\vec{v}_- \cdot \vec{v}_d}{2c^2} \right)$$

$$\rho_m = \rho_m^{+} + \rho_m^{-} = \rho + \frac{\rho \vec{v}_d \cdot \vec{v}_d + \rho_+ \vec{v}_+ \cdot \vec{v}_+ + \rho_- \vec{v}_- \cdot \vec{v}_- - 2\rho_+ \vec{v}_d \cdot \vec{v}_+ - 2\rho_- \vec{v}_d \cdot \vec{v}_-}{2c^2}$$

Alternatively:

$$\rho_m = \rho_m^+ + \rho_m^- = \rho_+ + \frac{1}{2c^2}\vec{J}_m^+ \cdot (\vec{v}_+ - \vec{v}_d) + \rho_- + \frac{1}{2c^2}\vec{J}_m^- \cdot (\vec{v}_- - \vec{v}_d) =$$
$$= \rho_+ + \frac{1}{2c^2}\vec{J}_m^+ \cdot \vec{v}_+ - \frac{1}{2c^2}\vec{J}_m^+ \cdot \vec{v}_d + \rho_- + \frac{1}{2c^2}\vec{J}_m^- \cdot \vec{v}_- - \frac{1}{2c^2}\vec{J}_m^- \cdot \vec{v}_d =$$
$$= \rho - \frac{1}{2c^2}\left(\vec{J}_m^+ + \vec{J}_m^-\right)\cdot \vec{v}_d + \frac{1}{2c^2}\vec{J}_m^+ \cdot \vec{v}_+ + \frac{1}{2c^2}\vec{J}_m^- \cdot \vec{v}_- =$$
$$= \rho - \frac{1}{2c^2}\vec{J}_m \cdot \vec{v}_d + \frac{1}{2c^2}\vec{J}_m^+ \cdot \vec{v}_+ + \frac{1}{2c^2}\vec{J}_m^- \cdot \vec{v}_-$$

Since $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, that means: $\frac{1}{2c^2 \varepsilon_0} = \frac{\mu_0}{2}$, it follows that:

$$\frac{\rho_m}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} + \frac{\mu_0}{2}\left(\rho \vec{v}_d \cdot \vec{v}_d + \rho_+ \vec{v}_+ \cdot \vec{v}_+ + \rho_- \vec{v}_- \cdot \vec{v}_- - 2\rho_+ \vec{v}_d \cdot \vec{v}_+ - 2\rho_- \vec{v}_d \cdot \vec{v}_-\right) \tag{31}$$

Or:

$$\frac{\rho_m}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} + \frac{\mu_0}{2}\left(-\vec{J}_m \cdot \vec{v}_d + \vec{J}_m^+ \cdot \vec{v}_+ + \vec{J}_m^- \cdot \vec{v}_-\right) \tag{32}$$

Referring to a generic IRF ***S***, in empty space the equations for the electromagnetic field, valid to first order of v/c, become:

$$\nabla \cdot \vec{E} = \frac{\rho_m}{\varepsilon_0} \tag{1Hm}$$
$$\nabla \cdot \vec{B} = 0 \tag{2H}$$
$$\nabla \wedge \vec{E} = -\frac{d\vec{B}}{d\,t} \tag{3H}$$
$$\nabla \wedge \vec{B} = \mu_0 \vec{J}_m + \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t} \tag{4H}$$

The force law: $$\vec{F} = q\,\vec{E} \tag{5H}$$

It should be noted that this formulation provides for the existence of forces exerted by currents on fixed electric charges.

Calculation of force between steady current elements, according to the modified theory

In $\boldsymbol{S_L}$, the system is described thus:

Source: the circuit Γ_S, in which I_S flows.
The element $d\vec{l}_S$ of Γ_S contains the negative charge dq_{S-} (conduction electrons moving at velocity $\vec{v}_{S-}$ in $\boldsymbol{S_L}$) and the positive charge dq_{S+} (positively charged metal ions, motionless in $\boldsymbol{S_L}$):

$$dq_{S-} = -I_S \; dt = -I_S \; \frac{dl_S}{\vec{v}_{S-}} \qquad\qquad dq_{S+} = -dq_{S-}$$

Detector 1: the negative charge dq_{d-} (conduction electrons moving at velocity $\vec{v}_{d1}$ in $\boldsymbol{S_L}$), associated with the infinitesimal portion $d\vec{l}_d$ of the circuit Γ_d.

$$dq_{d-} = -I_d \; dt = -I_d \; \frac{dl_d}{v_{d1}}$$

Detector 2: the positive charge dq_{d+} (metal ions, motionless in $\boldsymbol{S_L}$), associated with $d\vec{l}_d$.

$$dq_{d+} = -dq_{d-} = I_d \; \frac{dl_d}{v_{d1}}$$

In a generic IRF $\boldsymbol{S}$, the potential equations with the Lorenz-like gauge become:

$$\begin{cases} \nabla^2 \varphi - \mu_0 \varepsilon_0 \dfrac{d^2 \varphi}{dt^2} = -\dfrac{\rho_m}{\varepsilon_0} \\[2ex] \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}}{dt^2} = -\mu_0 \vec{J}_m \end{cases}$$

Referring to $\boldsymbol{S_{d\text{-}I}}$, i.e. the IRF instantaneously co-moving with the detector, this system becomes:

$$\begin{cases} \nabla^2 \varphi - \mu_0 \varepsilon_0 \dfrac{d^2 \varphi}{\left(d\,t^*\right)^2} = \nabla^2 \varphi - \mu_0 \varepsilon_0 \dfrac{\partial^2 \varphi}{\left(\partial\,t^*\right)^2} = -\dfrac{\rho_m}{\varepsilon_0} \\[2ex] \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}}{\left(d\,t^*\right)^2} = \nabla^2 \vec{A} - \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}}{\left(\partial\,t^*\right)^2} - \mu_0 \vec{J}_m \end{cases}$$

Two different IRFs will therefore be used, each co-moving with the respective detector.
Given their invariance properties, ρ_m and $\vec{J}_m$ can be calculated in any IRF, therefore also in $\boldsymbol{S_L}$:

For detector 1, in $\boldsymbol{S_L}$:

$$\rho = 0\,, \qquad \vec{v}_+ = 0\,, \qquad \vec{v}_- = \vec{v}_{S-} \qquad \vec{v}_d = \vec{v}_{d1}$$

$$\vec{J}^{\,+}_{m1} = \rho_+ \left(\vec{v}_+ - \vec{v}_{d1}\right) = -\rho_+ \vec{v}_{d1} \qquad \rightarrow \qquad \vec{J}^{\,+}_{m1} \cdot \vec{v}_+ = 0$$
$$\vec{J}^{\,-}_{m1} = \rho_- \left(\vec{v}_- - \vec{v}_{d1}\right) \qquad \rightarrow \qquad \vec{J}^{\,-}_{m1} \cdot \vec{v}_- = \rho_- \left(\vec{v}_- - \vec{v}_{d1}\right) \cdot \vec{v}_-$$
$$\vec{J}_{m1} = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho \vec{v}_{d1} = \rho_- \vec{v}_- \qquad \rightarrow \qquad \vec{J}_{m1} \cdot \vec{v}_{d1} = \rho_- \vec{v}_- \cdot \vec{v}_{d1}$$

$$\frac{\rho_{m1}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} + \frac{\mu_0}{2}\left(-\vec{J}_{m1} \cdot \vec{v}_{d1} + \vec{J}^{\,+}_{m1} \cdot \vec{v}_+ + \vec{J}^{\,-}_{m1} \cdot \vec{v}_-\right) = \frac{\mu_0}{2}\left[-\rho_- \vec{v}_- \cdot \vec{v}_{d1} + \rho_- \left(\vec{v}_- - \vec{v}_{d1}\right) \cdot \vec{v}_-\right] =$$
$$= \frac{\mu_0}{2}\left(-\rho_- \vec{v}_- \cdot \vec{v}_{d1} + \rho_- \vec{v}_- \cdot \vec{v}_- - \rho_- \cdot \vec{v}_{d1} \cdot \vec{v}_-\right) = \frac{\mu_0}{2} \rho_- \vec{v}_- \cdot \left(\vec{v}_- - 2\vec{v}_{d1}\right) = \frac{\mu_0}{2} \vec{J}_{m1} \cdot \left(\vec{v}_- - 2\vec{v}_{d1}\right)$$

So, in $\boldsymbol{S_{d\text{-}I}}$:

$$\begin{cases} \nabla^2 \varphi_1 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \varphi_1}{\left(\partial t^*\right)^2} = -\dfrac{\rho_{m1}}{\varepsilon_0} = \dfrac{\mu_0}{2} \vec{J}_{m1} \cdot \left(2\vec{v}_{d1} - \vec{v}_-\right) \\[2ex] \nabla^2 \vec{A}_1 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}_1}{\left(\partial t^*\right)^2} = -\mu_0 \vec{J}_{m1} \end{cases}$$

Therefore, the infinitesimal contribution to scalar potential perceived by detector 1, placed in position $\vec{r}_d^{\,*}$ and generated by the infinitesimal source placed in position $\vec{r}_S^{\,*}$, is:

$$d\varphi_1\left(\vec{r}_d^{\,*}, t\right) = \frac{\rho_{m1}}{4\pi\varepsilon_0} \frac{dV_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|} = \frac{-\mu_0 \vec{J}_{m1} \cdot \left(2\vec{v}_{d1} - \vec{v}_-\right)}{8\pi} \frac{dV_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|}$$

Considering that $\vec{J}_{m1}\; Sez\; dl_S = I_S\, d\vec{l}_S$, it follows that:

$$d\varphi_1\left(\vec{r}_d^{\,*}, t\right) = \frac{-\mu_0 \vec{J}_{m1} \cdot \left(2\vec{v}_{d1} - \vec{v}_-\right)}{8\pi} \frac{Sez\; dl_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|} = \frac{-\mu_0}{8\pi} \frac{I_S\, d\vec{l}_S \cdot \left(2\vec{v}_{d1} - \vec{v}_-\right)}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|}$$

Neglecting the propagation delay:

$$d\varphi_1\left(\vec{r}_d^{\,*}, t\right) \approx \frac{-\mu_0}{8\pi} \frac{I_S\, d\vec{l}_S \cdot \left(2\vec{v}_{d1} - \vec{v}_-\right)}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t)\right|}$$

The infinitesimal contribution to vector potential is:

$$d\vec{A}_1\left(\vec{r}_d^{\,*}, t\right) = \frac{\mu_0 \vec{J}_{m1}}{4\pi} \frac{dV_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|} = \frac{\mu_0 I_S}{4\pi} \frac{d\vec{l}_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t_R)\right|} \approx \frac{\mu_0 I_S}{4\pi} \frac{d\vec{l}_S}{\left|\vec{r}_d^{\,*}(t) - \vec{r}_S^{\,*}(t)\right|}$$

For detector 2, in $\boldsymbol{S_L}$:

$$\rho = 0\,, \qquad \vec{v}_+ = 0\,, \qquad \vec{v}_- = \vec{v}_{S-} \qquad \vec{v}_d = \vec{v}_{d2} = 0$$

$$\vec{J}^{+}_{m2} = \rho_+ \left(\vec{v}_+ - \vec{v}_{d2}\right) = 0 \qquad \rightarrow \qquad \vec{J}^{+}_{m2} \cdot \vec{v}_+ = 0$$

$$\vec{J}^{-}_{m2} = \rho_- \left(\vec{v}_- - \vec{v}_{d2}\right) = \rho_- \vec{v}_- \qquad \rightarrow \qquad \vec{J}^{-}_{m2} \cdot \vec{v}_- = \rho_- \vec{v}_- \cdot \vec{v}_-$$

$$\vec{J}_{m2} = \rho_+ \vec{v}_+ + \rho_- \vec{v}_- - \rho \vec{v}_{d2} = \rho_- \vec{v}_- = \vec{J}^{-}_{m2} \qquad \rightarrow \qquad \vec{J}_{m2} \cdot \vec{v}_{d2} = 0$$

$$\frac{\rho_{m2}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} + \frac{\mu_0}{2}\left(-\vec{J}_{m2} \cdot \vec{v}_{d2} + \vec{J}^{+}_{m2} \cdot \vec{v}_+ + \vec{J}^{-}_{m2} \cdot \vec{v}_-\right) = \frac{\mu_0}{2}\left(\vec{J}^{-}_{m2} \cdot \vec{v}_-\right) = \frac{\mu_0}{2} \rho_- \vec{v}_- \cdot \vec{v}_- = \frac{\mu_0}{2} \vec{J}_{m1} \cdot \vec{v}_-$$

Then, in $\boldsymbol{S_{d\text{-}1}}$, which is coincident with $\boldsymbol{S_L}$, one can formulate the following expressions:

$$\begin{cases} \nabla^2 \varphi_2 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \varphi_2}{\left(\partial t^{SL}\right)^2} = -\dfrac{\rho_{m2}}{\varepsilon_0} = -\dfrac{\mu_0}{2} \vec{J}_{m1} \cdot \vec{v}_- \\[2ex] \nabla^2 \vec{A}_2 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}_2}{\left(\partial t^{SL}\right)^2} = -\mu_0 \vec{J}_{m2} \end{cases}$$

Therefore, the infinitesimal contribution to scalar potential perceived by detector 2, is:

$$d\varphi_2\left(\vec{r}_d, t\right) = \frac{\rho_{m2}}{4\pi\varepsilon_0} \frac{dV_S}{\left|\vec{r}_d(t) - \vec{r}_S(t_R)\right|} = \frac{\mu_0 \vec{J}_{m1} \cdot \vec{v}_-}{8\pi} \frac{dV_S}{\left|\vec{r}_d(t) - \vec{r}_S(t_R)\right|}$$

$$d\varphi_2\left(\vec{r}_d, t\right) = \frac{\mu_0 \vec{J}_{m1} \cdot \vec{v}_-}{8\pi} \frac{Sez\ dl_S}{\left|\vec{r}_d(t) - \vec{r}_S(t_R)\right|} = \frac{\mu_0}{8\pi} \frac{I_S\, d\vec{l}_S \cdot \vec{v}_-}{\left|\vec{r}_d(t) - \vec{r}_S(t_R)\right|}$$

$$d\varphi_2\left(\vec{r}_d, t\right) \approx \frac{\mu_0}{8\pi} \frac{I_S\, d\vec{l}_S \cdot \vec{v}_-}{\left|\vec{r}_d(t) - \vec{r}_S(t)\right|}$$

The vector potential perceived by detector 2 is irrelevant for force calculation, given that its total time derivative is certainly zero, because of the time independence of the current that nullifies the partial derivative time in $\boldsymbol{S_L}$ and the stillness of the detector that nullifies the effect of $\vec{v}_d \cdot \nabla$ operator.

The force on detector 1 is:

$$d^2\vec{F}_{d1} = dq_{d1}\, d\vec{E} = -I_d\, \frac{dl_d}{v_{d1}}\left[-\nabla(d\varphi_1) - \frac{\partial\left(d\vec{A}_1\right)}{\partial t^*}\right] = I_d\, \frac{dl_d}{v_{d1}}\left[\nabla(d\varphi_1) + \frac{\partial\left(d\vec{A}_1\right)}{\partial t^{SL}} + (\vec{v}_{d1}\cdot\nabla)\left(d\vec{A}_1\right)\right]$$

$$d^2\vec{F}_{d1} = I_d\, \frac{dl_d}{v_{d1}}\left[\nabla(d\varphi_1) + (\vec{v}_{d1}\cdot\nabla)\left(d\vec{A}_1\right)\right]$$

$$d^2\vec{F}_{d1} = I_d\, \frac{dl_d}{v_{d1}}\left[\nabla\left(-\frac{\mu_0}{8\pi}\,\frac{I_S\, d\vec{l}_S\cdot(2\vec{v}_{d1}-\vec{v}_-)}{|\vec{r}_d-\vec{r}_S|}\right) + (\vec{v}_{d1}\cdot\nabla)\left(\frac{\mu_0\, I_S}{4\pi}\,\frac{d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right)\right] =$$
$$= \frac{\mu_0\, I_d\, I_S}{4\pi}\,\frac{dl_d}{v_{d1}}\left[-\nabla\left(\frac{d\vec{l}_S\cdot(2\vec{v}_{d1}-\vec{v}_-)}{2|\vec{r}_d-\vec{r}_S|}\right) + (\vec{v}_{d1}\cdot\nabla)\left(\frac{d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right)\right] =$$
$$= \frac{\mu_0\, I_d\, I_S}{4\pi}\,\frac{dl_d}{v_{d1}}\left[-\nabla\left(\frac{d\vec{l}_S\cdot\vec{v}_{d1}}{|\vec{r}_d-\vec{r}_S|}\right) + \nabla\left(\frac{d\vec{l}_S\cdot\vec{v}_-}{2|\vec{r}_d-\vec{r}_S|}\right) + (\vec{v}_{d1}\cdot\nabla)\left(\frac{d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right)\right] = \qquad ([a])$$
$$= \frac{\mu_0\, I_d\, I_S}{4\pi}\,\frac{dl_d}{v_{d1}}\left\{-\nabla\left(\frac{\vec{v}_{d1}\cdot d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right) + \nabla\left(\frac{\vec{v}_-\cdot d\vec{l}_S}{2|\vec{r}_d-\vec{r}_S|}\right) + \nabla\left(\frac{\vec{v}_{d1}\cdot d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right) - \vec{v}_{d1}\wedge\left[\nabla\wedge\left(\frac{d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right)\right]\right\} =$$
$$= \frac{\mu_0\, I_d\, I_S}{4\pi}\,\frac{dl_d}{v_{d1}}\left\{\frac{1}{2}\vec{v}_-\cdot d\vec{l}_S\nabla\left(\frac{1}{|\vec{r}_d-\vec{r}_S|}\right) - \vec{v}_{d1}\wedge\left[\nabla\wedge\left(\frac{d\vec{l}_S}{|\vec{r}_d-\vec{r}_S|}\right)\right]\right\} =$$
$$= \frac{\mu_0\, I_d\, I_S}{4\pi}\,\frac{dl_d}{v_{d1}}\left\{\frac{1}{2}\vec{v}_-\cdot d\vec{l}_S\left(-\frac{\vec{u}_{dS}}{|\vec{r}_{dS}|^2}\right) - \vec{v}_{d1}\wedge\left[-\frac{\vec{u}_{dS}}{|\vec{r}_{dS}|^2}\wedge d\vec{l}_S\right]\right\} =$$
$$= \frac{\mu_0\, I_S\, I_d}{4\pi\, r_{dS}^{\ 2}}\,\frac{dl_d}{v_{d1}}\left\{-\frac{1}{2}\vec{v}_-\cdot d\vec{l}_S\,\vec{u}_{dS} + \vec{v}_{d1}\wedge\left(\vec{u}_{dS}\wedge d\vec{l}_S\right)\right\} =$$
$$= \frac{\mu_0\, I_S\, I_d}{4\pi\, r_{dS}^{\ 2}}\left\{-\frac{dl_d}{2v_{d1}}\,\vec{v}_-\cdot d\vec{l}_S\,\vec{u}_{dS} - d\vec{l}_d\wedge\left(\vec{u}_{dS}\wedge d\vec{l}_S\right)\right\}$$

Since the moving charge is negative, $\vec{v}_d$ is oriented in opposition to the positive direction of the current.

