# SET OF ALL PAIRS OF TWIN PRIME NUMBERS IS INFINITE

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**Abstract**

In this paper we formulate an intuitive **Hypothesis** about a new aspect of a well known method called "Sieve of Eratosthenes" and then prove that set of natural numbers $N = \{1, 2, \ldots\}$ contains infinite number of pairs of twin primes.

**Introduction.** Following established tradition, as $p_i$, $i \in N$, we shall denote $i$-th prime ($p_1 = 2$). $N^+$ shall designate numerical series $N \cup \{0\}$. $r(\frac{a}{b})$, where $b \neq 0$, shall designate the remainder of division of $a$ by $b$, assuming $0 \leq r(\frac{a}{b}) < b$. Set of possible remainders from division by $b$, i.e. $\{0, 1, \ldots, b-1\}$, will be denoted as $R(b)$. If $A$ is a finite set of numbers, then $|A|$ shall mean the count of integers in $A$.

If $i \in N$ and $S_i \subseteq R(p_i)$, then

$$M_i(S_i) = \{x \in N \mid r(\frac{x}{p_i}) \in S_i\}.$$

It is clear that if $S_i = R(p_i)$ then $M_i(S_i) = N$, and if $S_i \subset R(p_i)$ then sets $M_i(S_i)$ and $N \setminus M_i(S_i)$ are infinite.

# 1 The Hypothesis

If $i \in N$, $S_i \subset R(p_i)$ and $|S_i| = 1$, then increasing sequence of all elements of the set

$$M_i(S_i) = \{x \in N \mid r(\frac{x}{p_i}) \in S_i\}$$

will be an arithmetic progression with common difference of $p_i$. Set of $M_i(S_i)$, where $1 < |S_i| < p_i$, which combines corresponding arithmetic progressions, will be called a group arithmetic progression with common difference of $p_i$.

It is clear that for every $i \in N$, if $S_i \subset R(p_i)$, then

$$\lim_{q \to \infty} \frac{|M_i(S_i) \cap \{x \in N \mid x \leq q\}|}{q} = \frac{|S_i|}{p_i}.$$

Let $n \in N$ and $S_i \subset R(p_i)$ for every $i \leq n$. Let $D_n$ is a set such that

$$D_n = \bigcap_{i=1}^{n} M_i(S_i) = \{x \in N \mid r(\frac{x}{p_1}) \in S_1, \ldots, r(\frac{x}{p_n}) \in S_n\},$$

and $D_n(q)$, where $q \in N$ and $q > p_n$, is a set such that

$$D_n(q) = D_n \cap \{x \in N \mid p_n < x \leq q\}.$$

Since all arithmetic progressions included in the group are disjoint, then for every $1 < i \leq n$

$$D_i = \bigcup_{k \in S_i} \{D_{i-1} \bigcap M_i(\{k\})\},$$

which, in view of mutual primeness of $p_i$ with $p_1, \ldots, p_{i-1}$, makes it quite probable that

$$\lim_{q \to \infty} |D_i(q)| = \lim_{q \to \infty} |D_{i-1}(q)| \cdot \frac{|S_i|}{p_i}.$$

Since all common differences $p_1, \ldots, p_n$ of arithmetic (possibly group) progressions included into $D_n$ are mutually prime, it is therefore very likely that

$$\lim_{q \to \infty} |D_n(q)| = \prod_{i=1}^{n} \frac{|S_i|}{p_i}.$$

**Hypothesis.**

$$|D_n(q)| \approx (q - p_n) \cdot \prod_{i=1}^{n} \frac{|S_i|}{p_i}.$$

(Approximation $\approx$ is due to finitness of the set $\{x \in N \mid p_n < x \leq q\}$.)

