Fractional Dirac Magnetic Monopole Charges without Observable Singularities

Jay R. Yablon
910 Northumberland Drive
Schenectady, New York 12309-2814
jyablon@nycap.rr.com

April 25, 2015

Abstract: It is widely believed that Dirac magnetic monopoles and their related electric charges must be quantized, and that any fractional charges one might posit cannot exist without creating forbidden observable singularities. Here, we explicitly present a vector potential for a Dirac monopole with fractional magnetic and electric charges whose curl is a Coulomb magnetic field and which potential has no observable singularities. We then demonstrate how these fractional charges are projected onto SO(3) from topological covering groups with generators which are the generalized m^{th} roots of the 2x2 identity matrix I, situated at various Euler angles on the complex plane of the covering group generators, all without observable singularities.

PACS: 11.15.-q; 14.80.Hv

Contents

1. Introduction

In 1931 Dirac [1] discovered that if magnetic charges with strength g were to hypothetically exist, then this would imply that electric charge strength e must be quantized. The relationship he found, often written as $2eg = n$ where n is a positive or negative integer or zero, came to be known as the Dirac Quantization Condition (DQC). In the mid-1970s, to remedy the need to resort to the fiction of Dirac's "nodal lines" which subsequently became known as Dirac strings, Wu and Yang [2], [3] developed an approach which does not at all make use of these strings. Its results are completely equivalent to Dirac's, with the only difference being that it is cast in the more-modern language of fiber bundles. In the Wu Yang approach, one uses $U(1)_{em}$ gauge theory to obtain the differential equation $e^{-i\Lambda} de^{i\Lambda} = i2egd\varphi$ where Λ is the gauge (really, phase) angle and φ is the geometric azimuth about the z-axis in the three dimensional physical space of the rotation group SO(3), which equation is easily seen to be solved for constant electric and magnetic charges by $\exp(i\Lambda) = \exp(i2eg\varphi)$. It has long been believed that the only solution to this latter Wu-Yang equation that is free of observable singularities, is $2eg = n$; and this is in fact true if one restricts the azimuth domain on SO(3) to $0 \le \varphi \le 2\pi$. But otherwise, as we show here, it becomes possible to widen this solution to include non-singular fractional charges of the form $2eg = n/m$ where m is also an integer.

In sections 2 through 4 we explicitly present a vector potential for a Dirac monopole with fractional magnetic and electric charges whose curl is a Coulomb magnetic field and which potential has no observable singularities using the standard Dirac condition. Thereafter, we show how if one instead moves this domain restriction out of SO(3) and into certain topological covering spaces $\tilde{G}$ for which the generators are generally the k^{th} roots $k = 1,2,3,4,5...$ of a triplet of 2x2 identity matrices I_i, $i = 1,2,3$, and then surjectively projects these groups homomorphically onto SO(3) via $\pi : \tilde{G} \to SO(3)$, the SO(3) domain is widened to $0 \le \varphi \le 2\pi k$. Most importantly, we show how with this widened domain, it then becomes possible to project Dirac monopoles onto SO(3) which adhere to the fractionalized quantization condition $2eg = n/m$, with $m = k$ arising from the expanded SO(3) domain, *which fractional charges do not give rise to any observable singularities*.

2. Local $U(1)_{em}$ Gauge Transformations, In General

We begin by considering a first electron wavefunction $\psi_-(x^\mu)$ which is related to a second electron wavefunction $\psi_+(x^\mu)$ by the local U(1) gauge transformation (throughout, we shall employ natural units $\hbar = c = 1$):

$$\psi_- \to \psi_+ = \psi_-' = \exp(i\Lambda)\psi_-, \tag{2.1}$$

where phase angle $\Lambda(x^\mu)$ varies locally as a function of the spacetime coordinates as do the wavefunctions. Because $U = \exp(i\Lambda)$ is unitary, $U^*U = 1$, the probability density

$\rho = \psi_+{}^\dagger \psi_+ = (U^* U)\psi_-{}^\dagger \psi_- = \psi_-{}^\dagger \psi_-$ is naturally invariant under this transformation. If we write the complex wavefunction $\psi_+(x^\mu) = A_+(x^\mu) + iB_+(x^\mu)$ and likewise $\psi_-(x^\mu) = A_-(x^\mu) + iB_-(x^\mu)$, this invariance means that $A_+{}^2(x^\mu) + B_+{}^2(x^\mu) = A_-{}^2(x^\mu) + B_-{}^2(x^\mu)$. And of course, we require that $\int \rho d^3x = 1$ to ensure that this ρ is indeed a probability density, which also serves to normalize ρ.

Next, we define a gauge potential $A_{-\mu}(x^\mu)$ to be the electromagnetic vector potential corresponding to the wavefunction ψ_-, and we then use this to define the gauge-covariant derivative $D_{-\mu} \equiv \partial_\mu + ieA_{-\mu}$ where e is the (running) electric charge strength, and where the sign of $ieA_{-\mu}$ is positive because we are using a Minkowski metric tensor $\mathrm{diag}(\eta_{\mu\nu}) = (1,-1,-1,-1)$ versus the oppositely-signed convention. Applying this derivative to each side of $\exp(i\Lambda)\psi_-$ in (2.1), we obtain:

$$\begin{aligned} &D_{-\mu}\left(\exp(i\Lambda)\psi_-\right) = \left(\partial_\mu + ieA_{-\mu}\right)\left(\exp(i\Lambda)\psi_-\right) \\ &= i\partial_\mu\Lambda \exp(i\Lambda)\psi_- + \exp(i\Lambda)\partial_\mu\psi_- + ieA_{-\mu}\exp(i\Lambda)\psi_- . \\ &= \exp(i\Lambda)\left[\partial_\mu\psi_- + \left[ieA_{-\mu} + i\partial_\mu\Lambda\right]\psi_-\right] \end{aligned} \tag{2.2}$$

Next, based on the inner bracketed expression in the bottom line above, we define a second gauge potential $A_{+\mu}$ corresponding with the wavefunction ψ_+ according to:

$$eA_{+\mu} \equiv eA_{-\mu} + \partial_\mu\Lambda . \tag{2.3}$$

With this, (2.2) may be written more compactly as:

$$D_{-\mu}\left(\exp(i\Lambda)\psi_-\right) = \exp(i\Lambda)\left(\partial_\mu + ieA_{+\mu}\right)\psi_- . \tag{2.4}$$

Then, defining the related covariant derivative $D_{+\mu} \equiv \partial_\mu + ieA_{+\mu}$, (2.4) further reduces to:

$$D_{-\mu}\left(\exp(i\Lambda)\psi_-\right) = \exp(i\Lambda)D_{+\mu}\psi_- . \tag{2.5}$$

If we now apply $D_{-\mu}$ to all expressions in the wavefunction transformation (2.1), the above allows us to write:

$$D_{-\mu}\psi_- \rightarrow D_{-\mu}\psi_+ = D_{-\mu}\psi'_- = D_{-\mu}\left(\exp(i\Lambda)\psi_-\right) = \exp(i\Lambda)D_{+\mu}\psi_- . \tag{2.6}$$

Multiplying the subset equality $D_{-\mu}\psi_+ = \exp(i\Lambda)D_{+\mu}\psi_-$ through by $\exp\left(-\tfrac{1}{2}i\Lambda\right)$, yields an expression which highlight the +/- symmetries:

$$\exp\left(-\tfrac{1}{2}i\Lambda\right)D_{-\mu}\psi_{+}=\exp\left(\tfrac{1}{2}i\Lambda\right)D_{+\mu}\psi_{-}, \tag{2.7}$$

or, directly in terms of the gauge potentials via $D_{-\mu}\equiv\partial_\mu+ieA_{-\mu}$ and $D_{+\mu}\equiv\partial_\mu+ieA_{+\mu}$:

$$\exp\left(-\tfrac{1}{2}i\Lambda\right)\left(\partial_\mu+ieA_{-\mu}\right)\psi_{+}=\exp\left(\tfrac{1}{2}i\Lambda\right)\left(\partial_\mu+ieA_{+\mu}\right)\psi_{-}. \tag{2.8}$$

The gauge transformation (2.3) acting on the gauge fields may readily be rewritten using the mathematical identity $i\partial_\mu\Lambda=e^{-i\Lambda}\partial_\mu e^{i\Lambda}$, as:

$$A_{+\mu}=A_{-\mu}+e^{-i\Lambda}\partial_\mu e^{i\Lambda}/ie. \tag{2.9}$$

Further, one may generally pack a vector potential into the differential one-form $A=A_\mu dx^\mu$. Therefore these two potentials have the associated one-forms $A_+=A_{+\mu}dx^\mu$ and $A_-=A_{-\mu}dx^\mu$, and (2.9) therefore compacts and rearranges into:

$$A_+-A_-=e^{-i\Lambda}de^{i\Lambda}/ie. \tag{2.10}$$

This tells us that these two gauge fields differ from one another by no more than a $U(1)_{em}$ gauge transformation.

3. A Coulomb Magnetic Field which is the Curl of a Vector Potential, i.e., a $U(1)_{em}$ Magnetic Monopole

The electromagnetic field strength two-form $F=\tfrac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ is in general related to the vector potential by $F=dA$, and so is a locally-exact two-form. Extracting the electric / magnetic bivector $F_{\mu\nu}$, the space components of the field strength tensor are $F_{ij}=\partial_i A_j-\partial_j A_i$. The magnetic field vector $F_{ij}=-\varepsilon_{ijk}B^k$ where ε_{ijk} is the antisymmetric Levi-Civita tensor and $\varepsilon_{123}=+1$, and where $B^k=\mathbf{B}=\left(B_x,B_y,B_z\right)$ in Cartesian coordinates. Likewise, using $\mathrm{diag}\left(\eta_{\mu\nu}\right)=\left(1,-1,-1,-1\right)$ to lower indexes in $A^\mu=\left(\phi,\mathbf{A}\right)=\left(\phi,A_x,A_y,A_z\right)$, and with $\partial_i=\nabla=\left(\partial_x,\partial_y,\partial_z\right)$, this means that $F_{ij}=-\varepsilon_{ijk}B^k=\partial_i A_j-\partial_j A_i$, or $\mathbf{B}=\nabla\times\mathbf{A}$. So whenever we have $F=dA$ in general for a given potential, the magnetic field $\mathbf{B}$ will be the curl of the vector potential, $\nabla\times\mathbf{A}$.