So: $$\frac{dl_d}{v_{d1}}\vec{v}_{d1} = -d\vec{l}_d$$

$$d^2\vec{F}_{d1} = \frac{\mu_0\, I_S\, I_d}{4\pi\, r_{dS}^{\ 2}}\left\{-\frac{dl_d}{2v_{d1}}\,\vec{v}_-\cdot d\vec{l}_S\,\vec{u}_{dS} + d\vec{l}_d\wedge\left(d\vec{l}_S\wedge\vec{u}_{dS}\right)\right\} \qquad (33)$$

[a] Using the vector identity: $(\vec{v}_d\cdot\nabla)\vec{C} = \nabla\left(\vec{v}_d\cdot\vec{C}\right) - \vec{v}_d\wedge\left(\nabla\wedge\vec{C}\right)$

The force on detector 2, remembering $dq_{d+} = -dq_{d-} = I_d \dfrac{dl_d}{v_{d1}}$, is:

$$d^2\vec{F}_{d2} = dq_{d2}\ d\vec{E} = I_d\ \frac{dl_d}{v_{d1}}\left[-\nabla(d\varphi_2) - \frac{\partial\left(d\vec{A}_2\right)}{\partial t^{SL}}\right] = -I_d\ \frac{dl_d}{v_{d1}}\nabla(d\varphi_2)$$

$$d^2\vec{F}_{d2} = -I_d\ \frac{dl_d}{v_{d1}}\nabla\left(\frac{\mu_0}{8\,\pi}\ \frac{I_S\ d\vec{l}_S \cdot \vec{v}_-}{|\,\vec{r}_d - \vec{r}_S\,|}\right) = -\frac{\mu_0\ I_d\ I_S}{4\,\pi}\ \frac{dl_d}{v_{d1}}\ \frac{1}{2}\nabla\left(\frac{d\vec{l}_S \cdot \vec{v}_-}{|\,\vec{r}_d - \vec{r}_S\,|}\right) =$$

$$= -\frac{\mu_0\ I_d\ I_S}{4\,\pi}\ \frac{dl_d}{v_{d1}}\vec{v}_- \cdot d\vec{l}_S\ \frac{1}{2}\nabla\left(\frac{1}{|\,\vec{r}_d - \vec{r}_S\,|}\right) = -\frac{\mu_0\ I_d\ I_S}{4\,\pi}\ \frac{dl_d}{v_{d1}}\vec{v}_- \cdot d\vec{l}_S\ \frac{1}{2}\left(-\frac{\vec{u}_{dS}}{|\,\vec{r}_{dS}|^2}\right)$$

So:

$$d^2\vec{F}_{d2} = \frac{\mu_0\ I_S\ I_d}{4\pi\, r_{dS}^{\ 2}}\left(\frac{dl_d}{2\,v_{d1}}\vec{v}_- \cdot d\vec{l}_S\right)\vec{u}_{dS} \tag{34}$$

The total force acting on the circuit portion dl_d, then, is:

$$d^2\vec{F}_d = d^2\vec{F}_{d1} + d^2\vec{F}_{d2} = \frac{\mu_0\ I_S\ I_d}{4\pi\, r_{dS}^{\ 2}}\left\{-\frac{dl_d}{2\,v_d}\vec{v}_- \cdot d\vec{l}_S\ \vec{u}_{dS} + d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right) + \frac{dl_d}{2\,v_d}\vec{v}_- \cdot d\vec{l}_S\ \vec{u}_{dS}\right\}$$

$$d^2\vec{F}_d = \frac{\mu_0\ I_S\ I_d}{4\pi\, r_{dS}^{\ 2}}\ d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right) \tag{35}$$

Expression (35) coincides with the expression of Grassmann's force:

$$d^2\vec{F}_d = d^2\vec{F}_d^{\,G}$$

Maxwell's theory and the modified version of Hertz-Phipps's theory predict the same forces between circuits in stationary conditions.
Instead, the two theories would make different predictions if one replaces the neutralized current of circuit Γ_d with an electron beam in vacuum.
If, however, one considers the case of a single point charge detector Q_d, motionless in $\boldsymbol{S_L}$, the new theory predicts that the element of the current $I_S\ d\vec{l}_S$ should exert a force on it equal to:

$$d\vec{F}_d = Q_d\ d\vec{E} = Q_d\left[-\nabla(d\varphi_2) - \frac{\partial\left(d\vec{A}_2\right)}{\partial t^{SL}}\right] = -Q_d\ \nabla(d\varphi_2)$$

$$d\vec{F}_d = -Q_d\nabla\left(\frac{\mu_0}{8\,\pi}\ \frac{I_S\ d\vec{l}_S \cdot \vec{v}_-}{|\,\vec{r}_d - \vec{r}_S\,|}\right) = \frac{\mu_0\ I_S}{8\pi\, r_{dS}^{\ 2}}\ Q_d\ v_-\ dl_S\ \vec{u}_{dS}$$

A second proposal for amendment

As already mentioned in the introductory part of this work, experimental results suggest the introduction of some kind of instantaneous interaction into the description of the electromagnetic phenomena. The existence of instantaneous interactions appears conceptually compatible with the non-locality introduced by using the total time derivative.
It is necessary to further modify the theory – described by the equations of fields (1Hm), (2H), (3H) and (4H), by the definition of force (5H) and by the descriptions of sources (25), (26), (27), (28), (29) and (30) – which only provides solutions with finite propagation speed.
The new equations are:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \qquad \text{(a1)}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 \qquad \text{(a2)}$$

$$\nabla \cdot \vec{E}_1 = \frac{\rho_m}{\varepsilon_0} \qquad \text{(a3)}$$

$$\nabla \wedge \vec{E}_1 = 0 \qquad \text{(a4)}$$

$$\nabla \cdot \vec{E}_2 = 0 \qquad \text{(a5)}$$

$$\nabla \wedge \vec{E}_2 = -\frac{d\vec{B}}{d\,t} \qquad \text{(a6)}$$

$$\nabla \cdot \vec{B}_1 = 0 \qquad \text{(a7)}$$

$$\nabla \wedge \vec{B}_1 = \mu_0 \vec{J}_m \qquad \text{(a8)}$$

$$\nabla \cdot \vec{B}_2 = 0 \qquad \text{(a9)}$$

$$\nabla \wedge \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t} \qquad \text{(a10)}$$

with:

$$\vec{J}_m^+ = \rho_+ \left(\vec{v}_+ - \vec{v}_d\right) \qquad \text{(a11)}$$

$$\vec{J}_m^- = \rho_- \left(\vec{v}_- - \vec{v}_d\right) \qquad \text{(a12)}$$

$$\vec{J}_m = \vec{J}_m^+ + \vec{J}_m^- \qquad \text{(a13)}$$

$$\rho_m^+ = \rho_+ + \frac{1}{2c^2}\vec{J}_m^+ \cdot \left(\vec{v}_+ - \vec{v}_d\right) \qquad \text{(a14)}$$

$$\rho_m^- = \rho_- + \frac{1}{2c^2}\vec{J}_m^- \cdot \left(\vec{v}_- - \vec{v}_d\right) \qquad \text{(a15)}$$

$$\rho_m = \rho_m^+ + \rho_m^- \qquad \text{(a16)}$$

Continuity equation

Differentiating the sum of (a3) and (a5) with respect to time:

$$\frac{d}{d\,t}\left(\nabla\cdot\vec{E}\right)=\frac{d}{d\,t}\left[\nabla\cdot\left(\vec{E}_1+\vec{E}_2\right)\right]=\nabla\cdot\frac{d\,\vec{E}_1}{d\,t}=\frac{1}{\varepsilon_0}\frac{d\,\rho_m}{d\,t} \qquad (\lozenge)$$

Calculating the divergence of the sum of (a8) and (a10):

$$\nabla\cdot\left(\nabla\wedge\vec{B}\right)=\nabla\cdot\left(\nabla\wedge\left(\vec{B}_1+\vec{B}_2\right)\right)=0=\mu_0\nabla\cdot\vec{J}_m+\mu_0\varepsilon_0\nabla\cdot\frac{d\,\vec{E}}{d\,t}$$ so:

$$\mu_0\nabla\cdot\vec{J}_m+\mu_0\varepsilon_0\nabla\cdot\frac{d\,\vec{E}_1}{d\,t}=0 \qquad (\blacklozenge)$$

Introducing (◊) into (♦): $\mu_0\nabla\cdot\vec{J}_m+\mu_0\varepsilon_0\dfrac{1}{\varepsilon_0}\dfrac{d\,\rho_m}{d\,t}=0$ so:

$$\nabla\cdot\vec{J}_m+\frac{d\,\rho_m}{d\,t}=0 \qquad \text{(a17)}$$

Poisson’s equations and wave equations

$$\nabla^2\vec{B}_1=-\mu_0\nabla\wedge\vec{J}_m$$ (a18) instantaneous

$$\nabla^2\vec{B}_2-\mu_0\varepsilon_0\frac{d^2\vec{B}_2}{dt^2}=\mu_0\varepsilon_0\frac{d^2\vec{B}_1}{dt^2}$$ (a19) propagative

$$\nabla^2\vec{E}_1=\frac{1}{\varepsilon_0}\nabla\rho_m$$ (a20) instantaneous

$$\nabla^2\vec{E}_2-\mu_0\varepsilon_0\frac{d^2\vec{E}_2}{dt^2}=\mu_0\frac{d\,\vec{J}_m}{dt}+\mu_0\varepsilon_0\frac{d^2\vec{E}_1}{dt^2}$$ (a21) propagative

Proof in Appendix - H.

$\vec{E}_1$ and $\vec{B}_1$ can be called the instantaneous components of the electric and magnetic fields.

$\vec{E}_2$ and $\vec{B}_2$ can be called the induced components of the electric and magnetic fields.

Absence of sources

Considering the fields' instantaneous components, the condition of "absence of sources" is true in every point of the space in which the charge density and current density are equal to zero.
In those regions:

$$\nabla^2 \vec{B}_1 = 0 \qquad \nabla^2 \vec{E}_1 = 0$$

Instead, considering the induced components of the fields, in every point in which the charge density and current density are equal to zero, the wave equations reduce to:

$$\nabla^2 \vec{B}_2 - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}_2}{dt^2} = \mu_0 \varepsilon_0 \frac{d^2 \vec{B}_1}{dt^2} \qquad \nabla^2 \vec{E}_2 - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}_2}{dt^2} = \mu_0 \varepsilon_0 \frac{d^2 \vec{E}_1}{dt^2}$$

The forcing terms are second order total time derivatives of the instantaneous fields and they are continuous functions which extend beyond the regions occupied by the charges.
So the wave equations do not become homogeneous immediately outside these regions, but only at distances that are large enough from the charges to render the contributions of these continuous functions negligible.
Thus, the radiation sources would not be point-like in the same way as the elementary charges which constitute the sources, but would be extended around the charges and described by continuous distributions, decreasing according to $1/r^2$.
At a great distance from the sources, the equations are approximated by their homogeneous forms:

$$\nabla^2 \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d^2 \vec{B}_2}{dt^2} \qquad \nabla^2 \vec{E}_2 = \mu_0 \varepsilon_0 \frac{d^2 \vec{E}_2}{dt^2}$$

which describe the far field propagation.

The instantaneous equations (a18) and (a20) remain unaltered in any IRF.

The propagation equations (a19) and (a21), for non-accelerated detectors, in $\boldsymbol{S_{d-I}}$, take the form:

$$\nabla^2 \vec{E}_2 - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_2}{\partial t^2} = \mu_0 \frac{\partial \vec{J}_m}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}_1}{\partial t^2}$$

$$\nabla^2 \vec{B}_2 - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}_2}{\partial t^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}_1}{\partial t^2}$$

With respect to the wave equation at a great distance from the source, the procedures followed using the previous version of the theory are still applicable.
Therefore, regarding aberration and Doppler effects, the relations (eo6), (eo7), (eo8), (eo9), (eo10), (eo11) and (eo12) are still valid.

Introduction of potentials

$\nabla \wedge \vec{E}_1 = 0$ means that $\vec{E}_1$ is conservative, so it can be expressed as:

$$\vec{E}_1 = -\nabla \varphi_1 \qquad \text{(a22)}$$

$$\nabla \cdot \vec{E}_1 = \frac{\rho_m}{\varepsilon_0} \quad \Rightarrow \quad \nabla \cdot \left(-\nabla \varphi_1\right) = \frac{\rho_m}{\varepsilon_0} \quad \Rightarrow \quad \nabla^2 \varphi_1 = -\frac{\rho_m}{\varepsilon_0}$$

$\nabla \cdot \vec{B}_1 = 0$ and $\nabla \cdot \vec{B}_2 = 0$ imply that $\vec{B}_1$ and $\vec{B}_2$ can be expressed by means of potential vectors:

$$\vec{B}_1 = \nabla \wedge \vec{A}_1 \qquad \text{(a23)}$$

$$\vec{B}_2 = \nabla \wedge \vec{A}_2 \qquad \text{(a24)}$$

So, calling: $\vec{A} = \vec{A}_1 + \vec{A}_2$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \nabla \wedge \vec{A}_1 + \nabla \wedge \vec{A}_1 = \nabla \wedge \left(\vec{A}_1 + \vec{A}_2\right) = \nabla \wedge \vec{A} \qquad \text{(a25)}$$

Introducing (a25) into (a6):

$$\nabla \wedge \vec{E}_2 = -\frac{d\vec{B}}{d\,t} = -\frac{d}{dt}\left(\nabla \wedge \vec{A}\right) = -\nabla \wedge \frac{d\vec{A}}{dt} \quad \Rightarrow \quad \nabla \wedge \left(\vec{E}_2 + \frac{d\vec{A}}{dt}\right) = 0$$

It means that $\vec{E}_2 + \frac{d\vec{A}}{dt}$ is conservative, so it can be expressed as: $\vec{E}_2 + \frac{d\vec{A}}{dt} = -\nabla \varphi_2$. So:

$$\vec{E}_2 = -\nabla \varphi_2 - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt} \qquad \text{(a26)}$$

Equations (a23) and (a24) leave the divergence of $\vec{A}_1$ and $\vec{A}_2$ indefinite.
This implies the possibility of adding gradients of any scalar function (nameable gauges) to potentials $\vec{A}_1$ and $\vec{A}_2$, without altering the corresponding magnetic fields $\vec{B}_1$ and $\vec{B}_2$.

Effect of a change in the gauge of $\vec{A}$

By applying a gauge variation: $\vec{A}_1^{\wedge} = \vec{A}_1 + \nabla \psi_1$ $\qquad$ $\vec{A}_2^{\wedge} = \vec{A}_2 + \nabla \psi_2$
So: $\vec{A}^{\wedge} = \vec{A} + \nabla \psi_1 + \nabla \psi_2$
and introducing this expression into equation (a6), it follows that:

$$\nabla \wedge \vec{E}_2 = -\frac{d\vec{B}}{d\,t} = -\frac{d}{dt}\left(\nabla \wedge \vec{A}^{\wedge}\right) = -\nabla \wedge \frac{d\vec{A}^{\wedge}}{dt}$$

$$\nabla \wedge \left(\vec{E}_2 + \frac{d\vec{A}^{\wedge}}{dt} \right) = 0 \qquad \Rightarrow \qquad \vec{E}_2 + \frac{d\vec{A}^{\wedge}}{dt} = -\nabla \varphi_2{}^{\wedge}$$

So:

$$\vec{E}_2 = -\nabla \varphi_2{}^{\wedge} - \frac{d\vec{A}^{\wedge}}{dt} = -\nabla \varphi_2{}^{\wedge} - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt} - \frac{d}{dt}\nabla \psi_1 - \frac{d}{dt}\nabla \psi_2 = -\nabla \left(\varphi_2{}^{\wedge} + \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt} \right) - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt}$$

From the comparison of the previous expression with (a26), it follows that:

$$\nabla \left(\varphi_2{}^{\wedge} + \frac{d\psi_1}{dt} + \frac{d\psi_2}{dt} \right) = \nabla \varphi_2$$

Consequently: $\qquad \varphi_2{}^{\wedge} = \varphi_2 - \frac{d\psi_1}{dt} - \frac{d\psi_2}{dt}$

which shows the effect of the choice of the vector potentials' gauge on the scalar potential φ_2.

