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Preface

Usually the models of real world problems in almost all disciplines like engineering, medical sciences, mathematics, physics, computer science, management sciences, operations research and artificial intelligence are mostly full of complexities and consist of several types of uncertainties while dealing them in several occasion. To overcome these diffi culties of uncertainties, many theories have been developed such as rough sets theory, probability theory, fuzzy sets theory, theory of vague sets, theory of soft ideals and the theory of intuitionistic fuzzy sets, theory of neutrosophic sets, Dezert-Smarandache Theory (DSmT), etc. Zadeh discovered the relationships of probability and fuzzy set theory which has appropriate approach to deal with uncertainties. Many authors have applied the fuzzy set theory to generalize the basic theories of Algebra. Mordeson et al. [26] has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy fi nite state mechanics and fuzzy languages and the use of fuzzifi cation in automata and formal language has widely been explored. Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming. It is worth mentioning that some recent investigations of l-semigroups are closely connected with algebraic logic and non-classical logics.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e. Moreover $ab = (ba)e$ for all elements a and b of the AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

In [28], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [32]. Bhakat and Das [1, 2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [14]. In [35] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [34] semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

In chapter one we have introduced the concept of $(\in, \in \vee q)$-fuzzy ideals in an AG-groupoid. We have discussed several important features of a completely regular AG-groupoid by using the $(\in, \in \vee q)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals and $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideals.

In chapter two, we investigate some characterizations of regular and intra-regular Abel-Grassmann's groupoids in terms of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

In chapter three we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in an AG-groupoid. We characterize intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals.

In chapter four we introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals in AG-groupoids. We characterize intra-regular AG-groupoids by the properties of these ideals.

In chapter five we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime (semiprime) ideals in AG-groupoids. We characterize intra regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

1

Generalized Fuzzy Interior Ideals of AG-groupoids

In this chapter, we have introduced the concept of $(\in, \in \vee q)$-fuzzy ideals in an AG-groupoid. We have discussed several important features of a completely regular AG-groupoid by using the $(\in, \in \vee q)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals and $(\in, \in \vee q)$-fuzzy $(1, 2)$-ideals. We have also used the concept of $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals $(\in, \in \vee q_k)$-fuzzy bi-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals in completely regular AG-groupoid and proved that the $(\in, \in \vee q_k)$-fuzzy left (right, two-sided), $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, and $(\in, \in \vee q_k)$-fuzzy interior ideals coincide in a completely regular AG-groupoid.

1.1 Introduction

Fuzzy set theory and its applications in several branches of Science are growing day by day. Since pacific models of real world problems in various fields such as computer science, artificial intelligence, operation research, management science, control engineering, robotics, expert systems and many others, may not be constructed because we are mostly and unfortunately uncertain in many occasions. For handling such diffi culties we need some natural tools such as probability theory and theory of fuzzy sets [40] which have already been developed. Associative Algebraic struc- tures are mostly used for applications of fuzzy sets. Mordeson, Malik and Kuroki [26] have discovered the vast fi eld of fuzzy semigroups, where the- oretical exploration of fuzzy semigroups and their applications are used in fuzzy coding, fuzzy fi nite-state machines and fuzzy languages. The use of fuzzification in automata and formal language has widely been explored. Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming.

The fundamental concept of fuzzy sets was fi rst introduced by Zadeh [40] in 1965. Given a set X, a fuzzy subset of X is, by defi nition an arbitrary mapping $f : X \to [0, 1]$ where $[0, 1]$ is the unit interval. Rosenfeld introduced the definition of a fuzzy subgroup of a group [33]. Kuroki initiated the theory of fuzzy bi ideals in semigroups [18]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [28]. The concept of quasi-coincidence of a fuzzy point to a fuzzy set was introduce in [32]. Jun and Song introduced (α, β)-fuzzy interior ideals in semigroups [14].

In [28], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [32]. Bhakat and Das [1, 2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [14]. In [35] regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [34] semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

In this paper, we have introduced the concept of $(\in, \in \vee q_k)$-fuzzy ideals in a new non-associative algebraic structure, that is, in an AG-groupoid and developed some new results. We have defined regular and intra-regular AG-groupoids and characterized them by $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e. Moreover $ab = (ba)e$ for all elements a and b of the AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

1.2 Abel Grassmann Groupoids

The concept of a left almost semigroup (LA-semigroup) [16] or an AG-groupoid was first given by M. A. Kazim and M. Naseeruddin in 1972. an AG-groupoid M is a groupoid having the left invertive law,

$$(ab)c = (cb)a, \text{ for all } a,\, b,\, c \in M. \tag{1}$$

In an AG-groupoid M, the following medial law [16] holds,

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\, b,\, c,\, d \in M. \tag{2}$$