Now, let us define the two four-vector potentials in A_- and A_+ of the last section *such that* these are the potentials for a *Coulomb magnetic field* $\mathbf{B}$ which is the curl of these vector potentials, $\mathbf{B}=\nabla\times\mathbf{A}$, that is, let us now define the potentials for a magnetic monopole. First, we posit a (running) magnetic charge strength g for such a monopole. Second, we write each of the potential one-forms A_- and A_+ in a polar coordinate basis as:

$$\begin{aligned} A_+ &\equiv g(\cos\theta+1)d\varphi \\ A_- &\equiv g(\cos\theta-1)d\varphi \end{aligned}. \tag{3.1}$$

Confining our domain to $0 \le \theta \le \pi$, we see that A_+ is a "southerly" potential defined everywhere except at $\theta = 0$, i.e., except due north of the origin, while A_- is "northerly" because it is defined everywhere except at $\theta = \pi$, i.e., except due south of the origin. We now show that these will indeed produce a Coulomb magnetic field for which $\mathbf{B} = \nabla \times \mathbf{A}$ for both of the vector potentials $\mathbf{A}_+$, $\mathbf{A}_-$.

First, we hold *g* constant, $dg = 0$, that is, we do not let *g* run over the region of spacetime in question. Now, because differential forms geometry teaches that $dd = 0$ in general and thus $dd\varphi = 0$ in this specific setting, this all means that:

$$F = dA_+ = dA_- = gd\cos\theta d\varphi . \tag{3.2}$$

Therefore, for either potential, the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}_- = \nabla \times \mathbf{A}_+$, as desired.

Of course, $dF = ddA_+ = ddA_- = 0$ via the same identity $dd = 0$, which means that *F* is closed and locally exact. But it is not globally exact. Specifically, if we integrate (3.2) over a closed two-dimensional surface with *g* still held constant, and if we also apply Gauss' / Stokes' theorem, then:

$$\iiint dF = \oiint F = \oiint gd\cos\theta d\varphi = g\int_0^\pi d\cos\theta \int_0^{2\pi} d\varphi = g\cos\theta\Big|_0^\pi \varphi\Big|_0^{2\pi} = -4\pi g . \tag{3.3}$$

The fact that we are holding *g* constant throughout the spacetime region under consideration is reflected by our having moved *g* outside the integral after the third equal sign above. Now let us specifically pinpoint the magnetic field.

To do so, we consider the circumstance for which the electric fields vanish, that is, for which $F_{0k} = -F_{k0} = \mathbf{E} = 0$. In this circumstance, $\oiint F = \oiint \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint \frac{1}{2} F_{ij} dx^i dx^j$. Then, using this in (3.3) also in view of $F_{ij} = -\varepsilon_{ijk} B^k$, we find that:

$$\oiint F = \oiint \tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint F_{12} dx^1 dx^2 + \oiint F_{23} dx^2 dx^3 + \oiint F_{31} dx^3 dx^1 = -\oiint \mathbf{B}\cdot\mathbf{dS} = -4\pi g . \tag{3.4}$$

So from the final equality above, this means that:

$$\oiint \mathbf{B}\cdot\mathbf{dS} = 4\pi g \equiv \mu , \tag{3.5}$$

where $\mu \equiv 4\pi g$ is defined as the total magnetic flux across the closed surface. Conversely, the magnetic charge strength $g = \mu / 4\pi$, when held constant $dg = 0$ in the integral (3.3), also represents the steradial density of magnetic flux across the closed surface. This, of course, is Gauss' law for magnetism in integral form, but with a non-zero magnetic flux μ across the closed surface. Thus, this is the integral formulation of Gauss' law for a magnetic monopole, for which there are no induced *electric* fields, which is confirmed because **E**=0 in (3.4), (3.5). As a result of there being no electric field induction, (3.5) describes this magnetic monopole at rest.

Now, in general, Coulomb's law cannot be derived from Gauss' law alone. However, if the magnetic monopole is stationary – which it is because $\mathbf{E} = 0$ in (3.4) and (3.5) – then the magnetic field **B** in (3.5) will be exactly spherically symmetric. As a result of this spherical symmetry, we may remove **B** from the integrand in (3.5), thus writing:

$$\mathbf{B} \oiint \mathbf{dS} = \mathbf{B} \cdot 4\pi r^2 = 4\pi g = \mu . \tag{3.6}$$

Because of the spherical symmetry, only the radial component B_r of **B** will be non-zero, that is, in spherical coordinates, we will have $\mathbf{B} = \left(B_r, B_\varphi, B_\theta\right) = \left(B_r, 0, 0\right)$. Therefore, (3.6) now yields:

$$B_r = \frac{g}{r^2} = \frac{\mu}{4\pi r^2} . \tag{3.7}$$

This is indeed a Coulomb magnetic field which has a magnetic charge strength *g*, and for which the total magnetic closed surface flux $\mu = 4\pi g$. Furthermore, this Coulomb magnetic field is the curl of the vector potentials, $\mathbf{B} = \nabla \times \mathbf{A}_- = \nabla \times \mathbf{A}_+$.

Now, we turn to examine the full set of conditions under which this Coulomb magnetic monopole with $\mathbf{B} = \nabla \times \mathbf{A}$ does not give rise to any observable singularities.

4. Conditions under which the U(1)$_{em}$ Magnetic Monopole has No Observable Singularities: The Standard Dirac Quantization Condition

Returning to (3.1), we first find that the difference:

$$A_+ - A_- = 2g d\varphi . \tag{4.1}$$

Combining the above with (2.10) then yields the differential equation:

$$e^{-i\Lambda} de^{i\Lambda} / ie = 2g d\varphi . \tag{4.2}$$

This differential equation clearly is solved for constant *e* and constant *g*, i.e., for $de = 0$ and $dg = 0$ by:

$$\exp(i\Lambda) = \exp(i2eg\varphi). \tag{4.3}$$

We then return to (2.1) and employ this solution to operate on ψ_-, thus writing:

$$\psi_- \to \psi_+ = \psi'_- = \exp(i\Lambda)\psi_- = \exp(i2eg\varphi)\psi_-. \tag{4.4}$$

Clearly, for $\varphi = 0$, we have $\psi_+ = \psi_-$. But as we move ψ_+ through the Coulomb magnetic field of (3.7), we must require that the wavefunction satisfy certain constraints. If we confine ourselves to the domain $0 \le \varphi \le 2\pi$, then to make ψ_+ single-valued for complete rotations through φ and thus avert string singularities, we are *required* to impose the condition:

$$2eg = n, \tag{4.5}$$

where *n* is a positive or negative integer, or zero. Using $g = \mu / 4\pi$, this may alternatively be expressed as

$$e\mu = 2\pi n. \tag{4.6}$$

These are two different but equivalent ways of stating the standard Dirac Quantization Condition (DQC). With this condition imposed, (4.4) becomes:

$$\psi_- \to \psi_+ = \psi'_- = \exp(i\Lambda)\psi_- = \exp(in\varphi)\psi_-. \tag{4.7}$$

Also, note the implied quantized relationship:

$$\Lambda = n\varphi \tag{4.8}$$

between the phase angle Λ and the azimuth angle φ.

Then, as we move ψ_+ over an entire, single closed curve from an azimuth $\varphi = 0$ to an azimuth $\varphi = 2\pi$, (4.6) above will become, for $\varphi = 2\pi$:

$$\psi_- \to \psi_+ = \psi'_- = \exp(i\Lambda)\psi_- = \exp(in\varphi)\psi_- = \exp(i2\pi n)\psi_- = \psi_-. \tag{4.9}$$

Therefore $\psi_+(\varphi = 2\pi)$ will have the single value $\psi_+ = \psi_-$ for any and all *n*. *This means that there will be no string singularities.* From (4.5), we see that the electric charge strength is quantized in units of $e = \frac{1}{2}n / g$, and reciprocally, that the magnetic charge strength is quantized in units of $g = \frac{1}{2}n / e$. Likewise, for $\varphi = 0$ wound to $\varphi = 2\pi$, (4.8) becomes:

$$\Lambda = 2\pi n \tag{4.10}$$

With the quantization condition (4.5) we may finally return to (3.1) and write the potentials as:

$$eA_{+} = \tfrac{1}{2} n\left(\cos\theta + 1\right) d\varphi \\ eA_{-} = \tfrac{1}{2} n\left(\cos\theta - 1\right) d\varphi , \qquad (4.11)$$

which complement the relationship $\psi_{+} = \exp(in\varphi)\psi_{-}$ in (4.7). It is sensible that for an electric charge strength which is quantized, the associated potentials will likewise be quantized as above.

The question we now raise is whether (4.5) is too restrictive, and in particular, a) what sorts of quantization conditions are permitted or required if we expand the azimuth domain to allow for $0 \le \varphi \le 2\pi k$, where k is any positive integer, b) what it actually means, topologically, to expand the azimuth domain in this fashion, c) how one might go about expanding this azimuth domain in a well-defined, unambiguous manner, d) whether there are other non-singular monopole charge solutions which are being overlooked in (4.5) and which are only revealed with this expanded domain, and e) what those overlooked non-singular solutions might be.

5. Extended Domain Non-Singular Conditions: The Fractional Dirac Quantization Condition

The standard DQC of (4.5), $2eg = n$, was a required condition for avoiding observable string singularities when we restricted our consideration to the azimuth domain $0 \le \varphi \le 2\pi$. Now we examine the question of what happens when we extend this domain to azimuths for which $0 \le \varphi \le 2\pi k$. To prepare for this examination, let us first postulate a replacement of the ordinary DQC with a more liberal Fractionalized Dirac Quantization Condition (FDQC):

$$2eg = \frac{n}{m} = \mathbb{Q} . \qquad (5.1)$$

Above, m is a positive integer, that is, $m = 1,2,3,4,5...$ and $\mathbb{Q}$ generally denotes any real number which can be written as a quotient n/m, i.e., any rational number. We then denote the set of irrational real numbers by $\bar{\mathbb{Q}}$. Under this liberalized condition the electric and magnetic charge strengths would become $e = \tfrac{1}{2} n/mg$ and $g = \tfrac{1}{2} n/me$, and so would not only be quantized in units of n, but would also be fractionalized by denominators of m. This also means that the vector potentials (3.1) would now become:

$$eA_{+} \equiv \frac{1}{2}\frac{n}{m}\left(\cos\theta + 1\right) d\varphi \\ eA_{-} \equiv \frac{1}{2}\frac{n}{m}\left(\cos\theta - 1\right) d\varphi , \qquad (5.2)$$

contrast (4.11). It is sensible that for fractionalized charges – to the extent that they can exist without singularities – the potential would also be cut by a commensurate fraction.

Now, we do not expect that (5.1) will avoid observable singularities without restriction. So our goal is to understand the circumstances under which (5.1) can hold without singularities, versus those under which it is excluded because of singularities. To do this we first explore what sorts of restrictions must be imposed upon (5.1) to maintain ψ_+ in (4.4) as a single-valued wavefunction and thus avoid any observable string singularities. Then, we turn to understanding the topological conditions that might support such fractional charges (5.1).