Gauge invariance

Gauge invariance applies. To verify this assertion, it is sufficient to express the electric field acting on a detector according to the new equations expressed in terms of potential, and apply a change of the gauge (details in Appendix - I).

Equations expressed through potentials

Introducing (a22) into (a3):

$$\nabla \cdot \left(-\nabla \varphi_1 \right) = \frac{\rho_m}{\varepsilon_0} \qquad \text{so} \qquad \nabla^2 \varphi_1 = -\frac{\rho_m}{\varepsilon_0}$$

Introducing (a26) into (a5):

$$\nabla \cdot \left(-\nabla \varphi_2 - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt} \right) = 0 \qquad \text{so} \qquad \nabla^2 \varphi_2 + \frac{d}{dt}\nabla \cdot \vec{A}_1 + \frac{d}{dt}\nabla \cdot \vec{A}_2 = 0$$

Introducing (a23) into (a8):

$$\nabla \wedge \left(\nabla \wedge \vec{A}_1 \right) = \mu_0 \vec{J}_m \qquad \nabla \left(\nabla \cdot \vec{A}_1 \right) - \nabla^2 \vec{A}_1 = \mu_0 \vec{J}_m \qquad \nabla^2 \vec{A}_1 - \nabla \left(\nabla \cdot \vec{A}_1 \right) = -\mu_0 \vec{J}_m$$

Introducing (a24) into (a10):

$$\nabla \wedge \left(\nabla \wedge \vec{A}_2 \right) = \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t} \qquad \text{so}$$

$$\nabla \left(\nabla \cdot \vec{A}_2 \right) - \nabla^2 \vec{A}_2 = \mu_0 \varepsilon_0 \frac{d}{d\,t} \left(-\nabla \varphi_1 - \nabla \varphi_2 - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt} \right)$$

$$\nabla^2 \vec{A}_2 - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}_2}{dt^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{A}_1}{dt^2} - \nabla\left(\nabla \cdot \vec{A}_2 + \mu_0 \varepsilon_0 \frac{d\varphi_1}{dt} + \mu_0 \varepsilon_0 \frac{d\varphi_2}{dt}\right) = 0$$

In this way one arrives at a system of four equations:

$$\begin{cases} \nabla^2 \varphi_1 = -\dfrac{\rho_m}{\varepsilon_0} \\ \nabla^2 \varphi_2 + \dfrac{d}{dt} \nabla \cdot \vec{A}_1 + \dfrac{d}{dt} \nabla \cdot \vec{A}_2 = 0 \\ \nabla^2 \vec{A}_1 - \nabla\left(\nabla \cdot \vec{A}_1\right) = -\mu_0 \vec{J}_m \\ \nabla^2 \vec{A}_2 - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}_2}{dt^2} - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}_1}{dt^2} - \nabla\left(\nabla \cdot \vec{A}_2 + \mu_0 \varepsilon_0 \dfrac{d\varphi_1}{dt} + \mu_0 \varepsilon_0 \dfrac{d\varphi_2}{dt}\right) = 0 \end{cases} \tag{a27}$$

The system can be rewritten in a simpler form by adopting the following choices of gauge:

$$\nabla \cdot \vec{A}_1 = 0 \tag{v1}$$

$$\nabla \cdot \vec{A}_2 = -\mu_0 \varepsilon_0 \frac{d\varphi_1}{dt} - \mu_0 \varepsilon_0 \frac{d\varphi_2}{dt} \tag{v2}$$

In such a case the system becomes:

$$\begin{cases} \nabla^2 \varphi_1 = -\dfrac{\rho_m}{\varepsilon_0} \\ \nabla^2 \varphi_2 - \mu_0 \varepsilon_0 \dfrac{d^2 \varphi_2}{dt^2} = \mu_0 \varepsilon_0 \dfrac{d^2 \varphi_1}{dt^2} \\ \nabla^2 \vec{A}_1 = -\mu_0 \vec{J}_m \\ \nabla^2 \vec{A}_2 - \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}_2}{dt^2} = \mu_0 \varepsilon_0 \dfrac{d^2 \vec{A}_1}{dt^2} \end{cases} \tag{a28}$$

which must be completed by the force law expressed in terms of potentials:

$$\vec{F} = q\vec{E} = q\left(\vec{E}_1 + \vec{E}_2\right) = q\left(-\nabla \varphi_1 - \nabla \varphi_2 - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt}\right) = q\left(-\nabla \varphi - \frac{d\vec{A}}{dt}\right) \tag{a29}$$

or:

$$\vec{F} = q\left(-\nabla \varphi_1 - \nabla \varphi_2 - \frac{\partial \vec{A}_1}{\partial t} - \nabla\left(\vec{v}_d \cdot \vec{A}_1\right) + \vec{v}_d \wedge \left(\nabla \wedge \vec{A}_1\right) - \frac{\partial \vec{A}_2}{\partial t} - \nabla\left(\vec{v}_d \cdot \vec{A}_2\right) + \vec{v}_d \wedge \left(\nabla \wedge \vec{A}_2\right)\right)$$

$$\vec{F} = q\left(-\nabla \varphi_1 - \nabla \varphi_2 - \frac{\partial \vec{A}_1}{\partial t} - \frac{\partial \vec{A}_2}{\partial t}\right) + q\, \vec{v}_d \wedge \left[\nabla \wedge \left(\vec{A}_1 + \vec{A}_2\right)\right] - q\, \nabla\left[\vec{v}_d \cdot \left(\vec{A}_1 + \vec{A}_2\right)\right] \tag{a30}$$

Solutions of system (a28)

For a non-accelerated detector, in $\boldsymbol{S_{d\text{-}I}}$, system (a28) becomes:

$$\begin{cases} \nabla^2 \varphi_1 = -\dfrac{\rho_m}{\varepsilon_0} \\ \nabla^2 \varphi_2 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \varphi_2}{\partial t^2} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \varphi_1}{\partial t^2} \\ \nabla^2 \vec{A}_1 = -\mu_0 \vec{J}_m \\ \nabla^2 \vec{A}_2 - \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}_2}{\partial t^2} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}_1}{\partial t^2} \end{cases} \tag{a31}$$

The instantaneous equations, which retain the same form in any IRF, have the instantaneous solutions:

$$\varphi_1(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{all\ space \\ at\, time\,(t)}} \frac{\rho_m(\vec{r}\,',t)}{|\vec{r} - \vec{r}\,'|} dV' \tag{a32}$$

$$\vec{A}_1(\vec{r},t) = \frac{\mu_0}{4\pi} \int_{\substack{all\ space \\ at\, time\,(t)}} \frac{\vec{J}_m(\vec{r}\,',t)}{|\vec{r} - \vec{r}\,'|} dV' \tag{a33}$$

The wave equations with non-local forcing terms have solutions with propagation speed equal to ***c***.
The forcing terms are not concentrated on charges but distributed in the space around them.
They are continuous functions, decreasing according to $1/r$.
The solutions are:

$$\varphi_2(\vec{r},t) = -\frac{\mu_0 \varepsilon_0}{4\pi} \int \frac{\dfrac{\partial^2 \varphi_1\left(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c\right)}{\partial t^2}}{|\vec{r} - \vec{r}\,'|} dV' = -\frac{\mu_0 \varepsilon_0}{4\pi} \int \frac{\dfrac{\partial^2 \varphi_1[\vec{r}\,'(t_R), t_R]}{\partial t^2}}{|\vec{r}(t) - \vec{r}\,'(t_R)|} dV' \tag{a34}$$

$$\vec{A}_2(\vec{r},t) = -\frac{\mu_0 \varepsilon_0}{4\pi} \int \frac{\dfrac{\partial^2 \vec{A}_1\left(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c\right)}{\partial t^2}}{|\vec{r} - \vec{r}\,'|} dV' = -\frac{\mu_0 \varepsilon_0}{4\pi} \int \frac{\dfrac{\partial^2 \vec{A}_1[\vec{r}\,'(t_R), t_R]}{\partial t^2}}{|\vec{r}(t) - \vec{r}\,'(t_R)|} dV' \tag{a35}$$

t_R is implicitly defined by: $t_R = t - |\vec{r} - \vec{r}\,'(t_R)|/c$

Some additional considerations on the new theory

The relativistic constraint which requires the speed limit ***c*** for the propagation of energy and information is consistent with the idea of locating the energy in the field.
It is assumed that the electromagnetic energy has the property of being placed with a certain density in the space between the charges.

$u = \frac{1}{2}\varepsilon_0 \, \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B}$ is intended as energy per unit volume;

$\vec{S} = \frac{1}{\mu_0} \vec{E} \wedge \vec{B}$ is intended as energy transferred per unit time per unit cross-sectional area;

$\vec{g} = \frac{1}{c^2} \vec{S}$ is intended as momentum per unit volume.

This interpretation shows its usefulness in different situations, but it is less than satisfactory because it is irreconcilable with a point-like description of the electric charges.
In fact, from the above definition of electromagnetic energy density, the energy associated with the field of a point charge has an infinite value.
Considering the charge dynamics, the need to exclude the (infinite) contribution of such energy from gravitational or inertial properties (i.e. the mass) of the charge shows the limits of consistency in the preceding assumptions.

In the new theory, assuming the existence of instantaneous interactions (action at a distance) and the fields' dependence on the detector's state of motion, the idea of spatially placeable electromagnetic energy seems to loose further credibility.
It seems more natural to interpret the energy as a relational property associated with the whole system of electric charges.
Only the phenomena of absorption and emission would have a location in space.

Possible extensions of the theory

Higher-order theory according to Phipps

Phipps proposed to extend the applicability of his theory beyond the limit of the first order (calling this version Neo-Hertzian), assuming the invariance of the following differentials:

1: $d\tau$ $$d\tau^2 = dt^2 - \frac{dr^2}{c^2}$$ with: $$dr^2 = dx^2 + dy^2 + dz^2$$

2: δr

In the first case, the differentials $d\tau$ and dr refer to coordinate increments associated with pairs of successive events, belonging to the trajectory of a single particle.
dt is the time interval between such events, measured by a clock at rest in the chosen IRF, called ***S***. It is therefore a differential of IRF coordinate time.
$d\tau$ is the time interval between such events, measured by a clock at rest with respect to the particle.
dr is the spatial distance between such events, evaluated in ***S***.
The invariance of $d\tau$ must be intended in the sense that all observers in whatever state of motion will agree on its numerical value.
The value of proper time τ may be (ideally) read on a clock co-moving with the particle.
d denotes a separation between events on the same worldline.

In the second case, δr indicates a spatial separation between points belonging to an extended structure (like a standard meter), which means a distance measured through rigid bodies at rest in an arbitrary IRF.
δ denotes a separation between events on different worldlines at the same coordinate time.

From the definition of $d\tau^2$ (considering the positive root only):

$$d\tau = \sqrt{dt^2 - \frac{dr^2}{c^2}} \qquad \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\frac{dr^2}{dt^2}} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}$$

The proper time used in electromagnetic theory is the proper time of the detector: $\tau = \tau_d$.

The new field equations are obtained by substituting the non-invariant time t with the assumed invariant time $\tau = \tau_d$.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \wedge \vec{E} = -\frac{d\vec{B}}{d\tau} \qquad \nabla \wedge \vec{B} = \mu_0 \vec{J}_m + \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\tau} \qquad \vec{J}_m = \vec{J} - \rho \vec{V}_d$$

Problematic nature of such choices

The adoption of a proper time in accordance with Phipps's assumptions leads to paradoxical predictions in respect to the assumption that absolute simultaneity exists between spatially separated events.

To clarify this statement, consider two generic inertial detectors ***d1*** and ***d2***, i.e. two classical particles with a uniform state of motion with respect to any IRF.

Let ***S1*** and ***S2*** be the IRFs respectively co-moving with ***d1*** and ***d2***.
Therefore, speaking of the detectors' proper times is equivalent to speaking of the coordinate time of ***S1*** and ***S2***.

To simplify the situation, assume that there is an intersection of the worldlines of the detectors ***d1*** and ***d2***.
This means that the trajectory of ***d1*** intersects the trajectory of ***d2***, and that the positions of the two particles coincide in the intersection.
Let us use this event of coincidence to make a univocal synchronization of the clocks carried by ***d1*** and ***d2***.
Consistent with Phipps's definition of proper time, one can say the following.

In ***S1***, the instant $\widehat{t}_1$ of its coordinate time must correspond to value $\widehat{t}_1$, read on the clock carried by ***d1***, and to value $\widehat{t}_2$ read on the clock carried by ***d2***, with:

$$\widehat{t}_2 = \frac{\widehat{t}}{\gamma_{d2\ in\ S1}} \qquad \gamma_{d2\ in\ S1} = 1 \Big/ \sqrt{1-\left(v_{d2\ in\ S1}\right)^2 \big/ c^2}$$

$$v_{d2\ in\ S1} \neq 0 \qquad \Rightarrow \qquad \gamma_{d2\ in\ S1} > 1 \qquad \Rightarrow \qquad \widehat{t}_2 < \widehat{t}_1$$

In ***S2***, the instant $\widehat{t}_2$ of its coordinate time must correspond to value $\widehat{t}_2$, read on the clock carried by ***d2***, and to value t_1^* read on the clock carried by ***d1***, with:

$$t_1^* = \frac{\widehat{t}_2}{\gamma_{d1\ in\ S2}} \qquad \gamma_{d1\ in\ S2} = 1 \Big/ \sqrt{1-\left(v_{d1\ in\ S2}\right)^2 \big/ c^2}$$

$$v_{d1\ in\ S12} \neq 0 \qquad \Rightarrow \qquad \gamma_{d1\ in\ S2} > 1 \qquad \Rightarrow \qquad t_1^* < \widehat{t}_2 < \widehat{t}_1 .$$

We may therefore conclude that, in the event of "the clock carried by **d2** marking the time $\widehat{t}_2$", the two inertial observers cannot agree on the time marked by the clock carried by ***d1***.

Since γ is a monotonically increasing function of the module of the detector-inertial observer's relative velocity, the proper time of any detector in motion with respect to the adopted IRF should flow slower than coordinate time.
And it must be true in every SRI, given the arbitrariness of its choice.
The adoption of these constraints rules out the possibility of a description of space and time which admits absolute simultaneity between distant events.

Searching for alternatives

The simplest alternative approach, which keeps the intake of invariance of the two differentials δr and $d\tau$, is the following.
It postulates the existence of a privileged reference frame ***S₀***, in respect to which the time flow is the maximum. Namely the beating of a clock at rest in ***S₀*** would have the maximum frequency.
The most natural candidate for such a privileged IRF would be one in which the dipole component of the cosmic background radiation vanishes.

Let t_0 be the coordinate time of $\boldsymbol{S_0}$.
Given a generic particle ***d***, identified in $\boldsymbol{S_0}$ by the vector $\vec{r}_{0\,d}\left(t_0\right)$, one can assume the invariance (in the sense that all observers will agree on its value) of:

$$d\tau_d = \sqrt{dt_0^{\ 2} - \frac{d\,r_{0\,d}^{\ \ 2}}{c^2}}$$

intended as the particle's proper time differential. It follows that:

$$\frac{d\tau_d}{dt_0} = \sqrt{1 - \frac{1}{c^2}\frac{d\,r_{0\,d}^{\ \ 2}}{dt_0^{\ 2}}} = \sqrt{1 - \frac{V_d^{\ 2}}{c^2}} = \frac{1}{\gamma_d}$$

V_d is the particle velocity in $\boldsymbol{S_0}$ (its absolute velocity).

$$d\tau_d = \sqrt{1 - \frac{V_d^{\ 2}}{c^2}}\; dt_0 = \frac{dt_0}{\gamma_d} \qquad\qquad \tau_d = \int \sqrt{1 - \frac{V_d^{\ 2}}{c^2}}\; dt_0$$

Using the coincidence of the proper time of a uniform moving particle and the coordinate time of the co-moving IRF, new transformation rules between IRF can be obtained.
Consider a generic IRF ***S***.
Let $\left(\vec{r}, t\right)$ and $\left(\vec{r}_0, t_0\right)$ be the spatial and temporal coordinates of the same event in ***S*** and $\boldsymbol{S_0}$.
Let $\vec{V}_S$ be the velocity of ***S*** in $\boldsymbol{S_0}$.
Let us choose the coordinate systems so that the origins of the axes in the two IRFs coincide at the chosen zero time.
The transformations can be expressed vectorially by:

$$\vec{r} = \vec{r}_0 - \vec{V}_S\; t_0 \qquad\qquad t = \frac{t_0}{\gamma_S} \qquad\qquad \text{(T1)}$$

The inverse transformations are:

$$\vec{r}_0 = \vec{r} + \vec{V}_S\; \gamma\, t \qquad\qquad t_0 = \gamma_S\; t \qquad\qquad \text{(T2)}$$

with: $$\gamma_S = 1\Big/\sqrt{1 - \frac{V_S^{\ 2}}{c^2}}$$

The origin ***O*** of the ***S***-axes is identified by the vectors:

$$\vec{r}(O) = 0, \quad \text{in } \boldsymbol{S} \qquad\qquad \vec{r}_0(O) = \vec{V}_S\ \gamma_S\ t = \vec{V}_S\ t_0, \quad \text{in } \boldsymbol{S_0}$$

Therefore, in $\boldsymbol{S_0}$ ***O*** is described as moving with velocity: $\vec{V}(O) = \dfrac{d\,\vec{r}_0(O)}{d\,t_0} = \vec{V}_S\,.$

The origin $\boldsymbol{O_0}$ of the $\boldsymbol{S_0}$-axes is identified by the vectors:

$$\vec{r}_0(O_0) = 0, \quad \text{in } \boldsymbol{S_0}\text{:} \qquad\qquad \vec{r}(O_0) = -\vec{V}_S\ t_0 = -\vec{V}_S\ \gamma_S\ t, \quad \text{in } \boldsymbol{S}$$

Therefore, in ***S*** $\boldsymbol{O_0}$ is described as moving with velocity: $\vec{v}(O_0) = \dfrac{d\,\vec{r}(O_0)}{d\,t} = -\vec{V}_S\ \gamma_S$

The speed of $\boldsymbol{O_0}$ evaluated by ***O*** is greater than the speed of ***O*** evaluated by $\boldsymbol{O_0}$.
Absolute simultaneity is recovered with these transformations, but the anti-symmetric property of the relative velocity is lost.
(T1) and (T2) allow linking the descriptions of the motion of a generic particle ***d*** in $\boldsymbol{S_0}$ and in ***S***.

