THERMODYNAMICS IN $f(T,\theta)$ GRAVITY

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In the present study, we discuss a non-equilibrium picture of thermodynamics at the apparent horizon of flat Friedmann-Roberton-Walker universe in $f(T,\theta)$ theory of gravity, where T is the torsion scalar and θ is the trace of the energy-momentum tensor. Mainly, we investigate the validity of the first and second laws of thermodynamics in this scenario. We consider two descriptions of the energy-momentum tensor of dark energy density and pressure and discuss that an equilibrium picture of gravitational thermodynamics can not be given in both cases. Furthermore, we also conclude that the second law of gravitational thermodynamics can be achieved both in phantom and quintessence phases of the universe.

Key words: Thermodynamics, dark matter.

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1. INTRODUCTION

Recently, Planck-2013 observations [1] of the cosmic microwave background has implied that the matter in our universe is dominated by two enigmatic components: dark energy (68.3 percent) and dark matter (26.8 percent). The remaining part (4.9 percent) is occupied by ordinary matters. It is commonly believed that our universe has a phase transition [2] from decelerating to accelerating and expands with accelerating velocity. This interesting feature of the universe is believed to be caused by mysterious dark contents. There are several proposals to be a candidate for dark part of the universe, but still the nature of dark universe is completely unknown [3]. One of the candidates is giving some unknown matters called dark energy in the framework of general relativity, another one is to modify the gravitational theory [4]. Adopting modified gravitation theories leads to very interesting conclusions at cosmological, galactic and solar systems scales. Besides, there is no definite physical criterion to select one of modified gravitation theories capable of matching the data at all scales [5]. Understanding the apparent acceleration of the universe is one of the most challenging topics for modern cosmology and fundamental physics today.

On the other hand, the gravitational thermodynamics is also one of the famous issues of current interest in modern cosmology. Cai and Kim [6] showed that, in general relativity, the Friedmann equation can be written in the form of the first law of thermodynamics

$$-dE = T_A dS_A \tag{1}$$

on the dynamical apparent horizon $\hat{r}_A$. In their work, Cai and Kim introduced that $T_A = (2\pi\hat{r}_A)^{-1}$, $S_A = \pi\hat{r}_A^2 G^{-1}$, dE are the Hawking temperature, the horizon entropy and the internal energy, respectively. Later, in general relativity, Friedmann equation was written in another form

$$dE = TdS + WdV \tag{2}$$

at dynamical apparent horizon, where $E = \rho V$, $W = \frac{1}{2}(\rho - p)$ are the internal energy and work density, respectively [7].

The gravitational thermodynamics has been studied in many modified gravity theories such as Gauss-Bonnet gravity [7], Lovelock gravity [8, 9], Braneworld gravity [10], non-linear gravity [11], scalar-tensor gravity [12], $f(R)$-gravity [13, 14], $f(T)$-gravity [15, 16] and $f(R,\theta)$-gravity (here θ is the trace of the energy-momentum tensor) [17]. It has been seen that, in scalar-tensor theory, $f(R)$-gravity, $f(T)$-gravity and $f(R,\theta)$-gravity, non-equilibrium definition of gravitational thermodynamics is required [14, 16–19]. These recent investigations have encouraged us to explore whether an equilibrium definition of thermodynamics can be obtained in the framework of $f(T,\theta)$-gravity. This work may provide some specific conclusions which would discriminate $f(T,\theta)$-gravity from various theories of modified gravity.

The plan of this paper is as follows: In the next section we introduce $f(T,\theta)$-gravity. In section 3, we mainly discuss whether an equilibrium definition of gravitational thermodynamics is possible in $f(T,\theta)$-gravity. The entropy at dynamical horizon is constructed from the first law of gravitational thermodynamics corresponding to the Friedmann equation. Next, we also investigate the second law of gravitational thermodynamics and obtain the validity condition. In section 4, we investigate another description of thermodynamics in $f(T,\theta)$-gravity. Finally, outlook of the work is given in section 5.

2. $f(T,\theta)$ THEORY OF GRAVITY

Throughout the work, we represent the space-time indices by Greek alphabet $(\mu,\nu,\alpha,\beta... = 0,1,2,3)$ and the Latin alphabet $(a,b,i,j....) = 0,1,2,3$ will be used to denote indices related to the tangent space.