The left identity in an AG-groupoid if exists is unique [27]. In an AG-groupoid M with left identity the following paramedial law holds [31],

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in M. \tag{3}$$

If an AG-groupoid M contains a left identity, then,

$$a(bc) = b(ac), \text{ for all } a,\ b,\ c \in M. \tag{4}$$

1.3 Preliminaries

Let S be an AG-groupoid. By an AG-subgroupoid of S, we means a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$) and it is called a two-sided ideal if it is both left and a right ideal of S. A non-empty subset A of an AG-groupoid S is called quasi-ideal of S if $SA \cap AS \subseteq A$. A non-empty subset A of an AG-groupoid S is called a generalized bi-ideal of S if $(AS)A \subseteq A$ and an AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A \subseteq A$. A non-empty subset A of an AG-groupoid S is called an interior ideal of S if $(SA)S \subseteq A$.

If S is an AG-groupoid with left identity e then $S = S^2$. It is easy to see that every one sided ideal of S is quasi-ideal of S. In [30] it is given that $L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (aS \cap Sa)$ are principal left ideal, principal two-sided ideal and principal quasi-ideal of S generated by a. Moreover using (1), left invertive law, paramedial law and medial law we get the following equations

$$a\,(Sa) = S(aa) = Sa^2,\ (Sa)a = (aa)S = a^2S \text{ and } (Sa)\,(Sa) = (SS)\,(aa) = Sa^2.$$

To obtain some more useful equations we use medial, paramedial laws and (1), we get

$$\begin{aligned}(Sa)^2 &= (Sa)(Sa) = (SS)a^2 = (aa)(SS) = S((aa)S)\\ &= (SS)((aa)S) = (Sa^2)SS = (Sa^2)S.\end{aligned}$$

Therefore

$$Sa^2 = a^2S = (Sa^2)S. \tag{2}$$

The following definitions are available in [26].

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x,\ y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$

$(f(xy) \geq f(x))$ for all $x,\ y \in S$. A fuzzy subset f of an AG-groupoid S is called a fuzzy two-sided ideal of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of an AG-groupoid S is called a fuzzy quasi-ideal of S if $f \circ C_S \cap C_S \circ f \subseteq f$. A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy bi-ideal of S if $f((xa)y) \geq f(x) \wedge f(y)$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of an AG-groupoid S is called a fuzzy interior ideal of S if $f((xa)y) \geq f(a)$, for all x, a and $y \in S$. Let f be a fuzzy subset of an AG-groupoid S, then f is called a fuzzy prime if $\max\{f(a), f(b)\} \geq f(ab)$, for all $a, b \in S$. f is called a fuzzy semiprime if $f(a) \geq f(a^2)$, for all $a \in S$.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by,

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b, c \in S, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

The symbols $f \cap g$ and $f \cup g$ will means the following fuzzy subsets of S

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x),\ \text{ for all } x \text{ in } S$$

and

$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x),\ \text{ for all } x \text{ in } S.$$

Let f be a fuzzy subset of an AG-groupoid S and $t \in (0, 1]$. Then $x_t \in f$ means $f(x) \geq t$, $x_t q f$ means $f(x) + t > 1$, $x_t \alpha \vee \beta f$ means $x_t \alpha f$ or $x_t \beta f$, where α, β denotes any one of $\in$, q, $\in \vee q$, $\in \wedge q$. $x_t \alpha \wedge \beta f$ means $x_t \alpha f$ and $x_t \beta f$, $x_t \overline{\alpha} f$ means $x_t \alpha f$ does not holds. Generalizing the concept of $x_t q f$, Jun [13, 14] defined $x_t q_k f$, where $k \in [0, 1)$, as $f(x) + t + k > 1$. $x_t \in \vee q_k f$ if $x_t \in f$ or $x_t q_k f$.

Let f and g be any two fuzzy subsets of an AG-groupoid S, then for $k \in [0, 1)$, the product $f \circ_k g$ is defined by,

$$(f \circ_k g)(a) = \begin{cases} \bigvee\limits_{a=bc} \left\{f(b) \wedge g(c) \wedge \frac{1-k}{2}\right\}, & \text{if there exist } b, c \in S, \text{ such that } a = bc. \\ 0, & \text{otherwise.} \end{cases}$$

The symbols $f \wedge g$ and $f \vee g$ will means the following fuzzy subsets of an AG-groupoid S.