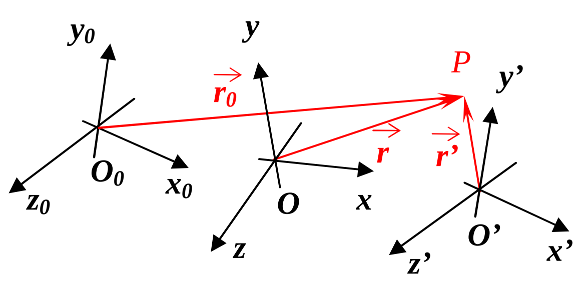
$$\vec{r}_d = \vec{r}_{0\,d} - \vec{V}_S\ t_0 = \vec{r}_{0\,d} - \vec{V}_S\ \gamma_S\ t$$

$$\vec{v}_d = \frac{d\,\vec{r}_d}{dt} = \frac{d\,\vec{r}_{0\,d}}{dt} - \vec{V}_S\ \gamma_S = \frac{d\,\vec{r}_{0\,d}}{dt_o}\frac{dt_o}{dt} - \vec{V}_S\ \gamma_S = \left(\vec{V}_d - \vec{V}_S\right)\gamma_S = \vec{V}_d\ \gamma_S + \vec{v}(O_0) \qquad \text{(T3)}$$

$$\vec{r}_{0\,d} = \vec{r}_d + \vec{V}_S\ \gamma_S\ t = \vec{r}_d + \vec{V}_S\ t_0$$

$$\vec{V}_d = \frac{d\,\vec{r}_{0\,d}}{dt_o} = \frac{d\,\vec{r}_d}{dt_o} + \vec{V}_S = \frac{d\,\vec{r}_d}{dt}\frac{dt}{dt_o} + \vec{V}_S = \frac{\vec{v}_d}{\gamma_S} + \vec{V}_S = \frac{\vec{v}_d}{\gamma_S} + \vec{V}(O) \qquad \text{(T4)}$$

Finally, considering a third IRF ***S'***, with the absolute velocity $\vec{V}_{S'}$, supposing the coincidence of the origins of all three IRFs at time zero, it follows:



$$\vec{r} = \vec{r}_0 - \vec{V}_S\ t_0 \qquad t = \frac{t_0}{\gamma_S}$$

$$\vec{r}_0 = \vec{r} + \vec{V}_S\ \gamma_S\ t \qquad t_0 = \gamma_S\ t$$

$$\vec{r}\,' = \vec{r}_0 - \vec{V}_{S'}\ t_0 \qquad t' = \frac{t_0}{\gamma_{S'}}$$

$$\vec{r}_0 = \vec{r}\,' + \vec{V}_{S'}\ \gamma_{S'}\ t' \qquad t_0 = \gamma_{S'}\ t'$$

Eliminating the variables with subscript 0:

$$\vec{r}\,' = \vec{r} + \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_S\ t \qquad t' = \frac{\gamma_S}{\gamma_{S'}}\,t \qquad \text{(T5)}$$

$$\vec{r} = \vec{r}\,' + \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_{S'}\ t' \qquad t = \frac{\gamma_{S'}}{\gamma_S}\,t' \qquad \text{(T6)}$$

Using (T5) and (T6) it is easy to express the relative motion of the origins of the axes of ***S*** and ***S'***:

$$\vec{r}\,'(O) = \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_{S'}\, t \qquad \vec{v}\,'_O = \frac{d\,\vec{r}\,'(O)}{dt'} = \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_{S'}$$

$$\vec{r}(O') = \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_S\, t \qquad \vec{v}_{O'} = \frac{d\,\vec{r}(O')}{dt} = \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_S$$

(T5) and (T6) allow linking the descriptions of the motion of a generic particle ***d*** in ***S*** and in ***S'***.

$$\vec{v}\,'_d = \frac{d\,\vec{r}_d\,'}{dt'} = \frac{d\,\vec{r}_d}{dt'} + \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_{S'} = \frac{d\,\vec{r}_d}{dt}\frac{d\,t}{dt'} + \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_{S'} = \vec{v}_d\,\frac{\gamma_{S'}}{\gamma_S} + \left(\vec{V}_S - \vec{V}_{S'}\right)\gamma_{S'} = \vec{v}_d\,\frac{\gamma_{S'}}{\gamma_S} + \vec{v}\,'_O \tag{T7}$$

$$\vec{v}_d = \frac{d\,\vec{r}_d}{dt} = \frac{d\,\vec{r}_d\,'}{dt} + \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_S = \frac{d\,\vec{r}_d\,'}{dt'}\frac{dt'}{dt} + \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_S = \vec{v}\,'_d\,\frac{\gamma_S}{\gamma_{S'}} + \left(\vec{V}_{S'} - \vec{V}_S\right)\gamma_S = \vec{v}\,'_d\,\frac{\gamma_S}{\gamma_{S'}} + \vec{v}_{O'} \tag{T8}$$

Such a formulation, although consistent and endowed with at least one interesting feature, has the theoretical disadvantage of erasing the principle of relativity and the practical defect of disagreeing with empirical evidence.
The interesting feature concerns the re-evaluation of the energy role.
In fact, the expression of the detector's proper time can be rewritten in the form:

$$d\tau_d = \sqrt{1 - \frac{V_d^{\,2}}{c^2}}\, dt_0 = \sqrt{1 - \frac{2\,T}{m_d\, c^2}}\, dt_0 \cong \left(1 - \frac{T}{m_d\, c^2}\right) dt_0$$

This formulation postulates a direct link between kinetic energy and the proper time of the particle.
Along with the usual properties of indestructibility, this formulation also attributes energy with the character of having an absolute value. It recovers energy's role of "fundamental substance".
However, the extension of the proper time's dependence on other forms of energy, such as the gravitational potential energy, does not seem compatible with experimental results (for example with data supplied from atomic clocks used in GPS satellites).

It has been experimentally established that clocks placed in fixed positions in a gravitational field with spherical symmetry show a dependence on Newtonian gravitational potential (with zero potential located to infinity) in sound agreement with the relation:

$$d\tau_{d\,0} \cong \sqrt{1 + \frac{2\,\Phi}{c^2}}\, d\tau_{d\,0\infty} \cong \left(1 + \frac{\Phi}{c^2}\right) d\tau_{d\,0\infty} \qquad \text{where:} \qquad \Phi(r) = -\frac{G\,M_S}{r}$$

$d\tau_{d\,0}$ is the proper time differential of a clock placed in a generic fixed position with respect to the gravitational field source.
$d\tau_{d\,0\infty}$ is the proper time differential of a clock at rest relative to the gravitational field source and positioned at spatial infinity.
M_S is the mass of the source.

Using the first expression for the differential to describe the behavior of a clock that is motionless relative to the source of the gravitational field and placed at infinity, we can say:

$$d\tau_{d\,0\,\infty} \cong \left(1-\frac{V_d^{\,2}}{2\,c^2}\right) dt_0 = \left(1-\frac{V_S^{\,2}}{2\,c^2}\right) dt_0$$

$V_S = V_d$ is the absolute velocity of the gravitational source.
Therefore, the behavior of a clock which is motionless relative to the source, but located at a generic distance r from it, would be described by:

$$d\tau_{d\,0} \cong \left(1+\frac{\Phi}{c^2}\right)\left(1-\frac{V_S^{\,2}}{2\,c^2}\right) dt_0 \cong \left(1+\frac{\Phi}{c^2}-\frac{V_S^{\,2}}{2\,c^2}\right) dt_0$$

To describe the behavior of a clock moving relative to the source, it is plausible to postulate:

$$d\tau_d \cong \sqrt{1+\frac{2\,\Phi}{c^2}-\frac{2\,T}{m_d\,c^2}}\; dt_0 \cong \left(1+\frac{\Phi}{c^2}-\frac{T}{m_d\,c^2}\right) dt_0 = \left(1-\frac{G\,M_S}{r\,c^2}-\frac{V_d^{\,2}}{2\,c^2}\right) dt_0$$

Let us use this expression to evaluate the time of a clock carried by a GPS satellite.
Consider a simplified system consisting of the Sun, Earth and a satellite.
The orbits are considered circular.
The absolute velocity of the satellite (detector) is: $\vec{V}_d = \vec{V}_{Sun} + \vec{V}_{Earth\_Sun} + \vec{V}_{Sat\_Earth}$

$\vec{V}_{Sun} =$ absolute velocity of the Sun.

$\vec{V}_{Earth\_Sun} =$ relative velocity the Earth-Sun in $\boldsymbol{S_0}$.

$\vec{V}_{Sat\_Earth} =$ relative velocity Satellite-Earth in $\boldsymbol{S_0}$.

$V_{Sun} \approx 370 \quad km/\sec$, on the basis of the dipole component of the cosmic background radiation.
$V_{Earth\_Sun} \approx 30 \quad km/\sec$
$V_{Sat\_Earth} \approx 3.9 \quad km/\sec$

So:

$$V_d^{\,2} = V_{Sun}^{\,2} + V_{Earth\_Sun}^{\,2} + V_{Sat}^{\,2} + 2\vec{V}_{Sun}\cdot\vec{V}_{Earth\_Sun} + 2\vec{V}_{Earth\_Sun}\cdot\vec{V}_{Sat\_Earth} + 2\vec{V}_{Sun}\cdot\vec{V}_{Sat\_Earth}$$

The mixed term $2\vec{V}_{Sun}\cdot\vec{V}_{Sat\_Earth}$ should produce a change in the rhythm of the satellite clock with amplitude in the order of $2\vec{V}_{Sun}\cdot\vec{V}_{Sat\_Earth}\big/2\,c^2 \approx 1.6 \quad 10^{-8}$.
The maximum value occurs when the plane of the satellite's orbit is parallel to the velocity vector of the Sun.
Such a variation would have a periodicity of half a sidereal day.

This prediction clearly conflicts with the available data, making it necessary to reject the previous approach.

Behavior of the clocks carried by the GPS satellites, according to GTR

Relativistic descriptions applied to the GPS system (see for example reference[8]) appear to agree with the experimental data, although they fail to justify the irrelevance of the solar and lunar gravitational potential in these descriptions.
These descriptions begin by applying the Schwarzschild metric in the so-called Earth Centered Inertial frame (ECI), a reference frame fixed to the Earth's center of gravity but not rotating.

The Schwarzschild metric is an exact solution of Einstein's equations, valid in the case of a single static and spherically symmetric gravitational source.
In this metric, the expression of the line element is:

$$ds^2 = -\left(1 - \frac{2\,G\,M_S}{r\,c^2}\right) c^2 dt^2 + \frac{1}{1 - \dfrac{2\,G\,M_S}{r\,c^2}}\, dr^2 + r^2 d\theta^2 + r^2 \sin^2(\vartheta)\, d\varphi^2$$

For weak fields, expanding the dr^2 coefficient (as a function of the variable M_S) into the Taylor series, the above formula can be approximated by the linearized Schwarzschild metric:

$$ds^2 \cong -\left(1 - \frac{2\,G\,M_S}{r\,c^2}\right) c^2 dt^2 + \left(1 + \frac{2\,G\,M_S}{r\,c^2}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2(\vartheta)\, d\varphi^2 =$$
$$= -\left(1 + \frac{2\,\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\,\Phi}{c^2}\right) dr^2 + r^2\left(d\theta^2 + \sin^2(\vartheta)\, d\varphi^2\right)$$

The presupposed spherical symmetry automatically excludes any role for sources other than the source placed in the origin of the coordinates.

The expression of the proper time differential, applicable to a satellite orbiting the Earth, obtained from this metric is:

$$d\tau_d \cong \left(1 + \frac{\Phi}{c^2} - \frac{{v_d}^2}{2\,c^2}\right) dt \qquad (♀1)$$

dt is the differential of a coordinate time, and is identical to the proper time of a clock in a fixed position with respect to the central source and located at infinity.
The coordinate time t differs from the GPS coordinate time by a scale factor, which serves to transform it into the proper time measured by a clock rigidly coupled to the reference geoid.

In the case of a circular orbit, the expression (♀1) implies two corrective terms of opposite sign for the time indicated by the orbiting clock, compared to the time indicated by a clock at rest in the ECI reference frame at ground level.
The first term means that time beats more quickly due to the fact that the module of gravitational potential at orbital altitude has a lower value compared to ground level.
The second term constitutes a slowdown in the beat of time due to the orbital velocity.
In the specific case of a GPS satellite orbit (orbital radius ≈ 26600 km), there is an overriding influence of the first term dictating an increase in speed (about 38 μs/day).
If one assumes a Keplerian orbit for the satellite, expression (♀1) also allows a correct prediction of the periodic variations in the proper time, associated with the orbital eccentricity.

Assuming that expression (♀1) is extendable to the case in which the potential is not only associated with the central source, but also includes a contribution from a remote source (such as the Sun), the solar potential should manifest measurable effects on the rhythm of the clock.
This means that if we assume the validity of the expression

$$d\tau_d \cong \left(1+\frac{\Phi_{Earth}+\Phi_{Sun}}{c^2}-\frac{{v_d}^2}{2\,c^2}\right)dt \qquad (♀2)$$

the effects caused by changes in the solar potential Φ_{Sun} around the satellite's orbital path – as a result of variations in the satellite-Sun distance – should be measurable for clocks orbiting in Keplerian motion.

On the contrary one could reject (♀2), since (♀2) is obtained from (♀1) by extending its validity to the total potential – an operation that necessarily destroys the assumed spherical symmetry.
In this case, however, it is unclear how to quantify the effects of the Sun's and the Moon's potentials.
In the relativistic literature concerning the GPS system, there is no satisfactory justification for the apparent irrelevance of the solar potential on the orbiting clocks.
Consider for example reference[9], which claims to explain the phenomenon by invoking the principle of equivalence and the free-falling state of the Earth and its system of satellites within the Sun's gravitational field.
If such considerations were correct, they could be applied to erroneously deduce the irrelevance of the terrestrial potential on a GPS satellite clock, since the satellite is free-falling around the Earth.

Assuming the validity of (♀2), it would be easier to justify the non-observation of satellites' proper time dependence on the varying distance from the Sun if we abandon the hypothesis of strictly Keplerian orbits.
Changes in a satellite's velocity, due to the presence of the Sun's gravitational field, should produce effects on the clock that compensate for those produced by the variation of the distance from the Sun.

In more general contexts, with multiple gravitational sources in relative motion, the possibility to express the proper time of a generic space probe – which is not constrained to follow a closed orbit around a dominant source – does not seem to be a solved problem according to the GTR.

The proposed new extension

Definitions and postulates

IRF: reference frame in uniform motion, far away from gravitational sources. Free-falling observers in regions with a gravity gradient are therefore excluded.

t : universal time; it is assumed to be common to each IRF.

τ_d : proper time of the detector.

In what follows, only gravitational sources consisting of a finite set of point sources with masses m_i were considered, but the generalization to continuous distributions is evident.

Proper time postulate

Consider a generic observer ***O***, with an associated non-rotating reference frame ***S***.
S is therefore a coordinate system, whose origin is located in the observer's point of view.
To give a physical interpretation to the spatial coordinates, we can understand them as describing the positions of a set of material points, with arbitrarily small masses, distributed in space and constrained to maintain constant their mutual distances.
The system of material points is therefore comparable to a rigid body.
If ***S*** is inertial, no action is required to ensure constancy in the mutual relations between the material points, which serve as a physical support for the coordinate system.
Otherwise, the rigidity of the system will be obtained by the application of forces.
The reference frame ***S*** will be used to describe the positions of all relevant entities.

It is postulated that the proper time τ_d of detector ***d*** – with charge q_d, (small) mass m_d, and moving with a velocity evaluated by ***O*** as being equal to $\vec{v}_d$ – is linked to the universal time by:

$$\frac{d\tau_d}{dt} = 1 + \frac{P_d - K_d}{c^2} = \frac{1}{\gamma_d} \qquad \text{(b1)}$$

with:

$$P_d = \sum_i \Phi_i \, \alpha_i = -G \sum_i \frac{m_i}{|\, \vec{r}_d - \vec{r}_i \,|} \, \alpha_i \qquad \text{(b2)}$$

$$K_d = \sum_i \frac{1}{2} \left| \vec{V}_d - \vec{V}_i \right|^2 \alpha_i = \sum_i \frac{1}{2} \left| \frac{d\,\vec{r}_d}{d\,t} - \frac{d\,\vec{r}_i}{d\,t} \right|^2 \alpha_i \qquad \text{(b3)}$$

$$\alpha_i = \frac{|\, \nabla\Phi_i \,|}{\left| \sum_k \nabla\Phi_k \right|} = \frac{|\, \nabla\Phi_i \,|}{|\, \nabla\Phi \,|} \qquad \text{(b4)}$$

The coefficient α_i "weighs" the relevance of the gravitational source with mass m_i in its (positional and kinematic) connection to the detector.
Definition (b4) of the weight functions α_i is merely one of the simplest alternative proposals for a weight function that could contemplate more complex relations.