The action of $f(T,\theta)$ gravity is given by

$$S = \int d^4xh \left[\frac{f(T,\theta)}{16\pi G} + L_m\right], \tag{3}$$

where L_m describes matter contents of the universe, $h = \sqrt{-g}$ is the determinant of tetrad field, g is the determinant of the metric tensor $g_{\mu\nu}$ and the metric tensor of the spacetime is related to the tetrad by $g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu$ [20].

The connection which is used in Einstein's theory of general relativity, is the Levi-Civita connection

$$\Gamma^\alpha_{\ \mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{4}$$

Levi-Civita connection leads us to vanishing torsion but non-zero spacetime curvature [21]. In torsion gravity (known also as teleparallel theory of gravity), tetrad fields give rise to the Weitzenbock connection which is given by

$$\widehat{\Gamma}^\alpha_{\ \mu\nu} = h^\alpha_i \partial_\nu h^i_\mu = -h^i_\mu \partial_\nu h^\alpha_i. \tag{5}$$

Thence, the covariant derivative of the tetrad field vanishes identically

$$\nabla_\mu h^\nu_i = h^\nu_i - \widehat{\Gamma}^\alpha_{\ \mu\nu} h^i_\alpha = 0. \tag{6}$$

This result gives us vanishing curvature and surviving torsion [20, 21]. The torsion and contortion are defined, respectively, by the following relations [22, 23]:

$$T^\alpha_{\ \mu\nu} = \widehat{\Gamma}^\alpha_{\ \nu\mu} - \widehat{\Gamma}^\alpha_{\ \mu\nu}, \tag{7}$$

$$\begin{aligned} K^\alpha_{\ \mu\nu} &= \widehat{\Gamma}^\alpha_{\ \mu\nu} - \Gamma^\alpha_{\ \mu\nu} \\ &= \frac{1}{2}(T_{\mu\ \nu}^{\ \alpha} + T_{\nu\ \mu}^{\ \alpha} - T^\alpha_{\ \mu\nu}). \end{aligned} \tag{8}$$

Next, we can define super-potential and the torsion scalar by

$$S_\alpha^{\ \mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}_{\ \ \ \alpha} + \delta^\mu_\alpha T^{\xi\nu}_{\ \ \ \xi} - \delta^\nu_\alpha T^{\xi\mu}_{\ \ \ \xi}), \tag{9}$$

$$\begin{aligned} T &\equiv S_\alpha^{\ \mu\nu} T^\alpha_{\ \mu\nu} \\ &= \frac{1}{4} T^{\xi\mu\nu} T_{\xi\mu\nu} + \frac{1}{2} T^{\xi\mu\nu} T_{\nu\mu\xi} - T_{\xi\mu}^{\ \ \ \xi} T^{\nu\mu}_{\ \ \ \nu}, \end{aligned} \tag{10}$$

respectively.

The generalized field equations of $f(T,\theta)$-gravity are extracted by varying the action (3) with respect to tetrad field as follows

$$\begin{aligned} f_T G_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(f - T f_T) + S_{\nu\mu\rho}(f_{TT} \nabla^\rho T + f_{T\theta} \nabla^\rho \theta) \\ = 8\pi G \theta_{\mu\nu} - f_\theta \theta_{\mu\nu} - f_\theta B_{\mu\nu}, \end{aligned} \tag{11}$$

here we denote that $f_T = \partial f/\partial T$, $f_\theta = \partial f/\partial \theta$, $f_{TT} = \partial^2 f/\partial T^2$ and $f_{T\theta} = \partial^2 f/\partial T \partial \theta$ and we also have

$$B^\mu_a = \frac{h^k_\alpha \delta \theta^\alpha_k}{\delta h^a_\mu}, \tag{12}$$

and

$$\theta \equiv h^a_\mu \theta^\mu_a = 3p_m - \rho_m. \tag{13}$$

The choice of $f(T,\theta) \equiv f(T)$ results in the field equations of $f(T)$-gravity. The energy-momentum tensor of perfect fluid is defined as

$$\theta_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \tag{14}$$

where u_μ is the four-velocity of the fluid. Hence, one can obtain that

$$B_{\mu\nu} = \theta_{\mu\nu} - 2p_m g_{\mu\nu}. \tag{15}$$

We assume only the non-relativistic matter (cold dark matter and baryons) with $p_m = 0$, consequently the contribution of T comes only from ordinary matters.

