# Demonstrating Lorenz Curve Distribution and Increasing Gini Coefficient with the Iterating (Koch Snowflake) Fractal Attractor.

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Abstract

Global income has increased exponentially over the last two hundred years; while, and at the same time respective Gini coefficients have also increased: this investigation tested whether this pattern is a property of the mathematical geometry termed a fractal attractor. The Koch Snowflake fractal was selected and inverted to best model economic production and growth: all triangle area sizes in the fractal grew with iteration-time from an arbitrary size – growing the total set. Area of triangle the 'bits' represented wealth. Kinematic analysis – velocity and acceleration – was undertaken, and it was noted growing triangles propagate in a sinusoidal spiral. Using Lorenz curve and Gini methods, bit size distribution – for each iteration-time – was graphed. The curves produced matched the regular Lorenz curve shape and expanded out to the right with fractal growth – increasing the corresponding Gini coefficients: contradicting Kuznets cycles. The 'gap' between iteration triangle sizes (wealth) was found to accelerate apart, just as it is conjectured to do so in reality. It was concluded the wealth (and income) Lorenz distribution – along with acceleration properties – is an aspect of the fractal. Form and change of the Lorenz curve are inextricably linked to the growth and development of a fractal attractor; and from this – given real economic data – it can be deduced an economy – whether cultural or not – behaves as a fractal and can be explained as a fractal. Questions of the discrete and wave properties and the accelerated expansion – similar to that of trees and the conjectured growth of universe at large – of the fractal growth, were discussed.

**Keywords:**

**Fractal geometry, Lorenz curve, Gini Coefficient, Wealth distribution**

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## 1 INTRODUCTION

The increasing spread of income and wealth distribution – and its possible acceleration (Piketty 2016) – has long been at the centre of controversy from both sides of the economic – and political – spectrum. The disparity is almost always presented as a negative consequence of the 'free' market and of economic growth and viewed as 'a wrong': something that should be controlled or stemmed. The OECD even purport it to be the cause of 'falls in economics growth' (OECD 2014). What if could be shown to be an aspect of a greater mathematical geometry: revealing it to be a universal, natural phenomenon – a property of nature, and indeed 'the market' itself? Would this relieve us of our concerns and many conjectures, much like Kepler's elliptical geometry did in the 17$^{th}$ Centaury – completing the problem of planetary orbits?

There is a geometry that may stand as a candidate, offering similar potential to the ellipse, it is 'the fractal' and fractal geometry. Developed during the last decades of the 19$^{th}$ Centenary: fractals generally known to be one of the best ways to describe nature.

The father of fractal geometry, Benoit Mandelbrot (1924-1910) – interestingly – said of himself to be 'a would be Kepler – of complexity' (Wolfram 2012). In his early work before fractals Mandelbrot wrote on powers laws and showed the income and wealth disparity to follow a typical ('natural') Pareto power law pattern (B. Mandelbrot 1960, *The Guardian* 2011), and in his original work on fractals, he used this geometry to describe the shape and behaviour of market prices through time (B. B. Mandelbrot 2008). In this paper I would like to continue his work, and show that not only are prices fractal, but indeed income distribution is too, and both are aspects of a (greater) fractal – one we term 'a market'. I believe there is a fundamental connection between power laws, fractals and income and wealth distribution.

To investigate the above, an emergent fractal attractor was tested to whether it produces a Lorenz curve and respective Gini coefficient – and if so, do they change with growth and with time. Is inequality inextricably linked to growth? The hypothesis of this study was yes they are linked: inequality is an aspect or property of the fractal, and an economy – whether cultural or not – is a fractal phenomenon, best described by fractal geometry.

## 1.1 The Lorenz Curve

Income and wealth distribution was first represented graphically by M. O. Lorenz's 1905 and the curve he produced – the Lorenz curve – named after him (Lorenz 1905). Shown in figure 1 below: the percentage of households is on the x-axis, and the percentage of income or wealth on the y-axis. The Lorenz curve shows the distribution of individuals' income (or wealth) for a given population. The Lorenz curve always falls below the line 'perfect equality' (a line of equal distribution throughout the population) and the Gini coefficient (termed 'Ginis' for short) tells the ratio between the Lorenz curve and the line of equality (area A), and the total (triangle) area under the line of equality (areas A and B).

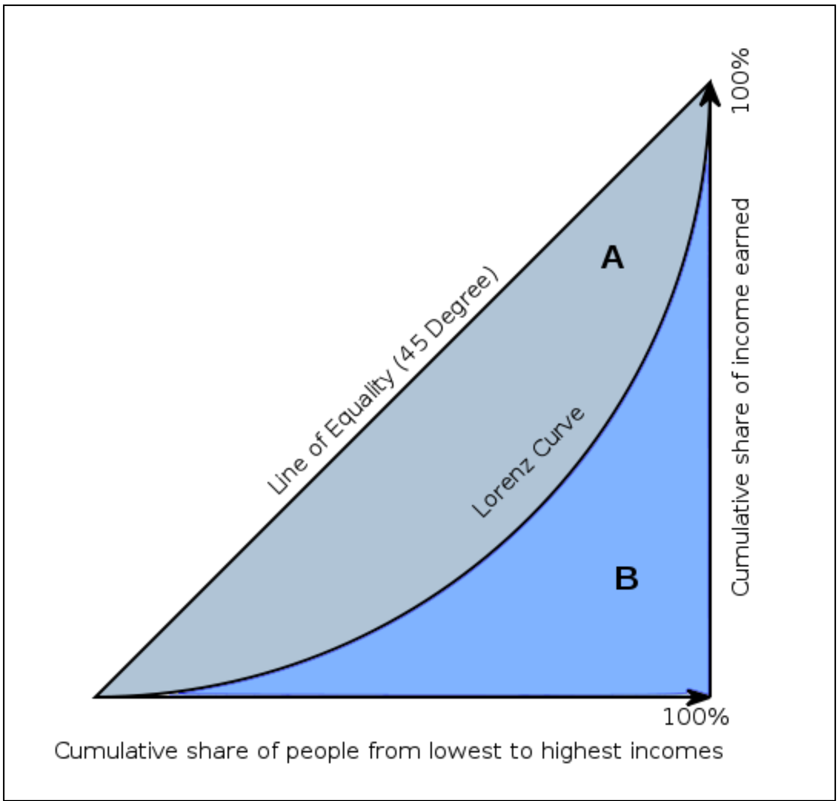


**Figure 1. Lorenz Diagram.** The graph shows that the Gini coefficient is equal to the area marked A divided by the sum of the areas marked A and B. that is, Gini = A / (A + B). It is also equal to 2*A due to the fact that A + B = 0.5 (since the axes scale from 0 to 1)**("Gini Coefficient" 2015)**.

Lorenz distribution has already been shown to be universal or fractal: it is a 'natural' scale invariant property observed in any wealth or income distribution of any population – whether cultural or not (Damgaard and Weiner 2000), and is as relevant to the natural sciences as it is to the classical economics. Ecologist Geerat J. Vermeij remarks: " One of the most pervasive and far-reaching realities in economic systems is inequality, the tendency for one party involved in an interaction over resources to gain more, or less, than its rival." (Vermeij 2009)

## 1.2 High Wealth Gini and Exponential Income Growth

Wealth is a stock concept and income a flow, and both maybe argued to be causal to each other: income creates wealth; wealth creates income. Parkin explains the difference between the wealth Lorenz and the income Lorenz curves is the wealth excludes human capital: 'wealth and income are just different ways of looking at the same thing…because the national survey of wealth excludes human capital, income distribution is a more accurate measure of income inequality' (Parkin, Powell, and Matthews 2008). Not withstanding this, both wealth and income produce comparable Lorenz curves – even if the wealth Gini coefficients are higher than income. In this study the focus was on wealth distribution – due directly to the 'stock' nature of the fractal: for this reason, the relationship between wealth and income is assumed inextricable and equivalent.

Real data shows local – by country –wealth Ginis increase only to be curtailed by a possible 'Kuznets effect' – where Ginis rise and then are fall with economic growth and development due intervention – or as Milanovic posits it, technology, openness and politics (TOP) (Milanovic 2016b). For America alone, Ginis have increased (Fortune 2015), and the wealth proportion of 'the 1%' also increased in America from 1980 to 2000 (Domhof 2016) – a period of market liberalisation. This increase is also evident in China from the period 1967 to 2007 – see appendix figure 4 below. Most other developed nations have also had their Gini increase in recent years ("Distribution of Wealth" 2015) as shown below in appendix figure 5.

If Milanovic is right, and the up and down Ginis – so called Kznet waves – persist, what then is their cause? This is to Ferreira ' the great epistemological challenge of inequality analysis' (Ferreira 2016). At least for the 'ups', can the fractal offer a solution?

On long term income growth: data for the last two Centuries show global national income increased exponentially, while corresponding income Ginis over the same period also show increase (though at a decreasing rate) – figure 2 below (Milanovic 2009). Assuming income and wealth are the same, does this exponential growth of income and concurrent increasing Gini coefficients concur with the fractal development?