From (b1): $dt = \gamma_d \; d\tau_d$

and: $\gamma_d = 1 \Big/ \left(1 + \dfrac{P_d - K_d}{c^2}\right) = 1 \Big/ \left[\; 1 + \dfrac{1}{c^2} \sum_i \left(\Phi_i - \dfrac{1}{2} \left| \vec{V}_d - \vec{V}_i \right|^2 \right) \alpha_i \;\right]$

Applying the previous expression in regions of space where the inertial reference frames are definable, namely in regions distant from the sources where $\alpha_i \to 0$, it follows that: $d\tau = dt$.

Equation (b1) theorizes that the variation of the detector's proper time in relation to the universal time depends exclusively on the position and velocity of the detector with respect to the gravitational sources.

<u>Postulates on Dynamics</u>

It is postulated that a particle's dynamics can be formulated according to the following definitions:

momentum: $$\vec{p} \equiv m \frac{d\,\vec{r}}{d\,t} = \frac{m}{\gamma} \frac{d\,\vec{r}}{d\tau} \qquad \text{(b5)}$$

force: $$\vec{F} \equiv \frac{d\,\vec{p}}{d\tau} = \gamma \frac{d\,\vec{p}}{d\,t} = \gamma\, m \frac{d^2\vec{r}}{d\,t^2} = \gamma\, m\, \vec{a} \qquad \text{(b6)}$$

and by assuming the validity of the universal gravitational law:

$$\vec{F}_g = -\nabla\Phi \qquad \text{(b7)}$$

Draft of an electromagnetic theory with the use of proper time

Applying (b1) to a generic observer ***O*** and to detector ***d***:

$$d\tau_O = \left\{ 1 + \frac{1}{c^2} \sum_i \left[\left(\Phi_i - \frac{1}{2} \left| \vec{V}_i \right|^2 \right) \frac{\left| \nabla \Phi_i \right|}{\left| \nabla \Phi \right|} \right] \right\} dt = \frac{dt}{\gamma_O} \qquad \text{(b8)}$$

$$d\tau_d = \left\{ 1 + \frac{1}{c^2} \sum_i \left[\left(\Phi_i - \frac{1}{2} \left| \vec{V}_d - \vec{V}_i \right|^2 \right) \frac{\left| \nabla \Phi_i \right|}{\left| \nabla \Phi \right|} \right] \right\} dt = \frac{dt}{\gamma_d} \qquad \text{(b9)}$$

Where:

$$\vec{V}_i = \frac{d\,\vec{r}_i}{d\,t} \qquad\qquad \vec{V}_d = \frac{d\,\vec{r}_d}{d\,t}$$

$\vec{V}_O = 0$, since ***O*** defines the origin of the chosen reference frame.

Using the following notation:

$\vec{V}_d = \dfrac{d\,\vec{r}_d}{d\,t}$ detector's velocity with respect to ***O***, calculated using universal time.

$\vec{v}_d = \dfrac{d\,\vec{r}_d}{d\tau_O}$ detector's velocity with respect to ***O***, calculated using the proper time of ***O***.

In short, $\vec{v}_d$ is the speed of the detector evaluated by ***O***.

$\vec{v}_{d\_d} = \dfrac{d\,\vec{r}_d}{d\tau_d}$ detector's velocity with respect to ***O***, calculated using the detector's proper time.

It follows that:

$$\vec{V}_d = \frac{d\,\vec{r}_d}{d\,t} = \frac{d\,\vec{r}_d}{\gamma_d\ d\tau_d} = \frac{d\,\vec{r}_d}{\gamma_O\ d\tau_O} \qquad \rightarrow \qquad \vec{V}_d = \frac{\vec{v}_{d\_d}}{\gamma_d} = \frac{\vec{v}_d}{\gamma_O}$$

$$\vec{V}_i = \frac{d\,\vec{r}_i}{d\,t} = \frac{d\,\vec{r}_i}{\gamma_d\ d\tau_d} = \frac{d\,\vec{r}_i}{\gamma_O\ d\tau_O} \qquad \rightarrow \qquad \vec{V}_i = \frac{\vec{v}_{i\_d}}{\gamma_d} = \frac{\vec{v}_i}{\gamma_O}$$

Other useful expressions:

$$dt = \gamma_O \; d\tau_O = \gamma_d \; d\tau_d \qquad \rightarrow \qquad \frac{d\tau_O}{d\tau_d} = \frac{\gamma_d}{\gamma_O}$$

$$\vec{V}_d = \frac{\vec{v}_{d\_d}}{\gamma_d} = \frac{\vec{v}_d}{\gamma_O} \qquad \rightarrow \qquad \gamma_O \; \vec{V}_d = \vec{v}_d$$

$$\rightarrow \qquad \frac{\gamma_O}{\gamma_d} \vec{v}_{d\_d} = \vec{v}_d$$

It is assumed that the detector's proper time must be used in place of the coordinate time to formulate the differential equations of the fields.

It is appropriate to offer some clarifications in this regard.

Given a generic observer ***O***, associated with a non-rotating reference frame ***S*** and a particle, which constitutes the real detector in motion with respect to ***O*** with velocity $\vec{V}_d$ (evaluated using universal time), the proper time of such a detector is expressed by (b9).

In spatial positions different from those occupied by the real detector, we should instead consider a virtual detector which shares the same velocity of the real detector with respect to ***O*** (again assessed through universal time).

Because a virtual detector will "see" different gravitational potentials and gradients of these potentials compared to those "seen" by the real detector (since the two have different positions), the γ_d of the virtual detector is generally different from that of the real detector.
Therefore, the proper time of a virtual detector does not necessarily coincide with the proper time of the real detector to which it is associated.

The proposition is therefore that of a mechanism which mixes local influences (the potential in the virtual detector's position) with non-local influences (the speed of the real detector, placed elsewhere).

Total derivative with respect to the detector's proper time

The "total proper time derivative" is intended as the limit of the ratio between the variation of a quantity to follow the motion of a detector and the proper time interval of this variation, when this interval tends towards zero.

If $x_d(\tau_d)$, $y_d(\tau_d)$ and $z_d(\tau_d)$ are the coordinates of a detector as a function of its own proper time, it follows that:

$$\frac{d\,f}{d\,\tau_d}=\frac{\partial\,f}{\partial\,\tau_d}+\frac{\partial\,f}{\partial\,x}\frac{d\,x_d}{d\,\tau_d}+\frac{\partial\,f}{\partial\,y}\frac{d\,y_d}{d\,\tau_d}+\frac{\partial\,f}{\partial\,z}\frac{d\,z_d}{d\,\tau_d}=\frac{\partial\,f}{\partial\,\tau_d}+\left(\vec{v}_{d\_d}\cdot\nabla\right)f$$

$$\frac{d\,f}{d\,\tau_d}=\frac{\partial\,f}{\partial\,\tau_O}\frac{d\,\tau_O}{d\,\tau_d}+\frac{\partial\,f}{\partial\,x}\frac{d\,x_d}{d\,\tau_O}\frac{d\,\tau_O}{d\,\tau_d}+\frac{\partial\,f}{\partial\,y}\frac{d\,y_d}{d\,\tau_O}\frac{d\,\tau_O}{d\,\tau_d}+\frac{\partial\,f}{\partial\,z}\frac{d\,z_d}{d\,\tau_O}\frac{d\,\tau_O}{d\,\tau_d}=\frac{\gamma_d}{\gamma_O}\left[\frac{\partial\,f}{\partial\,\tau_O}+\left(\vec{v}_d\cdot\nabla\right)f\right]$$

$$\frac{d\,f}{d\,\tau_d}=\frac{\partial\,f}{\partial\,t}\frac{d\,t}{d\,\tau_d}+\frac{\partial\,f}{\partial\,x}\frac{d\,x_d}{d\,t}\frac{d\,t}{d\,\tau_d}+\frac{\partial\,f}{\partial\,y}\frac{d\,y_d}{d\,t}\frac{d\,t}{d\,\tau_d}+\frac{\partial\,f}{\partial\,z}\frac{d\,z_d}{d\,t}\frac{d\,t}{d\,\tau_d}=\gamma_d\left[\frac{\partial\,f}{\partial\,t}+\left(\vec{V}_d\cdot\nabla\right)f\right]$$

and:

$$\frac{d^2}{d\tau_d^{\ 2}}=\frac{d}{d\tau_d}\left(\frac{d}{d\tau_d}\right)=\gamma_d\left\{\frac{\partial}{\partial t}\left[\gamma_d\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)\right]+\left(\vec{V}_d\cdot\nabla\right)\left[\gamma_d\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)\right]\right\}$$

$$\frac{d^2}{d\tau_d^{\ 2}}=\gamma_d\left\{\gamma_d\left[\frac{\partial^2}{\partial t^2}+\frac{\partial\vec{V}_d}{\partial t}\cdot\nabla+\left(\vec{V}_d\cdot\nabla\right)\frac{\partial}{\partial t}+\frac{\partial\gamma_d}{\partial t}\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)\right]+\right.$$
$$\left.+\gamma_d\left(\vec{V}_d\cdot\nabla\right)\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)+\left(\vec{V}_d\cdot\nabla\right)\gamma_d\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)\right\}$$

$$\frac{d^2}{d\tau_d^{\ 2}}=\gamma_d\left\{\gamma_d\left[\frac{\partial^2}{\partial t^2}+\left(\frac{\partial\vec{V}_d}{\partial t}\cdot\nabla\right)+2\left(\vec{V}_d\cdot\nabla\right)\frac{\partial}{\partial t}+\left(\vec{V}_d\cdot\nabla\right)\left(\vec{V}_d\cdot\nabla\right)\right]+\right.$$
$$\left.+\frac{\partial\gamma_d}{\partial t}\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)+\left(\vec{V}_d\cdot\nabla\right)\gamma_d\left(\frac{\partial}{\partial t}+\vec{V}_d\cdot\nabla\right)\right\}$$

In many cases: $\frac{\partial\gamma_d}{\partial t}\approx 0$ and $\left(\vec{V}_d\cdot\nabla\right)\gamma_d\approx 0$.

So:

$$\frac{d^2}{d\tau_d^{\ 2}}\approx\gamma_d\left\{\gamma_d\left[\frac{\partial^2}{\partial t^2}+\left(\frac{\partial\vec{V}_d}{\partial t}\cdot\nabla\right)+2\left(\vec{V}_d\cdot\nabla\right)\frac{\partial}{\partial t}+\left(\vec{V}_d\cdot\nabla\right)\left(\vec{V}_d\cdot\nabla\right)\right]\right\}=\gamma_d^{\ 2}\,\frac{d^2}{dt^2}$$

Equations of the electromagnetic field in empty space, in a generic non-rotating reference frame ***S***, associated with observer ***O***:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \qquad \text{(c1)}$$

$$\vec{B} = \vec{B}_1 + \vec{B}_2 \qquad \text{(c2)}$$

$$\nabla \cdot \vec{E}_1 = \frac{\rho_m}{\varepsilon_0} \qquad \text{(c3)}$$

$$\nabla \wedge \vec{E}_1 = 0 \qquad \text{(c4)}$$

$$\nabla \cdot \vec{E}_2 = 0 \qquad \text{(c5)}$$

$$\nabla \wedge \vec{E}_2 = -\frac{d\vec{B}}{d\tau_d} \qquad \text{(c6)}$$

$$\nabla \cdot \vec{B}_1 = 0 \qquad \text{(c7)}$$

$$\nabla \wedge \vec{B}_1 = \mu_0 \vec{J}_m \qquad \text{(c8)}$$

$$\nabla \cdot \vec{B}_2 = 0 \qquad \text{(c9)}$$

$$\nabla \wedge \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d\vec{E}}{d\tau_d} \qquad \text{(c10)}$$

$$\vec{J}_m^+ = \rho_+ \left(\vec{V}_+ - \vec{V}_d\right) = \rho_+ \frac{\vec{v}_+ - \vec{v}_d}{\gamma_O} = \rho_+ \frac{\vec{v}_{+\_d} - \vec{v}_{d\_d}}{\gamma_d} \qquad \text{(c11)}$$

$$\vec{J}_m^- = \rho_- \left(\vec{V}_- - \vec{V}_d\right) = \rho_- \frac{\vec{v}_- - \vec{v}_d}{\gamma_O} = \rho_- \frac{\vec{v}_{-\_d} - \vec{v}_{d\_d}}{\gamma_d} \qquad \text{(c12)}$$

$$\vec{J}_m = \vec{J}_m^+ + \vec{J}_m^- \qquad \text{(c13)}$$

$$\rho_m^+ = \rho_+ + \frac{\vec{J}_m^+}{2c^2} \cdot \left(\vec{V}_+ - \vec{V}_d\right) = \rho_+ \left(1 + \frac{\left|\vec{v}_+ - \vec{v}_d\right|^2}{2c^2\ \gamma_O{}^2}\right) \qquad \text{(c14)}$$

$$\rho_m^- = \rho_- + \frac{\vec{J}_m^-}{2c^2} \cdot \left(\vec{V}_- - \vec{V}_d\right) = \rho_- \left(1 + \frac{\left|\vec{v}_- - \vec{v}_d\right|^2}{2c^2\ \gamma_O{}^2}\right) \qquad \text{(c15)}$$

$$\rho_m = \rho_m^+ + \rho_m^- \qquad \text{(c16)}$$

$\vec{v}_+$ ($\vec{v}_-$) is the average speed of the positive (negative) charges contained in dV, evaluated by observer ***O***.

Continuity equation

Differentiating the sum of (c3) and (c5) with respect to the detector's proper time, we can say that:

$$\frac{d}{d\tau_d}\left(\nabla\cdot\vec{E}\right)=\frac{d}{d\tau_d}\left[\nabla\cdot\left(\vec{E}_1+\vec{E}_2\right)\right]=\nabla\cdot\frac{d\vec{E}_1}{d\tau_d}=\frac{1}{\varepsilon_0}\frac{d\rho_m}{d\tau_d} \qquad (♂)$$

Calculating the divergence of the sum of (c8) and (c10):

$$\nabla\cdot\left(\nabla\wedge\vec{B}\right)=\nabla\cdot\left(\nabla\wedge\left(\vec{B}_1+\vec{B}_2\right)\right)=0=\mu_0\nabla\cdot\vec{J}_m+\mu_0\varepsilon_0\nabla\cdot\frac{d\vec{E}}{d\tau_d} \qquad (♀)$$

Introducing (♂) into (♀): $\mu_0\nabla\cdot\vec{J}_m+\mu_0\varepsilon_0\frac{1}{\varepsilon_0}\frac{d\rho_m}{d\tau_d}=0$

So:

$$\nabla\cdot\vec{J}_m+\frac{d\rho_m}{d\tau_d}=0 \qquad \text{(c17)}$$

Poisson's equations and wave equations

Using the same procedure followed in the first-order version, one obtains:

$$\nabla^2\vec{B}_1=-\mu_0\nabla\wedge\vec{J}_m \qquad \text{(c18)}$$

$$\nabla^2\vec{B}_2-\mu_0\varepsilon_0\frac{d^2\vec{B}_2}{d\tau_d^{\ 2}}=\mu_0\varepsilon_0\frac{d^2\vec{B}_1}{d\tau_d^{\ 2}} \qquad \text{(c19)}$$

$$\nabla^2\vec{E}_1=\frac{1}{\varepsilon_0}\nabla\rho_m \qquad \text{(c20)}$$

$$\nabla^2\vec{E}_2-\mu_0\varepsilon_0\frac{d^2\vec{E}_2}{d\tau_d^{\ 2}}=\mu_0\frac{d\vec{J}_m}{d\tau_d}+\mu_0\varepsilon_0\frac{d^2\vec{E}_1}{d\tau_d} \qquad \text{(c21)}$$

At a great distance from the sources, the wave equations are approximated by their homogeneous forms:

$$\nabla^2\vec{B}_2=\mu_0\varepsilon_0\frac{d^2\vec{B}_2}{d\tau_d^{\ 2}} \qquad\qquad \nabla^2\vec{E}_2=\mu_0\varepsilon_0\frac{d^2\vec{E}_2}{d\tau_d^{\ 2}}$$

If: $\frac{\partial\gamma_d}{\partial t}\approx 0$ and $\left(\vec{V}_d\cdot\nabla\right)\gamma_d\approx 0$

then: $\frac{d^2}{d\tau_d^{\ 2}}\approx\gamma_d^{\ 2}\,\frac{d^2}{dt^2}$

In such a case, the homogeneous wave equations can be approximated by:

$$\nabla^2 \vec{B}_2 = \mu_0 \varepsilon_0 \ \gamma_d{}^2 \ \frac{d^2 \vec{B}_2}{dt^2} \qquad\qquad \nabla^2 \vec{E}_2 = \mu_0 \varepsilon_0 \ \gamma_d{}^2 \ \frac{d^2 \vec{E}_2}{dt^2}$$

Considering the propagation of an electromagnetic disturbance associated with a detector at rest with respect to a single dominant gravitational source, from such equations one can deduce that the speed of the electromagnetic front decreases where the gravitational field is more intense.