The flat Friedmann-Robertson-Walker spacetime is described by the line-element

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{16}$$

where $a(t)$ defines the scale factor. In this background, the gravitational field equations of $f(T,\theta)$-gravity transform

$$3H^2 f_T - \frac{1}{2}(f - Tf_T) = \rho_m(8\pi G - 2f_\theta), \tag{17}$$

$$-(3H^2 + 2\dot{H})f_T + \frac{1}{2}(f - Tf_T) + H\dot{T}f_{TT} - Hf_{T\theta}\dot{\rho}_m = 0. \tag{18}$$

Here dot means derivative with respect to cosmic time t. These can be rewritten as

$$3H^2 = 8\pi G_{eff}(\rho_m + \rho_d), \tag{19}$$

$$-2\dot{H} = 8\pi G_{eff}(\rho_m + \rho_d + p_d), \tag{20}$$

where ρ_d and p_d are the energy density and pressure of dark components

$$\rho_d = \frac{1}{8\pi G - 2f_\theta}\frac{f - Tf_T}{2}, \tag{21}$$

$$p_d = \frac{-1}{8\pi G - 2f_\theta}\left(\frac{f - Tf_T}{2} + H\dot{T}f_{TT} - Hf_{T\theta}\frac{3H\dot{H}f_T - \theta\dot{T}f_{T\theta}}{4\pi G - \frac{5}{4}f_\theta - \theta f_{\theta\theta}}\right), \tag{22}$$

and

$$G_{eff} \equiv \frac{1}{f_T}(G - \frac{f_\theta}{4\pi}). \tag{23}$$

The equation of state parameter of dark fluid ω_d is obtained as ($p_d = \omega_d \rho_d$)

$$\omega_d = -1 + \frac{f_{T\theta}\frac{3H\dot{H}f_T - \theta\dot{T}f_{T\theta}}{4\pi G - \frac{5}{4}f_\theta - \theta f_{\theta\theta}} - \dot{T}f_{TT}}{\frac{1}{2H}(f - Tf_T)}. \tag{24}$$

The semi-conservation equation of ordinary matter [17] is given by

$$\dot{\rho}+3H\rho=q. \tag{25}$$

Similarly, the energy-momentum tensor of dark components satisfy the following conservation laws

$$\dot{\rho}_d+3H(\rho_d+p_d)=q_d, \tag{26}$$

$$\dot{\rho}_t+3H(\rho_t+p_t)=q_t, \tag{27}$$

where $\rho_t=\rho_m+\rho_d$, $p_t=p_d$ and $q_t=q+q_d$. Here, q and q_t represent the energy exchange term and the total energy exchange term, respectively. Substituting equations (19) and (20) in relation (27), one can obtain

$$q_t=\frac{3H^2}{8\pi G}\partial_t\left(\frac{f_T}{1-\frac{f_\theta}{4\pi G}}\right)=\frac{-T}{16\pi G}\partial_t\left(\frac{f_T}{1-\frac{f_\theta}{4\pi G}}\right). \tag{28}$$

We can recover the relation of total energy exchange term in $f(T)$-gravity if $f_\theta=0$. In teleparallel gravity, $q_t=0$ for the choice $f(T,\theta)=T$.

3. LAWS OF GRAVITATIONAL THERMODYNAMICS

In this part of the work, we investigate the validity of the first and second laws of the gravitational thermodynamics in $f(T,\theta)$ theory of gravity for the flat Friedmann-Robertson-Walker spacetime.

3.1. THE FIRST LAW

In this sub-section, we examine the validity of the first law of gravitational thermodynamics in $f(T,\theta)$-gravity at the apparent horizon of flat Friedmann-Robertson-Walker universe.