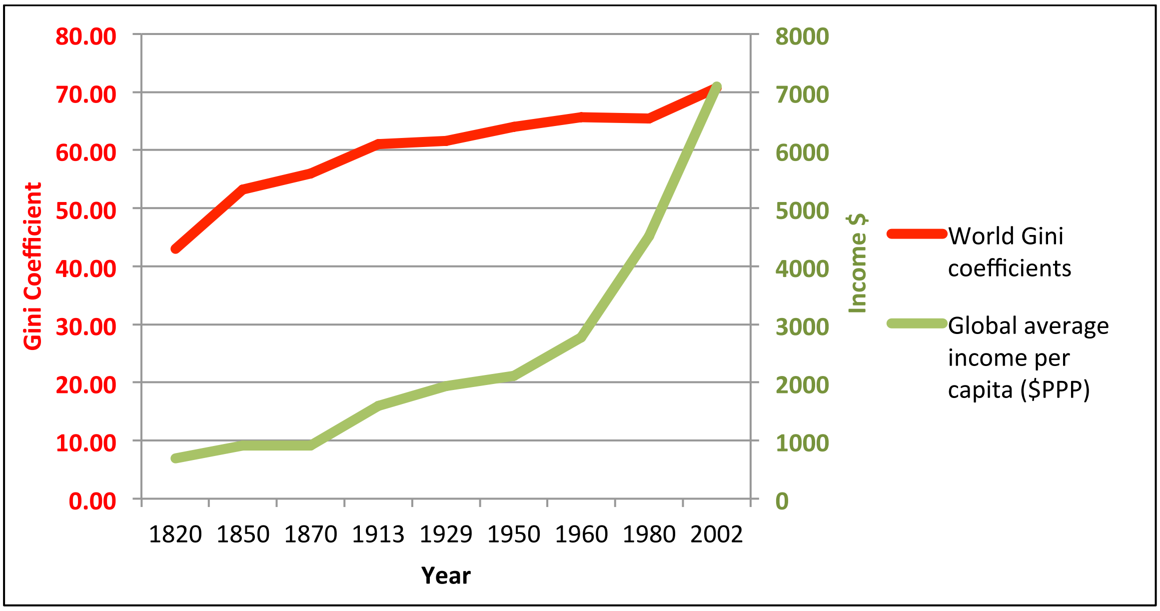


**Figure 2 Global Income and Gini Coefficients for last 2 Centuries.** Over the last two centuries global income growth (green) per capita has been exponential and respective Gini coefficient (red) increasing (at a decreasing rate). (Milanovic 2009)

## 1.3 The Classical Fractal

Fractals – also described as L-systems – are emergent objects that develop and (or) grow with the iteration of a simple rule. They possess self-similarity at all scales and can be observed as being regular but irregular (same but different) objects. They are classically demonstrated by the original Mandelbrot Set (B. B. Mandelbrot 1980), and – in one of their most simplest forms – the Koch Snowflake (Figure 1 below, A and B respectively). Familiar examples of them in reality are clouds, waves, coastlines and trees. All fractals have a defined 'fractal dimension'. Stewart in his book on chaos (and fractals) 'God does not play dice' – said: '..coastlines and Koch Snowflakes are equally rough' (Stewart 1997). Indeed, the very close fractal dimension values of both the Koch Snowflake and the coastlines of islands (Great Britain) 1.26, and between 1.15 and 1.2 respectively – stands as testament of their power to best model nature and reality.

Figure 1 B below shows the classical view of fractal growth (or development) – achieved by the iteration of a simple rule, by adding more (triangle) bits – of diminishing size – to the previous triangle, originating from an initial (iteration 0) bit – also known as the 'axiom' in L-system theory. The snowflake shape is formed at and around 5 or 6 iterations; from this point on – to the observer – it no longer changes shape. This growth to 'equilibrium' observation is not lost to this economist, and will be the subject of another future investigation; but it is the comparison or distribution of the size of triangles – from the small to the original large – that is the focus of this study.

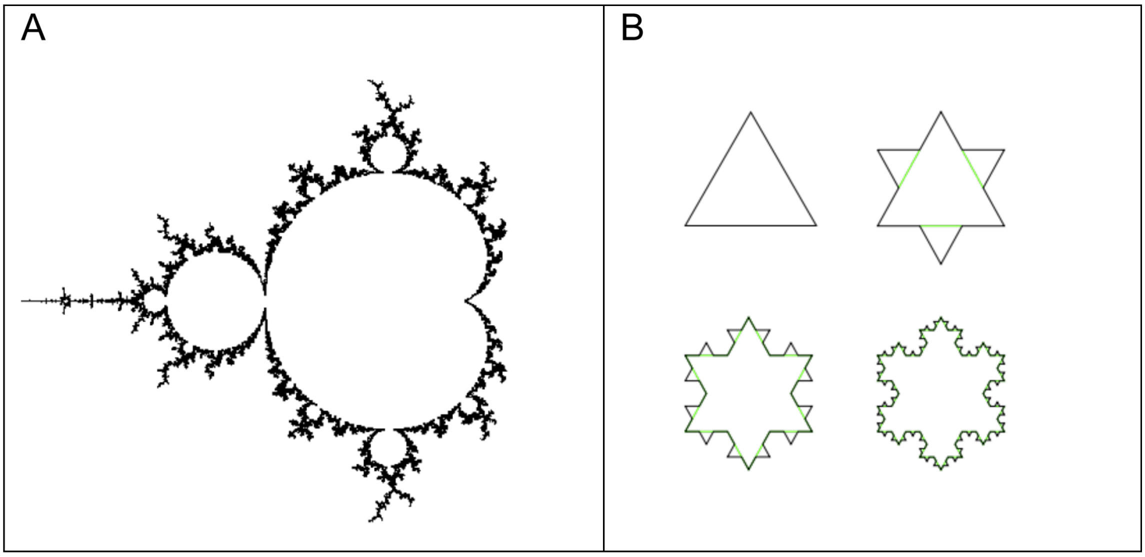


**Figure 3. (Classical) Fractals.** (A) boundary of the Mandelbrot set; (B) The Koch Snowflake fractal from iteration-time (*t*) 0 to 3. Reference: (A) (Prokofiev 2007); (B) **("Koch Snowflake" 2014)**.

Fractals are by their nature complex objects – but this property of complexity poses a problem, and it was for this reason the Koch Snowflake fractal attractor was chosen for analysis. The Koch Snowflake has regular-regularity rather than regular-irregularity (or complexity).

Testing was done using standard Lorenz curve and Gini coefficient methods. The distribution of triangle areas were graphed and Gini coefficients calculated for each iteration time as the fractal grew (or developed).

Area of the triangle bits was assumed to stand for an individual's wealth.

## 1.4 Production, and the Inverted Fractal

To 'produce' a Lorenz curve with a fractal attractor, we first need to determine what it meant by production and thus growth of the fractal. When we attempt to do this, we find there is a paradox – there are two views that conflict with each other: one where the original triangle bit size remains constant, and the new bits diminish in size as the fractal iterates – this is termed (A) 'a consumption perspective'; and (B) where it is the new triangle bits size remain constant, and all earlier triangle bits expand and grow as the fractal iterates – termed 'a production perspective'. A is the classical view – as shown in figure 3B (and A in figure 4 below) and B the 'inverted', shown in figure 4 B. Both of these views, A and B, are relative views of the same process: both are true, but only one really describes the production and the growth from production, and for this study it is assumed to be the inverted view B.

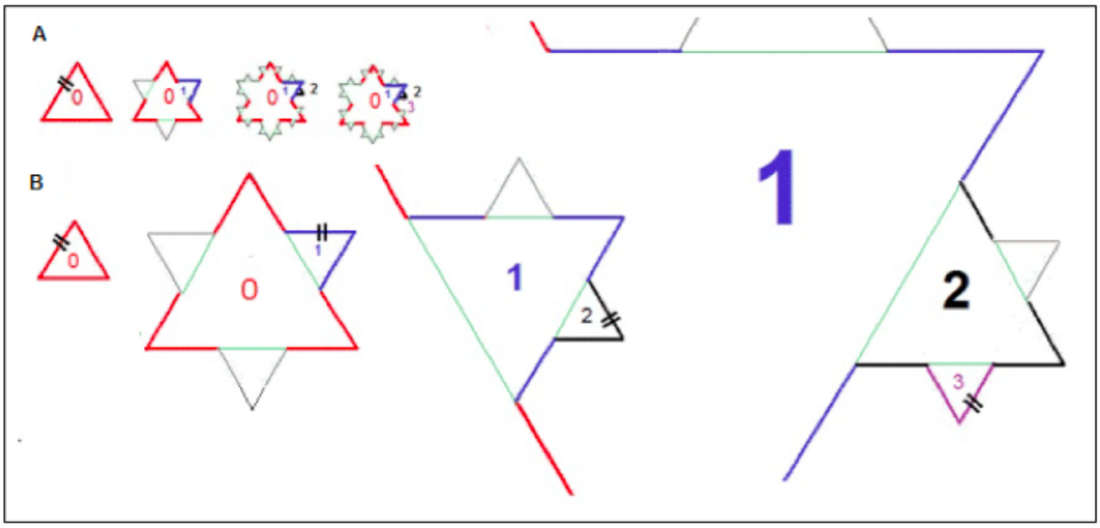


**Figure 4. Expansion of the inverted Koch Snowflake fractal (fractspansion).** The schematics above demonstrate fractal development by (A) the classical Snowflake perspective, where the standard sized thatched (iteration '0') is the focus, and the following triangles diminish in size from colour red iteration 0 to colour purple iteration 3; and (B) the inverted, fractspanding perspective where the new (thatched) triangle is the focus and held at standard size while the original red iteration 0 triangle expands in area – as the fractal iterates.

There is a practical reason for this: the 'inverted' production growth can be 'observed' in real life nature fractals, trees. With trees (thus all plants – in principle) it is the trunk of the tree that grows, and the new branch size that remains constant. This is converse to 'our' static observation of trees, where the (larger) constant sized trunk is observed with diminishing sized branches 'protruding' – self similar – from it.

### 1.5 Acceleration

With inversion (figure 4 B) it can be shown as the fractal grows larger and larger with respect to iteration time, but as it does the distance between centre-points of the original thatched triangle and the older (from iteration 0 to iteration 3) accelerate apart. To support this conjecture – and the inversion view – a recent study of trees has shown tree's growth – and thus all plants – accelerate with age (Stephenson et al. 2014) – again matching this inversion perspective. To test and analyse the consequences of exponential (income or wealth) growth and the affect of 'groups' kinematic analysis was conducted on the fractal. If the fractal behaves as an economy and produces a Lorenz curve, groups will accelerate apart.

The relationship between these 'observation' and inverted 'production' views of the fractal – with reference to economic theory – must be addressed, but much of this is outside the scope of this investigation, and so for the purposes of this investigation the fractal was 'inverted' and analysed. It should be made clear, inversion or not, it is the same geometry analysed, just with different perspectives: one from the top looking down, the other from the bottom looking up.

## 2 METHODS

To analyse the inverted fractal two spread sheet models were developed, and stored in a repository: one to analyse the inverted growth behaviour of the Koch Snowflake fractal (Macdonald 2016); and another to model the Lorenz and Gini behaviour (Macdonald 2015).