If we utilize the FDQC of (5.1) rather than the DQC of (4.5) and simultaneously consider the extended domain $0 \le \varphi \le 2\pi k$, then (4.4) becomes:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i2eg\varphi)\psi_- = \exp\left(i\frac{n}{m}\varphi\right)\psi_- . \tag{5.3}$$

Of course, for $\varphi = 0$ we still have $\psi_+ = \psi_-$, which is single valued.

Now, let us now suppose that we again move the electron from $\varphi = 0$ to $\varphi = 2\pi$ over a single rotation about the magnetic monopole field (3.7). With $\varphi = 2\pi$, (5.3) yields:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i4\pi eg)\psi_- = \exp\left(i\frac{1}{m}2\pi n\right)\psi_- . \tag{5.4}$$

This will be single-valued for all *n*, *if and only if* $m = 1$, which is the standard DQC of (4.5). In other words, when the azimuth domain is restricted to $0 \le \varphi \le 2\pi$, we *must* employ the standard DQC $2eg = n$ without fractionalization, which is contained within (5.4) for this required *m*=1. No fractional charges may be admitted with a $0 \le \varphi \le 2\pi$ domain. And this is where the prevailing view and understanding of Dirac monopoles ends.

But now, rather than performing a single rotation, let us revolve the electron about the monopole twice, from $\varphi = 0$ to $\varphi = 4\pi$. That is, let us now consider the domain $0 \le \varphi \le 4\pi$ for which *k*=2. Then, (5.3) yields:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i8\pi eg)\psi_- = \exp\left(i\frac{1}{m}4\pi n\right)\psi_- . \tag{5.5}$$

If *m*=1 this will be single-valued for all *n*, which of course is trivial, because *m*=1 used in (5.1) is still just the standard DQC. However, here, we may also employ *m*=2 without having an observable singularity. If *m*=2, then (5.5) will become $\psi_+ = \exp(i2\pi n)\psi_-$, and we will also retain $\Lambda = 2\pi n$ from (3.10). Clearly, $\psi_+ = \exp(i2\pi n)\psi_-$ is still single-valued for any and all *n* and thus produces no observable singularities. In sum: for k=2, we can set either $m=1$ or $m=2$ in (5.1) and remain free of singularities.

Next, let us wind the electron about the monopole three times, from $\varphi=0$ to $\varphi=6\pi$. Now the domain is $0\le\varphi\le 6\pi$ with k=3. For this domain (5.3) yields:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i12\pi eg)\psi_- = \exp\left(i\frac{1}{m}6\pi n\right)\psi_- . \tag{5.6}$$

This wavefunction ψ_+ will be single-valued and thus non-singular for m=1 and m=3. However, it is not single-valued for m=2, because in this event, $\psi_+ = \exp(i3\pi n)\psi_- = \exp(i\pi n)\psi_- = \mp\psi_-$, which is two-valued, with the coefficient ± 1. For odd $n=1,3,5...$ $\psi_+ = -\psi_-$ while for even $n=0,2,4,6...$ $\psi_+ = +\psi_-$. Thus, for k=3, the fractions $m=1,3$ are non-singular and so are permitted, but m=2 would lead to an observable singularity and so is excluded.

For a k=4 quadruple revolution over the domain $0\le\varphi\le 8\pi$, we have:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i16\pi eg)\psi_- = \exp\left(i\frac{1}{m}8\pi n\right)\psi_- . \tag{5.7}$$

This will remain single-valued thus non-singular for $m=1,2,4$. However, m=3 is excluded, and it is interesting to see why.

For the excluded fraction m=3, (5.7) becomes:

$$\psi_+(m=3) = \exp\left(i\frac{8}{3}\pi n\right)\psi_- = \exp\left(i\frac{2}{3}\pi n\right)\psi_- = \left[\cos\left(\frac{2}{3}\pi n\right) + i\sin\left(\frac{2}{3}\pi n\right)\right]\psi_- . \tag{5.8}$$

For n=1, m=3 the argument of these periodic functions becomes $2\pi/3 = 120°$, which sits in the upper-left quadrant of the complex $a+bi$ plane. We can use a $30°-60°-90°$ triangle to ascertain that $\cos(2\pi/3) = -\tfrac{1}{2}$ and $\sin(2\pi/3) = \sqrt{3}/2$. So for n=1, (5.8) becomes:

$$\psi_+(n=1, m=3) = \left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]\psi_- . \tag{5.9}$$

For n=2, m=3 the argument becomes $4\pi/3 = 240°$, which uses the same $30°-60°-90°$ triangle simply in a different (lower-left) quadrant of the complex plane where the sine and cosine are both negative. Now:

$$\psi_+(n=2,m=3)=\left[-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]\psi_- . \qquad (5.10)$$

For n=3, m=3 the argument becomes 2π and therefore $\psi_+(n=3,m=3)=+1\cdot\psi_-$. So m=3 is excluded from the k=4, $0\le\varphi\le 8\pi$ circumstance because ψ_+ becomes triple-valued, with the coefficients in (5.9) and (5.10) as well as the coefficient +1. For higher n, these same results merely recycle themselves. Importantly, as we shall shortly develop in depth, on close perusal, we realize that *these coefficients are identical with the cubed roots of unity*.

The k=4 domain in (5.7) is also the first domain for which k is not a prime number, and this is responsible for the fact that the fraction 2, which is a prime factor of 4, is also permitted amongst $m=1,2,4$ without singularity. Specifically, when we set m=2 in (5.7), we obtain:

$$\psi_+=\exp(i\Lambda)\psi_-=\exp(i16\pi eg)\psi_-=\exp(i4\pi n)\psi_-=\psi_- , \qquad (5.11)$$

which is clearly single-valued therefore non-singular and permitted.

In the foregoing, we now see that the Euler relation $\sqrt[m]{1}=\exp i\vartheta=\exp(i2\pi n/m)$ for the m^{th} roots of unity plays a pivotal role in weeding out singular from non-singular fractionalized solutions. At (5.6), for k=3, we excluded m=2 because it yielded the two-valued wavefunction coefficient ± 1. But this coefficient contains no more and no less than the square roots of unity $\sqrt[2]{1}=\exp i\vartheta=\exp(i\pi n)=\pm 1$ with the Euler angles $\vartheta=\pi n=180°,360°$ in the complex plane for $n=1,2$. Then, in (5.9) and (5.10) for k=4, we excluded m=3 because this yielded the three-valued wavefunction coefficient $\sqrt[3]{1}=\exp i\vartheta=\exp(i2\pi n/3)$ with $\vartheta=2\pi n/3=120°,240°,360°$ associated with the cubed roots of unity for $n=1,2,3$. At the same time, $(k,m)=(4,2)$ is permitted, because 2 is an evenly-divisible factor of k=4. Let us now continue to some larger k:

For five k=5 rotations of the wavefunction about the monopole, $0\le\varphi\le 10\pi$ we have:

$$\psi_+=\exp(i\Lambda)\psi_-=\exp(i20\pi eg)\psi_-=\exp\left(i\frac{1}{m}10\pi n\right)\psi_- . \qquad (5.12)$$

Here, the only permitted non-singular fractions are $m=1,5$. For the excluded m=2 fraction the above would yield the two-valued $\psi_+=\exp(i5\pi n)\psi_-=\exp(i\pi n)\psi_-=\sqrt[2]{1}\psi_-$. The excluded m=3 fraction $\psi_+=\exp(i10\pi n/3)\psi_-=\exp(i4\pi n/3)\psi_-$ has $\vartheta=4\pi n/3=240°,480°,720°$ for $n=1,2,3$ which replicates the three-valued $\psi_+=\sqrt[3]{1}\psi_-$ for the cubed roots of unity. The excluded m=4 fraction yields $\psi_+=\exp(i10\pi n/4)\psi_-=\exp(i\pi n/2)\psi_-=\sqrt[4]{1}\psi_-$ which is quadruple-valued based on the fourth roots of unity $\sqrt[4]{1}=\pm 1,\pm i$ with

$\vartheta = \pi n/2 = 90°, 180°, 270°, 360°$ for $n = 1, 2, 3, 4$. So we see that it is easy to summarize the excluded states simply by using the m^{th} roots of unity $\sqrt[m]{1}$ as the wavefunction coefficient.

For six k=6 rotations with domain $0 \le \varphi \le 12\pi$ we have:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i24\pi eg)\psi_- = \exp\left(i\frac{1}{m}12\pi n\right)\psi_- . \tag{5.13}$$

This permits $m = 1, 2, 3, 6$, because it is single-valued and so non-singular for any of these. It will be readily seen that the excluded $m = 4, 5$ yield the respective four-valued and five-valued results $\psi_+(m=4) = \sqrt[4]{1}\psi_-$ and $\psi_+(m=5) = \sqrt[5]{1}\psi_-$ involving the fourth and fifth roots of unity. Of course, 6 =2x3 is not a prime number, and we see that its prime factors are precisely those $m = 2, 3$ fractions which are also permitted. For m=2 the above becomes:

$$\psi_+ = \exp(i6\pi n)\psi_- = \psi_- , \tag{5.14}$$

and for m=3 (5.13) becomes:

$$\psi_+ = \exp(i4\pi n)\psi_- = \psi_- . \tag{5.15}$$

For k=7 seven revolutions over $0 \le \varphi \le 14\pi$, we have:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i28\pi eg\varphi)\psi_- = \exp\left(i\frac{1}{m}14\pi n\right)\psi_- . \tag{5.16}$$

The only permitted fractions are $m = 1, 7$, and this is because 7 is a prime number. The excluded fractions yield the m-valued $\psi_+(m) = \sqrt[m]{1}\psi_-$ for $m = 2, 3, 4, 5, 6$.

For an octuplet of revolutions, k=8 over $0 \le \varphi \le 16\pi$, we have:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i32\pi eg)\psi_- = \exp\left(i\frac{1}{m}16\pi n\right)\psi_- . \tag{5.17}$$

The permitted fractions are $m = 1, 2, 4, 8$, which represent the prime factorization of k=8. The wavefunction $\psi_+(m) = \sqrt[m]{1}\psi_-$ is multivalued for the excluded fractions / unity roots $m = 3, 5, 6, 7$.

For nine revolutions, k=9 over $0 \le \varphi \le 18\pi$, we have:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i36\pi eg)\psi_- = \exp\left(i\frac{1}{m}18\pi n\right)\psi_- . \tag{5.18}$$

This admits the non-singular $m=1,3,9$, which again are the prime factors, this time of *k*=9. All other excluded fractions $m=2,4,5,6,7,8$ yield multivalued $\psi_+(m)=\sqrt[m]{1}\psi_-$.

Finally, for ten revolutions over $0\le\varphi\le 20\pi$ we have:

$$\psi_+=\exp(i\Lambda)\psi_-=\exp(i40\pi eg)\psi_-=\exp\left(i\frac{1}{m}20\pi n\right)\psi_- . \tag{5.19}$$

This allows the fractions $m=1,2,5,10$ which the prime factors of *k*=10, and the excluded states $m=3,4,6,7,8,9$ continue to be multi-valued with $\psi_+(m)=\sqrt[m]{1}\psi_-$.