Conclusions

Driven by the desire to seek alternatives to the space-time description proposed by the theory of relativity, this paper has critically reviewed Hertz-Phipps's electromagnetic theory, which is considered interesting due to its properties of invariance under Galilean transformation laws.
It has been shown that this theory is incompatible with experience regarding interactions between electrical circuits in a stationary state. An amendment, which seems to solve these predictive discrepancies, has been proposed.
The modified theory involves the coexistence of instantaneous and delayed interactions.
The adequacy of this new theory regarding the totality of known electromagnetic phenomena remains an open question.
In the more general formulation, the theory draws on concepts of a material particle's universal time and proper time, the latter being defined as the time marked by a clock transported from the particle.
The universal time coincides with the proper time of a clock in uniform motion and placed at a great distance from all gravitational sources.
It is hypothesized that the beating of such clock is at its maximum pace.
In every other case, a particle's proper time is assumed to be an exclusive function of its position and speed with respect to each gravitational source.
Unfortunately, such a formulation does not seem to be deducible by an elegant principle, as in the case of the constancy of the speed of light for all inertial observers. Instead, the theory adopts a phenomenological approach.
The theory can be called relativistic, since it does not presuppose the existence of a privileged reference frame and because only relative positions and speeds have relevance.

It is my hope that this work will be deemed worthy of a theoretical critique and that it may provide a stimulus towards the implementation of wider experimental activities in order to confirm or disprove the existence of instantaneous interactions.

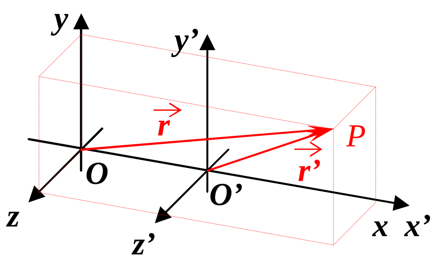
Appendix

A. Lorentz Transformations

Given the two IRFs ***S*** and ***S'***, let $(\vec{r},t)$ and $(\vec{r}\,',t')$ be the spatial and temporal coordinates of the same event in the two reference frames. Let $\vec{v}$ be the velocity of ***S'*** in relation to ***S***.
Expressed in vector form, the Lorentz transformations are:

$$\vec{r}\,'=\vec{r}+(\gamma-1)\frac{\vec{v}\cdot\vec{r}}{v^2}\,\vec{v}-\gamma\vec{v}\,t \qquad t'=\gamma\left(t-\frac{\vec{v}\cdot\vec{r}}{c^2}\right) \qquad \text{with:}\quad \gamma=1\Big/\sqrt{1-v^2/c^2}$$

They assume the simplest form when they refer to Cartesian coordinates with appropriate choices regarding the orientation of the axes, the location of the origins and the choice of synchronization of the two observers' clocks.
Let ***O***, ***x***, ***y***, ***z***, and ***O'***, ***x'***, ***y'***, ***z'*** be the Cartesian coordinates associated with ***S*** and ***S'*** respectively, chosen so that the axes ***x*** and ***x'*** are superimposed and oriented in the same direction as the relative speed, while ***y*** is parallel to ***y'*** and ***z*** parallel to ***z'***.



Let the zero of time be chosen at the overlapping instant of ***O*** and ***O'***.
Then Lorentz Transformations can be expressed in the form:

direct transformations	inverse transformations
$x'=\gamma\,(x-vt)$	$x=\gamma\,(x'+vt)$
$y'=y$	$y'=y$
$z'=z$	$z'=z$
$t'=\gamma\left(t-\dfrac{v}{c^2}x\right)$	$t=\gamma\left(t'+\dfrac{v}{c^2}x\right)$

Considering only transformations of the first order in v/c (the first-order Taylor series approximation of transformation equations), we can say that:

$$\begin{aligned} x' &\cong x-vt \\ y' &= y \\ z' &= z \\ t' &\cong t-\frac{v}{c^2}x \end{aligned}$$

It must be noted that, even in this special case, there is a deviation from the Galilean transformation regarding the time coordinate, a non-conformity that increases with distance.
It means that low-speed LT only coincide with the GT locally.

B. **Galilean transformations and operators**

$$\nabla' = \nabla \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

The proof of the above expressions is simple considering the fact that $\vec{v}$ is constant and that the chain rule applies for partial differential operators that operate on the fields, as functions of x, y, z, t.

Given a generic vector field $\vec{C}$, which is a function of the spatial coordinates and time, it follows that:

$$\frac{\partial \vec{C}}{\partial x'} = \frac{\partial \vec{C}}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial \vec{C}}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial \vec{C}}{\partial z}\frac{\partial z}{\partial x'} = \frac{\partial \vec{C}}{\partial x}$$

Similarly: $\frac{\partial \vec{C}}{\partial y'} = \frac{\partial \vec{C}}{\partial y}$ $\qquad$ $\frac{\partial \vec{C}}{\partial z'} = \frac{\partial \vec{C}}{\partial z}$

it follows that: $\nabla' \vec{C} = \nabla \vec{C}$

Since: $x' = x - v_x\, t$ $\qquad$ $y' = y - v_y\, t$ $\qquad$ $z' = z - v_z\, t$

it follows that: $\frac{\partial x'}{\partial t} = -v_x$ $\qquad$ $\frac{\partial y'}{\partial t} = -v_y$ $\qquad$ $\frac{\partial z'}{\partial t} = -v_z$

Since: $x = x' + v_x\, t'$ $\qquad$ $y = y' + v_y\, t'$ $\qquad$ $z = z' + v_z\, t'$

it follows that: $\frac{\partial x}{\partial t'} = v_x$ $\qquad$ $\frac{\partial y}{\partial t'} = v_y$ $\qquad$ $\frac{\partial z}{\partial t'} = v_z$

Therefore:

$$\frac{\partial \vec{C}}{\partial t'} = \frac{\partial \vec{C}}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial \vec{C}}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial \vec{C}}{\partial z}\frac{\partial z}{\partial t'} + \frac{\partial \vec{C}}{\partial t}\frac{\partial t}{\partial t'} = \frac{\partial \vec{C}}{\partial x} v_x + \frac{\partial \vec{C}}{\partial y} v_y + \frac{\partial \vec{C}}{\partial z} v_z + \frac{\partial \vec{C}}{\partial t}$$

i.e.: $\frac{\partial \vec{C}}{\partial t'} = \frac{\partial \vec{C}}{\partial t} + \left(\vec{v} \cdot \nabla\right)\vec{C}$

(q.e.d.)

C. **Expansion on the total time derivative expression in wave equations in *S***

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2} = -\mu_0 \nabla \wedge \vec{J}_m$$

$$\frac{d\,\vec{B}}{d\,t} = \frac{\partial\,\vec{B}}{\partial\,t} + (\vec{v}_d \cdot \nabla)\vec{B}$$

$$\frac{d^2\,\vec{B}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{B}}{dt}\right) = \frac{\partial}{\partial\,t}\left(\frac{d\vec{B}}{dt}\right) + (\vec{v}_d \cdot \nabla)\left(\frac{d\vec{B}}{dt}\right)$$

$$\frac{d^2\,\vec{B}}{dt^2} = \frac{\partial}{\partial\,t}\left[\frac{\partial\vec{B}}{\partial\,t} + (\vec{v}_d \cdot \nabla)\vec{B}\right] + (\vec{v}_d \cdot \nabla)\left[\frac{\partial\vec{B}}{\partial\,t} + (\vec{v}_d \cdot \nabla)\vec{B}\right]$$

$$\frac{d^2\,\vec{B}}{dt^2} = \frac{\partial^2\,\vec{B}}{\partial t^2} + \frac{\partial}{\partial\,t}\left[(\vec{v}_d \cdot \nabla)\vec{B}\right] + (\vec{v}_d \cdot \nabla)\frac{\partial\,\vec{B}}{\partial\,t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla)\vec{B}$$

So:

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{\frac{\partial}{\partial t}\left[(\vec{v}_d \cdot \nabla)\vec{B}\right] + (\vec{v}_d \cdot \nabla)\frac{\partial\,\vec{B}}{\partial\,t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla)\vec{B}\right\} = -\mu_0 \nabla \wedge \vec{J}_m$$

Similarly:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2} = \mu_0 \frac{d\,\vec{J}_m}{d\,t} + \frac{1}{\varepsilon_0}\nabla\rho$$

$$\frac{d^2\,\vec{E}}{dt^2} = \frac{\partial^2\,\vec{E}}{\partial t^2} + \frac{\partial}{\partial\,t}\left[(\vec{v}_d \cdot \nabla)\vec{E}\right] + (\vec{v}_d \cdot \nabla)\frac{\partial\,\vec{E}}{\partial\,t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla)\vec{E}$$

So:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \varepsilon_0 \left\{\frac{\partial}{\partial t}\left[(\vec{v}_d \cdot \nabla)\vec{E}\right] + (\vec{v}_d \cdot \nabla)\frac{\partial\,\vec{E}}{\partial\,t} + (\vec{v}_d \cdot \nabla)(\vec{v}_d \cdot \nabla)\vec{E}\right\} = \mu_0 \frac{d\,\vec{J}_m}{d\,t} + \frac{1}{\varepsilon_0}\nabla\rho$$

D. **Wave equation and its monochromatic solutions**

Consider the homogeneous wave equation: $\nabla^2\vec{E}-\mu_0\varepsilon_0\dfrac{d^2\vec{E}}{dt^2}=0$

The solution we seek takes the form: $\vec{E}=\vec{E}(p)$

Where the phase p is: $p=\vec{k}\cdot\vec{r}-\omega t=x\,k_x+y\,k_y+z\,k_z-\omega t$

which corresponds to a phase velocity: $v_{ph}=\omega/k$

The following expression therefore applies:

$$\frac{\partial p}{\partial x}=k_x \qquad \frac{\partial p}{\partial y}=k_y \qquad \frac{\partial p}{\partial z}=k_z \qquad \frac{\partial p}{\partial t}=-\omega$$

Introducing the solution in the wave equation and expanding on the terms, it follows that:

$$\nabla^2\vec{E}(p)=\left[\frac{\partial^2E_x(p)}{\partial x^2}+\frac{\partial^2E_x(p)}{\partial y^2}+\frac{\partial^2E_x(p)}{\partial z^2}\right]\vec{u}_x+\left[\frac{\partial^2E_y(p)}{\partial x^2}+\frac{\partial^2E_y(p)}{\partial y^2}+\frac{\partial^2E_y(p)}{\partial z^2}\right]\vec{u}_y+$$
$$+\left[\frac{\partial^2E_z(p)}{\partial x^2}+\frac{\partial^2E_z(p)}{\partial y^2}+\frac{\partial^2E_z(p)}{\partial z^2}\right]\vec{u}_z$$

$$\frac{\partial^2E_x(p)}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial E_x(p)}{\partial p}\frac{\partial p}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial E_x(p)}{\partial p}k_x\right)=\frac{\partial^2E_x(p)}{\partial p^2}\frac{\partial p}{\partial x}k_x=\frac{\partial^2E_x(p)}{\partial p^2}k_x^2\text{, etc.}$$

So:

$$\nabla^2\vec{E}(p)=\left[\frac{\partial^2E_x(p)}{\partial p^2}k_x^2\right]\vec{u}_x+\left[\frac{\partial^2E_y(p)}{\partial p^2}k_y^2\right]\vec{u}_y+\left[\frac{\partial^2E_z(p)}{\partial p^2}k_z^2\right]\vec{u}_z=\left(k_x^2+k_y^2+k_z^2\right)\frac{\partial^2\vec{E}}{\partial p^2}=k^2\frac{\partial^2\vec{E}}{\partial p^2}$$

(res. 1)

$$\frac{d\,\vec{E}(p)}{d\,t}=\frac{\partial\,\vec{E}(p)}{\partial t}+(\vec{v}_d\cdot\nabla)\vec{E}(p)$$

$$\frac{d^2\,\vec{E}(p)}{dt^2}=\frac{d}{dt}\left(\frac{d\vec{E}}{dt}\right)=\frac{\partial}{\partial t}\left(\frac{d\vec{E}}{dt}\right)+(\vec{v}_d\cdot\nabla)\left(\frac{d\vec{E}}{dt}\right)=\frac{\partial}{\partial t}\left[\frac{\partial\vec{E}}{\partial t}+(\vec{v}_d\cdot\nabla)\vec{E}\right]+(\vec{v}_d\cdot\nabla)\left[\frac{\partial\vec{E}}{\partial t}+(\vec{v}_d\cdot\nabla)\vec{E}\right]=$$
$$=\frac{\partial^2\vec{E}}{\partial t^2}+\frac{\partial}{\partial t}\left[(\vec{v}_d\cdot\nabla)\vec{E}\right]+(\vec{v}_d\cdot\nabla)\frac{\partial\vec{E}}{\partial t}+(\vec{v}_d\cdot\nabla)(\vec{v}_d\cdot\nabla)\vec{E}$$

Since: $\dfrac{\partial}{\partial t}\left[(\vec{v}_d\cdot\nabla)\vec{E}\right]=\vec{i}\left(\left(\dfrac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)E_x\right)+\vec{j}\left(\left(\dfrac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)E_y\right)+\vec{k}\left(\left(\dfrac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)E_z\right)+$

$$+\vec{i}\left((\vec{v}_d\cdot\nabla)\frac{\partial}{\partial t}E_x\right)+\vec{j}\left((\vec{v}_d\cdot\nabla)\frac{\partial}{\partial t}E_y\right)+\vec{k}\left((\vec{v}_d\cdot\nabla)\frac{\partial}{\partial t}E_z\right)=\left(\frac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)\vec{E}+(\vec{v}_d\cdot\nabla)\frac{\partial}{\partial t}\vec{E}$$

it follows that: $\frac{d^2\vec{E}}{dt^2} = \frac{\partial^2\vec{E}}{\partial t^2} + \left(\frac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)\vec{E} + 2\left(\vec{v}_d\cdot\nabla\right)\frac{\partial}{\partial t}\vec{E} + \left(\vec{v}_d\cdot\nabla\right)^2\vec{E}$

The various terms can be expressed as:

$$\frac{\partial^2\vec{E}}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial\vec{E}}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial t}\right] = \frac{\partial^2\vec{E}}{\partial p^2}\left(\frac{\partial p}{\partial t}\right)^2 = \frac{\partial^2\vec{E}}{\partial p^2}\omega^2$$

$$\left(\frac{\partial\vec{v}_d}{\partial t}\cdot\nabla\right)\vec{E} = a_{dx}\frac{\partial\vec{E}}{\partial x} + a_{dy}\frac{\partial\vec{E}}{\partial y} + a_{dz}\frac{\partial\vec{E}}{\partial z} = a_{dx}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial x} + a_{dy}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial y} + a_{dz}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial z} =$$
$$= a_{dx}\frac{\partial\vec{E}}{\partial p}k_x + a_{dy}\frac{\partial\vec{E}}{\partial p}k_y + a_{dz}\frac{\partial\vec{E}}{\partial p}k_z = \vec{a}_d\cdot\vec{k}\,\frac{\partial\vec{E}}{\partial p}$$

$$2\left(\vec{v}_d\cdot\nabla\right)\frac{\partial}{\partial t}\vec{E} = 2\left(\vec{v}_d\cdot\nabla\right)\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial t} = 2\left[v_{dx}\frac{\partial}{\partial x}\left(\frac{\partial\vec{E}}{\partial p}\right) + v_{dy}\frac{\partial}{\partial y}\left(\frac{\partial\vec{E}}{\partial p}\right) + v_{dz}\frac{\partial}{\partial z}\left(\frac{\partial\vec{E}}{\partial p}\right)\right]\frac{\partial p}{\partial t} =$$
$$= -2\left[v_{dx}\frac{\partial}{\partial p}\left(\frac{\partial\vec{E}}{\partial p}\right)\frac{\partial p}{\partial x} + v_{dy}\frac{\partial}{\partial p}\left(\frac{\partial\vec{E}}{\partial p}\right)\frac{\partial p}{\partial y} + v_{dz}\frac{\partial}{\partial p}\left(\frac{\partial\vec{E}}{\partial p}\right)\frac{\partial p}{\partial z}\right]\omega =$$
$$= -2\left[v_{dx}\frac{\partial^2\vec{E}}{\partial p^2}k_x + v_{dy}\frac{\partial^2\vec{E}}{\partial p^2}k_y + v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_z\right]\omega = -2\vec{v}_d\cdot\vec{k}\,\frac{\partial^2\vec{E}}{\partial p^2}\,\omega$$

$$\left(\vec{v}_d\cdot\nabla\right)^2\vec{E} = \left(\vec{v}_d\cdot\nabla\right)\left(\vec{v}_d\cdot\nabla\right)\vec{E} = \left(\vec{v}_d\cdot\nabla\right)\left[v_{dx}\frac{\partial\vec{E}}{\partial x} + v_{dy}\frac{\partial\vec{E}}{\partial y} + v_{dx}\frac{\partial\vec{E}}{\partial z}\right] =$$
$$= \left(\vec{v}_d\cdot\nabla\right)\left[v_{dx}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial x} + v_{dy}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial y} + v_{dz}\frac{\partial\vec{E}}{\partial p}\frac{\partial p}{\partial z}\right] = \left(\vec{v}_d\cdot\nabla\right)\left[v_{dx}\frac{\partial\vec{E}}{\partial p}k_x + v_{dy}\frac{\partial\vec{E}}{\partial p}k_y + v_{dz}\frac{\partial\vec{E}}{\partial p}k_z\right] =$$
$$= v_{dx}\frac{\partial}{\partial p}\left[v_{dx}\frac{\partial\vec{E}}{\partial p}k_x + v_{dy}\frac{\partial\vec{E}}{\partial p}k_y + v_{dz}\frac{\partial\vec{E}}{\partial p}k_z\right]\frac{\partial p}{\partial x} + v_{dy}\frac{\partial}{\partial p}\left[v_{dx}\frac{\partial\vec{E}}{\partial p}k_x + v_{dy}\frac{\partial\vec{E}}{\partial p}k_y + v_{dz}\frac{\partial\vec{E}}{\partial p}k_z\right]\frac{\partial p}{\partial y} +$$
$$+ v_{dz}\frac{\partial}{\partial p}\left[v_{dx}\frac{\partial\vec{E}}{\partial p}k_x + v_{dy}\frac{\partial\vec{E}}{\partial p}k_y + v_{dz}\frac{\partial\vec{E}}{\partial p}k_z\right]\frac{\partial p}{\partial z} =$$
$$= \left[v_{dx}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_x + v_{dx}v_{dy}\frac{\partial^2\vec{E}}{\partial p^2}k_y + v_{dx}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_z\right]k_x + \left[v_{dx}v_{dy}\frac{\partial^2\vec{E}}{\partial p^2}k_x + v_{dy}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_y + v_{dy}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_z\right]k_y +$$
$$+ \left[v_{dx}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_x + v_{dy}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_y + v_{dz}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_z\right]k_z =$$
$$= v_{dx}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_x^{\ 2} + v_{dx}v_{dy}\frac{\partial^2\vec{E}}{\partial p^2}k_xk_y + v_{dx}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_xk_z + v_{dx}v_{dy}\frac{\partial^2\vec{E}}{\partial p^2}k_xk_y + v_{dy}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_y^{\ 2} + v_{dy}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_yk_z +$$
$$+ v_{dx}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_xk_z + v_{dy}v_{dz}\frac{\partial^2\vec{E}}{\partial p^2}k_yk_z + v_{dz}^{\ 2}\frac{\partial^2\vec{E}}{\partial p^2}k_z^{\ 2} =$$
$$= \left(v_{dx}^{\ 2}k_x^{\ 2} + 2v_{dx}v_{dy}k_xk_y + 2v_{dx}v_{dz}k_xk_z + v_{dy}^{\ 2}k_y^{\ 2} + 2v_{dy}v_{dz}k_yk_z + v_{dz}^{\ 2}k_z^{\ 2}\right)\frac{\partial^2\vec{E}}{\partial p^2} = \left(\vec{v}_d\cdot\vec{k}\right)^2\frac{\partial^2\vec{E}}{\partial p^2}$$