The dynamical apparent horizon can be obtained by using [17]

$$h^{\mu\nu}\partial_\mu\hat{r}\partial_\nu\hat{r}=0. \tag{29}$$

This relation leads us to the radius of dynamical apparent horizon for the flat Friedmann-Robertson-Walker spacetime $\hat{r}_A=\frac{1}{H}$. Next, the corresponding Hawking temperature on the apparent horizon [6] is described by

$$T_A=\frac{1}{2\pi\hat{r}_A}\frac{2H\hat{r}_A-\dot{\hat{r}}_A}{2H\hat{r}_A}, \tag{30}$$

where $2H\hat{r}_A>\dot{\hat{r}}_A$ provides that the temperature is positive. Cai *et al.* [24], in 2009, showed that the apparent horizon of the Friedmann-Robertson-Walker spacetime has an associated Hawking temperature.

In general relativity, the horizon (apparent) entropy is described by the Bekenstein-Hawking equation $S_A = \frac{A}{4G}$, where $A = 4\pi\hat{r}_A^2$ is the area of the apparent horizon [25–27]. The horizon entropy in $f(R)$-gravity [28, 29] is given by $S_A = \frac{Af_R}{4G}$. Next, in $f(T)$-gravity, the horizon entropy is defined as $S_A = \frac{Af_T}{4G}$ [30]. In the $f(T,\theta)$ theory of gravity, the horizon entropy is expressed as

$$S_A = \frac{A}{4G}\frac{f_T}{\Sigma}, \tag{31}$$

where

$$\Sigma = 1 - \frac{f_\theta}{4\pi G}. \tag{32}$$

Considering equation (20) and the definition of dynamical apparent horizon, we find

$$f_T\frac{d\hat{r}_A}{dt} = 4\pi G\hat{r}_A^3(\rho_t + p_t)H\Sigma. \tag{33}$$

Now, using equations (32) and (33) it follows that

$$\frac{1}{2\pi\hat{r}_A}dS_A = 4\pi\hat{r}_A^3(\rho_t + p_t)Hdt + \frac{\hat{r}_A}{2G\Sigma}df_T + \frac{\hat{r}_A f_T}{2G}d\left(\frac{1}{\Sigma}\right). \tag{34}$$

After multiplying both sides of this relation with a factor $\frac{2H\hat{r}_A - \dot{\hat{r}}_A}{2H\hat{r}_A}$, we obtain

$$T_A dS_A = 4\pi\hat{r}_A^3(\rho_t + p_t)Hdt - 2\pi\hat{r}_A^2(\rho_t + p_t)d\hat{r}_A + \frac{\pi\hat{r}_A^2 T_A}{G\Sigma}df_T + \frac{\pi\hat{r}_A^2 T_A f_T}{G}d\left(\frac{1}{\Sigma}\right) \tag{35}$$

On the other hand, in the general relativity, the energy of the universe within the apparent horizon is defined by the Misner-Sharp [31, 32] relation $E = \frac{\hat{r}_A}{2G}$. In $f(T,\theta)$ theory of gravity, we define the energy of the universe within the apparent horizon as

$$E = \frac{\hat{r}_A}{2G}\frac{f_T}{\Sigma}. \tag{36}$$

It is important to mention here that $E > 0$ if $G_{eff} > 0$, hence the effective gravitational coupling constant in $f(T,\theta)$-gravity should be positive [17]. By taking time derivative of equation (36) we find

$$dE = -4\pi\hat{r}_A^3(\rho_t + p_t)Hdt + 4\pi\hat{r}_A^2\rho_t d\hat{r}_A + \frac{\hat{r}_A}{2G\Sigma}df_T + \frac{\hat{r}_A f_T}{2G}d\left(\frac{1}{\Sigma}\right). \tag{37}$$

Then, using this result in equation (35) it follows that

$$T_A dS_A = -dE + WdV + \frac{1 + 2\pi\hat{r}_A T_A}{2G\Sigma}\hat{r}_A df_T + \frac{1 + 2\pi\hat{r}_A T_A}{2G}\hat{r}_A f_T d\left(\frac{1}{\Sigma}\right), \tag{38}$$

where

$$W = \frac{1}{2}(\rho_t - p_t) \tag{39}$$

and

$$V = \frac{4}{3}\pi \hat{r}_A^3 \tag{40}$$

are the work density [17, 33, 34] and volume, respectively. Furthermore, we can rewrite equation (40) in the following form:

$$T_A dS_A + T_A d_i \widetilde{S} = -dE + W dV, \tag{41}$$

where

$$T_A d_i \widetilde{S} = -\frac{1+2\pi \hat{r}_A T_A}{2G\hat{r}_A^{-1}} d\left(\frac{f_T}{\Sigma}\right). \tag{42}$$