So we see that as a general rule, if we move an electron from $\varphi=0$ to $\varphi=2\pi k$ where *k* is an integer denoting the number of revolutions about the monopole, the fractions *m* in (5.1) which are permitted without singularity will be $m=1$ and $m=k$ if *k* is a prime number, and additionally, all integers in the prime factorization of *k* if *k* is not prime. The fractions which are excludes are those with multivalued m^{th} roots of unity operating on the wavefunction according to $\psi_+(m)=\sqrt[m]{1}\psi_-$.

Seeing that the root of unity relationship:

$$\sqrt[m]{1}=\exp i\vartheta=\exp\left(i2\pi\frac{n}{m}\right)=\cos\left(2\pi\frac{n}{m}\right)+i\sin\left(2\pi\frac{n}{m}\right)=\cos(2\pi\mathbb{Q})+i\sin(2\pi\mathbb{Q}) \tag{5.20}$$

plays a fundamental mathematical role in characterizing and understanding when the fractional charge condition of (5.1) will and will not create observable singularities, it becomes apparent that it will generally be desirable to evaluate sin and cos functions in which the Euler angle $\vartheta=2\pi\mathbb{Q}$ is a rational multiple of 2π. For simple angles such as $2\pi/3=120°$ and $4\pi/3=240°$ with $\mathbb{Q}=1/3$ and $\mathbb{Q}=2/3$ as in (5.9) and (5.10), one can draw suitable triangles and obtain these sines and cosines in terms of roots of integers. But as the fractional *m* become larger integers, it becomes difficult, and in many cases impossible, to draw a regular polygon and then start manipulating subset triangles. The preferred approach, which can be used for *any* fraction *m*, is to instead write these roots as $x^m=1$ i.e., as the polynomial equation $x^m-1=0$, and then to find each of the *m* values of *x* which are roots of this polynomial. Of course, one of these *m* roots is always 1 itself, so $x-1=0$ can always be factored out. It is then readily seen with this factorization that this polynomial may be written as:

$$x^m-1=\left(x^{m-1}+x^{m-2}+x^{m-3}...+x^3+x^2+x+1\right)(x-1)=(x-1)\sum_{i=0}^{m-1}x^i=0 . \tag{5.21}$$

So the *m*-1 m^{th} roots of unity aside from 1 itself are generally found by solving the polynomial:

$$\sum_{i=0}^{m-1} x^i = 0 . \tag{5.22}$$

Of course, for large *m*, this is not a trivial polynomial to solve. But in principle, this makes it possible to find any and all roots that may be desired. So, for example, for the cubed roots of unity used in (5.9) and (5.10), the polynomial (here, quadratic) is $x^2+x+1=0$, which is readily solved as $x=\left(-1\pm\sqrt{1-4}\right)/2=\left(-1\pm i\sqrt{3}\right)/2$ and which indeed corresponds to (5.9) and (5.10). The third of these three roots is the trivial root $\sqrt[3]{1}=1$.

From (5.3), the phase is now related to the azimuth by:

$$\Lambda = \frac{n}{m}\varphi . \tag{5.23}$$

So for $\varphi = 2\pi k$ in general, this phase is given by:

$$\Lambda = \frac{k}{m} 2\pi n . \tag{5.24}$$

Whenever *m*=*k*, the phase result (5.24) becomes $\Lambda = 2\pi n$, which is no different at all from the result (4.10) for the standard DQC.

Now let us see how to make topological sense of these non-singular, fractional charge solutions $2eg = n/m = \mathbb{Q}$ of (5.1) which do admit fractions with $m \ge 2$, but which still maintain a single-valued wavefunction and so avert any observable singularities, when the domain $0 \le \varphi \le 2\pi k$ runs to a higher upper limit than 2π.

6. Using *k*th Root-of-Unity Covering Groups with 2π Domain Limitations to Project a 2π*k* Domain onto SO(3) and thereby Permit *m*=*k* Fractional Dirac Charges without Observable Singularities

We begin with (5.5) in which an electron is moved through two azimuth rotations about the monopole, from $\varphi = 0$ to $\varphi = 4\pi$. When we take an integral such as (3.3) over the domains of $0 \le \theta \le \pi$ and $0 \le \varphi \le 4\pi$, it is clear that we are double-covering the rotation group SO(3) in the physical space of spacetime. This covering may be topologically described by a two-to-one mapping of the elements of SU(2) onto those of SO(3), i.e., by a surjective homomorphism described by the projection $\pi : SU(2) \to SO(3)$. Suppose that we now wish to limit the domain to $0 \le \varphi \le 2\pi$ and still double-cover SO(3) over $0 \le \varphi \le 4\pi$ so as to permit a fractional charge with *m*=2 and have no observable singularity. How do we do this? If we impose this domain limitation on SU(2) rather than on SO(3), then a 2π rotation in SU(2) will projectively map onto a 4π rotation for SO(3) and so we can double cover SO(3) over the larger domain $0 \le \varphi \le 4\pi$ and so admit a non-singular *m*=2 fractional charge, while at the same time restraining SU(2) to remain within the $0 \le \varphi \le 2\pi$ domain. Simply put, and as is well-known, with $i = 1,2,3$

corresponding to the three physical space dimensions, any rotation over some angle θ_i in SU(2) maps onto a rotation through $2\theta_i$ in SO(3).

This is also responsible for so-called orientation-entanglement whereby a spinor will reverse sign under a 2π rotation in SO(3) which is a 1π rotation in SU(2), and only have its sign restored under a 4π rotation in the same SO(3) which is a 2π rotation in SU(2). Specifically, using the rotation operator $R = \exp(i\mathbf{J}\cdot\boldsymbol{\theta})$ defined on SO(3) and the unitary operator $U = \exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2)$ defined on the universal cover SU(2), when we increase one of the angles in $\boldsymbol{\theta} = \theta^i$ by 2π, this sign change *does* appear in SU(2) whereby $U \to -U$, but it *does not* appear in SO(3) whereby $R \to R$. This means that there is something captured by SU(2) that is missing from SO(3). This in turn means that SO(3) is only an approximate symmetry, whereas the true, exact, operative symmetry which records this sign change is seen only when we employ the universal cover $\tilde{G}_U = SU(2)$ and project this via $\pi : SU(2) \to SO(3)$ onto SO(3).

So now we return to (5.5) for a $\varphi = 0$ to $\varphi = 4\pi$ double rotation / double cover of SO(3), with the domain $0 \le \varphi \le 4\pi$. As already seen, this does admit both of the non-singular fractional values $m = 1$ and $m = 2$ from the Fractional DQC $2eg = n/m$ of (5.1) by maintaining a single-valued wavefunction for all *n* in either case. Of course, the non-singular *m*=1 is just the standard DQC; our present interest is in *m*=2 because this represents a half-unit charge and because this too has no observable singularities.

The vector potential for such a half-unit fractional charge is represented by (5.2) with *m*=2, and it certainly makes sense that a monopole with half the charge strength will have a potential which is likewise cut by a factor of two. And via (5.5), see also (5.24), we see that the *m*=2 phase solution is still $\Lambda = 2\pi n$, just as for the standard DQC, see (4.10). So if we wish to restrict our domain to $0 \le \varphi \le 2\pi$ yet capture all of the operative symmetries and still admit this half-unit charge without singularity, we can go into SU(2), limit the domain to $0 \le \varphi \le 2\pi$ in SU(2), and ensure that when projected onto SO(3) this will yield a $0 \le \varphi \le 4\pi$ domain. The spinor sign change will be seen under the exact SU(2) symmetry but will be missing from and not seen in the approximate symmetry of SO(3). Most importantly, by virtue of $\pi : SU(2) \to SO(3)$ projecting the domain $0 \le \varphi \le 4\pi$ into SO(3) even though the SU(2) domain is $0 \le \varphi \le 2\pi$, the *m*=2 fractional charge will be permitted to exist without singularity as seen in (5.5) because the wavefunction remains single-valued. So this tells us how, in principle, we may use SU(2) to admit a quantized fractional charge $2eg = n/m$ with *m*=2, without singularity. But what about the higher fractions with $m \ge 3$? Let's start with $m = 3$, then generalize.

In (5.6) we see that a non-singular *m*=3 fraction is permitted, but only for a triple cover from $\varphi = 0$ to $\varphi = 6\pi$. As before, we would like to start with a domain restricted to $0 \le \varphi \le 2\pi$, and so we need to find some covering group $\tilde{G}$ of SO(3) whereby $\pi : \tilde{G} \to SO(3)$ projects a $0 \le \varphi \le 2\pi$ domain in $\tilde{G}$ onto a $0 \le \varphi \le 6\pi$ domain in SO(3). In short, we need a group $\tilde{G}$ which provides a triple cover of SO(3). So, how do we go about finding such a group, as well as other groups $\tilde{G}$ which can quadruple, quintuple, pentuple, sextuple, etc., cover SO(3) to lay the groundwork for even larger fractional denominators?

In a very basic sense, we can regard SU(2) as the "square root" of SO(3). So now, we must find a "cubed root" of SO(3), and even higher roots for larger fractions. So, what does it really mean to take such roots, and how do we formalize this? For SU(2) which is the "square root" and a universal cover of SO(3), working from (5.5), we may write:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i8\pi eg)\psi_- = \exp\left(i2\pi\frac{n}{m}\right)\exp\left(i2\pi\frac{n}{m}\right)\psi_- . \tag{6.1}$$

As already seen, m=2 for a half-unit charge is permitted, because the wavefunction remains single-valued. With m=2, each of the factors $\exp(i2\pi n/m)$ in the above becomes $\exp(i\pi n)$ which for n=1 corresponds to the Euler angle $\vartheta = \pi = 180°$. And the potential (5.2) will be cut down to one-half of its whole-integer value which relates to our also observing a charge that is cut down by half. But the double multiplication of these factors $\exp(i\pi n)$ together still maintains a single-valued wavefunction by arriving at a total angle $\vartheta = 2\pi = 360°$ in the expression $\exp(i2\pi n)$ operating on ψ_-, such that $\psi_+ = \exp(i2\pi n)\psi_- = \psi_-$ for any and all n and thereby averts observable singularities. At the same time, the exact symmetry group SU(2) will "see" a spinor sign reversal that is not "seen" by the approximate symmetry group SO(3), which sign reversal represents the square roots of unity $\sqrt[2]{1} = \pm 1$ which are more generally represented by this same $\exp(i\pi n) = \pm 1$ for all n.

