Quantum Group $SU_q(2)$ as the proper group to describe the symmetry structure of the nucleus

Syed Afsar Abbas

Jafar Sadiq Research Institute
AzimGreenHome, NewSirSyed Nagar, Aligarh-202002, India
drafsarabbas@gmail.com

Abstract

The nucleus displaying both the single particle aspects as well as the collective aspects simultaneously, does not seem to be amenable to a simple group theoretical structure to explain its existence. The isospin group SU(2) takes account of the single particle aspects quite well but the collectivity is basically put in by hand. The point is that, is there some inherent symmetry connecting the single particle aspect and the collective aspects through some group theoretical structure. We do consistent and exact matching of the deformed and the superdeformed bands in various nuclei. Thus we shall show that the Quantum Group $SU_q(2)$ fulfills this requirement - not in any approximate manner, but in an exact manner.

Keywords: quantum group, Lie group, superdeformation, nuclear-symmetry, isospin-symmetry

Is the nucleus amenable to a group theoretical description? What it means is that, is there an intrinsic underlying symmetry of the nucleus which may demand a group theory to describe it? What we have in mind is the kind of description that the SU(3)- flavour model provides for a comprehensive description of all the hadronic fermions and bosons at low energies. At a little higher energies, it is the SU(4)-flavour group which describes successfully the corresponding symmetry. Also the Standard Model group structure of $SU(N_c) \otimes SU(2)_L \otimes U(1)_Y$ successfully describes all the elementary particles known to us at present,

However the complexity of the nuclear system seems to point that the answer to the above question may be in the negative. However, one should not forget that the SU(2) isospin symmetry of the nucleus, by invoking the Generalized Pauli Exclusion Principle and through a self-consistent Hartree-Fock calculation, does lead to a consistent understanding of the single particle character of the nuclear system. Thus it should be construed to mean that the SU(2) group does provide the proper group to understand the single particle character of the nucleus [1]. However collectivity is another fundamental characteristic of the nucleus. But that has to be put in by hand, and lies outside the purview of the SU(2) model of the nucleus.

The Interacting Boson Model (IBM) was an attempt to seek a group theoretical basis for the nuclear collective motion [2]. However, with time, a large number of variants of the IBM have been developed.

Here we shall show that the Quantum Group $SU_q(2)$ fulfills the above requirement of being the proper group to describe the basic and inherent symmetries of nuclei, both of the single particle and of the collective characters. And this is shown here not as an approximation, but to hold true in an exact manner.

The quantum groups [3-7] arise naturally in several areas of physics [8] and hence is the focus of much ongoing effort. The quantum algebra $SU_q(2)$ has already been applied to nuclear physics [9-12]. But in all these applications in nuclear physics, the quantum group $SU_q(2)$ is used to describe collective behaviour only as a good approximation. Hence as per these calculations, this quantum group is not taken as a group to describe the nucleus, but only provides an approximate description of the nuclear reality. Below we shall show how this quantum group is actually an exact symmetry of the nucleus. This is in conflict with what some others have expressed. For example in ref. [13] they say, "For deformed nuclei, perhaps it is more appropriate to invoke the q-deformations of higher dimensional groups (i.e. higher than $SU_q(2)$)". We show why this statement is not correct and that indeed $SU_q(2)$ itself is the right group to display the basic symmetry of the nucleus.

The quantum algebra $SU_q(2)$ is defined by the generators $J_+, J_- and J_0$ which satisfy the commutation relations

$$[J^0, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = [2J_0] = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}} \tag{2}$$

where the term on the right hand is the q-number [x]. When $q = e^t$ (with t real) it becomes$[x] = sinh(tx)/sinh(t)$ and for $q = e^{it}$ (with t real) it is $[x] = sin(tx)/sin(t)$. As q approaches 1 (or as t goes to 0) the q-numbers become the standard numbers and the $SU_q(2)$ algebra goes to the classical algebra $SU(2)$. This last point is most important. The quantum algebra here has built into it the isospin group SU(2) structure. Thus the quantum group group $SU_q(2)$ should be considered as a larger group than the group SU(2) as it contains it as subset of its parameters in a non-linear manner. This is somewhat akin to the statement that the SU(2) is a subgroup of the bigger SU(3) group