This extra term is the main reason for the violation of the first law of gravitational thermodynamics in $f(T,\theta)$-gravity. All of the matter fields see the same horizon and Hawking temperature, but some gravitational degrees of freedom in $f(T,\theta)$ theory of gravity feel a different background metric, horizon and Hawking temperature. Black holes in such a case cannot be in equilibrium. From this point of view, the first law of gravitational thermodynamics in equilibrium is violated. Dubovski and Sibiryakov [35] have discussed the effect of spontaneous breaking of Lorentz invariance on black hole thermodynamics, and they found a similar conclusion. On the other hand, when we compare the cosmological setup of $f(T,\theta)$-gravity with general relativity, teleparallel gravity, Gauss-Bonnet gravity and Lovelock gravity, we have an additional term in the first law of thermodynamics [7–9, 17]. Next, we may also interpret this additional term as the entropy production term developed due to the non-equilibrium framework in $f(T,\theta)$-gravity. Furthermore, if we assume $f(T,\theta) = T$, then we achieve the traditional first law of thermodynamics in general relativity and teleparallel gravity.

3.2. THE SECOND LAW

The entropy of the matter inside the apparent horizon is defined by the following Gibbs' equation [36]

$$\bar{T} dS_m = dE_m + p_m dV, \tag{43}$$

where $E_m = \rho_m V$ is the internal energy and $\bar{T}$ is the temperature of total energy inside the horizon. Here, we consider that $\bar{T}$ is proportional to horizon temperature [14, 19], *i.e.* $\bar{T} = \gamma T_A$, where $0 < \gamma < 1$ to ensure that temperature being positive and smaller than the temperature of dynamical apparent horizon [17]. Taking the time derivative of equation (43), one can get (remember we have $p_m = 0$)

$$\bar{T}\frac{dS_m}{dt} = \rho_m \left(\frac{dV}{dt} - 3HV\right). \tag{44}$$

In the previous sub-section, we gave the evolution of horizon entropy S_A and auxiliary entropy $\widetilde{S}$. Hence, we can write

$$T_A \frac{dS_A}{dt} = 4\pi \hat{r}_A^3(\rho_t + p_t)H - 2\pi \hat{r}_A^2(\rho_t + p_t)\frac{d\hat{r}_A}{dt} + \frac{\pi \hat{r}_A^2 T_A}{G\Sigma}\frac{df_T}{dt} + \frac{\pi \hat{r}_A^2 T_A f_T}{G}\partial_t\left(\frac{1}{\Sigma}\right), \quad (45)$$

$$T_A \frac{d_i\widetilde{S}}{dt} = -\frac{1+2\pi \hat{r}_A T_A}{2G\hat{r}_A^{-1}}\partial_t\left(\frac{f_T}{\Sigma}\right). \quad (46)$$

Now, using the relations (44), (45) and (46), we find

$$\frac{dS_t}{dt} = \frac{24\pi}{\gamma T \hat{r}_A}\left[(1-\gamma)\dot{\rho}_t V + (1-\frac{\gamma}{2})(\rho_t + p_t)\dot{V}\right] \geq 0, \quad (47)$$

where

$$S_t = S_m + S_A + \widetilde{S} \quad (48)$$

is the total entropy. The result given by equation (47) describes the validity of the second law of gravitational thermodynamics, *i.e.* $S_t \geq 0$. Hence, using (19) and (20), equation (47) can be reduced to the following result

$$\frac{dS_t}{dt} = \frac{12\pi}{\gamma T G \Sigma H^3}\frac{\Omega}{H} \geq 0, \quad (49)$$

where

$$\frac{\Omega}{H} = 2(1-\gamma)\dot{H}H^2 f_T + (2-\gamma)\dot{H}^2 f_T + (1-\gamma)\Sigma H^3 \partial_t\left(\frac{f_T}{\Sigma}\right) \quad (50)$$