So for a covering group ${}_3\tilde{G}$ which is a "cubed root" of SO(3) as designated by the left subscript "3" that we have now introduced, let us now write (5.6) as:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i12\pi eg)\psi_- = \exp\left(i2\pi\frac{n}{m}\right)\exp\left(i2\pi\frac{n}{m}\right)\exp\left(i2\pi\frac{n}{m}\right)\psi_- . \tag{6.2}$$

For the m=3 solution of interest, each of these factors becomes $\exp(i2\pi n/3)$ which for n=1 corresponds to the Euler angle $\vartheta = 2\pi/3 = 120°$. But here, the triple multiplication will again maintain a non-singular single-valued wavefunction, and the potential (5.2) will be cut down to one-third of its whole-integer value. Indeed, when we take this triple product, the three Euler angles of $120°$ apiece now add up to $\vartheta = 2\pi = 360°$. There is a 2π in each factor to maintain a $0 \le \varphi \le 2\pi$ domain in ${}_3\tilde{G}$, but when we multiply everything together we arrive at an overall 6π factor which yields a $0 \le \varphi \le 6\pi$ domain on SO(3). And, just as SU(2) "sees" a spinor sign reversal based on $\sqrt[2]{1} = \pm 1 = \exp(i2\pi n/2)$ which SO(3) does not see, ${}_3\tilde{G}$ will "see" some coefficients based on the cubed roots of unity $\sqrt[3]{1} = \exp(i2\pi n/3)$ that SO(3) does not see, namely, the roots $\left(-1 \pm i\sqrt{3}\right)/2$ of (5.9) and (5.10) that we also obtained from (5.21).

Now, contrasting (6.1) and (6.2), and extrapolating the same calculation to fourth roots in (5.7) and fifth roots in (5.12) and so on, we see that in all cases, the relationships of section 4 and any higher-fraction relationships for the domain $0 \le \varphi \le 2\pi k$ on SO(3) can all be written as:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i4\pi keg)\psi_- = \exp\left(i\frac{n}{m}2\pi k\right)\psi_- = \exp\left(i2\pi\frac{n}{m}\right)^k \psi_- . \tag{6.3}$$

But we now also recognize, very importantly, that $\sqrt[m]{1} = (1)^{\frac{1}{m}} = \exp i\vartheta = \exp(i2\pi n/m)$ appearing in the final expression above is the very same fundamental mathematical relationship (5.20) of Euler, which is used to write the m^{th} roots of unity. Therefore, we may rewrite the above as:

$$\psi_+ = \exp(i\Lambda)\psi_- = \exp(i4\pi eg)^k \psi_- = \exp\left(i2\pi\frac{n}{m}\right)^k \psi_- = \exp(i2\pi n)^{\frac{k}{m}} \psi_- = (1)^{\frac{k}{m}} \psi_- , \tag{6.4}$$

where in the third term we have used $(i4\pi keg) = (i4\pi eg)^k$ to move k into the exponent. Embedded in the above, we may also now consolidate the phase relationship down to:

$$\exp(i\Lambda) = (1)^{\frac{k}{m}} . \tag{6.5}$$

So as we see, a domain of $0 \le \varphi \le 2\pi$ in the group ${}_k\tilde{G}$ associated with the factor $\exp(i2\pi n/m)$ in (6.3) will project a domain of $0 \le \varphi \le 2\pi k$ onto SO(3) via $\pi : {}_k\tilde{G} \to SO(3)$, thus permitting the fractional charge $4\pi eg = e\mu = 2\pi n/m$ with $m = k$, as well as other integers m in the prime factorization of k, without observable singularity.

Referring now to (6.4), we see that when $k = \mathbb{P}$ *is* a prime number, the wavefunction will be single-valued and so there will be no observable singularities *if and only if* one of two conditions is satisfied: First, when $m = 1$ which of course is trivial and corresponds to the non-fractionalized charges of the ordinary DQC. Second, when $m = k$ in which case there are fractional charges yet the wavefunction still remains single-valued. When k *is not* a prime number, $k \ne \mathbb{P}$, then there will be other charges besides $m = k$ which via $(1)^{k/m} = 1 = \exp(i2\pi n)^{k/m}$ will maintain a single-valued coefficient in all situations. We saw this in the last section, but $\psi_+ = (1)^{k/m}\psi_-$ in (6.4) enables us to summarize this much more compactly: For the smallest non-prime $k = 4$ we may have $m = 1, 2, 4$ which represents the prime factorization of 4. For non-prime $k = 6$ we may have all of $m = 1, 2, 3, 6$ which represents the prime factorization of 6. For non-prime k=8 the non-singular fractions are $m = 1, 2, 4, 8$ which are the prime factorizations of 8. And for k=9 which is the first odd number that is not prime, we may have $m = 1, 3, 9$ which are the prime factorizations of 9. Finally, for k=10 we may have $m = 1, 2, 5, 10$, once again, the prime factorization. This carries forward *ad infinitum*.

We also see from this that the quantization condition (5.1) *must be restricted in all circumstances* to rational numbers $2eg = \mathbb{Q}$ and *must of necessity exclude* all irrational numbers $\bar{\mathbb{Q}}$. Why? To maintain a single-valued wavefunction, (6.3) teaches that it is necessary that there exist some integer *k* such that $\exp(i2\pi n/m)^k = \exp(i2\pi\mathbb{Q})^k = 1$, because the unitarity of this result is what yields the single valued wavefunction $\psi_+ = \psi_-$ following a $\varphi = 2\pi k$ azimuth rotation and thus avoids observable singularities. So long as $\mathbb{Q} = n/m$ is indeed a rational number, this can be achieved by at least one choice of *k*, namely, *k*=*m*, and for non-prime *k*, by *m* being equal to one of the integers in the prime factorization of *k*.

So as a proof by contradiction, let us suppose that we were to employ a condition $2eg = \bar{\mathbb{Q}}$ using a number $\bar{\mathbb{Q}}$ posited to be *irrational*. Then the requirement to avoid observable singularities would become $\exp\left(i2\pi\bar{\mathbb{Q}}\right)^k = \exp\left(i2\pi k\bar{\mathbb{Q}}\right) = 1$. This would mean that there must be some value of *k* for which $k\bar{\mathbb{Q}} = \mathbb{Z}$ is an integer $\mathbb{Z}$, so that we could have $\exp\left(i2\pi\mathbb{Z}\right) = 1$. In other words, $\bar{\mathbb{Q}} = \mathbb{Z}/k$ would have to be the rational number $\mathbb{Z}/k$. But this contradicts our positing $\bar{\mathbb{Q}}$ to be an irrational number, and so proves that the Dirac condition must be restricted in all circumstances to $2eg = \mathbb{Q}$ with $\mathbb{Q} = n/m$ rational, if singularities are to be avoided.

One can also think about this geometrically using a unit circle: to maintain a single-valued wavefunction, we must be able to take some Euler angle ϑ and multiply this angle by some integer *k* such that $k\vartheta = 2\pi n$ rotates to an angle which is a whole-integer multiple of $2\pi = 360°$. Therefore, to "fit" this onto the unit circle, the original angle must be $\vartheta = 2\pi n/k = 2\pi\mathbb{Q}$, and so an irrational $\bar{\mathbb{Q}}$ in $\vartheta = 2\pi\bar{\mathbb{Q}}$ could never work to provide the correct fit to the unit circle. So we see that the Dirac *Quantization* Condition, really generalizes to a Dirac *Rationality* Condition.

From there, our task is to find these root of unity groups which map onto SO(3) via $\pi : {}_k\tilde{G} \to SO(3)$, for which the domain of ${}_k\tilde{G}$ runs from $0 \le \varphi \le 2\pi$ and the domain projected onto SO(3) then runs from $0 \le \varphi \le 2\pi k$ and therefore admits fractional charges with $m = k$ and *m* equal to integers in the prime factorization of *k*, because these are the conditions under which $\psi_+ = \exp(i2\pi n/m)^k \psi_- = \exp(i2\pi n)^{k/m} \psi_- = (1)^{k/m} \psi_- = \psi_-$ remains single-valued and thus has no observable singularities. More simply put: given that $k \ge m$, the non-singular solutions must all have $(1)^{k/m} = 1$, a.k.a. $k/m = \mathbb{Z}$ where $\mathbb{Z}$ is an integer.

This in turn teaches us that the *kernel* of this mapping must be equal to the m^{th} roots of unity, that is, we must have $\ker\pi = \sqrt[m]{1}$. Therefore, extracting the key items of information from (6.4) by factoring out the wavefunctions and then taking the k^{th} root of every term in (6.4) which contains *k* in an exponent, we may write this required kernel in several interrelated formulations, also using $g = \mu/4\pi$ from (3.5), as:

$$\ker \pi = \sqrt[m]{1} = (1)^{\frac{1}{m}} = \exp i\vartheta = \exp\left(i2\pi\frac{n}{m}\right) = \exp(i2\pi n)^{\frac{1}{m}} = \exp(i4\pi eg) = \exp(ie\mu), \tag{6.6}$$

Now we shall develop these root-of-unity covering groups in detail. The most important aid that we have to perform this development, is the group SU(2) which is the universal cover of SO(3). For, as we shall see, the development of these root groups, fundamentally, boils down to spotting SU(2) at the Euler angle $\vartheta = \pi$, spotting SO(3) at $\vartheta = 2\pi$, and then developing the other ${}_k\tilde{G}$ by rotating the SU(2) generators through the unit circle in the complex plane to other angles $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$ which are rational multiples $2\pi\mathbb{Q}$ of 2π. Henceforth, we shall denote these root-of-unity covering groups as ${}^n_m\tilde{G}$ to represent the key parameters *n* and *m* in the Euler angle $\vartheta = 2\pi n/m$ associated with each such group. And because the generators of these groups these will be 2x2 matrices formed from the three Pauli matrices σ_i, we shall further denote these as ${}^n_m\tilde{G}(2)$ to represent that these groups also use 2x2 matrices as their generators. We omit the "special" prefix "S," however, because as we shall see, these generators will not necessarily be traceless.

7. Generators for the Root-of-Unity Covering Groups which Project the Fractional Dirac Charges onto SO(3) via the Universal Cover SU(2); and Euler Angles in the Complex Generator Plane

We stated prior to (6.1) that in a very basic sense, SU(2) is the "square root" of SO(3). Let is now formalize that sense more precisely. Mathematically, it first became apparent back at (5.9) and (5.10) that roots of unity are essential for characterizing situations under which the fractional charges $2eg = n/m$ yield single-valued or multivalued wavefunctions, and thus, the circumstances under which Dirac string singularities are observable thus forbidden, or not observable thus permitted. Then, in the last section we laid out how these roots of unity are in fact fundamental to developing the covering groups through which $\pi : {}^n_m\tilde{G}(2) \to SO(3)$ lays the topological groundwork to support these fractional charges. Specifically, we established at (6.6) that $\ker \pi = \sqrt[m]{1}$ must itself be an m^{th} root of unity for any given ${}^n_m\tilde{G}(2)$. What now changes from here, mathematically, is really very simple: Whereas we have heretofore concerned ourselves with roots of the scalar number 1, we shall now concern ourselves with roots of the 2x2 identity matrix $\text{diag}(I) = (1,1)$.