As in the usual Lie algebra the irreducible representations D(j) of $SU_q(2)$ are determined by the highest weight states with j=0,1/2,1,... The basis states $|j, m > \{-j \leq m \leq j\}$ are connected to the highest weight states as follows

$$\mid j, m\rangle = \sqrt{\frac{[j+m]!}{[2j]![j-m]!}} (J_-)^{j-m} \mid j, j\rangle \tag{3}$$

where [n]!=[1][2]..[n] and $J_+|j, j >= 0, < j, j|j, j >= 1$. Also $J_0|j, m >= m|j, m >$ and

$$J_\pm \mid j, m\rangle = \sqrt{[j \pm m][j \pm m + 1]} \mid j, m \pm 1\rangle \tag{4}$$

The Casimir invariant operator is given as

$$C_2^q = J_- J_+ + [J_0][J_0 + 1] \tag{5}$$

whose eigenvalue is

$$C_2^q \mid j, m\rangle =\mid [j][j+1] \mid j, m\rangle \tag{6}$$

We wish to use this language for the deformed nuclei. As an extension of the standard rotor model we formulate a q-rotor model based on the group $SU_q(2)$. It is a system whose Hamiltonian is given as

$$H_q = \frac{C_2^q}{2I} + E_0 \tag{7}$$

Here I is the moment of inertia and E_0 is the band head energy For the case of q being a phase, for

$$E_j = \frac{[j][j+1]}{2I} + E_0 \tag{8}$$

we get

$$E_j = \frac{Sin(tj)Sin(t(j+1))}{2I * Sin^2 t} + E_0 \tag{9}$$

For small t we can Taylor expand the series to look like [10]

$$E_j = E_0 + \frac{[(j_0(r))j(j+1) - tj_1(r)(j(j+1))^2 + (2/3)t^2 j_2(r)j(j+1))^3 - ..]}{2I(j_0(r))^2} \tag{10}$$

Now this is of the form

$$E_j = E_0 + A(j(j+1)) + B(j(j+1))^2 + C(j(j+1)^3 + .. \tag{11}$$

This is an empirical series used in fitting rotational bands in nuclei.It is found that A,B,C.. have alternating signs and that B/A is apprx 10^-3 This is also similar to the VMI model of nuclear physics [10]. Note that the VMI model in nuclear physics is entirely a phenomenological model with no group theoretical basis. Here it turns out that the the quantum group $SU_q(2)$ provides a very good group theoretical justification of the same.

The authors of ref. [10] have used the above expansion eqn.(10) as a series in powers of j(j+1) as a good approximation of the quantum group $SU_q(2)$ to study deformation and superdeformation in nuclei. They got pretty good fits and that prompted them to go to higher quantum group as they said, "The study of the quantum algebra $SU_q(3)$ and its use for the description of the rotational spectra is therefore of interest and it is under investigation."

So they clearly thought that the quantum group $SU_q(2)$ was not big enough to describe the nuclear collective effects in a satisfactory manner. They perhaps thought the freedom of more free parameters in a larger group may turn out to be better. But note that this logic is not always correct. The SU(3) description of the u-, d- and s-quarks does not improve by going to the bigger SU(4) group!

Perhaps also the above attitude was due to the fact the energy spectrum of $SU_q(2)$ as given in eqn. (9) is non-linear. To simplify it they linearized it by taking the Taylor series and getting the approximate expression as given in eqn. (10). Thus no fundamentality is being displayed here.

But let us take the $SU_q(2)$ prediction more seriously. The above non-linear expression eqn. (9) is unique and an exact prediction of $SU_q(2)$ as to what the deformed and superdeformed bands in nucleus should be - no approximation and this should be exactly true. If this $SU_q(2)$ were the proper group theoretical structure of the nucleus and which consistently takes care of the non-linearity inherent in the collective motions in the nucleus, then this should hold good for the nucleus in an exact manner and not as any approximation.

With the above attitude we did the fittings of the same nuclei as given in ref. [10], but now with the exact non-linear expression for deformed and superdeformed nuclei. We give in Table 1 our prediction versus theirs [10] for $^{194}Hg(2)$. The goodness of the fit is determined by the root-mean-square deviation:

$$\sigma = [1/N\Sigma[\frac{E_j(cal) - E_j(exp)^2}{E_j(exp)}]^{1/2} \tag{12}$$

Note that our calculation with exact prediction of the $SU_q(2)$ group has a far superior fit as per the root-mean-square deviation. Hence we take this as a confirmation of the goodness of the group $SU_q(2)$ itself as displaying the proper symmetry structure of the nucleus. Hence in as much as collectivity is being mapped properly and correctly by the non-linear quantum group $SU_q(2)$ and as SU(2) is the proper substructure of it, this should be taken as the proper group to describe the intrinsic symmetry structure of the nucleus.