Let us consider two extreme cases - the minimum and the maximum sets $D_n$:

**Case 1.**

Let for any $i \leq n$ set $S_i = \{0\}$, then

$$D_n = \bigcap_{i=1}^{n} M_i(S_i) = \{x \in N \mid r(\frac{x}{p_1}) = 0, \ldots, r(\frac{x}{p_n}) = 0\}.$$

It is sufficient to mention the well known arithmetic fact, that the intersection of two arithmetic progressions with mutually prime common differences $p_x, p_y$ is always an arithmetic progression with common difference of $p_x \times p_y$. Therefore, this part of the **Hypothesis**, having the form

$|D_n(q)| \approx (q-p_n) \cdot \dfrac{1}{p_1 \times \cdots \times p_n}$, is provable in Peano Arithmetic.

**Case 2.**

Let $S_1 = R(p_1) \setminus \{0\}, \ldots, S_n = R(p_n) \setminus \{0\}$, then

$$D(n) = \bigcap_{i=1}^{n} M_i(S_i) = \{x \in N \mid r(\frac{x}{p_1}) \neq 0, \ldots, r(\frac{x}{p_n}) \neq 0\}$$

and by **Hypothesis**

$$|D_n(q)| \approx (q-p_n) \cdot \prod_{i=1}^{n} \frac{|S_i|}{p_i}.$$

It is clear that set $D_n$ is a set of numbers that are mutually prime with $p_1, p_2, \ldots, p_n$. It is also clear that $D_n(p_n^2)$ is a set of approximately all primes that are greater than $p_n$ and less than $p_n^2$. If we follow the trend $Td_n$ of changes of $|D_n(p_n^2)|$ with increasing $n$, we will notice that

$$Td_n = \frac{|D_{n+1}(p_{n+1}^2)|}{|D_n(p_n^2)|} = \frac{p_{n+1}^2 - p_{n+1}}{p_n^2 - p_n} \cdot \frac{p_{n+1}-1}{p_{n+1}},$$

substituting $p_{n+1} = p_n + c$:

$$Td_n = \frac{(p_n+c)^2 - (p_n+c)}{p_n^2 - p_n} \cdot \frac{p_n+c-1}{p_n+c} = \frac{p_n^2 + 2p_n \cdot c + c^2 - p_n - c}{p_n^2 - p_n} \cdot \frac{p_n+c-1}{p_n+c}.$$

Since $c \geq 2$ for any $n > 1$, therefore $Td_n > 1$ for any $n > 1$. Taking into account the fact that $|D_2(3^2)| > 0$, this means that there are infinitely many prime numbers in $N$ (which is proved in Peano Arithmetic).

Consistency with the axioms of Peano Arithmetic of the two extreme cases of **Hypothesis** suggests that all intermediate cases of the **Hypothesis** are consistent with axioms of Peano Arithmetic.

## 2 Twin prime numbers

It is well known that all prime numbers greater than 3 are numbers of the form $6m-1$ or $6m+1$. Therefore, every pair of twin prime numbers "surrounds" some integer which is a multiple of six, which becomes the "middle" of the pair. Taking this fact into account, let us define set $D_n, n > 2$ as follows:

$$D_n = \{x \in N \mid r(\frac{x}{p_1}) = r(\frac{x}{p_2}) = 0\& \bigwedge_{i=3}^{n} (r(\frac{x}{p_i}) \neq 1\& r(\frac{x}{p_i}) \neq p_i - 1)\}.$$

It is clear that set $D_n$ contains only numbers $6m, m > 1$; moreover, the numbers $6m-1$ and $6m+1$ are mutually prime with $p_1, \ldots, p_n$. It is also clear that set $D_n(p_n^2)$ is a set of approximately all the "middles" of pairs of twin prime numbers laying between $p_n$ and $p_n^2$. If we follow the trend $Td_n$ of changes of $|D_n(p_n^2)|$ with increasing $n$ ($n \geq 3$), we will notice that ($c \geq 2$)

$$Td_n = \frac{|D_{n+1}(p_{n+1}^2)|}{|D_n(p_n^2)|} = \frac{(p_n+c)^2 - (p_n+c)}{p_n^2 - p_n} \cdot \frac{p_n+c-2}{p_n+c} > 1.$$

Taken together with the apparent $|D_3(p_3^2)| > 0$, this fact means that there are infinitely many pairs of twin primes in the set of natural numbers $N$.

## References


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