To create a quantitative data series for analysis (used in both spread sheet models) the classical Koch Snowflake area equations were adapted to invert the fractal. A data table was produced to calculate the area growth at each and every iteration of a single triangle. Area was calculated from the following formula (1) measured in standard (arbitrary) centimetre units (cm)

$$A = \frac{l^2\sqrt{3}}{4} \tag{1}$$

where (***A***) was the area of a single triangle, and where ***l*** was the triangle's base length. Length in Table 1 was set to 15.1967128766173 $cm^{-1}$ so the area of the initial (base) triangle (***i***=0) approximated an arbitrary area of 100 $cm^{-2}$. To expand the triangle with iteration (and thus invert the fractal) the base length was multiplied by 3 (instead of dividing it by 3 as used in the classical snowflake method). The iteration-time (***i***) number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Iteration '0' refers to a pre iteration size; the first position. Calculations were made to the arbitrary 12th iteration, and the results graphed.

## 2.1 Lorenz Curve

A method to construct a Lorenz curve by speadsheet is demonstrated by (arnoldhite 2016): this method was followed. A table was created ranking triangles by their size at each iteration time – in ascending order. At each ranked quantity the following was calculated:

1. a percentage quantity (Quantity/ Total Quantity) – for the line of equality;
2. a percentage area (Area/ Total Area) – for the Lorenz curve;
3. Cumulative percentage Area;
4. and finally – for the calculation of the Gini Coefficient – the area under the Lorenz Curve.

To clarify, the first triangle represents the initial condition for iteration: it is the first individual.

## 2.2 Gini Coefficient

An elementary method to calculate the Gini coefficient by spreadsheet is demonstrated by (arnoldhite 2014) – this method was followed. The Gini Coefficient is a calculated by dividing the area between the line of equality and the Lorenz curve (area A in figure 1 above) by the total area under the line of equality:

$$\frac{\boldsymbol{A}}{(\boldsymbol{A}+\boldsymbol{B}).} \tag{2}$$

Summing all the areas under the Lorenz Curve gives the area of B.

Gini Coefficients were calculated for each iteration-time, and for each iteration-time a new table was created.

## 2.3 Analysis of Area Expansion of the Total Inverted Fractal

To analyse the velocity and acceleration of expansion of inverted fractal to an observer within the fractal another spread sheet model was developed, and stored in a repository (Macdonald 2016).

With iteration new triangles are (in discrete quantities) introduced into the set – at an exponential rate. While the areas of new triangles remain constant, the earlier triangles expand, and by this the total fractal set expands. To calculate the area change of a total inverted fractal (as it iterated), the area of the single triangle (at each iteration time) was multiplied by its corresponding quantity of triangles (at each iteration time).

Two data tables (tables 3 and 4 in the 'inverted fractal' spread sheet file) were developed. Table 3 columns were filled with the calculated triangle areas at each of the corresponding iteration time – beginning with the birth of the triangle and continuing to iteration ten. In table 4, triangle areas of table 3 were multiplied by the number of triangles in the series corresponding with their iteration time.

Values calculated in table 3 and 4 were totalled and analysed in a new table (table 5). Analysed were: total area expansion per iteration, expansion ratio, expansion velocity, expansion acceleration, and expansion acceleration ratio. Calculations in the columns used kinematic equations developed below.

## 2.4 Kinematics

Classical physics equations were used to calculate velocity and acceleration of: the receding points (table 2) and the increasing area (table 5).

### 2.4.1 Velocity

Velocity ($\boldsymbol{v}$) was calculated by the following equation

$$\boldsymbol{v} = \frac{\Delta \boldsymbol{d}}{\Delta \boldsymbol{i}} \tag{3}$$

where classical time was exchanged for iteration time ($\boldsymbol{i}$). Velocity is measured in standard units per iteration $\boldsymbol{cm^{-1}\, i^{-1}}$ for receding points and $\boldsymbol{cm^{-2}\, i^{-1}}$ for increasing area.

#### 2.4.2 Acceleration

Acceleration ($\boldsymbol{a}$) was calculated by the following equation

$$\boldsymbol{a} = \frac{\Delta \boldsymbol{v}}{\Delta \boldsymbol{i}} \tag{4}$$

Acceleration is measured in standard units per iteration $\boldsymbol{cm^{-1}\, i^{-2}}$ and $\boldsymbol{cm^{-2}\, i^{-2}}$.

### 2.5 Velocity-Distance ('gap') Diagram

To test for increasing wealth velocity against distance (the gap) between groups – analogist to cosmology's Hubble's Law – a Hubble 'like' a scatter graph diagram titled the 'Fractal/ Hubble-Wealth diagram' was constructed from the results of the recession velocity and distance calculations (in table 2 of the 'inverted fractal' spread sheet file). On the x-axis was the total distance (not displacement from observer) of triangle centre points at each iteration time from $\boldsymbol{i_0}$; and on the y-axis the expansion velocity at each iteration time. A best fitting linear regression line was calculated and an equation (5) was derived

$$\boldsymbol{v} = \boldsymbol{K_0 D} \tag{5}$$

where $\boldsymbol{K_0}$ the (present) Hubble (like) wealth constant (the gradient), and $\boldsymbol{D}$ the distance.

### 2.6 Acceleration-Distance Diagram

Using the same methods as used to develop the Fractal/ Hubble-Wealth diagram (as described above in 2.5) an acceleration vs. distance diagram was created and an expansion constant derived.

## 2.7 Spiral Propagation though Time

The propagation of triangles in the (inverted) Koch Snowflake fractal, is not linear but in the form of a logarithmic spiral – as shown in Figure 2B (above), and in Appendix Figure 1. The method given thus far assumes, and calculates the linear circumference of this spiral and not the true displacement (the radius). This method was justified by arguing the required radius (or displacement) of the logarithmic spiral calculation was too complex to calculate, (and beyond the scope of this investigation), and that expansion inferences from inverted fractal could be made from the linear circumference alone. This been said, a spiral model was created independently, and radii measured to test whether spiral results were consistent with the linear results in the investigation. Measurements were made using TI – Nspire™ geometric software (see Appendix I Figure 1). Displacements, and the derived Hubble diagram from this radius model were expected to show significantly lower values than the above (calculated) circumference non-vector method, but nonetheless share the same (exponential) behaviour. Appendix Figure 1 shows in the distance between centre points, and in blue the displacement.

See Appendix Figures 1, and 2, and Table 1 for results.

# 3 RESULTS

Figure 5 below is a composite diagram of Lorenz curves from iteration-time 1 (the line of perfect equality) to iteration time 5. All curves are derived from the spreadsheet model.

## 3.1 Lorenz Curves

From figure 5 below: as the fractal iterated, the derived Lorenz curves expended out or shifted to the right of the 'line of equality'. At iteration-time 1 – the line of equality – distribution is homogenous; one triangle with one area. With iteration-time and growth the second iteration time ($i$=2) Lorenz showed 25% of the area is found in 75% of the triangles; and conversely the remaining 75% of the area is found in the remaining 25% of the triangles. At iteration-time 5 ($i$=5) this distribution has expanded and shows only a small percentage of the triangles account for some 99% of the area.

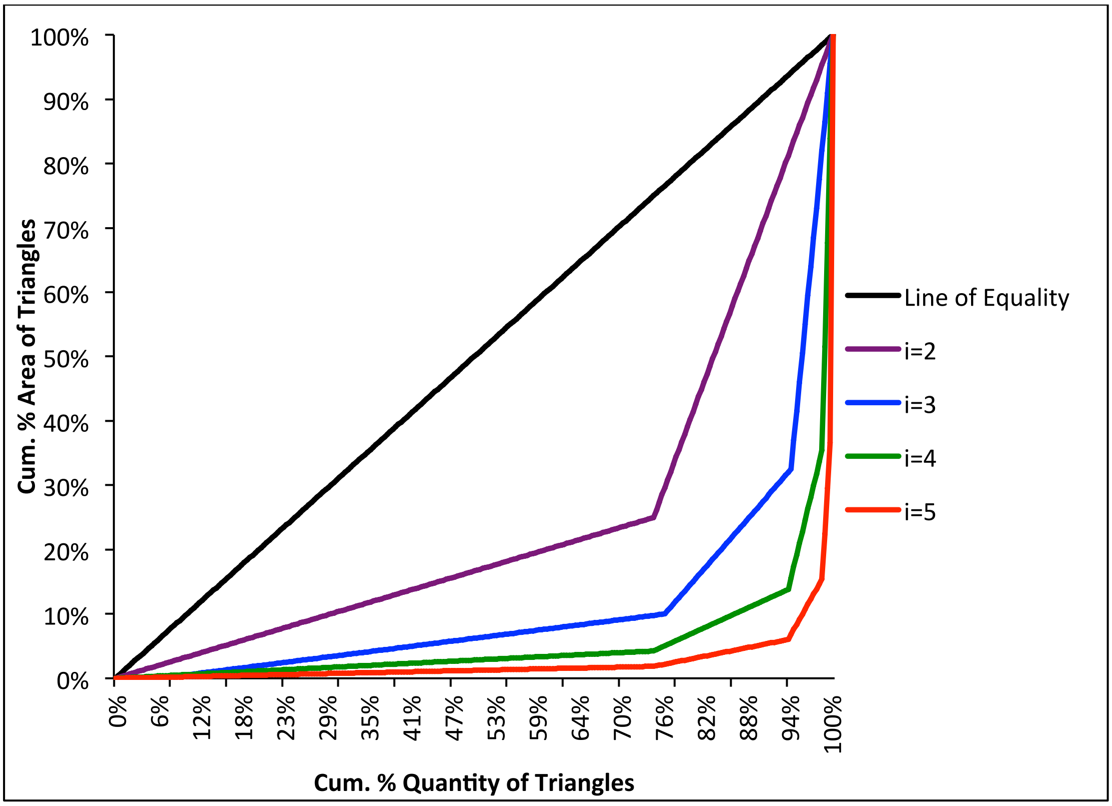


**Figure 5. Fractal (Koch Snowflake) Lorenz Curves Expansion.** Lorenz Curves are produced for the Koch Snowflake fractal from iteration (i) 2 to iteration 5. The Lorenz curve expands out to the right with growth and time as a result of increasing spread and quantity of triangles sizes.