It follows that:

$$\frac{d^2\vec{E}}{dt^2}=\frac{\partial^2\vec{E}}{\partial p^2}\omega^2+\vec{a}_d\cdot\vec{k}\ \frac{\partial\vec{E}}{\partial p}-2\vec{v}_d\cdot\vec{k}\ \frac{\partial^2\vec{E}}{\partial p^2}\ \omega+\left(\vec{v}_d\cdot\vec{k}\right)^2\ \frac{\partial^2\vec{E}}{\partial p^2}=$$

$$=\left[\ \omega^2-2\vec{v}_d\cdot\vec{k}\ \omega+\left(\vec{v}_d\cdot\vec{k}\right)^2\right]\frac{\partial^2\vec{E}}{\partial p^2}+\vec{a}_d\cdot\vec{k}\ \frac{\partial\vec{E}}{\partial p}=$$

$$=\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2\frac{\partial^2\vec{E}}{\partial p^2}+\vec{a}_d\cdot\vec{k}\ \frac{\partial\vec{E}}{\partial p} \qquad \text{(res. 2)}$$

Using (res. 1) and (res. 2):

$$\nabla^2\vec{E}-\mu_0\varepsilon_0\frac{d^2\vec{E}}{dt^2}=k^2\ \frac{\partial^2\vec{E}}{\partial\ p^2}-\mu_0\varepsilon_0\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2\frac{\partial^2\vec{E}}{\partial p^2}+\mu_0\varepsilon_0\ \vec{a}_d\cdot\vec{k}\ \frac{\partial\vec{E}}{\partial p}=0$$

which means:

$$\left\{\ k^2-\frac{1}{c^2}\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2\ \right\}\frac{\partial^2\vec{E}}{\partial p^2}+\frac{1}{c^2}\ \vec{a}_d\cdot\vec{k}\ \frac{\partial\vec{E}}{\partial p}=0$$

Limiting ourselves to the consideration of inertial detectors only, where $\vec{a}_d=0$, it follows that:

$$\left\{\ k^2-\frac{1}{c^2}\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2\ \right\}\frac{\partial^2\vec{E}}{\partial p^2}=0$$

Consequently:

$$k^2-\frac{1}{c^2}\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2=0 \qquad\qquad c^2k^2-\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2=0$$

$$c^2k^2=\left[\omega-\left(\vec{v}_d\cdot\vec{k}\right)\right]^2 \qquad\qquad \pm\,c\,k=\omega-\left(\vec{v}_d\cdot\vec{k}\right)$$

So:

$$\omega=\pm\,c\,k+\vec{k}\cdot\vec{v}_d$$

This means that, for inertial detectors, the phase propagation speed is:

$$v_{ph}=\frac{\omega}{k}=\pm\,c+\frac{\vec{k}}{k}\cdot\vec{v}_d$$

In addition:

$$p=\vec{k}\cdot\vec{r}-\omega t=\ \vec{k}\cdot\vec{r}-\left(\pm\,c\,k+\vec{k}\cdot\vec{v}_d\right)t$$

Note

Returning to the general case of non-inertial detectors, it has been shown that the wave equation, for monochromatic solutions, takes the form:

$$\left\{ k^2 - \frac{1}{c^2}\left[\omega - \left(\vec{v}_d \cdot \vec{k}\right)\right]^2 \right\} \frac{\partial^2 \vec{E}}{\partial p^2} + \frac{1}{c^2}\, \vec{a}_d \cdot \vec{k}\, \frac{\partial \vec{E}}{\partial p} = 0$$

The fulfillment of this constraint is equivalent to the contemporary fulfillment of:

$$k^2 - \frac{1}{c^2}\left[\omega - \left(\vec{v}_d \cdot \vec{k}\right)\right]^2 = 0 \qquad \text{and} \qquad \vec{a}_d \cdot \vec{k} = 0$$

The second constraint is not compatible with the arbitrariness of $\vec{a}$.

This result simply demonstrates the incompatibility between accelerated detectors and a monochromatic description of the wave.
Since the frequency perceived by the detector depends on its velocity, if the detector has variable velocity, the frequency cannot be described by a single value, so the wave cannot be described as monochromatic.

E. **Curl and the total time derivative operators' permutability**

$$\frac{d}{d\,t}=\frac{\partial}{\partial t}+\left(\vec{v}_d\cdot\nabla\right) \qquad \nabla\equiv\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$$

$$\vec{v}_d\cdot\nabla\equiv v_{d\,x}\frac{\partial}{\partial x}+v_{d\,y}\frac{\partial}{\partial y}+v_{d\,z}\frac{\partial}{\partial z} \qquad \frac{d}{dt}\left(\nabla\wedge\vec{A}\right)=\frac{\partial}{\partial t}\left(\nabla\wedge\vec{A}\right)+\left(\vec{v}_d\cdot\nabla\right)\left(\nabla\wedge\vec{A}\right)$$

$$\nabla\wedge\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\vec{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\vec{j}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\vec{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$\begin{aligned}
&\left(\vec{v}_d\cdot\nabla\right)\left(\nabla\wedge\vec{A}\right)=v_{d\,x}\frac{\partial}{\partial x}\left(\nabla\wedge\vec{A}\right)+v_{d\,y}\frac{\partial}{\partial y}\left(\nabla\wedge\vec{A}\right)+v_{d\,z}\frac{\partial}{\partial z}\left(\nabla\wedge\vec{A}\right)=\\
&=\vec{i}\left[v_{d\,x}\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+v_{d\,y}\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+v_{d\,z}\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\right]+\\
&-\vec{j}\left[v_{d\,x}\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+v_{d\,y}\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+v_{d\,z}\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\right]+\\
&+\vec{k}\left[v_{d\,x}\frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+v_{d\,y}\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+v_{d\,z}\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right]=\\
&=\vec{i}\left[v_{d\,x}\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_y}{\partial x\partial z}\right)+v_{d\,y}\left(\frac{\partial^2 A_z}{\partial y^2}-\frac{\partial^2 A_y}{\partial y\partial z}\right)+v_{d\,z}\left(\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2 A_y}{\partial z^2}\right)\right]\\
&-\vec{j}\left[v_{d\,x}\left(\frac{\partial^2 A_z}{\partial x^2}-\frac{\partial^2 A_x}{\partial x\partial z}\right)+v_{d\,y}\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y\partial z}\right)+v_{d\,z}\left(\frac{\partial^2 A_z}{\partial x\partial z}-\frac{\partial^2 A_x}{\partial z^2}\right)\right]\\
&+\vec{k}\left[v_{d\,x}\left(\frac{\partial^2 A_y}{\partial x^2}-\frac{\partial^2 A_x}{\partial x\partial y}\right)+v_{d\,y}\left(\frac{\partial^2 A_y}{\partial x\partial y}-\frac{\partial^2 A_x}{\partial y^2}\right)+v_{d\,z}\left(\frac{\partial^2 A_y}{\partial x\partial z}-\frac{\partial^2 A_x}{\partial y\partial z}\right)\right]
\end{aligned}$$

(res. a)

$$\begin{aligned}
&\left(\vec{v}_d\cdot\nabla\right)\vec{A}=v_{d\,x}\frac{\partial}{\partial x}\vec{A}+v_{d\,y}\frac{\partial}{\partial y}\vec{A}+v_{d\,z}\frac{\partial}{\partial z}\vec{A}=\\
&=v_{d\,x}\frac{\partial}{\partial x}\left(\vec{i}A_x+\vec{j}A_y+\vec{k}A_z\right)+v_{d\,y}\frac{\partial}{\partial y}\left(\vec{i}A_x+\vec{j}A_y+\vec{k}A_z\right)+v_{d\,z}\frac{\partial}{\partial z}\left(\vec{i}A_x+\vec{j}A_y+\vec{k}A_z\right)=\\
&=\vec{i}\left(v_{d\,x}\frac{\partial A_x}{\partial x}+v_{d\,y}\frac{\partial A_x}{\partial y}+v_{d\,z}\frac{\partial A_x}{\partial z}\right)+\\
&+\vec{j}\left(v_{d\,x}\frac{\partial A_y}{\partial x}+v_{d\,y}\frac{\partial A_y}{\partial y}+v_{d\,z}\frac{\partial A_y}{\partial z}\right)+\\
&+\vec{k}\left(v_{d\,x}\frac{\partial A_z}{\partial x}+v_{d\,y}\frac{\partial A_z}{\partial y}+v_{d\,z}\frac{\partial A_z}{\partial z}\right)=\\
&=\vec{i}\;p_1+\vec{j}\;p_2+\vec{k}\;p_3
\end{aligned}$$

It follows that:

$$\nabla \wedge \left[\left(\vec{v}_d \cdot \nabla \right) \vec{A} \right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p_1 & p_2 & p_3 \end{vmatrix} = \vec{i} \left(\frac{\partial\, p_3}{\partial y} - \frac{\partial\, p_2}{\partial z} \right) - \vec{j} \left(\frac{\partial\, p_3}{\partial x} - \frac{\partial\, p_3}{\partial z} \right) + \vec{k} \left(\frac{\partial\, p_2}{\partial x} - \frac{\partial\, p_3}{\partial y} \right) =$$

$$= \vec{i} \left[\left(v_{dx} \frac{\partial^2 A_z}{\partial x \partial y} + v_{dy} \frac{\partial^2 A_z}{\partial y^2} + v_{dz} \frac{\partial^2 A_z}{\partial y \partial z} \right) - \left(v_{dx} \frac{\partial^2 A_y}{\partial x \partial z} + v_{dy} \frac{\partial^2 A_y}{\partial y \partial z} + v_{dz} \frac{\partial^2 A_y}{\partial z^2} \right) \right]$$
$$- \vec{j} \left[\left(v_{dx} \frac{\partial^2 A_z}{\partial x^2} + v_{dy} \frac{\partial^2 A_z}{\partial x \partial y} + v_{dz} \frac{\partial^2 A_z}{\partial x \partial z} \right) - \left(v_{dx} \frac{\partial^2 A_x}{\partial x \partial z} + v_{dy} \frac{\partial^2 A_x}{\partial y \partial z} + v_{dz} \frac{\partial^2 A_x}{\partial z^2} \right) \right]$$
$$+ \vec{k} \left[\left(v_{dx} \frac{\partial^2 A_y}{\partial x^2} + v_{dy} \frac{\partial^2 A_y}{\partial x \partial y} + v_{dz} \frac{\partial^2 A_y}{\partial x \partial z} \right) - \left(v_{dx} \frac{\partial^2 A_x}{\partial x \partial y} + v_{dy} \frac{\partial^2 A_x}{\partial y^2} + v_{dz} \frac{\partial^2 A_x}{\partial y \partial z} \right) \right] \qquad \text{(res. b)}$$

By comparing the terms of (res. a) and (res. b), one can deduce the identity of the two expressions. Therefore:

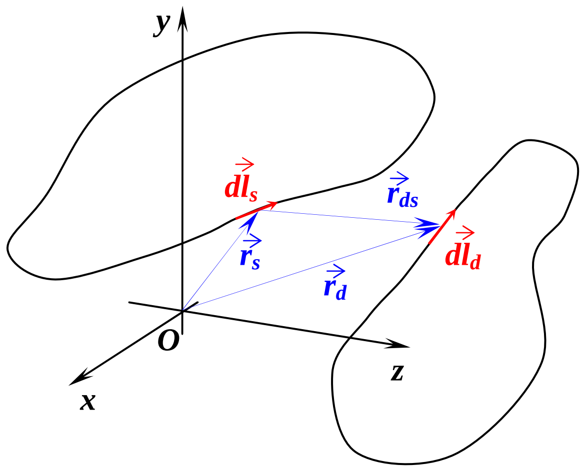
$$\left(\vec{v}_d \cdot \nabla \right) \left(\nabla \wedge \vec{A} \right) = \nabla \wedge \left[\left(\vec{v}_d \cdot \nabla \right) \vec{A} \right]$$

It follows that:

$$\frac{d}{dt} \left(\nabla \wedge \vec{A} \right) = \frac{\partial}{\partial t} \left(\nabla \wedge \vec{A} \right) + \left(\vec{v}_d \cdot \nabla \right) \left(\nabla \wedge \vec{A} \right) = \nabla \wedge \frac{\partial \vec{A}}{\partial t} + \nabla \wedge \left[\left(\vec{v}_d \cdot \nabla \right) \vec{A} \right] = \nabla \wedge \left[\frac{\partial \vec{A}}{\partial t} + \left(\vec{v}_d \cdot \nabla \right) \vec{A} \right] = \nabla \wedge \frac{d \vec{A}}{dt}$$

(q.e.d.)

F. **Regarding expressions of the force between current elements**



In the historical development of electromagnetism, two alternative laws have been proposed in magnetostatics to describe the force between constant current elements.
These laws are known as the Ampere force and the Grassmann force (see reference[10]).
Let $d\vec{l}_S$ be an infinitesimal stretch of a circuit in which the continuous current I_S flows.
Let $d\vec{l}_d$ be an infinitesimal stretch of a second circuit in which the continuous current I_d flows.
We shall use the following notation:

$$\vec{r}_{dS} = \vec{r}_d - \vec{r}_S$$

$$\vec{u}_{dS} = \vec{r}_d - \vec{r}_S / r_{dS}$$

$$\vec{u}_{Sd} = \vec{r}_S - \vec{r}_d / r_{dS} = -\vec{u}_{dS}$$

Let $d^2\vec{F}_d$ be the force exerted on $I_d d\vec{l}_d$ by $I_S d\vec{l}_S$, and $d^2\vec{F}_S$ be the force exerted on $I_S d\vec{l}_S$ by $I_d d\vec{l}_d$.

According to Ampere:

$$d^2\vec{F}_d^A = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\left[3\left(d\vec{l}_S \cdot \vec{u}_{dS}\right)\left(d\vec{l}_d \cdot \vec{u}_{dS}\right) - 2\left(d\vec{l}_S \cdot d\vec{l}_d\right)\right]\vec{u}_{dS}$$

$$d^2\vec{F}_S^A = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2}\left[3\left(d\vec{l}_S \cdot \vec{u}_{Sd}\right)\left(d\vec{l}_d \cdot \vec{u}_{Sd}\right) - 2\left(d\vec{l}_S \cdot d\vec{l}_d\right)\right]\vec{u}_{Sd}$$

Therefore: $\quad d^2\vec{F}_d^A = -d^2\vec{F}_S^A$

The forces $d^2\vec{F}_d^A$ and $d^2\vec{F}_S^A$ also share the same line of application.
Thus the principle of action and reaction (Newton's third law) applies.

According to Grassmann:

$$d^2\vec{F}_d^G = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2} d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right)$$

$$d^2\vec{F}_S^G = \frac{\mu_0 I_S I_d}{4\pi r_{dS}^2} d\vec{l}_S \wedge \left(d\vec{l}_d \wedge \vec{u}_{Sd}\right)$$

The Grassmann force is adopted by Maxwell's electromagnetism.
It also appears in the literature with the denominations of the Biot-Savart force, because it is obtainable by means of the two Laplace formulas, the first of which is a generalization of the Biot-Savart law and the second is equivalent to the Lorentz force.
Indeed:

$$d^2\vec{F}_d^G = I_d\ d\vec{l}_d \wedge d\vec{B}_d$$ Second Laplace Formula (Lorentz force)

$$d\vec{B}_d = d\vec{B}(\vec{r}_d) = \frac{\mu_0\, I_S}{4\pi}\frac{d\vec{l}_S \wedge (\vec{r}_d - \vec{r}_S)}{|\vec{r}_d - \vec{r}_S|^3}$$ First Laplace Formula, a generalization of the Biot-Savart law for the evaluation of the magnetic field generated by a rectilinear conductor.

