On a Formalization of Natural Arithmetic Theory

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Abstract

In this paper we define an arithmetic theory PAM, which is an extension of Peano arithmetic PA, and prove that theory PAM has only one (up to isomorphism) model, which is the standard PA–model.

Peano arithmetic PA is a well known formal theory of the signature $\sigma =< 0, s, +, \cdot >$, where 0 — the symbol of the constant, s — the symbol of the one-place successor function, $+$ and $\cdot$ — symbols of two–place functions of addition and multiplication respectively.

Here we assume a fixed Gödel numbering by natural numbers (described by the theory PA) of all formulas (of signature σ) and sequences of these formulas. The set of standard natural numbers is denoted by ω.

It is known [1,2,3,4] that Gödel numbering of formulas (their sequences) is always, by definition, such that for any initial concepts which characterize syntactic properties of formulas (their sequences) and which are clearly (recursively) formulated in the metalanguage, there are PA–concepts, which characterize theoretic–numerical properties of Gödel numbers of formulas (their sequences) and which are interpreted in the metalanguage by corresponding initial concepts.

If a number n, $n \in \omega$, occurs in a formula, then this number n denotes the (σ–)term $s(s(\ldots s(0)\ldots))$, where the number of symbols "s" equals to n. The Gödel number of a formula φ is denoted by $\ulcorner\varphi\urcorner$. By T_A, where A — a set of (σ–)proposals, we denote a theory of signature σ, for which A is the set of its axioms. By $Prov_A(x, y)$ we denote a formula which states that x is the Gödel number of a T_A–proof of the formula φ and $\ulcorner\varphi\urcorner = y$.

Definition. Arithmetic theory PAM is a formal theory of signature σ with the axioms of the theory PA and metaaxioms (the scheme of metaaxioms)

$$MA(\varphi) : \exists x Prov_A(x, \ulcorner\varphi\urcorner) \to \varphi,$$

where A is the set of PA–axioms and φ is a σ–sentence.

If $\mathfrak{M}(\varphi, T)$ is a model (proposal, theory) of signature σ, then "$\mathfrak{M} \models \varphi$" shall mean that sentence φ is true in the model $\mathfrak{M}$, "$\mathfrak{M} \models T$" shall mean that $\mathfrak{M}$ is a model of theory T, and "$T \vdash \varphi$" shall mean that sentence φ is provable in theory T.

Theorem 1. If theory PA has a standard model, then this system is a model of theory PAM.

Proof. Let Ω — the standard model of PA. Since the axioms of PA are true in the model Ω, it is sufficient to prove that all metaaxioms of PAM are true in the model Ω.

Let $MA(\varphi) : \exists x Prov_A(x, \ulcorner\varphi\urcorner) \to \varphi$ be a metaaxiom of PAM. If $\Omega \not\models \exists x Prov_A(x, \ulcorner\varphi\urcorner)$, then, clearly, $\Omega \models MA(\varphi)$. Let $\Omega \models \exists x Prov_A(x, \ulcorner\varphi\urcorner)$ and a number q such that $\Omega \models Prov_A(q, \ulcorner\varphi\urcorner)$. Then, in view of $q \in \omega$, $PA \vdash \varphi$. Consequently, $\Omega \models \varphi$, and hence, $\Omega \models MA(\varphi)$.

Lemma. In theory PAM, it is provable Con_{PA} such that $Con_{PA} \leftrightarrow \forall x \neg Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$.

Proof. Suppose $\neg Con_{PA}$, i.e. $\exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$. Then, by the metaaxiom $MA(0 \neq 0)$, we have $0 \neq 0$. A contradiction.

Theorem 2. There exists a non–standard model of PA such that this system and all of its supersystems, which are models of PA, and all systems isomorphic to them, are not models of theory PAM.

Proof. After the famous Gödel's works, it became clear that theory PA is incomplete and for this theory exist a standard (implicit) model Ω and a non–standard model Ω^*, whose domain is a set of numbers ω^* such that $\omega \subset \omega^*$, $\omega^* \setminus \omega \neq \varnothing$ and $\forall x, y(x \in \omega \& y \in \omega^* \setminus \omega \rightarrow x < y)$, and it is known that $\Omega^* \models \exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$ (in short,$\Omega^* \models \neg Con_{PA}$). By Lemma, Ω^* is not a model of theory PAM.