## 3.2 Gini Coefficient

Gini coefficients at each iteration are shown below in figure 6. The Gini coefficient increased at a decreasing rate – approximating a value of 1 – with iteration-time.

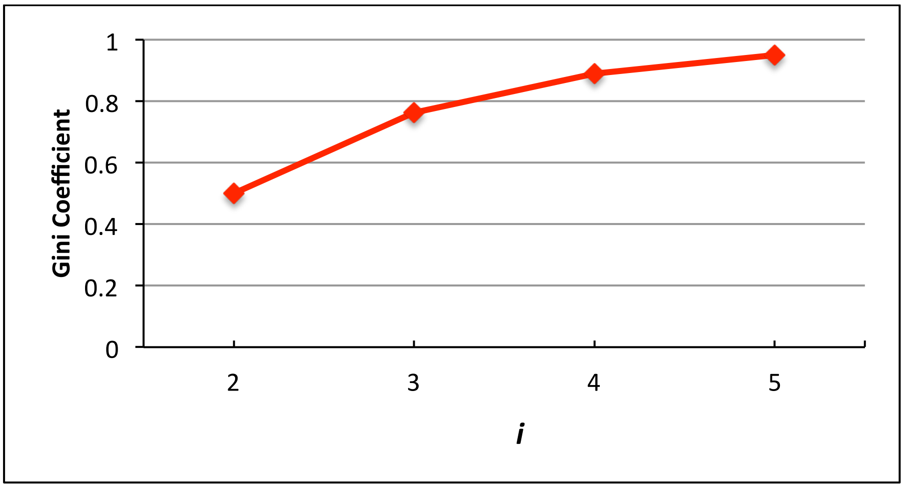


**Figure 6. Koch Snowflake Gini Coefficient by Iteration-time (*i*).** The Gini coefficient is a measure of the area between the line of equality and the Lorenz curve in relation to an area of perfect equality. As the (Koch Snowflake) fractal grows – and/or develops – with iteration time, the Gini Coefficient increases.

### 3.3 Expansion of Initial Triangle

The area of the initial triangle of the inverted Koch Snowflake fractal increased exponentially – shown here in Figure 7.

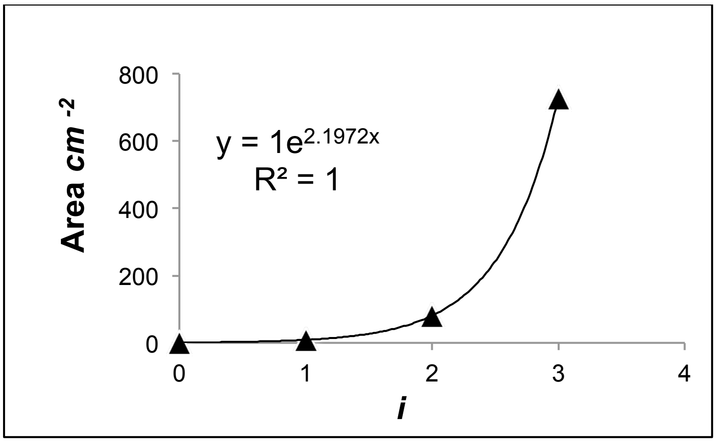


**Figure 7. Area Expansion of a single triangle.** iteration time (i). cm = centimetres.

This expansion with respect to iteration time is written as

$$\boldsymbol{A = 1e^{2.1972i}} \tag{6}$$

### 3.4 Total Fractal Expansion

The area of the total fractal (Figure 8A) and the distance between points (Figure 8B) of the inverted fractal also expanded exponentially.

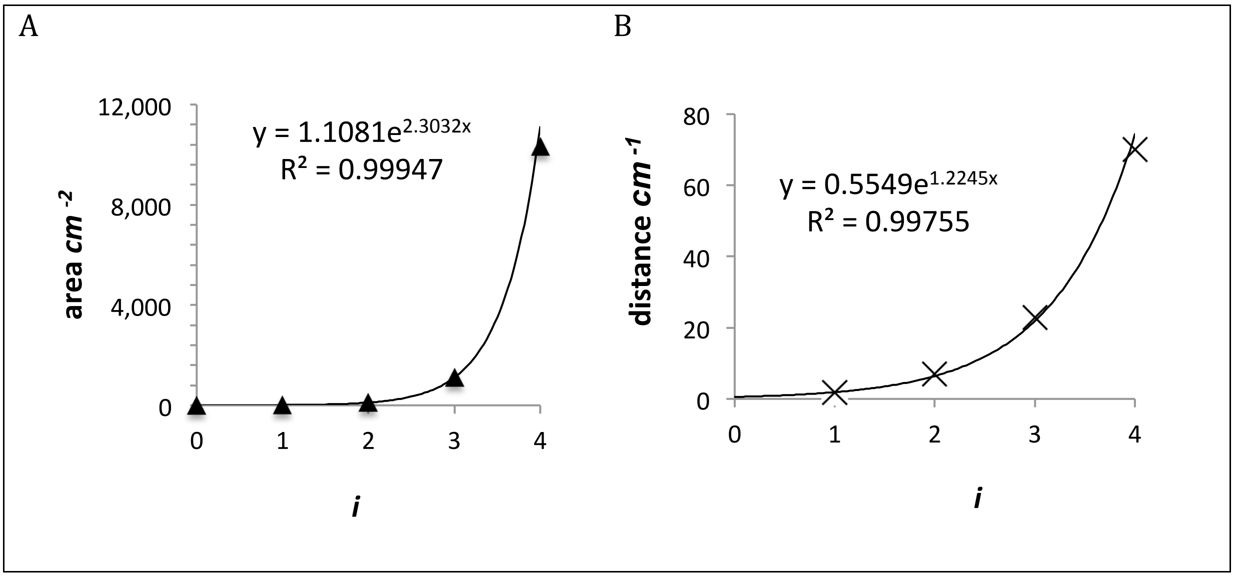

**Figure 8. Inverted Koch Snowflake fractal expansion per iteration time (i).** (A) total area expansion and (B) distance between points. cm = centimetres.

The expansion of the total area ($\boldsymbol{A}^{\boldsymbol{T}}$) is described as

$$\boldsymbol{A^T = 1.1081e^{2.3032i}} \tag{7}$$

The expansion of distance between points ($\boldsymbol{D}$) is described by the equation

$$\boldsymbol{D = 0.5549e^{1.2245i}} \tag{8}$$

## 3.5 Velocity

The (recession) velocities for both total area and distance between points (Figures 9A and 9B respectively) increased exponentially per iteration time.

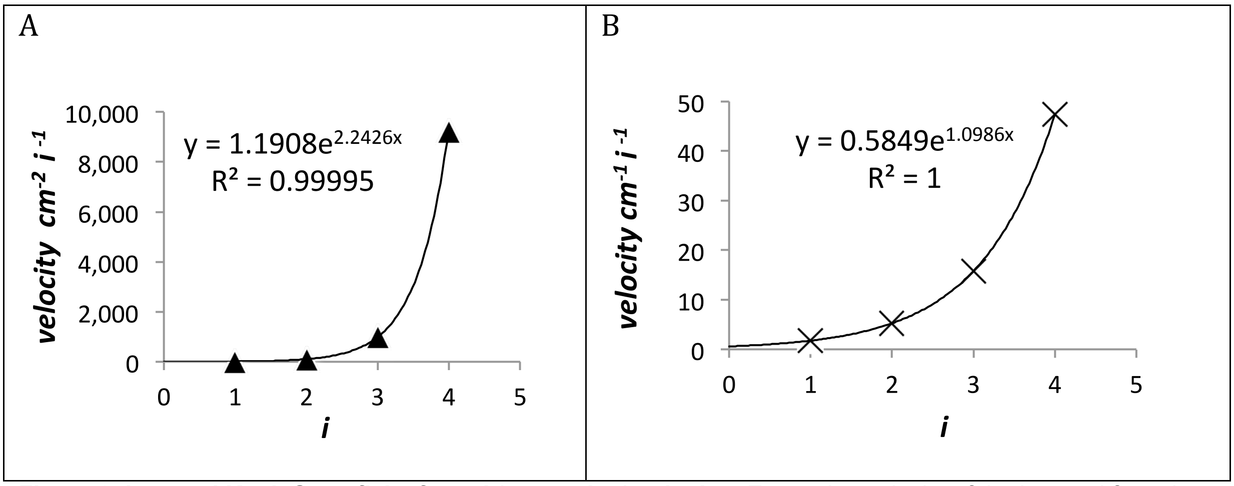


**Figure 9. Inverted Koch Snowflake fractal (expansion) velocity**. Expansion velocity of the inverted fractal at each corresponding iteration time (i): (A) expansion of total area, and (B) distance between points. cm = centimetres.

Velocity is described by the following equations respectively

$$\boldsymbol{v = 1.1908e^{2.3032i}} \tag{9}$$

$$\boldsymbol{v^T = 0.5549e^{1.0986i}} \tag{10}$$

where $v^T$ is the (recession) velocity of the total area; and $v$ the (recession) velocity of distance between points.

### 3.6 The Fractal/ Hubble-Wealth Diagram

As the distance between centre points increases (at each corresponding iteration time), so too does the recession velocity of the points – as shown in Figure 10 below.