Therefore the condition to satisfy the second law of gravitational thermodynamics is equivalent to $\Omega \geq 0$ and it shows the validity of the second law of thermodynamics depends on $H > 0$, $\dot{H} > 0$ and the $f(T,\theta)$-gravity model. Also, Σ and f_T are positive in order to keep $E > 0$. On the other hand, if we assume $\gamma = 1$ which means the temperature between inside and outside the dynamical apparent horizon remains the same then the second law of gravitational thermodynamics is valid only if

$$\widetilde{\Omega} = \frac{\dot{H}^2 f_T}{\Sigma} \geq 0. \quad (51)$$

In the flat Friedmann-Robertson-Walker universe, the effective equation-of-state parameter is given as

$$\omega_{eff} = -1 - \frac{2\dot{H}}{3H^2}. \quad (52)$$

Here $\dot{H} < 0$, $\omega_{eff} > -1$, corresponds to quintessence (non-phantom) region of the universe while $\dot{H} > 0$, $\omega_{eff} < -1$, represents the phantom phase. It seems $\widetilde{\Omega} \geq 0$ in both phantom and quintessence phases. As a result, the second law of gravitational thermodynamics in $f(T,\theta)$-gravity is satisfied in both phantom and quintessence regions of the universe. In literature, Bamba and Geng found that the second law of gravitational thermodynamics holds in $f(R)$-gravity and $f(T)$-gravity [14, 16] and Sharif and Zubair obtained the same conclusion in $f(R,\theta)$ theory of gravity [17].

4. ANOTHER POINT OF VIEW

In section 3, we have seen that there is an additional non-equilibrium entropy production term $d_i\widetilde{S}$ in laws of gravitational thermodynamics. In this section, we discussed, by redefining the dark energy density and dark pressure, whether the extra entropy production term can be removed. It has been shown so far that the equilibrium definition of thermodynamics can be obtained in modified theories of gravity and there can be no extra entropy production term [14, 16, 19].

4.1. REDEFINING THE DARK COMPONENTS

Gravitational field equations (19) and (20) can be rewritten as

$$3H^2 = 8\pi\hat{G}_{eff}(\rho_m + \hat{\rho}_d), \tag{53}$$

$$-2\dot{H} = 8\pi\hat{G}_{eff}(\rho_m + \hat{\rho}_d + \hat{p}_d), \tag{54}$$

where $\hat{\rho}_d$ and $\hat{p}_d$ are redefinitions of the dark energy density and dark pressure and the dark components are redefined as

$$\hat{\rho}_d = \frac{1}{8\pi G - 2f_\theta}\left[\frac{f - Tf_T}{2} + 3(1-f_T)H^2\right], \tag{55}$$

$$\hat{p}_d = \frac{-1}{8\pi G - 2f_\theta}\left(\frac{f - Tf_T}{2} + H\dot{T}f_{TT} + (1-f_T)(3H^2 + 2\dot{H}) - Hf_{T\theta}\frac{3H\dot{H}f_T - \theta\dot{T}f_{T\theta}}{4\pi G - \frac{5}{4}f_\theta - \theta f_{\theta\theta}}\right), \tag{56}$$

and

$$\hat{G}_{eff} \equiv G - \frac{f_\theta}{4\pi}. \tag{57}$$

These new definitions of dark components lead us to the following expression of total energy exchange

$$\hat{q}_t = \frac{-T}{8\pi G}\partial_t\left(\frac{1}{\Sigma}\right) \neq 0. \tag{58}$$

Since $\partial_t[f_\theta(T,\theta)] \neq 0$ we get $\Sigma \neq 0$. Hence $\hat{q}_t$ does not equal to zero. Thus, we may not define the equilibrium picture of gravitational thermodynamics in this modified theory of gravity, and we should consider the non-equilibrium description of thermodynamics. This conclusion differ from other modified gravity theories due to the matter dependence of Lagrangian density [17].

Now, we re-investigate the validity of the first and second laws of gravitational thermodynamics in this point of view.