Let's start with the square root of the 2x2 *I*, that is, $\sqrt[2]{I}$. If we restrict our consideration to traceless Hermitian matrices, we know immediately that aside from *I* itself for which $I^2 = I$, there are three other matrices which fit this bill, namely, the Pauli matrices σ_i for which $\sigma_1{}^2 = \sigma_2{}^2 = \sigma_3{}^2 = I$, $\sigma_i{}^\dagger = \sigma_i$, and $\text{Tr}(\sigma_i) = 0$. These are the traceless, Hermitian square roots $\sigma_i = \sqrt[2]{I_i}$ of a triplet I_i of 2x2 identity matrices. If we want to formalize the mathematics a bit more, we can write these square root relations using the anti-commutator relation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$.

Of course, these matrices also have the non-zero commutator $\left[\sigma_i, \sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$. And it is also very well-known that we obtain $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$ by combining these relationships, and that this expression and the related expression $\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu}$ obtained from combining the Dirac matrix relations $\left\{\gamma_\mu, \gamma_\nu\right\} = 2g_{\mu\nu}$ and $\left[\gamma_\mu, \gamma_\nu\right] = -2i\sigma_{\mu\nu}$ are central to the Gordon decomposition by which the electron spin and magnetic moment is separated from orbital angular momentum.

So when we say that SU(2) is the "square root" of SO(3), this is highlighted at several levels. First, of course, is the fact that the generators $\sigma_i = \sqrt[2]{I_i}$ are indeed square roots of the identity matrix triplet. But the $\pi : SU(2) \rightarrow SO(3)$ mapping is made evident rather simply when we form $\sigma_i x^i$ from the space coordinates $x^i = \left(x, y, z\right)$ and then square this to obtain $\sigma_i\sigma_j x^i x^j = \left(\delta_{ij} + i\varepsilon_{ijk}\sigma_k\right)x^i x^j = x^2 + y^2 + z^2 = r^2$, wherein the Pythagorean length *r* is the defining invariant of the rotation group SO(3). Without the "square root" Pauli generators, the way to take a square root is to write $r = \pm\sqrt{x^2 + y^2 + z^2}$, and the two-valuedness of taking square *m*=2 roots shows up in the $\pm$ sign, which more deeply, is really $\exp i\vartheta = \exp\left(i2\pi n / m\right)$ for *m*=2 and $n = 1, 2$, i.e., for $\vartheta = 180°, 360°$. With the Pauli generators, this two-valuedness instead shows up in the fact that the eigenvalues $\lambda = \pm 1$ are obtained from the characteristic equation $\left|\sigma_i - \lambda\right| = 0$ for all three σ_i. There are other ways to illustrate how SU(2) is the square root of SO(3), including by using the spinors in $\sigma_i x^i = -\xi\xi^\dagger$ which we shall examine a bit later. But for the moment, the foregoing provides us with a simple point of departure to now more generally consider the m^{th} roots of the identity matrix, $\sqrt[m]{I}$. By doing so, we are able to obtain the generators for these m^{th} root of unity covering groups ${}^n_m\tilde{G}(2)$.

We start with the Pauli matrices σ_i themselves, posit three associated angles θ_i in physical space, and form the unitary matrices $U_i = \exp\left(i\sigma_i\theta_i\right)$, thus $U^\dagger U = 1$ given $\sigma_i^\dagger = \sigma_i$, for SO(3) rotations through respective angles $\theta_i = \theta_x, \theta_y, \theta_z$ about each of the x, y, z axes. It is well-known how to use the series $\exp\left(ix\right) = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 ...$ together with the fact that $\sigma_i^{2n} = I_i$ and $\sigma_i^{2n+1} = \sigma_i$ to flesh out these unitary matrices into the well-known:

$$
\begin{aligned}
U_1 &= \exp\left(i\sigma_1\theta_1\right) = \begin{pmatrix} \cos\theta_1 & i\sin\theta_1 \\ i\sin\theta_1 & \cos\theta_1 \end{pmatrix} \\
U_2 &= \exp\left(i\sigma_2\theta_2\right) = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \\
U_3 &= \exp\left(i\sigma_3\theta_3\right) = \begin{pmatrix} \cos\theta_3 + i\sin\theta_3 & 0 \\ 0 & \cos\theta_3 - i\sin\theta_3 \end{pmatrix}
\end{aligned}
\quad . \tag{7.1}
$$

Now, it happens that with a judicious choice of these angles θ_i we can cause each of these U_i to be identical to the corresponding σ_i up to an overall constant factor. Specifically, if we choose each of these angles such that $\theta_i = \pi/2$, we readily see that:

$$U_1(\pi/2) = \exp\left(i\sigma_1\frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2) & i\sin(\pi/2) \\ i\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1$$
$$U_2(\pi/2) = \exp\left(i\sigma_2\frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2 \quad .(7.2)$$
$$U_3(\pi/2) = \exp\left(i\sigma_3\frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2)+i\sin(\pi/2) & 0 \\ 0 & \cos(\pi/2)-i\sin(\pi/2) \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3$$

Consolidating, we see that $U_i(\pi/2) = \exp(i\sigma_i\pi/2) = i\sigma_i$ in general, which we rewrite as:

$$\sigma_i = -i\exp\left(i\sigma_i\frac{\pi}{2}\right). \tag{7.3}$$

So now we can square this expression, and because $\sigma_i{}^2 = I_i$, we can write the identity matrix triplet I_i as:

$$I_i = (-i)^2\exp(i\sigma_i\pi). \tag{7.4}$$

We deliberately do *not* turn $(-i)^2 \to -1$ because when we later take square roots of this, we want to recover $-i$ alone, and not extraneously introduce a two-valued $\pm i = \sqrt{-1}$. Of course, the identity matrix taken to any integer power n is still the identity matrix $I_i{}^n = I_i$, so the most general expression for this triplet of identity matrices is:

$$I_i = I_i{}^n = (-i)^{2n}\exp(i\sigma_i\pi n) = (-i)^{2n}\left[\cos(\sigma_i\pi n) + i\sin(\sigma_i\pi n)\right]. \tag{7.5}$$

Now that we have the identity matrices represented in this form, it is an easy matter to obtain their generalized m^{th} roots, $\sqrt[m]{I_i}$. There are simply:

$${}_m^n\tau_i \equiv \sqrt[m]{I_i} = (-i)^{\frac{2n}{m}}\exp\left(i\sigma_i\pi\frac{n}{m}\right) = (-i)^{\frac{2n}{m}}\left[\cos\left(\sigma_i\pi\frac{n}{m}\right) + i\sin\left(\sigma_i\pi\frac{n}{m}\right)\right]. \tag{7.6}$$

In the above, we have defined ${}_m^n\tau_i \equiv \sqrt[m]{I_i}$ for each of these roots, which we now explain: First, each of these $\sqrt[m]{I_i}$ is a triplet $i = 1,2,3$ of 2x2 matrices, just like the Pauli σ_i themselves. These

$\sqrt[m]{I_i}$ are parameterized by the two integers *n* and *m* which enter (7.6) in the ratio $\mathbb{Q} = n/m$ of a rational number. Analogously to how σ_i are the generators of SU(2), these $\sqrt[m]{I_i}$ are the generators of the root covering group ${}^n_m\tilde{G}(2)$ for the m^{th} root of I_i raised to the n^{th} power. Thus, for example, $\sqrt[3]{I_i}^{\,2} = \left(I_i\right)^{2/3}$ with *n*=2 and *m*=3 is the 2/3 root of I_i, which explains why we denote ${}^n_m\tilde{G}(2)$ with a left superscript *n* and left subscript *m*. Thus, analogizing to σ_i used to represent $\sqrt[2]{I_i}$, we use τ_i to generally represent the root-of-unity generators for these groups ${}^n_m\tilde{G}(2)$. And because it is important to know the *n* and *m* integers associated with any of these τ_i, we write these as ${}^n_m\tau_i$ so as to provide a simple shorthand for knowing at a glance that ${}^n_m\tau_i$ are the generators for the $n/m^{\text{ th}}$ root-of-unity group ${}^n_m\tilde{G}(2)$.

So as a test, to confirm that (7.6) is correct for the square root of unity (namely SU(2)), we may set *m*=2 in (7.6) to obtain:

$$ {}^n_2\tau_i = \left(-i\right)^n \exp\left(i\sigma_i \pi n/2\right). \tag{7.7}$$

Referring to (7.5) we see that ${}^n_2\tau_i = I_i$ for $n = 0, 2, 4, 6...$, which recovers the identity matrices. And referring to (7.3), we see that ${}^1_2\tau_i = \sigma_i$ for $n = 1$, while (7.6) shows that for successive $n = 3, 5, 7...$ the sign flip in $\left(-i\right)^n$ will be precisely offset by a sign flip in $\exp\left(i\sigma_i \pi n/2\right)$, so that ${}^n_2\tau_i = \sigma_i$ for $n = 1, 3, 5, 7...$ generally. But this exercise also alerts us, while ${}^1_2\tau_i = {}^1_2\tau_i^{\dagger}$ and $\mathrm{Tr}\left({}^1_2\tau_i\right) = 0$ in the special case ${}^1_2\tau_i = \sigma_i$, that in general the ${}^n_m\tau_i$ are neither Hermitian nor traceless. The ${}^2_2\tau_i = I_i$ generators, for example, certainly are not traceless, but rather have $\mathrm{Tr}\left({}^2_2\tau_i\right) = 2$. And by definition they will commute with any other 2x2 matrices and are their own inverses. Further, from (7.6), taking the Hermitian conjugate, we find:

$$ {}^n_m\tau_i^{\dagger} = \left(i\right)^{\frac{2n}{m}} \exp\left(-i\sigma_i \pi \frac{n}{m}\right) = \left(i\right)^{\frac{2n}{m}} \left[\cos\left(\sigma_i \pi \frac{n}{m}\right) - i \sin\left(\sigma_i \pi \frac{n}{m}\right)\right]. \tag{7.8}$$

So as a general rule, ${}^n_m\tau_i \neq {}^n_m\tau_i^{\dagger}$, that is, these are not Hermitian. However, these ${}^n_m\tau_i$ are unitary, $\left({}^n_m\tau_i\right)^{\dagger}\left({}^n_m\tau_i\right) = I_i$.

It is also useful to use the Euler formulation $-i = \exp\left(i3\pi/2\right)$ to write:

$$ \left(-i\right)^{\frac{2n}{m}} = \exp\left(i3\pi \frac{n}{m}\right), \tag{7.9}$$

and then use this in (7.6) to write:

$$ {}_m^n\tau_i = \exp\left(i3\pi \frac{n}{m} \right) \exp\left(i\sigma_i \pi \frac{n}{m} \right) = \exp\left(i\pi \frac{n}{m} \left(\sigma_i + 3I_i \right) \right). \qquad (7.10) $$

The resultant $\sigma_i + 3I_i$ does have a trace, which is another view of how in general, $\mathrm{Tr}\left({}_m^n\tau_i \right) \neq 0$.