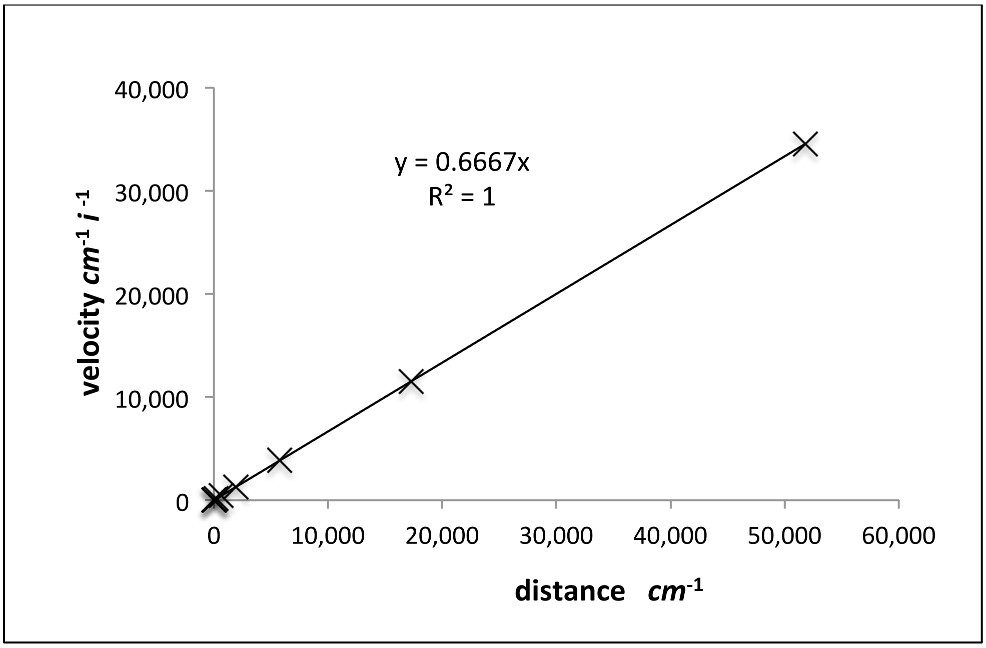


**Figure 10. The Fractal/ Hubble-Wealth diagram.** As distance between triangle geometric centres increases with iteration time (i), the recession velocity of the points increases. cm = centimetres.

Recession velocity vs. distance of the fractal is described by the equation

$$\boldsymbol{v = 0.6679D} \tag{11}$$

where the constant factor is measured in units of $cm^{-1}\ i^{-1}\ cm^{-1}$.

### 3.7 Acceleration of Area and Distance Between Points

The accelerations for both total area and (recession) distance between points (Figure 11A and 11B respectively) increased exponentially per iteration time.

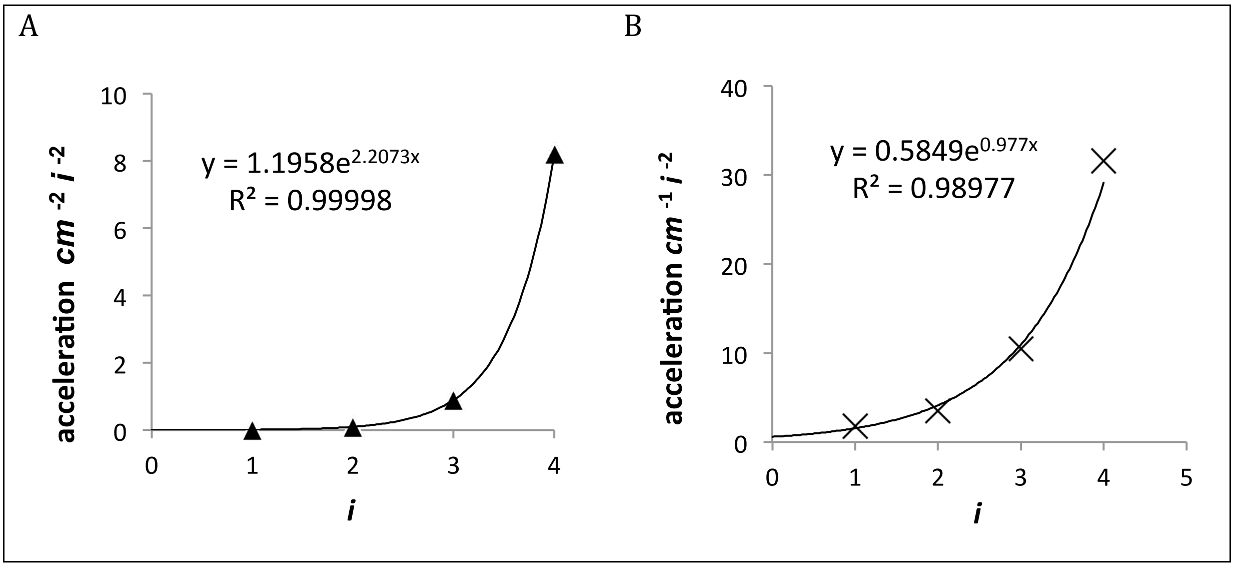


**Figure 11. Inverted Koch Snowflake fractal (expansion) acceleration.** Acceleration of the inverted fractal at each corresponding iteration time (i): (A) expansion of total area, and (B) distance between points. cm = centimetres.

Acceleration is described by the following equations respectively

$$\boldsymbol{a^T = 1.1958e^{2.2073i}} \tag{12}$$

$$\boldsymbol{a = 0.5849e^{0.977i}} \tag{13}$$

where $\boldsymbol{a^T}$ is the (recession) acceleration of the total area; $\boldsymbol{a}$ the (recession) acceleration of distance between points.

As the distance of centre points increases (at each corresponding iteration time) from an observer, so does the recession acceleration of the points (expanding away) – as shown in Figure 12 below.

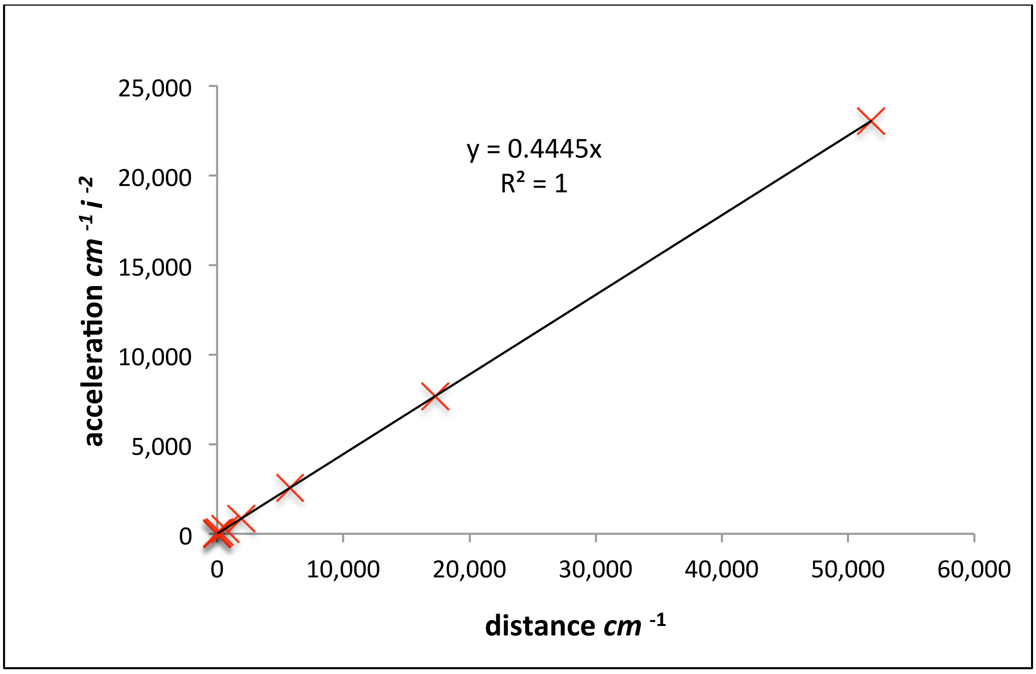


**Figure 12. Recessional acceleration vs. distance on the inverted Koch Snowflake fractal.** As distance between triangle geometric centres increases with iteration time (***i***), the recession acceleration of the points increases. ***cm*** = centimetres.

The recession acceleration of points at each iteration time at differing distances on the inverted fractal is described by the equation

$$\boldsymbol{a} = \mathbf{0.4447}\boldsymbol{D} \tag{14}$$

where the constant factor is measured in units of $cm^{-1}$ $\boldsymbol{i}^{-2}$ $cm^{-1}$. $\boldsymbol{a}$ = acceleration; $\boldsymbol{D}$ = distance.

## 4 DISCUSSIONS

In this study, the Koch Snowflake fractal attractor was modelled to describe an aspect of an economy – specifically growth and wealth distribution. There are however, many other aspects and insights to be taken from the same fractal – with respect to the market and knowledge – and all with broad implications. Some of these insights have been touched upon here, though in a superficial manner, because they are important to the topic – others not. The following discussions are what are considered most relevant to the topic of distribution; but together, in time – with the other potential insights on the fractal exposed – the fractal may sound the what we term 'the market' to be a fractal object, only separated in our reality by its intangibility and abstractness, but none the less, real.

## 4.1 A Fractal Lorenz Curve

The study demonstrated the distribution of bit area sizes on a fractal is not equal, and becomes more unequal as bits are produced with iteration-time: this appears to match real life Lorenz income distribution curves and how they are derived.

The iterating inverted Koch Snowflake fractal attractor produced a clear Lorenz curve (figure 5): one similar to any produced from real economic wealth and income data. As this pattern can also be produced from the static classical fractal, the study showed the Lorenz distribution analysis – the Lorenz curve – is an aspect or property of a fractal attractor and can be derived from all things fractal in nature. From this it was deduced the cultural/human market economy behaves as a fractal. Inequality of income or wealth distribution is often seen as an undesirable trade-off of the 'free market'; however, as revealed in this fractal experiment, it maybe more true to say this inequality is a natural phenomenon, a property of the 'market' – not unusual or special, but typical.

Insights from this model will be present in all structures that iterate (with time) and show fractal/ hierarchical structure – from clouds, to trees, to information, and even to the abundance of the elements universe. As an aside to further support the model, Lorenz curves were created from the branch weight distribution of a typical tree (Macdonald 2016a) and for the abundance of elements in the near 13.8 billion year old universe (see appendix figure 3)(Macdonald 2016b). Both of these experiments produced curves matching the fractal Lorenz curve and the economics Lorenz curves. Trees are clear fractals; and the universe is conjectured to be fractal. Fractal cosmology is a recognised body of science with it's own unique theory on the structure of the universe (Pietronero 1987).