4.2. NEW FORM OF THE FIRST LAW

In this new scenario, the time derivative of dynamical apparent horizon is written as

$$\frac{d\hat{r}_A}{dt} = 4\pi\hat{r}_A^3 HG\Sigma(\hat{\rho}_t + \hat{p}_t) \tag{59}$$

Now, the horizon entropy is given as

$$S_A = \frac{A}{4G}\frac{1}{\Sigma}. \tag{60}$$

Considering equation (59) the horizon entropy takes the following form

$$\frac{1}{2\pi\hat{r}_A}\frac{dS_A}{dt} = 4H\pi\hat{r}_A^3(\hat{\rho}_t + \hat{p}_t) + \frac{\hat{r}_A}{2G}\partial_t\left(\frac{1}{\Sigma}\right). \tag{61}$$

Next, multiplying both sides of this relation with a factor $\frac{2H\hat{r}_A - \dot{\hat{r}}_A}{2H\hat{r}_A}$ gives

$$T_A\frac{dS_A}{dt} = 4H\pi\hat{r}_A^3(\hat{\rho}_t + \hat{p}_t) - 2\pi\hat{r}_A^2(\hat{\rho}_t + \hat{p}_t)\frac{d\hat{r}_A}{dt} + \frac{\pi\hat{r}_A^2 T_A}{G}\partial_t\left(\frac{1}{\Sigma}\right). \tag{62}$$

Introducing the following Misner-Sharp energy relation

$$E = \frac{\hat{r}_A}{2G}\frac{1}{\Sigma} = V\hat{\rho}_t, \tag{63}$$

we find

$$\frac{dE}{dt} = -4H\pi\hat{r}_A^3(\hat{\rho}_t + \hat{p}_t) + 4\pi\hat{r}_A^2\hat{\rho}_t\frac{d\hat{r}_A}{dt} + \frac{\hat{r}_A}{2G}\partial_t\left(\frac{1}{\Sigma}\right). \tag{64}$$

Now, by combining equations (62) and (64), we can obtain the following result for the first law of gravitational thermodynamics

$$T_A dS_A + T_A d_i\widetilde{S} = -dE + WdV, \tag{65}$$

where

$$T_A d_i\widetilde{S} = -\frac{1 + 2\pi\hat{r}_A T_A}{2G\hat{r}_A^{-1}}d\left(\frac{1}{\Sigma}\right) = -\Sigma\left[\frac{E}{T_A} + S_A\right]d\left(\frac{1}{\Sigma}\right). \tag{66}$$

is the additional entropy term which is produced due to the matter contents of the universe. Here, in $f(T,\theta)$-gravity, the first law of thermodynamics does not hold due to the presence of extra entropy term $T_A d_i\widetilde{S}$. But, this term vanishes if we take

$f(T,\theta) = f(T)$ which leads to the equilibrium description of gravitational thermodynamics in $f(T)$ theory of gravity. Under this limit we get the same result [16] obtained by Bamba and Geng.

4.3. NEW FORM OF THE SECOND LAW

To establish the new form of second law of thermodynamics in $f(T,\theta)$-gravity, we consider the Gibbs relation in terms of all matter and energy components

$$T_t dS_t = d(\hat{\rho}_t V) + \hat{p}_t dV. \tag{67}$$

In this scenario, the second law of gravitational thermodynamics can be defined as

$$\dot{S}_A + \dot{S}_m + \dot{\widetilde{S}} \geq 0, \tag{68}$$

which implies that

$$\frac{12\pi}{\gamma T G \Sigma H^3} \frac{\Lambda}{H} \geq 0, \tag{69}$$

where

$$\frac{\Lambda}{H} = 2(1-\gamma)\dot{H}H^2 + (2-\gamma)\dot{H}^2 + (1-\gamma)\Sigma H^3 \partial_t \left(\frac{1}{\Sigma}\right). \tag{70}$$

Hence, the second law of gravitational thermodynamics can be achieved if we have the conditions $\partial_t(\frac{1}{\Sigma}) \geq 0$, $H \geq 0$ and $\dot{H} \geq 0$. Note that in both descriptions of dark contents, the second law of thermodynamics is valid both in phantom and quintessence regions of the universe.