Since $\Omega^* \models \exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$, then in ω^* exists a non–standard number r^* such that the "formula" $Prov_A(r^*, \ulcorner 0 \neq 0 \urcorner)$ is true in Ω^*. Note that r^* is the Gödel number of a sequence of formulas of (although) non–standard length yet satisfying the definition of the "proof in theory PA".

Clearly, the number r^* exists in domain of any supersystem of the system Ω^*, which is a model of theory PA. Therefore, in such supersystems the "formula" $Prov_A(r^*, \ulcorner 0 \neq 0 \urcorner)$ is true, and consequently $\exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$, i.e. $\neg Con_{PA}$. The latter, by Lemma, means that all such supersystems and systems isomorphic to them are not models of theory PAM.

Theorem 3. Theory PAM has only one (up to isomorphism) model, which is the standard model of theory PA.

Proof. Consider non–standard PA–model Ω^* from the proof of Theorem 2. Since $\Omega^* \models \exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$, then there is a number in ω^*, which is a Gödel number $\ulcorner D^* \urcorner (= r^*)$ of a sequence of formulas $D^* : \varphi_0, \varphi_1, \ldots, \varphi_{q^*}$, which is a non–standard proof (of length $q^* + 1$) of the formula $0 \neq 0$.

Since the set of theorems in theory PA is countable and every proof of a theorem in PA is finite, then without loosing generality, we assume that the beginning of the sequence D^*, namely $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$, where $n \in \omega$, contains all formulas provable in PA with their proofs.

Consider a sequence of formulas R^*, obtained from D^* by replacing in the latter every formula φ_y, where $y \in \omega^* \setminus \omega$ and $y \leq q^*$, by the formula $0 \neq 0$. Clearly, the sequence of formulas R^* is a non–standard proof of the formula $0 \neq 0$, since such formula is derivable from the formula $0 \neq 0 \to 0 \neq 0$ provable in PA and the previous formula $0 \neq 0$ by *Modus ponens*. It is also clear that the Gödel number $\ulcorner R^* \urcorner$ of sequence R^* belongs to ω^*.

By Theorem 2, it suffices to show that all non–standard subsystems of Ω^*, which are models of theory PA, are not models of theory PAM. Now we show that it is so.

Let Ω^*_y be a non–standard subsystem of the system Ω^*, which is a model of theory PA whose domain is the set ω^*_y ($\subset \omega^*$). Let a non–standard p^* be such that the Gödel number of the sequence of formulas $R^*_y : \varphi_0, \varphi_1, \ldots, \varphi_{p^*}$ (which is a restriction of the sequence R^* by the formula φ_{p^*}) belongs to the domain ω^*_y (such p^* can always be chosen by means of its sufficient decrease). Sequence of formulas R^*_y is non–standard proof of the formula $0 \neq 0$ (φ_{p^*}). Since the Gödel number $\ulcorner R^*_y \urcorner$ of this sequence of formulas R^*_y belongs to the domain of ω^*_y, then $\Omega^*_y \models \exists x Prov_A(x, \ulcorner 0 \neq 0 \urcorner)$, i.e. $\Omega^*_y \models \neg Prov_{PA}$. In view of the Lemma, the system Ω^*_y and all its isomorphic systems are not models of theory PAM.

This result is consistent with [2, page 294]: "all models of elementary arithmetic whose universe contains only these numbers are indeed isomorphic".

References

[1].*S.K. Kleene*, Introduction to Metamathematics, Oxford.,1957.

[2].*A.A. Fraenkel, Y. Bar-Hillel*, Foundations of Set Theory, NORTH-HOLLAND PUBLISHING COMPANY, Amsterdam, New York, Oxford, 1958.

[3].*J.L. Ershov, E.A. Palyutin*, Mathematical Logic, Moscow, Science, 1979.

[4].*K. Smorynski*, The incompleteness Theorems, Handbook of Mathematical Logic, NORTH-HOLLAND PUBLISHING COMPANY, Amsterdam, New York, Oxford, 1977, p.821-865.