Now, while these ${}_m^n\tau_i$ were developed in order to accommodate fractional Dirac charges, the existence of these ${}_m^n\tau_i$ as the n/m^{th} roots of the identity triplet I_i is *independent* of our wanting to lay the topological groundwork for these fractional charges. If one had set out to find generators which are the generalized roots of the 2x2 I_i, one could have done so as shown here without any reference to or thought about Dirac monopoles or the DQC or fractional charges. The point of contact to formally accommodate fractional charges is now found in the kernel expression (6.6), and specifically, in its embedded relationships:

$$ \vartheta = 2\pi \frac{n}{m} = 4\pi e g = e\mu . \qquad (7.11) $$

So if we now use (7.11) divided through by 2 in (7.10), we may obtain:

$$ \begin{aligned} {}_m^n\tau_i &= \exp\left(i\frac{3\vartheta}{2} \right) \exp\left(i\sigma_i \frac{\vartheta}{2} \right) = \exp\left(i3\pi \frac{n}{m} \right) \exp\left(i\sigma_i \pi \frac{n}{m} \right) \\ &= \exp\left(i6\pi e g \right) \exp\left(i\sigma_i 2\pi e g \right) = \exp\left(i\frac{3}{2} e\mu \right) \exp\left(i\sigma_i \frac{e\mu}{2} \right) \end{aligned} . \qquad (7.12) $$

The universal cover SU(2) has the generators $\sigma_i = {}_2^1\tau_i$ as already discussed. If we want an easy way to think out this, we can simply use *n*=1 and *m*=2 in (7.11) to find that $\vartheta = \pi$. So surely enough, as laid out at the end of section 5, we spot the SU(2) generators at $\vartheta = \pi$ in the complex plane, and immediately know that when we square these generators, we will double the angle, and thereby end up with SO(3) spotted at $\vartheta = 2\pi$. Then, when thinking about the other root generators, it is easiest to simply think about the angle at which those generators are disposed. The non-trivial cubed-root generators, for example, will be at $\vartheta = 120°, 240°$, so that when cubed they will yield either of $\vartheta = 360°, 720°$. For the fourth root the non-trivial generators will be spotted at $\vartheta = 90°, 180°, 270°$ and when raised to the fourth power will yield $\vartheta = 360°, 720°, 1080°$. The pentuple generators will be at $\vartheta = 72°, 144°, 216°, 288°$ and when raised to the fifth power will again recover an integer multiple of $360°$. And so on.

Further, as we saw in the various relationships $\psi_+ = \exp\left(i2\pi n/m \right)^k \psi_-$ throughout section 4, see (6.3), it is this angle of $360°$, and its integer multiples, via $\psi_+ = \exp\left(i2\pi n \right)^{k/m} \psi_- = \left(1 \right)^{k/m} \psi_-$, see (6.4), which keeps the wavefunctions single-valued and so avoids observable singularities even with fractional charges. Consequently, the Euler angle in (7.11) provides a very powerful vehicle to cut through all the algebra of these root covering groups, and think about these groups and their operations very simply in terms of orientations and rotations of the Euler angle ϑ on the unit circle in the complex plane in which the generators ${}_m^n\tau_i$ are spotted.

From this view, the simplest portion of (7.12) is that which contains these ϑ:

$$\tau_i(\vartheta) = \exp\left(i\frac{3\vartheta}{2}\right)\exp\left(i\sigma_i\frac{\vartheta}{2}\right). \tag{7.13}$$

In this form, we do not even need to explicitly display the *n* and *m* parameters in ${}_m^n\tau_i = \tau_i(\vartheta)$, because via (7.11) these are incorporated into the angle $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$. In this form, $\sigma_i = \tau_i(\pi)$ and $I_i = \tau_i(2\pi)$. From this view, the SU(2) group of σ_i is a universal cover because any other set of generators including the unity matrices I_i can be obtained merely by rotating the angle of these generators from $\vartheta = \pi$ to the pertinent rational multiple of $360°$, i.e., to $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$.

8. How Fractional Dirac Charges are Topologically Mapped from the Root-of-Unity Covering Groups onto SO(3)

Having developed these generators τ_i in (7.13) for the root-of-unity covering groups groups ${}_m^n\tilde{G}(2)$ which in view of $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$ we now designate as $\tilde{G}(2,\vartheta)$, it remains to explore the surjective homomorphic mapping $\pi : \tilde{G}(2,\vartheta) \to SO(3)$ which projects these fractional charges onto SO(3) from the root space $\tilde{G}(2,\vartheta)$. To do this, it is helpful to develop the commutators $\left[\tau_i, \tau_j\right]$ for any given $\tau_i(\vartheta)$.

First, working from (7.13) we construct:

$$\left[\tau_i, \tau_j\right] = \exp(i3\vartheta)\left[\exp\left(i\sigma_i\frac{\vartheta}{2}\right), \exp\left(i\sigma_j\frac{\vartheta}{2}\right)\right]. \tag{8.1}$$

To evaluate this, it helps to also construct the commutators $\left[U_i, U_j\right]$ of the unitary matrices (7.1). This exercise is straightforward and yields:

$$\begin{aligned}
\left[U_1,U_2\right] &= \left[\exp\left(i\sigma_1\theta_1\right),\exp\left(i\sigma_2\theta_2\right)\right] = -2i\sin\theta_1\sin\theta_2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = -2i\sin\theta_1\sin\theta_2\sigma_3\\
\left[U_2,U_3\right] &= \left[\exp\left(i\sigma_2\theta_2\right),\exp\left(i\sigma_3\theta_3\right)\right] = -2i\sin\theta_2\sin\theta_3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = -2i\sin\theta_2\sin\theta_3\sigma_1\ .\\
\left[U_3,U_1\right] &= \left[\exp\left(i\sigma_3\theta_3\right),\exp\left(i\sigma_1\theta_1\right)\right] = -2i\sin\theta_3\sin\theta_1\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = -2i\sin\theta_2\sin\theta_3\sigma_2
\end{aligned} \tag{8.2}$$

In the circumstance where $\theta \equiv \theta_1 = \theta_2 = \theta_3$ this consolidates to:

$$\left[U_i,U_j\right] = \left[\exp\left(i\sigma_i\theta\right),\exp\left(i\sigma_j\theta\right)\right] = -2i\sin^2\theta\varepsilon_{ijk}\sigma_k\,. \tag{8.3}$$

Thus, if we set $\theta = \vartheta/2$ and also apply the half angle $\sin^2\left(\vartheta/2\right) = \left(1-\cos\vartheta\right)/2$, (8.3) becomes:

$$\left[\exp\left(i\sigma_i\frac{\vartheta}{2}\right),\exp\left(i\sigma_j\frac{\vartheta}{2}\right)\right] = -2i\sin^2\frac{\vartheta}{2}\varepsilon_{ijk}\sigma_k = -i\left(1-\cos\vartheta\right)\varepsilon_{ijk}\sigma_k\,. \tag{8.4}$$

Combining this with (8.1) and also applying $\left[\sigma_i,\sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$ then finally yields:

$$\left[\tau_i,\tau_j\right] = -i\exp\left(i3\vartheta\right)\left(1-\cos\vartheta\right)\varepsilon_{ijk}\sigma_k = -\tfrac{1}{2}\exp\left(i3\vartheta\right)\left(1-\cos\vartheta\right)\left[\sigma_i,\sigma_j\right]. \tag{8.5}$$

Isolating σ_i with some simple re-indexing and reverting $2\sin^2\left(\vartheta/2\right) = \left(1-\cos\vartheta\right)$, this may be written as:

$$\sigma_i = \tfrac{1}{4}i\exp\left(-i3\vartheta\right)\csc^2\left(\vartheta/2\right)\varepsilon_{ijk}\left[\tau_j,\tau_k\right]. \tag{8.6}$$

To confirm for SU(2), we know that the σ_i are spotted at $\vartheta = \pi$. At this orientation, we obtain $\sigma_i = \tfrac{1}{4}i\exp\left(-i3\pi\right)\csc^2\left(\pi/2\right)\varepsilon_{ijk}\left[\tau_j,\tau_k\right] = -\tfrac{1}{4}i\varepsilon_{ijk}\left[\tau_j,\tau_k\right]$. Because $\left[\sigma_i,\sigma_j\right] = 2i\varepsilon_{ijk}\sigma_k$ is readily rewritten as $\sigma_1 = -\tfrac{1}{4}i\varepsilon_{ijk}\left[\sigma_j,\sigma_k\right]$, we confirm that $\tau_i = \sigma_i$ at $\vartheta = \pi$. We also know that $\csc^2\left(\vartheta/2\right) = \infty$ at $\vartheta = 2\pi k$. But this singular behavior of (8.6) makes perfect sense when viewed via (8.5). Because $1-\cos\vartheta = 0$ at $\vartheta = 2\pi k$, likewise $\left[\tau_i,\tau_j\right] = 0$, i.e., the $\vartheta = 2\pi k$ generators do commute for all $\vartheta = 2\pi k$. This reconfirms that $\tau_i = I_i$, the identity matrix triplet, for all $\vartheta = 2\pi k$. It is for these very same angles, that the wavefunctions become single-valued and so the fractional monopolies avoid observable singularities.