5. OUTLOOK

$f(T,\theta)$-gravity is the generalization of $f(T)$-gravity and can be applied to investigate many issues of modern cosmology and astrophysics. We focus on laws of gravitational thermodynamics of a spherical symmetric Friedmann-Robertson-Walker spacetime containing only the ordinary matter and we assume the boundary of the universe is enclosed by apparent horizon with the Hawking temperature. We observe that, in the first law of gravitational thermodynamics, an auxiliary entropy term $d_i\widetilde{S}$ is produced as compared both in general relativity and teleparallel gravity. On the other hand, we derive a general circumstance for the second law of gravitational thermodynamics. Bamba and Geng [16], in $f(T)$ theory of gravity, discussed the first and second laws of thermodynamics at the dynamical apparent horizon of Friedmann-Robertson-Walker universe with both non-equilibrium and equilibrium descriptions. By assuming $f(T,\theta) = f(T)$, our results give $d_i\widetilde{S} = 0$ which means one can define an equilibrium description of thermodynamics in $f(T)$-gravity. In general case, we conclude that (i) the description of equilibrium thermodynamics is not feasible in $f(T,\theta)$ theory of gravity even if we redefine the dark energy density and dark pressure, (ii) in thermal equilibrium, the second law of gravitational thermodynamics is satisfied in both phantom and quintessence phases.

REFERENCES

1. Planck Collaboration, arXiv:1303.5072.
2. C. Chawla, R. K. Mishra and A. Pradhan, Eur. Phys. J. Plus **127**, 137 (2012).
3. S. Chakraborty and A. Biswas, Astrophys. Space Sci. **343**, 791 (2013).
4. K. Bamba, C.-Q. Geng and T. Tsujikawa, Phys. Lett. B **688**, 101 (2010).
5. S. Capozziello, R. Cianci, C. Stornaiolo and S. Vignolo, Phys. Scripta **78**, 065010 (2008).
6. R. G. Cai and S. P. Kim, JHEP **02**, 050 (2005).
7. M. Akbar and R.-G. Cai, Phys. Rev. D **7**, 084003 (2007).
8. R.-G. Cai and L.-M. Cao, Phys. Rev. D **75**, 064008 (2007).
9. R.-G. Cai, L.-M. Cao, Y.-P. Hu and S. P. Kim, Phys. Rev. D **78**, 124012 (2008).
10. A. Sheykhi, B. Wang and R. G. Cai, Phys. Rev. D **76**, 023515 (2007).
11. C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. **86**, 121301 (2006).
12. S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, Class. Quantum Grav. **25**, 235018 (2008).
13. K. Karami, M. S. Khaledian and N. Abdollahi, EPL **98**, 30010 (2012).
14. K. Bamba and C.-Q. Geng, JCAP **06**, 014 (2010).
15. K. Karami and A. Abdolmaleki, JCAP **04**, 007 (2012).
16. K. Bamba and C.-Q. Geng, JCAP **11**, 008 (2011).
17. M. Sharif and M. Zubair, JCAP **03**, 028 (2012).
18. Y. G. Gong and A. Wang, Phys. Rev. Lett. **99**, 211301 (2007).
19. K. Bamba and C.-Q. Geng, Phys. Lett. B **679**, 282 (2009).
20. F. Kiani and K. Nozari, Phys. Lett. B **728**, 554 (2014).
21. D. Liu and M. L. Reboucas, Phys. Rev. D **86**, 083515 (2012).
22. K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
23. K. Hayashi and T. Shirafuji, Phys. Rev. D **24**, 3312 (1981).
24. R. G. Cai, L. M. Cao and Y. P. Hu, Class. Quant. Grav. **26**, 55018 (2009).
25. J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
26. J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
27. S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
28. R. M. Wald, Phys. Rev. D **48**, 3427 (1993).
29. G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP **02**, 010 (2005).
30. R. X. Miao, M. Li and Y. G. Miao, JCAP **11**, 033 (2011).
31. C. W. Misner and D. H. Sharp, Phys. Rev. **B571**, 136 (1964).
32. D. Bak and S.-J. Rey, Class. Quant. Grav. **17**, L83 (2000).
33. S. A. Hayward, Class. Quant. Grav. **15**, 3147 (1998).
34. S. A. Hayward, S. Mukohyama and M. Ashworth, Phys. Lett. A **256**, 347 (1999).
35. S. Dubovsky and S. Sibiryakov, Phys. Lett. B **638**, 509 (2006).
36. G. Izquierdo and D. Pavon, Phys. Lett. B **639**, 1 (2006).