Below we give more calculations confirming the veracity of the above statement. Starting with the discovery [14] of discrete line superdeformed band in 152-Dy, much work has been done to study superdeformed bands in nuclei [15,16].

We find that our method (by using the exact expression eqn. (9)) is a marked improvement over the previous attempts in any fits of the superdeformed bands. So much so that we can make predictions for the spin of the superdeformed bands in nuclei. The spins for the nuclei in the region A=150 are hard to predict because of the high values. However our predictions are always better than those of the others [9,10]

However note that for higher j the above series, eqn. (10), does not converge fast enough and hence if we use it to fit superdeformed band with high j the fits are likely to be of poor quality. What we have done is to use the above philosophy that we have the exact nonlinear expression for E_j, eqn. (9) and this we can justifiably use for any j (howsoever high). We therefore use the exact expression of E_j (which has two parameters I and t) to fit the deformed and superdeformed bands in nuclei. As expected we find that we immediately make fits of superior quality than done earlier. For example for 192-Hg (2164 kev) we obtain a fit where the Standard Deviation is 3.53 kev compared to the earlier fit [12] of Standard Deviation 8.26 kev. This is general conclusion that always the fits that we get are superior to the one obtained by the Taylor series expansion method. Here as a sample we show how our fits are for a pair of identical superdeformed bands in two neighbouring nuclei as displayed in Table 2.

Here we also address ourselves to the important and open problem of the spin determination of superdeformed bands in nuclei. Experimentally what one determines are the transition energies $E_j = E_j - E_{j-2}$.This superdeformed band cascade is least square fit to our nonlinear expression for E_j. How good an agreement one gets with the experimental numbers depends upon the assigned level spin. We vary the lowest spin J_0 and find that for a particular value of it the fit is much superior compared to when we change it by one unit or more.

Next we study the difficult problem of the spin determination of superdeformed bands in the region $A \sim 150$. The problem has to do with the fact that in these nuclei J_0 is high ~ 30. The tabulation of the superdeformed bands in various nuclei has been published [17]. Tentative assignment for J_0 based on collective experimental information have been made there [17] for various bands. But as they state in these determinations "An 1h-2h uncertainty is expected" [17]. So to say these estimates are not final but too much variations are not expected. However in literature we note that on the basis of some theoretical predictions there are large variations. For example in ref. [12] they obtain J_0=36.5 for Tb^{151}(yrast) - to be compared to 25.5 of [17]; J_0=38 for Gd^{150} (excited:770kev) - to be compared to 31 of [17]; J_0=29 for Gd^{148} (701kev) - to be compared to 24 of [17].

How do our predictions come out for these nuclei? We list them in Table 3. For comparison we list the predictions from [17] also. Note that our predictions are closer to those of [17] and hence are "much better' than those say, of [12].

So all this confirms that the goodness of the use of the exact $SU_q(2)$ prediction of eqn. (9), supports the fact that this indeed is the justified candidate to be the proper group accounting for the intrinsic symmetry structure of the nucleus.

Without getting into irrelevant details, let us understand that the essential and intrinsic characteristics, that define the many-body character of the nucleus are two fold: firstly, it has single particle structure and secondly, it has collective structure. The group which properly accounts for these conflicting aspects of the nucleus (but which indeed is its intrinsic symmetry structure), as shown here, is the non-linear quantum group $SU_q(2)$. It describes all collective aspects of the nucleus and in a certain linear limit leads to the standard Lie group SU(2) which describes the single particle character of the nucleus.

REFERENCES

[1]. Wong S S M (1990) "Introductory Nuclear Physics", Prentice Hall

[2]. Iachello F and Arima A (1988), "Interacting Boson Model", Cambridge University Press, Cambridge