Now we are in a position to explicitly display the mapping $\pi:\tilde{G}(2,\vartheta)\to SO(3)$ for all $\tilde{G}(2,\vartheta)$. First, we nominally re-index (8.5) into the form:

$$\varepsilon_{ijk}\left[\tau_j,\tau_k\right]=-2i\exp\left(i3\vartheta\right)\left(1-\cos\vartheta\right)\sigma_i\,. \tag{8.7}$$

Now, using the space coordinates $x^i=\left(x,y,z\right)$ and forming $\sigma_i x^i$ we can use (8.7) to write:

$$\varepsilon_{ijk}\left[\tau_j,\tau_k\right]x^i=-2i\exp\left(i3\vartheta\right)\left(1-\cos\vartheta\right)\sigma_i x^i=-2i\exp\left(i3\vartheta\right)\left(1-\cos\vartheta\right)\begin{pmatrix} z & x-iy \\ x+iy & -z\end{pmatrix}. \tag{8.8}$$

Then, restructuring to isolate the $\sigma_i x^i$ matrix and also making use of the spinor relationships:

$$x=\tfrac{1}{2}\left(\xi_2{}^2-\xi_1{}^2\right);\quad y=\tfrac{1}{2i}\left(\xi_1{}^2+\xi_2{}^2\right);\quad z=\xi_1\xi_2 \tag{8.9}$$

as well as the cross product:

$$\varepsilon_{ijk}\left[\tau_j,\tau_k\right]x^i=2\left[\boldsymbol{\tau}\times\boldsymbol{\tau}\right]\cdot\mathbf{x}\,, \tag{8.10}$$

we may now write:

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{x}=\sigma_i x^i&=\begin{pmatrix} z & x-iy \\ x+iy & -z\end{pmatrix}=\begin{pmatrix} \xi_1\xi_2 & -\xi_1{}^2 \\ \xi_2{}^2 & -\xi_1\xi_2\end{pmatrix}=\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\begin{pmatrix}\xi_2 & -\xi_1\end{pmatrix}=-\xi\xi^\dagger \\ &=\tfrac{1}{4}i\exp\left(-i3\vartheta\right)\csc^2\left(\vartheta/2\right)\varepsilon_{ijk}\left[\tau_j,\tau_k\right]x^i=\tfrac{1}{2}i\exp\left(-i3\vartheta\right)\csc^2\left(\vartheta/2\right)\left[\boldsymbol{\tau}\times\boldsymbol{\tau}\right]\cdot\mathbf{x}\end{aligned}\,. \tag{8.11}$$

Of course, the determinant $\left|\boldsymbol{\sigma}\cdot\mathbf{x}\right|=x^2+y^2+z^2=r^2$ is the Pythagorean invariant of rotation under SO(3) transformations, which are equivalent to SU(2) transformations on the transposed complex spinor doublet $\xi^T=\left(\xi_1,\xi_2\right)^T$. So taking the determinant of all the main expressions in (8.11) we obtain:

$$r^2=x^2+y^2+z^2=\left|\sigma_i x^i\right|=\left|-\xi\xi^\dagger\right|=\left|\tfrac{1}{4}i\exp\left(-i3\vartheta\right)\csc^2\left(\vartheta/2\right)\varepsilon_{ijk}\left[\tau_j,\tau_k\right]x^i\right|, \tag{8.12}$$

with $\tau_i\left(\vartheta\right)$ given in (7.13) and ϑ given in (7.12). This explicitly illustrates the covering projection $\pi:\tilde{G}(2,\vartheta)\to SO(3)$, showing all of $r^2=x^2+y^2+z^2$ from SO(3), the SU(2) spinor relationships $\left|\sigma_i x^i\right|=\left|-\xi\xi^\dagger\right|$, and the $\tilde{G}(2,\vartheta)$ root group generators also operating on the space coordinates **x**.

Having made these connections through the Euler angle ϑ, we can now use (7.11) to re-express the above in terms of the rational parameters $\mathbb{Q}=n/m$, and most importantly, the electric and magnetic charge strengths. Using $\vartheta=2\pi n/m$ the above becomes:

$$r^2 = x^2 + y^2 + z^2 = \left|\sigma_i x^i\right| = \left|-\xi\xi^\dagger\right| = \left|\tfrac{1}{4} i \exp\left(-i6\pi\frac{n}{m}\right)\csc^2\left(\pi\frac{n}{m}\right)\varepsilon_{ijk}\left[\tau_j, \tau_k\right]x^i\right|. \tag{8.13}$$

We again emphasize that as we did prior to (7.11), that the relationships above are *entirely independent* of anything having to do with Dirac monopole charges or whether these can exist in fractional states without singularity. These simply specify relationships among various fractional generators τ_i, the Pauli matrices σ_i, and rotationally-invariant lengths *r* in SO(3).

It is via the final set of connections $\vartheta = 4\pi eg = e\mu$ that the electric and magnetic charges explicitly enter, whereby (8.12) now becomes, in terms of *g*:

$$r^2 = x^2 + y^2 + z^2 = \left|\sigma_i x^i\right| = \left|-\xi\xi^\dagger\right| = \left|\tfrac{1}{4} i \exp\left(-i12\pi eg\right)\csc^2\left(2\pi eg\right)\varepsilon_{ijk}\left[\tau_j, \tau_k\right]x^i\right|, \tag{8.14}$$

and in terms of μ:

$$r^2 = x^2 + y^2 + z^2 = \left|\sigma_i x^i\right| = \left|-\xi\xi^\dagger\right| = \left|\tfrac{1}{4} i \exp\left(-i3e\mu\right)\csc^2\left(e\mu/2\right)\varepsilon_{ijk}\left[\tau_j, \tau_k\right]x^i\right|. \tag{8.15}$$

It is worth noting that these m^{th} root geometries of their very nature, give rise to fractional denominators *m* which need not be equal to 1, and which generally are not equal to 1. So this raises an interesting point about the symmetries of these fractional charges. If, hypothetically, fractional charges were not permitted, the Dirac condition would of course be $2\pi n = 4\pi eg = e\mu$ as is presently believed to be the case. But this would *not* remove the *m* denominator from (7.11) because that denominator arises from root generators and generally from the Euler relation $\sqrt[m]{1} = \exp\left(i2\pi n/m\right)$ which is mathematically true no matter what the correct state of affairs might be for Dirac monopole charges. This means that were the Dirac monopole to be truly restricted to $2\pi n = 4\pi eg = e\mu$ as is presently believed, then (7.11) would have to be modified, *not by setting m*=1, but rather, by dividing the third and fourth expressions through by *m*. That is, (7.11) would have to become:

$$\vartheta = 2\pi\frac{n}{m} = 4\pi\frac{eg}{m} = \frac{e\mu}{m}. \tag{7.11a}$$

This in turn would mean upon substitution into (8.12), that in lieu of (8.15) we would have:

$$r^2 = x^2 + y^2 + z^2 = \left|\sigma_i x^i\right| = \left|-\xi\xi^\dagger\right| = \left|\tfrac{1}{4} i \exp\left(-i3\frac{e\mu}{m}\right)\csc^2\left(\frac{e\mu}{2m}\right)\varepsilon_{ijk}\left[\tau_j, \tau_k\right]x^i\right|. \tag{8.15a}$$

So if fractional Dirac charges did not exist, as is presently the prevailing view, then (8.15a) rather than (8.15) would describe the projection $\pi : \tilde{G}(2,\vartheta) \to SO(3)$ of these root of unity groups onto SO(3). Contrasting, we see that (8.15) expresses the invariant rotational length *r* entirely in terms of the product $e\mu$ *without any explicit appearance of the quantum numbers n* *or m*. So the form of (8.15) is *invariant* with respect to the rational number $\mathbb{Q} = n/m$. On the other hand, (8.15a) does not have this same symmetry, and is in fact weaker. Rather, (8.15a) contains an explicit appearance of *m* in the term $e\mu/m = e\mu\mathbb{Q}/n$. So the form of (8.15a) is *not invariant* with respect to *n* and *m*, but quite explicitly requires that one or the other of these integers appear explicitly in the $\pi : \tilde{G}(2,\vartheta) \to SO(3)$ mapping. This means that the Fractional DQC actually has a higher degree of symmetry than the standard DQC.

9. Summary and Conclusion

To summarize, when we take a domain over $0 \le \varphi \le 2\pi k$ in SO(3), there is nothing to distinguish the $0 \le \varphi \le 2\pi$ domain from the $2\pi \le \varphi \le 4\pi$ domain . . . from the $2\pi(k-2) \le \varphi \le 2\pi(k-1)$ domain from the $2\pi(k-1) \le \varphi \le 2\pi k$ domain. But when we take a domain $0 \le \varphi \le 2\pi$ in the root group $\tilde{G}(2,\vartheta) = {}^{n}_{m}\tilde{G}(2)$ with generators ${}^{n}_{m}\tau_i$, there are a total of *m* distinct m^{th} roots spotted along the unit circle with $1 \le n \le m$ each of which has the distinct kernel $(1)^{n/m}$. This is simply an extension, to higher roots, of how SU(2) "sees" the $\pm$ sign in a spinor which SO(3) does not. Then, all of the 2π-domain covers associated with each *n* in the generators $(I_i)^{n/m}$ then get patched together onto a union $(I_i)^{1/m} \cup (I_i)^{2/m} \cup ... (I_i)^{(m-1)/m} \cup (I_i)^{m/m}$ on SO(3), and as a result the domain of SO(3) will run over $0 \le \varphi \le 2\pi k$. Importantly, however, each 2π subset of this enlarged domain will have been mapped *from a different one* of the ${}^{n}_{m}\tilde{G}(2)$ associated with $(I_i)^{n/m}$ with $1 \le n \le m$ and thus will have a distinguishing symmetry feature – namely the distinct root of unity which provides its kernel and associated distinct generators – which SO(3) alone does not have absent this domain mapping. It is this enlarged domain on SU(3) with unique ${}^{n}_{m}\tau_i = (I_i)^{n/m}$ for each of the $k \times 2\pi$ domains in $0 \le \varphi \le 2\pi k$ which then provides the freedom for fractional Dirac charges to arise unambiguously and without observable singularities.

The scope of this paper, as summarized, has been limited to the question of whether fractional $U(1)_{em}$ magnetic monopole charges with $2eg = n/m$ can exist without observable singularities. Given the showing here that fractional charges with $2eg = n/m$ can indeed be projected onto SO(3) without observable singularities, and given how (8.15) which includes these fractional charges has a much stronger symmetry than (8.15a) which excludes them, it appears that the prevailing view that fractional Dirac charges are unable to exist free of observable singularities will have to be changed.

Once it is understood that these fractional charges are not excluded on the ground of giving rise to observed singularities, and that they possess a stronger symmetry than the quantized-only charges of the standard DQC, and that they are related closely to mathematical roots of unity, one will then need to take with utmost seriousness the possibility that these fractional Dirac charges do exist in the natural world. Especially, as one starts to sort out *primitive* from non-primitive roots of unity and more directly study orientation / entanglement which we have not done here, this will then open for serious study, the question whether these fractional Dirac charges are in fact what is being observed in the Fractional Quantum Hall Effect (FQHE) [4] observed at ultra-low-temperatures near 0K, in which case the FQHE would be the first experimental evidence of magnetic monopoles. Further, if the FQHE can be understood in this way, and because the FQHE occurs only at such low temperatures, this will also open for serious study whether the existence of fractional charges emanating from $U(1)_{em}$ gauge theory which appear only at low temperatures points toward a fundamental unification between electrodynamics and thermodynamics.

References

[1] Dirac, P.A.M., *Quantized Singularities in the Electromagnetic Field*, Proceedings of the Royal Society A 133 (821): 60–72, (September 1931)
[2] Wu, T. T. and Yang, C.N., *Concept of non-integrable phase factors and global formulation of gauge fields,* Phys. Rev. D 12 (1975) 3845
[3] Wu, T. T. and Yang, C. N., *Dirac Monopole without Strings: Classical Lagrangian theory,* Phys. Rev. D 14 (1976) 437
[4] Laughlin, R. B. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, *Physical Review Letters* 50 18 (1983)