### 4.1.1 Regular Regularity Assumption

The unrealistic 'regular-regular' assumption (as described in the introduction) demonstrated by the Koch Snowflake – as opposed to a more realistic rough 'regular-irregular' fractal like the Mandelbrot set – reveals only one structural aspect of the fractal attractor: it does not show the irregular chaotic or diverse aspects as revealed in individual countries Lorenz curves, and respective Gini coefficients – both static and temporal. It shows an aspect or property of all fractals, part of their structure. Reality is by no means regular; for instance, wealth is not added in equal bits as assumed by equal triangles, and wealth does not always grow as triangles grow, it can decrease.

#### 4.1.2 Growth and Recession

While the iterating fractal demonstrates growth, receding or recession would imply the process is reversed and all bit sizes decrease in size with iteration-time.

### 4.2 Gini Coefficients

The calculated area Gini coefficients from the fractal model also match real wealth Gini values – noticeably greater than real income Gini coefficients and thus more skewed to the right.

### 4.3 Shifting Lorenz Curve and Increasing Gini coefficient with Growth

Area growth of the fractal and inequality between area sizes are inextricability linked as a function of time: this appears to be a law of the fractal. If an economy is assumed to behave as a fractal attractor, we can expect its respective Lorenz curve will shift out to the right and the distribution of wealth, and or income will become – inextricable to growth – more unequal with time.

The outward shifting of the Lorenz curves and thus increasing fractal Gini coefficients (figure 6) is due to the exponential growth of the fractal (figure 8A). This may directly explain the increase in global income Gini coefficients with time along with the exponential (income) growth – observed over the last two Centuries (as introduced in figure 2).

### 4.4 Gini – Development Contradictions

As a result of growth, the fractal produced increasing fractal-Gini coefficients at – approaching a value of 1 (figure 6) – with respect to iteration-time. From this, using this fractal model it can be inferred the degree to which a real economy is developed by its Gini coefficient alone – given an fractal becomes more 'developed' and unequal as it grows – with time. This property suggests a low iteration-time (young) economy pertains to less inequality, and conversely, an economy with larger iteration-time a high inequality. This may however be contradicted by real income Gini data (though it is more supported for wealth distribution alone). Many less economically developed economics (LEDCs), have higher income Gini coefficients than more economically developed economies (MEDCs). LEDGs; however, maybe atypical, and may not have achieved these 'high' Gini values by growing (or emerging) in a fractal 'economic' manner with respect to iteration-time, but instead had their Gini coefficients anomalously distorted – to the high – only as a result of high degrees of corruption, or low degrees of technology, openness as posited by Milanovic .

## 4.5 Kuznets Curve Cycles

Continuing directly from the above, the fractal-Lorenz curve model, and its increasing Gini with time, directly contradicts the income Kuznets curve – and Kuznets cycles – where the Kuznets curve shows (income) Ginis first rise with economic growth and then fall. This result may suggest (again as mentioned above) the – decreasing Gini – Kuznets curve and Kuznets cycles behaviour is atypical – a cultural phenomenon, brought about by intervention. Indeed this – intervention – is how the Kuznets curve is generally reasoned: 're-distribution is achieved through progressive taxes and welfare payments, and job creation' (economics online 2016). This sustained increase of the fractal is in disagreement with Milanovic and others. Milanovic's puts the decrease – and dip in the middle class (appendix figure 6) – down to technology openness and politics: however, politics may well be the sole reason as technology and openness each on their own should lead to growth and thus wealth diversion, not the reverse. It maybe concluded the downward side of the Kuznets cycles are solely a product of intervention and the upward side a product of 'freer' liberalised markets associated with growth.

### 4.5.1 Matching Real Ginis

Real data showing increasing wealth and income Gini coefficients points to – or matches – freer markets and this matches how the fractal Gini grows. Ginis have grown as the world's income economy has grown over the last two Centuries (figure 2); and, in periods of 'freer markets' or during the recent time of globalisation (appendix figure 5), Ginis have also – on the whole – increased, this is especially evident in China (appendix figure 4).

## 4.6 Growth and Wealth from an Inverted Perspective

Unlike the classical Koch Snowflake fractal formation (as described in the introduction), the inverted Snowflake is not a static model; it is dynamic and it demonstrates growth. The following are the key insights from the model as a consequence of this growth.

### 4.6.1 Bit Size – Aggregate – Growth

The model demonstrated the simultaneous and inextricable growth of all bit sizes relative to each other with respect to time:  no matter their (arbitrary) starting size to grow the whole (aggregate) geometric set size, all the parts (all triangles) grow. In principle, any single bit size will in time grow to be equal to its predecessors: a branch on a tree – a real fractal – will in time grow to be the size of the trunk of the same tree. Of course when the 'single' branch achieves this size, the trunk, and the predecessors will have again grown.

### 4.6.2 Relative Position and Distribution

The inverted Koch Snowflake is in principle infinite in size – and thus in time. From this it can be inferred: wherever one's observation point is within the fractal set – this is to say, if a triangle bit is chosen at random as a observing position in the set, and an observer observes, from this bit, and looks forward and back – there will always be small 'bits' ahead, and always large 'bits' behind. Within the infinite set, the observer will never be the largest. This insight has relevance to one's position within an economic system: wherever one is on the 'wealth/income ladder', there will always be someone 'wealthier' above, and someone 'poorer' below – unless you are first and the largest. Of course at the scale and in the context of all 'complex' life on Earth, wealth is finite, but the range of sizes of wealth are extremely diverse – from arguably we humans, through to microbes, maybe molecules, and even – as discussed above – the elements (and beyond). Whatever the range, if something is fractal by nature (and nature appears to be fractal wherever we look), the distribution of sizes will fit a Lorenz distribution structure.

This principle is well described – indirectly – by Professor Hans Rosling of Gapminder in this BBC lecture on population. At (time 35:42) he shows the relative perspectives of looking up to the wealthy, and down to the poor: the very poorest think of the next group just as the middle class do the most richest (BBC 2016).

This relative position will be returned to when acceleration is discussed – below.

### 4.6.3 Refuting the 'Zero Sum Game' and Other Myths

From the above (4.6.1), the fractal model refutes conjectures like 'the rich get richer and the poor get poorer', or 'the zero sum game' of wealth growth. For the fractal to grow, all bit sizes grow, and this only happens when new bits are added – or new branches are formed.

If bit sizes were of equal size – and thus equally distributed – growth would be limited to one or two iteration-times, and arguably the fractal set would not grow at all. Further to this, a branch on a tree cannot hold branches of a similar size. Nowhere is this observed in nature, if it were, it would not be fractal. For the same reasons, an economy cannot be equally distributed.

### 4.6.4 Symbiosis

A (fractal-structured) tree demonstrates an act of internal symbiosis: the unequal branch size structure of branches allows for greater light capture – for the productive leaves. If the woody stem of a tree supports and services the leaves to gain more productive light, the wealth structure in an economy may serve a similar role supporting productive enterprises: without this it will not 'hold up'. From this it is deduced the fractal 'market' structure is the most efficient means to grow. If the trunk of the tree were to redistribute its area or volume to the outer (smaller) branches in 'an act of fairness', the fractal geometry (tree) would collapse. The same would result if the outer branches were to cannibalise (or tax) the trunk. It may be interesting – given the fractal has no bounds for examples in 'nature' – to investigate the degree to which other biological systems redistribute wealth or income within their own system by means of tax or regulation – examples are not obvious, and there is no mention of this in Vermeij's: Nature, An Economic History. In his book he goes to great length describing the universality of trade and exchange – shared in all economies.

### 4.6.5 Age Order

The fractal derived Lorenz curve reveals an age order structure of the wealth distribution with growth: the oldest have the largest proportion. This agrees with real tree development and the age of the elements in the universe on the periodic table: the trunk is the oldest 'bit'; and helium and hydrogen were the first elements formed straight after the big bang. Subatomic particles came before helium and hydrogen; and the larger – and fewer – elements like gold, came after. Whether this age order is prevalent in the economy maybe unclear to see at first, but the rule should hold on examination.

## 4.7 Velocity and Acceleration of Wealth

The inverted fractal model demonstrated exponential velocity of area growth (figure 9 above); this concurs with the long-term global income growth.

Wealth or income acceleration (shown figure 11 above); however, at least in the field of economics, is not so clear to see by real data – though it is often cased as a conjecture or as threat from the more left sided commentators. From this study, this conjecture can be relegated to a prediction. We can predict wealth and/or income will accelerate with time, as will the 'gap' between population groups (more on this below).

Appendix figure 6 (Milanovic 2016a) shows the global percentage growth at each percentile of the global population between 1988 and 2008. Apart from the (USA) middle class (which demands an explanation as to why not), all percentiles are growing by positive percentage and thus are – by compounding growth – accelerating not only together, but also accelerating apart from one another in nominal income terms.

If long-term growth acceleration were true – with time, this pattern would not be unique to an economy: the plants, and possibly even the universe as a whole have been found to accelerate in similar manner. Trees, 97% of them – taken from a study of some 600,000, on six continents, and 100 year of data collection– have also been shown to accelerate with growth (Stephenson et al. 2014). Trees are the obvious fractal structure; they have a fractal dimension and are regularly used as fractal examples. I have been able claim, using the inverted fractal model, plants – in their tree form – also comply to fractal geometry, and will accelerate with growth. With the universe, current observations have revealed it too to be not only (Hubble) expanding, but also – doing so

– at an accelerating rate. Both of these observations – expanding and accelerating – have no obvious explanation. The universe is said to be: 'been pulled apart' by a mysterious 'dark energy'. In my original study on the inverted fractal I applied this model same to this problem claiming the universe's accelerating expansion (the 'dark energy') and associated problems, maybe best described by (inverted) fractal geometry (Macdonald 2014). If these biological and cosmological acceleration observations are the same as this wealth and income acceleration, they may all be unified and explained by fractal geometry.