So: $$d^2\vec{F}_d^G = I_d\ d\vec{l}_d \wedge \frac{\mu_0\, I_S}{4\pi}\frac{d\vec{l}_S \wedge \vec{r}_{dS}}{|\vec{r}_{dS}|^3} = \frac{\mu_0\, I_S\, I_d}{4\pi\, r_{dS}{}^2}\, d\vec{l}_d \wedge \left(d\vec{l}_S \wedge \vec{u}_{dS}\right)$$

Similarly:

$$d^2\vec{F}_S^G = I_S\ d\vec{l}_S \wedge d\vec{B}_S$$

$$d\vec{B}_S = d\vec{B}(\vec{r}_S) = \frac{\mu_0\, I_d}{4\pi}\frac{d\vec{l}_d \wedge (\vec{r}_S - \vec{r}_d)}{|\vec{r}_S - \vec{r}_d|^3}$$

So: $$d^2\vec{F}_S^G = I_S\ d\vec{l}_S \wedge \frac{\mu_0\, I_d}{4\pi}\frac{d\vec{l}_d \wedge \vec{r}_{Sd}}{|\vec{r}_{Sd}|^3} = \frac{\mu_0\, I_S\, I_d}{4\pi\, r_{dS}{}^2}\, d\vec{l}_S \wedge \left(d\vec{l}_d \wedge \vec{u}_{Sd}\right)$$

Since $d^2\vec{F}_d^G$ and $d^2\vec{F}_S^G$ are respectively perpendicular to $d\vec{l}_d$ and $d\vec{l}_S$, we can clearly see that such forces do not necessarily share the same line of application.

So, in general: $$d^2\vec{F}_d^G \neq -d^2\vec{F}_S^G$$

Consequently, Newton's third law is not respected.

It has been shown[11] that by evaluating the forces acting between complete circuits (necessarily closed), the Ampere and Grassmann laws produce the same results.
On the other hand, these two laws envisage different results when considering the forces exerted on limited sections of the circuits.

G. **Gradient and curl of the function** $\dfrac{1}{|\vec r-\vec r\,'|}$

$$\frac{1}{|\vec r-\vec r\,'|}=\left[(x-x')^2+(y-y')^2+(z-z')^2\ \right]^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{|\vec r-\vec r\,'|}\right)=-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\ \right]^{-\frac{3}{2}}2(x-x')=-\frac{x-x'}{|\vec r-\vec r\,'|^3}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{|\vec r-\vec r\,'|}\right)=-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\ \right]^{-\frac{3}{2}}2(y-y')=-\frac{y-y'}{|\vec r-\vec r\,'|^3}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{|\vec r-\vec r\,'|}\right)=-\frac{1}{2}\left[(x-x')^2+(y-y')^2+(z-z')^2\ \right]^{-\frac{3}{2}}2(z-z')=-\frac{z-z'}{|\vec r-\vec r\,'|^3}$$

It follows that:

$$\nabla\frac{1}{|\vec r-\vec r\,'|}=\frac{\partial}{\partial x}\left(\frac{1}{|\vec r-\vec r\,'|}\right)\vec i+\frac{\partial}{\partial y}\left(\frac{1}{|\vec r-\vec r\,'|}\right)\vec j+\frac{\partial}{\partial z}\left(\frac{1}{|\vec r-\vec r\,'|}\right)\vec k=-\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3}=-\frac{\vec u_{\vec r-\vec r\,'}}{|\vec r-\vec r\,'|^2}$$

Since:

$$\nabla\wedge\vec A=\begin{vmatrix}\vec i & \vec j & \vec k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\vec i\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\vec j\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\vec k\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

it follows that:

$$\nabla\wedge\left(\frac{d\vec l\,'}{|\vec r-\vec r\,'|}\right)=\vec i\left[\frac{\partial}{\partial y}\left(\frac{dl_z'}{|\vec r-\vec r\,'|}\right)-\frac{\partial}{\partial z}\left(\frac{dl_y'}{|\vec r-\vec r\,'|}\right)\right]+$$
$$-\vec j\left[\frac{\partial}{\partial x}\left(\frac{dl_z'}{|\vec r-\vec r\,'|}\right)-\frac{\partial}{\partial z}\left(\frac{dl_x'}{|\vec r-\vec r\,'|}\right)\right]+\vec k\left[\frac{\partial}{\partial x}\left(\frac{dl_y'}{|\vec r-\vec r\,'|}\right)-\frac{\partial}{\partial y}\left(\frac{dl_x'}{|\vec r-\vec r\,'|}\right)\right]$$

$$\nabla\wedge\left(\frac{d\vec l\,'}{|\vec r-\vec r\,'|}\right)=\vec i\left[-\frac{y-y'}{|\vec r-\vec r\,'|^3}dl_z'+\frac{z-z'}{|\vec r-\vec r\,'|^3}dl_y'\right]+$$
$$-\vec j\left[-\frac{x-x'}{|\vec r-\vec r\,'|^3}dl_z'+\frac{z-z'}{|\vec r-\vec r\,'|^3}dl_x'\right]+\vec k\left[-\frac{x-x'}{|\vec r-\vec r\,'|^3}dl_y'+\frac{y-y'}{|\vec r-\vec r\,'|^3}dl_x'\right]$$

$$\nabla\wedge\left(\frac{d\vec l\,'}{|\vec r-\vec r\,'|}\right)=\frac{-1}{|\vec r-\vec r\,'|^3}\left\{\vec i\left[(y-y')dl_z'-(z-z')dl_y'\right]+\right.$$
$$\left.-\vec j\left[(x-x')dl_z'-(z-z')dl_x'\right]+\vec k\left[(x-x')dl_y'-(y-y')dl_x'\right]\right\}$$

Given: $\vec{a} = a_x \, \vec{i} + a_y \, \vec{j} + a_z \, \vec{k}$ $\quad\quad$ $\vec{b} = b_x \, \vec{i} + b_y \, \vec{j} + b_z \, \vec{k}$

it follows that: $\vec{a} \wedge \vec{b} = \left(a_y b_z - a_z b_y \right) \vec{i} + \left(a_z b_x - a_x b_z \right) \vec{j} + \left(a_x b_y - a_y b_x \right) \vec{k}$

Given that: $\vec{r} - \vec{r}\,' = (x - x')\, \vec{i} + (y - y')\, \vec{j} + (z - z')\, \vec{k}$

and: $d\vec{l}\,' = dl_x' \, \vec{i} + dl_y' \, \vec{j} + dl_z' \, \vec{k}$

it follows that:

$$\left(\vec{r} - \vec{r}\,' \right) \wedge d\vec{l}\,' = \vec{i} \left[(y - y') dl_z' - (z - z') dl_y' \right] - \vec{j} \left[(x - x') dl_z' - (z - z') dl_x' \right] + \vec{k} \left[(x - x') dl_y' - (y - y') dl_x' \right]$$

So:

$$\nabla \wedge \left(\frac{d\vec{l}\,'}{\left| \vec{r} - \vec{r}\,' \right|} \right) = \frac{-1}{\left| \vec{r} - \vec{r}\,' \right|^3} \left(\vec{r} - \vec{r}\,' \right) \wedge d\vec{l}\,' = - \frac{\vec{u}_{\vec{r} - \vec{r}\,'}}{\left| \vec{r} - \vec{r}\,' \right|^2} \wedge d\vec{l}$$

H. **Poisson and wave equations for instantaneous and induced components**

Taking the rotor of (a8) and using the vector identity $\nabla \wedge \left(\nabla \wedge \vec{V}\right) = \nabla\left(\nabla \cdot \vec{V}\right) - \nabla^2 \vec{V}$, it follows that:

$$\nabla \wedge \left(\nabla \wedge \vec{B}_1\right) = \nabla \wedge \left(\mu_0 \vec{J}_m\right) \qquad \Rightarrow \qquad \nabla\left(\nabla \cdot \vec{B}_1\right) - \nabla^2 \vec{B}_1 = \mu_0 \nabla \wedge \vec{J}_m$$

Considering (a7): $-\nabla^2 \vec{B}_1 = \mu_0 \nabla \wedge \vec{J}_m$

Equation (a18) follows as a consequence.

Taking the rotor of (a10) and using the previous identity, we can say that:

$$\nabla \wedge \left(\nabla \wedge \vec{B}_2\right) = \nabla \wedge \left(\mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t}\right) \qquad \Rightarrow \qquad \nabla\left(\nabla \cdot \vec{B}_2\right) - \nabla^2 \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d}{d\,t}\left(\nabla \wedge \vec{E}\right)$$

Considering (a1): $\nabla\left(\nabla \cdot \vec{B}_2\right) - \nabla^2 \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d}{d\,t}\left(\nabla \wedge \vec{E}_1 + \nabla \wedge \vec{E}_2\right)$

Considering (a9), (a4) and (a6): $-\nabla^2 \vec{B}_2 = \mu_0 \varepsilon_0 \frac{d}{d\,t}\left(-\frac{d\vec{B}}{d\,t}\right)$

Considering (a2): $-\nabla^2 \vec{B}_2 = -\mu_0 \varepsilon_0 \frac{d}{d\,t}\left(\frac{d\vec{B}_1}{d\,t}\right) - \mu_0 \varepsilon_0 \frac{d}{d\,t}\left(\frac{d\vec{B}_2}{d\,t}\right)$

Equation (a19) can therefore be deduced.

Taking the rotor of (a4), it follows that:

$$\nabla \wedge \left(\nabla \wedge \vec{E}_1\right) = 0 \qquad \Rightarrow \qquad \nabla\left(\nabla \cdot \vec{E}_1\right) - \nabla^2 \vec{E}_1 = 0$$

Considering (a3), equation (a20) follows as a consequence.

Taking the rotor of (a6), we can say:

$$\nabla \wedge \left(\nabla \wedge \vec{E}_2\right) = -\nabla \wedge \frac{d\vec{B}}{d\,t} \qquad \Rightarrow \qquad \nabla\left(\nabla \cdot \vec{E}_2\right) - \nabla^2 \vec{E}_2 = -\frac{d}{dt}\left(\nabla \wedge \vec{B}\right)$$

Considering (a2): $\nabla\left(\nabla \cdot \vec{E}_2\right) - \nabla^2 \vec{E}_2 = -\frac{d}{dt}\left(\nabla \wedge \vec{B}_1 + \nabla \wedge \vec{B}_2\right)$

Considering (a5): $\nabla^2 \vec{E}_2 = \frac{d}{dt}\left(\nabla \wedge \vec{B}_1\right) + \frac{d}{dt}\left(\nabla \wedge \vec{B}_2\right)$

Considering (a8) and (a10): $\nabla^2 \vec{E}_2 = \frac{d}{dt}\left(\mu_0 \vec{J}_m\right) + \frac{d}{dt}\left(\mu_0 \varepsilon_0 \frac{d\vec{E}}{d\,t}\right)$

Considering (a1): $\nabla^2 \vec{E}_2 = \mu_0 \frac{d\,\vec{J}_m}{dt} + \mu_0 \varepsilon_0 \frac{d}{dt}\left[\frac{d}{d\,t}\left(\vec{E}_1 + \vec{E}_2\right)\right]$

Expressed differently:

$$\nabla^2 \vec{E}_2 - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}_2}{dt^2} = \mu_0 \frac{d\,\vec{J}_m}{dt} + \mu_0 \varepsilon_0 \frac{d^2 \vec{E}_1}{dt^2}$$

Equation (a21) follows as a consequence.

I. Gauge invariance in the proposed theory with instantaneous and induced components

$$\vec{E} = -\nabla\varphi_1 - \nabla\varphi_2 - \frac{d\vec{A}_1}{dt} - \frac{d\vec{A}_2}{dt} \qquad\qquad \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_d \cdot \nabla)$$

So: $$\vec{E} = -\nabla\varphi_1 - \nabla\varphi_2 - \frac{\partial\vec{A}_1}{\partial t} - (\vec{v}_d \cdot \nabla)\vec{A}_1 - \frac{\partial\vec{A}_2}{\partial t} - (\vec{v}_d \cdot \nabla)\vec{A}_2$$

Since:

$$(\vec{v}_d \cdot \nabla)\vec{A}_1 = \nabla(\vec{v}_d \cdot \vec{A}_1) - \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) \qquad\qquad (\vec{v}_d \cdot \nabla)\vec{A}_2 = \nabla(\vec{v}_d \cdot \vec{A}_2) - \vec{v}_d \wedge (\nabla \wedge \vec{A}_2)$$

it follows that:

$$\vec{E} = -\nabla\varphi_1 - \nabla\varphi_2 - \frac{\partial\vec{A}_1}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_1) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) - \frac{\partial\vec{A}_2}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_2) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2)$$

Applying a gauge variation:

$$\vec{A}_1^{\wedge} = \vec{A}_1 + \nabla\psi_1 \qquad \vec{A}_2^{\wedge} = \vec{A}_2 + \nabla\psi_2 \qquad \varphi_2^{\wedge} = \varphi_2 - \frac{d\psi_1}{dt} - \frac{d\psi_2}{dt}$$

we obtain:

$$\vec{E}^{\wedge} = -\nabla\varphi_1 - \nabla\varphi_2^{\wedge} - \frac{\partial\vec{A}_1^{\wedge}}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_1^{\wedge}) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1^{\wedge}) - \frac{\partial\vec{A}_2^{\wedge}}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_2^{\wedge}) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2^{\wedge})$$

$$\vec{E}^{\wedge} = -\nabla\varphi_1 - \nabla\left(\varphi_2 - \frac{d\psi_1}{dt} - \frac{d\psi_2}{dt}\right) - \frac{\partial(\vec{A}_1 + \nabla\psi_1)}{\partial t} - \nabla(\vec{v}_d \cdot (\vec{A}_1 + \nabla\psi_1)) + \vec{v}_d \wedge (\nabla \wedge (\vec{A}_1 + \nabla\psi_1)) +$$

$$- \frac{\partial(\vec{A}_2 + \nabla\psi_2)}{\partial t} - \nabla(\vec{v}_d \cdot (\vec{A}_2 + \nabla\psi_2)) + \vec{v}_d \wedge (\nabla \wedge (\vec{A}_2 + \nabla\psi_2)) =$$

$$= -\nabla\varphi_1 - \nabla\varphi_2 + \frac{d\nabla\psi_1}{dt} + \frac{d\nabla\psi_2}{dt} - \frac{\partial\vec{A}_1}{\partial t} - \frac{\partial\nabla\psi_1}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_1) - \nabla(\vec{v}_d \cdot \nabla\psi_1) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) +$$

$$+ \vec{v}_d \wedge (\nabla \wedge \nabla\psi_1) - \frac{\partial\vec{A}_2}{\partial t} - \frac{\partial\nabla\psi_2}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_2) - \nabla(\vec{v}_d \cdot \nabla\psi_2) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2) + \vec{v}_d \wedge (\nabla \wedge \nabla\psi_2) =$$

$$= -\nabla\varphi_1 - \nabla\varphi_2 + \frac{\partial\nabla\psi_1}{\partial t} + (\vec{v}_d \cdot \nabla)(\nabla\psi_1) + \frac{\partial\nabla\psi_2}{\partial t} + (\vec{v}_d \cdot \nabla)(\nabla\psi_2) - \frac{\partial\vec{A}_1}{\partial t} - \frac{\partial\nabla\psi_1}{\partial t} +$$

$$- \nabla(\vec{v}_d \cdot \vec{A}_1) - \nabla(\vec{v}_d \cdot \nabla\psi_1) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) + \vec{v}_d \wedge (\nabla \wedge \nabla\psi_1) - \frac{\partial\vec{A}_2}{\partial t} - \frac{\partial\nabla\psi_2}{\partial t} +$$

$$- \nabla(\vec{v}_d \cdot \vec{A}_2) - \nabla(\vec{v}_d \cdot \nabla\psi_2) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2) + \vec{v}_d \wedge (\nabla \wedge \nabla\psi_2) =$$

$$= -\nabla\varphi_1 - \nabla\varphi_2 - \frac{\partial\vec{A}_1}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_1) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) - \frac{\partial\vec{A}_2}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_2) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2)$$

$$+ (\vec{v}_d \cdot \nabla)(\nabla\psi_1) + (\vec{v}_d \cdot \nabla)(\nabla\psi_2) - \nabla(\vec{v}_d \cdot \nabla\psi_1) - \nabla(\vec{v}_d \cdot \nabla\psi_2) + \vec{v}_d \wedge (\nabla \wedge \nabla\psi_1) + \vec{v}_d \wedge (\nabla \wedge \nabla\psi_2)$$

Since: $(\vec{v}_d \cdot \nabla)(\nabla\psi) = \nabla(\vec{v}_d \cdot \nabla\psi)$ it follows that:

$$\vec{E}^{\wedge} = -\nabla\varphi_1 - \nabla\varphi_2 - \frac{\partial\vec{A}_1}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_1) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_1) - \frac{\partial\vec{A}_2}{\partial t} - \nabla(\vec{v}_d \cdot \vec{A}_2) + \vec{v}_d \wedge (\nabla \wedge \vec{A}_2)$$

This means that: $\vec{E}^{\wedge} = \vec{E}$ $\qquad$ $\vec{F}^{\wedge} = \vec{F}$ $\qquad$ (q.e.d.)

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