[3]. Kulish P P and Reshetikhin N Yu (1981) Zapiski Semenarov LOMI **101**, 101

[4]. Sklyanin E K (1982) Funct. Anal. Appl. **16**, 262

[5]. Drinfeld V G (1985) Dokladi A N USSR **282**, 1060

[6]. Jimbo M (1986) Lett. Math. Phys. **11**, 247

[7]. Macfarlane A J (1989) J. Phys, A: Math. Gen. **22**, 4581

[8]. Biedenharn L C (1990) Lecture Notes Phys. **37**, 67

[9]. Raychev P P et al (1990) J. Phys. G:Nucl. Part. Phys. **16**, L137

[10]. Bonatsos D et al (1991) J. Phys. G: Nucl. part. Phys. **17**, L67

[11]. Zeng J Y et al (1991) Phys. Rev. C **44** R1745

[12]. Wu C S et al (1992) Phys. Rev. C **45** 2507

[13]. Gupta R K et al (1992) J. Phys. G: Nucl.Part. Phys. **18**, L73

[14]. Twin P J (1986) Phys. Rev. Lett. **57**, 911

[15]. Sharpey-Schafer J F (1992) Prog. Part. Nucl.Phys. **28**, 187

[16]. Janssens R V F and Khoo T L (1991) Ann. Rev. Part. Nucl. Sci. **41**, 321

[17]. Han X-L and Wu C-L (1993) At. Data and Nucl. Data Tables

J	Expt[Mev]	Bonatsos et.al.	Our fit
10	0.201	0.201	0.200
12	0.444	0.443	0.442
14	0.727	0.726	0.724
16	1.052	1.049	1.047
18	1.415	1.413	1.410
20	1.818	1.815	1.812
22	2.259	2.257	2.253
24	2.737	2.735	2.731
26	3.252	3.251	3.247
28	3.802	3.803	3.798
30	4.387	4.389	4.385
32	5.007	5.009	5.006
34	5.659	5.662	5.659
36	6.344	6.347	6.345
38	7.062	7.061	7.061
40	7.809	7.805	7.807

Table 1: Experimental and theoretical data for superdeformed bands in $^{194}Hg(2)$ as given in Ref.[10] and compared with our own result obtained with exact expression eqn. (9). σ=2.092 kev, t=0.0107 and 1/2I=532 kev for Bonatsos et. al. [10] and σ=1.051 kev, t=0.011 and 1/2I=530 kev for our result Note $J = J_0 + 2$.

	$^{152}D_y$				^{151}Tb
J	Expt[kev]	Our fit	J	Expt[kev]	Our fit
26	602.2	598.46			
28	647.2	645.4	27.5	647.0	640.31
30	692.2	692.34	29.5	692.0	687.69
32	737.5	739.27	31.5	738.0	735.06
34	783.5	786.21	33.5	783.0	782.42
36	829.2	883.15	35.5	828.0	892.76
38	876.1	880.09	37.5	876.0	877.09
40	923.1	927.03	39.5	992.0	921.41
42	970.0	973.96	41.5	970.0	971.72
44	1017.0	1020.9	43.5	1016.0	1019.0
46	1064.8	1067.84	43.5	1963.0	1066.28
48	1112.7	1114.78	47.5	1112.0	1113.53
50	1160.8	1161.72	49.5	1158.0	1160.77
52	1208.7	1208.65	51.5	1207.0	1207.99
54	1256.6	1255.59	53.5	1256.0	1255.19
56	1304.7	1302.53	55.5	1305.0	1302.37
58	1353.0	1349.47	57.5	1353.0	1349.53
60	1401.7	1396.41			
62	1449.4	1443.34			

Table 2: Fitting of identical bands in the neighbouring nuclei ^{152}Dy and $^{151}Tb^{*}$. σ is the rms deviation as defined in the text. For $^{152}Dy\sigma$ =3.3x10^{-3}, t=1x10^{-5} and 1/2I=587. For $^{151}Tb^{*}$ σ=3.67x10^{-3}, t=9.9x10−4 and 1/2I=593. Note $J = J_0 + 2$.

Nucleus	Data Table [17]	Our Prediction	Others[11,12]
$^{152}Dy(602.2)$	22	24	25
$^{151}Tb(647.0)$	24.5	25.5	27.5
$^{151}Tb(728.0)$	25.5	29.5	36.5
$^{150}Gd(770.0)$	28	31	38
$^{150}Tb(598.0)$	21	21	23
$^{149}Gd(859.9)$	37.5	31.5	34.5
$^{146}Gd(826.0)$	30	28	34
$^{147}Gd(779.1)$	27.5	26.5	31.5
$^{147}Gd(663.9)$	22.5	23.5	27.5
$^{148}Gd(701.0)$	24	24	29

Table 3: Predictions for the spin J_0 of the SD bands for nuclei in the region $A \sim 150$