## 4.8 Fractal Hubble-Wealth Diagram

In the cosmology investigation using the same expanding inverted fractal I was able to produce a Hubble diagram (showing increasing recession velocity of galaxies with distance). As the model deriving this fractal Lorenz curve is the same as the inverted fractal model, I believe, given the matching principles of the model, this recessional velocity and acceleration between points can equally apply to an economy. The Fractal Hubble-Wealth diagram (figure 10 above) predicts as the velocity of wealth creation increases with 'distance' (or the gap) between the observer or individual and another group or individual. When velocity ($\boldsymbol{v}$) is plotted against distance of points ($\boldsymbol{D}$) (figure 10, and Appendix figure 2) the inverted fractal demonstrates Hubble's Law described by the equation

$$\boldsymbol{v} = \boldsymbol{F_v D} \tag{15}$$

where ($\boldsymbol{F_v}$) is the slope of the line of best fit – the fractal (Hubble-Wealth) recession velocity constant.

### 4.8.1 Exponentially Accelerating Gap

In the same cosmological study using the inverted fractal, a complementary diagram to the fractal Hubble diagram was produced showing acceleration by distance: accordingly figure 12 shows the 'gap' accelerates apart: the acceleration of wealth creation increases with 'distance' (or the gap) between the observer and another group or individual. Groups – or 'the gap' – will accelerate apart with growth/time.

### 4.8.2 Non-Inertial Frames of Reference.

Continuing with this topic of acceleration and further refuting of the claim 'the rich get richer, while the poor get poor' as discussed above (4.7.1): the 'poor getting poorer' maybe only perceived to be so and is an illusion of the acceleration. Observations within the fractal set or economy are made from – what is termed in physics – a 'non-inertial or accelerating frame of reference'. This may account for the illusion – from a fixed position in the set – the rich appear to be getting richer (ahead in the set), while (behind) the poor appear to be getting poorer; but in reality, the poor are also accelerating, only they appear – falsely – to be receding away.

### 4.9 Wave Properties

Another property of the fractal growth – demonstrated in methods 2.1 – is the sinusoidal spiralling of the discrete component bits 'though' time – triangles in the case of the Koch Snowflake demonstrated in figures 4B and appendix figure 1. This (vector) spiralling – which is not obvious to the observer as it is the linier displacement that is observed – suggests economic growth is as a result of a propagating transverse wave. The fractal economy may have a wave function property. Further analysis of this property is beyond the scope of this investigation but will be addressed in a complementary study. Suffice to say, this occurrence of wave propagation of discrete 'bits' along with concepts of time and growth is not lost on the author and may stand as a candidate or window into quantum mechanical theory – where all things are described by a wave function. Put another way, to describe the growth behaviour of the fractal may require mathematics derived from quantum mechanics, and it maybe that the 'market' structure – with its Lorenz wealth and income distribution – is a standing wave phenomenon, and all a consequence or aspect of iteration – of time.

## 5 CONCLUSIONS

This investigation found the Lorenz curve distribution is a property of a fractal and from this the market maybe best described by fractal geometry. The economy behaves as a fractal. This property is revealed in all things fractal such as trees, clouds, possibly the universe, and economies – whether cultural or not. Lorenz distribution is inextricably linked to growth and development of the fractal (or economy). As the fractal grows and develops – with time – the area of component bit sizes – its wealth – distribution increases resulting with the Lorenz curve shifting out to the right and the respective Gini coefficient increasing. Exponential income growth and Gini coefficient globally over the last two Centauries conform to modelled fractal growth – given income is related to wealth creation. The wealth gap – or distance between area sizes/wealth groups – expands apart at an accelerating rate (exponentially) becoming increasingly more unequal with time. The velocity and acceleration of area increase increases as the gap or distance increases – similar, if not identical to the cosmological Hubble's law and 'dark energy' (conjecture). This cosmological similarity, along with the wave like properties of the growing fractal, maybe revealing properties beyond the scope of economic studies, and these are yet to be explored. The iterating fractal, growing or emerging – bit-by-bit – may offer insights in the quantum aspects of our reality.

# 6 APPENDIX

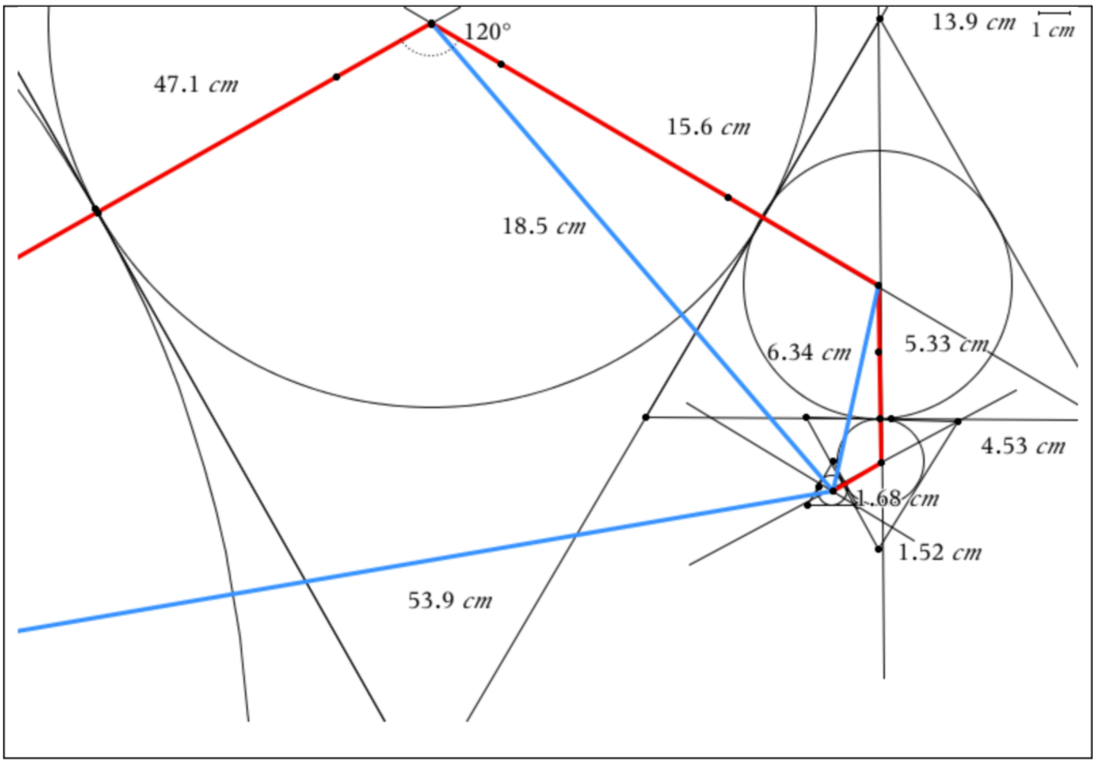


**Figure 1. Displacement measurements from radii on the iterating Koch Snowflake created with TI-Nspire ™ software.** Displacement is measured between (discrete) triangle centres and used in the calculation of the

fractal/Hubble constant. The red line traces the circumference (the distance) of the fractal spiral, and the blue line the displacement of the fractal spiral from an arbitrary centre of observation. cm = centimetres.

**Table 1.** Displacement records taken from direct radius measurements and calculations from the iterating Koch Snowflake fractal spiral (Appendix Figure 1).

| *i* | *Displacement: cm* | *Total Displacement: cm* | *Expansion Ratio:* | *Velocity:* $cm\ i^{-1}$ | *Acceleration:* $cm\ i^{-2}$ |
|---|---|---|---|---|---|
| *0* | | | - | | |
| *1* | *1.68* | *1.68* | - | *1.68* | *1.68* |
| *2* | *4.66* | *6.34* | *3.77* | *4.66* | *2.98* |
| *3* | *12.16* | *18.5* | *2.92* | *12.16* | *7.50* |
| *4* | *35.4* | *53.9* | *2.91* | *35.40* | *23.24* |

**cm = centimetres. i = iteration time.**

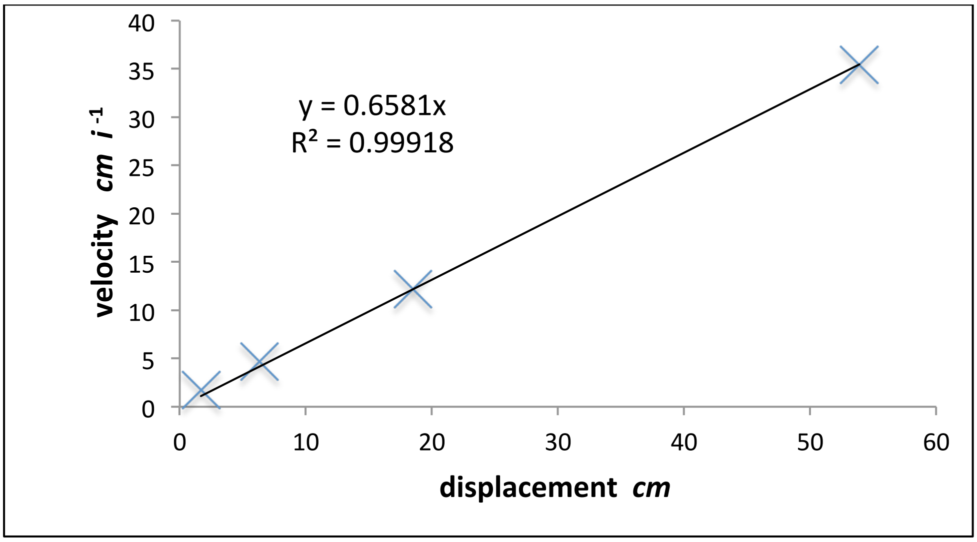


**Figure 2. The Fractal Hubble-Wealth Diagram.** Recessional velocity vs. distance from radius measurements (Appendix Figure 1). From an arbitrary observation point on the inverted (Koch Snowflake) fractal: as the distance between triangle geometric centres points increases, the recession velocity of the points receding away increases. cm = centimetres. i = iteration time.

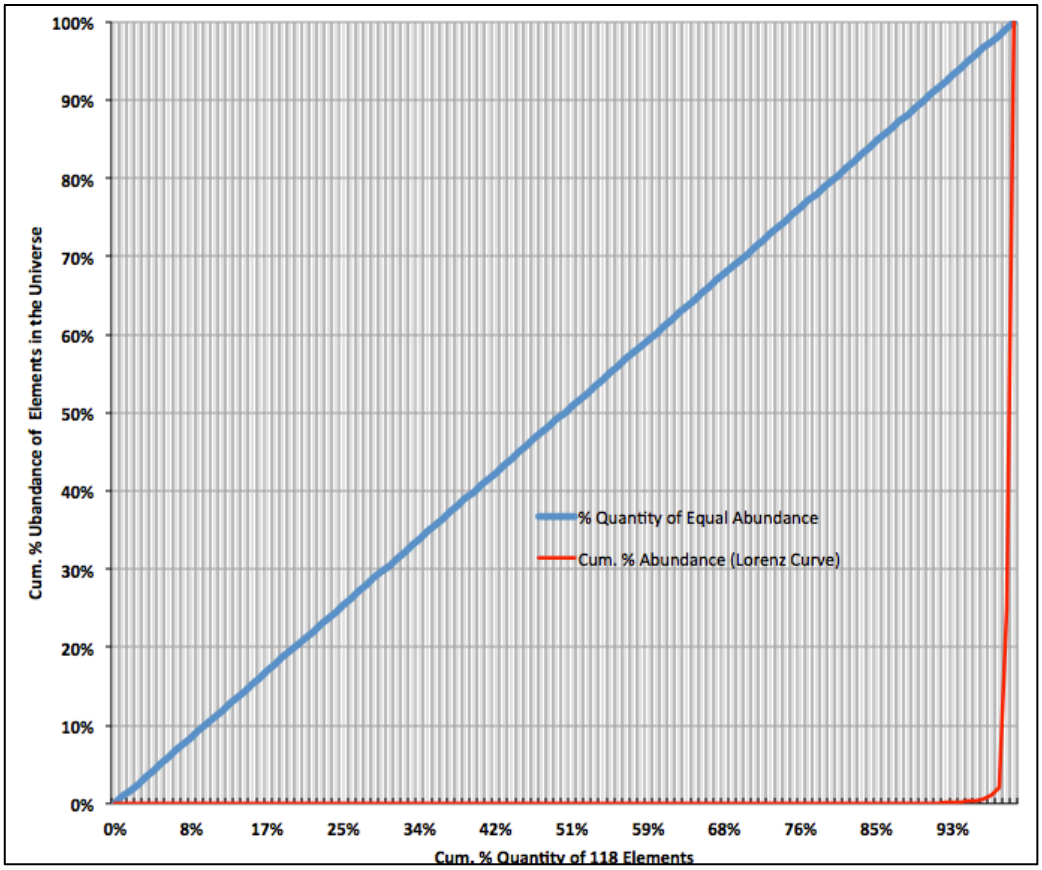


**Figure 3. Lorenz Curve of the (118) Elements from the Periodic Table.** Using Lorenz curve method the distribution of elements curve was produced. The Gini coefficient was calculated: it is equal to 0.9864 (4sf) (Macdonald 2016b)

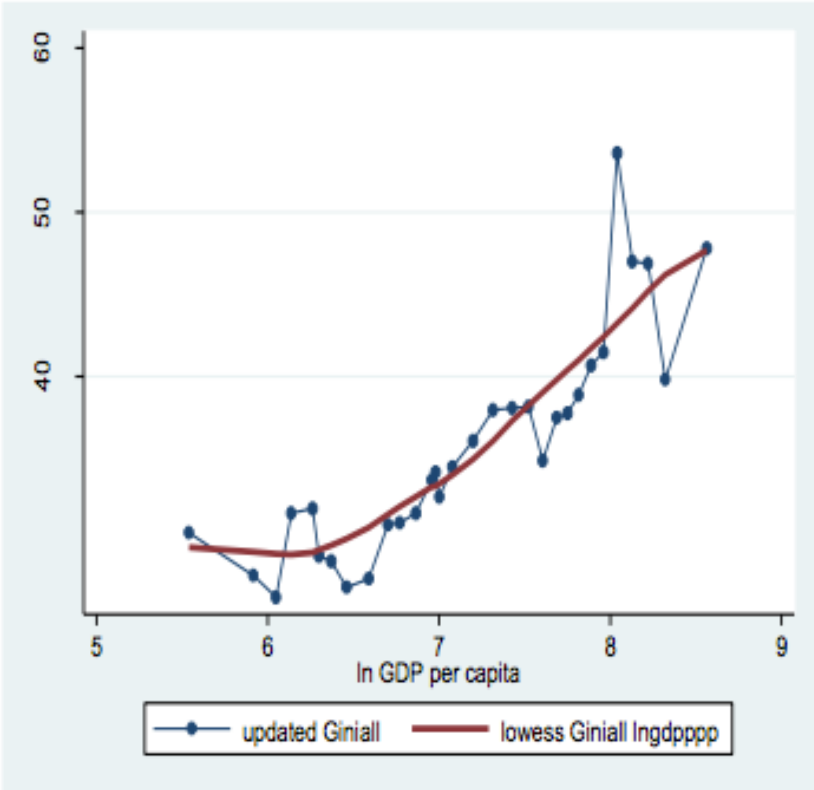


**Figure 4. Gini (Kuznets curve) with GDP per capita for China 1967-2007(Milanovic 2012)**

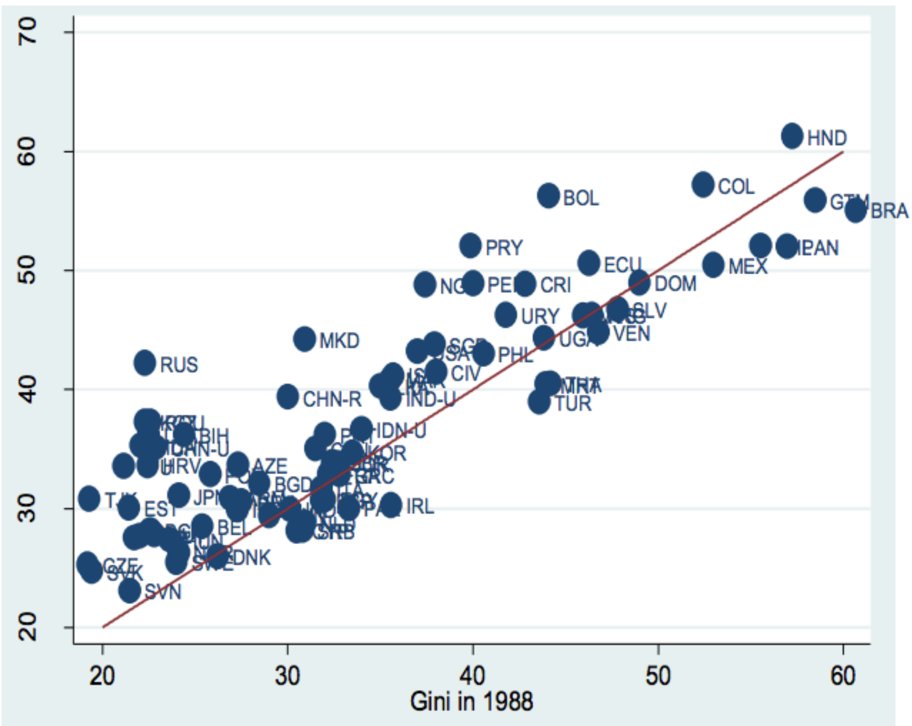


**Figure 5. Income Gini Change from 1988 to 2008. (Milanovic 2012)**

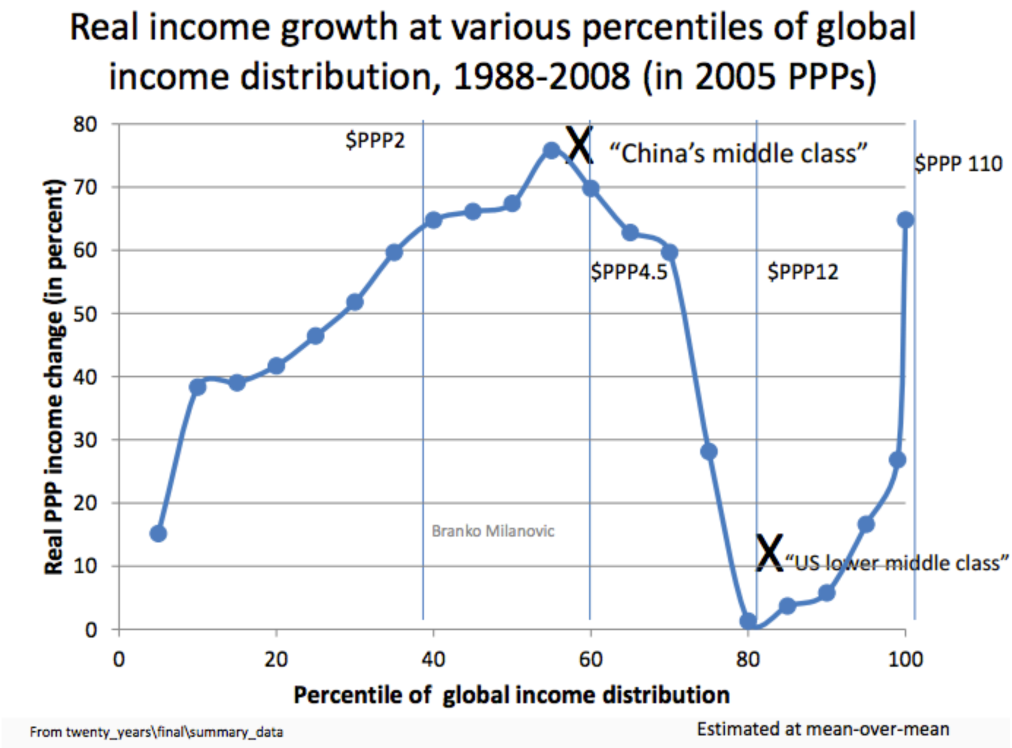


**Figure 6. Global growth by percentile 1988 to 2008. (Milanovic 2016a)**

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Demonstrating Lorenz Curve Distribution and Increasing Gini Coefficient with the Iterating (Koch Snowflake) Fractal Attractor.

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