$(f \wedge g)(x) = \min\{f(x), g(x)\}$ for all x in S.

$(f \vee g)(x) = \max\{f(x), g(x)\}$ for all x in S.

Definition 1 *A fuzzy subset f of an AG-groupoid S is called fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $k \in [0, 1)$ such that $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$.*

Definition 2 *A fuzzy subset f of an AG-groupoid S is called fuzzy left (right) ideal of S if for all $x, y \in S$ and $k \in [0,1)$ such that $f(xy) \geq \min\{f(y), \frac{1-k}{2}\}$ $(f(xy) \geq \min\{f(x), \frac{1-k}{2}\})$.*

A fuzzy subset f of an AG-groupoid S is called fuzzy ideal if it is fuzzy left as well as fuzzy right ideal of S.

Definition 3 *A fuzzy subset f of an AG-groupoid S is called fuzzy quasi ideal of S, if*

$f(a) \geq \min\{(f \circ \varsigma)(a), (\varsigma \circ f)(a), \frac{1-k}{2}\}$. where ς is the fuzzy subset of S mapping every element of S on 1.

Definition 4 *A fuzzy subset f is called a fuzzy generalized bi-ideal of S if $f((xa)y) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all x, a and $y \in S$. A fuzzy AG-subgroupoid f of S is called a fuzzy bi-ideal of S if $f((xa)y) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$, for all $x, a, y \in S$ and $k \in [0,1)$.*

Definition 5 *An $(\in, \in \vee q_k)$-fuzzy subset f of an AG-groupoid S is called prime if for all $a, b \in S$ and $t \in (0,1]$, it satisfies,*

$(ab)_t \in f$ implies that $a_t \in \vee q_k f$ or $b_t \in \vee q_k f$.

Theorem 6 *An $(\in, \in \vee q_k)$-fuzzy ideal f of an AG-groupoid S is prime if for all $a, b \in S$, it satisfies,*

$\max\{f(a), f(b)\} \geq \min\{f(ab), \frac{1-k}{2}\}$.

Proof. It is straightforward. ■

Definition 7 *A fuzzy subset f of an AG-groupoid S is called $(\in, \in \vee q_k)$-fuzzy semiprime if it satisfies,*

$a_t^2 \in f$ this implies that $a_t \in \vee q_k f$ for all $a \in S$ and $t \in (0,1]$.

Theorem 8 *An $(\in, \in \vee q_k)$-fuzzy ideal f of an AG-groupoid S is called semiprime if for any $a \in S$ and $k \in [0,1)$, if it satisfies,*

$f(a) \geq \min\{f(a^2), \frac{1-k}{2}\}$.

Proof. It is easy. ■

Definition 9 *For a fuzzy subset F of an AG-groupoid M and $t \in (0,1]$, the crisp set $U(F;t) = \{x \in M$ such that $F(x) \geq t\}$ is called a level subset of F.*

Definition 10 *A fuzzy subset F of an AG-groupoid M of the form*

$$F(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Lemma 11 *A fuzzy subset F of an AG-groupoid M is a fuzzy interior ideal of M if and only if $U(F;t)\,(\neq \emptyset)$ is an interior ideal of M.*

Definition 12 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy interior ideal of M if for all $t, r \in (0,1]$ and $x, a, y \in M$.*
$(A1)$ *$x_t \in F$ and $y_r \in F$ implies that $(xy)_{\min\{t,r\}} \in \vee qF$.*
$(A2)$ *$a_t \in F$ implies $((xa)y)_t \in \vee qF$.*

Theorem 13 *For a fuzzy subset F of an AG-groupoid M. The conditions $(A1)$ and $(A2)$ of Definition 4, are equivalent to the following,*
$(A3)$ $(\forall x, y \in M) F(xy) \geq \min\{F(x), F(y), 0.5\}$
$(A4)$ $(\forall x, a, y \in M) F((xa)y) \geq \min\{F(a), 0.5\}$.

Lemma 14 *A fuzzy subset F of an AG-groupoid M is an $(\in, \in \vee q)$-fuzzy interior ideal of M if and only if $U(F;t)\,(\neq \emptyset)$ is an interior ideal of M, for all $t \in (0, 0.5]$.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of M. Let $x, y \in U(F;t)$ and $t \in (0, 0.5]$, then $F(x) \geq t$ and $F(y) \geq t$, so $F(x) \wedge F(y) \geq t$. As F is an $(\in, \in \vee q)$-fuzzy interior ideal of M, so

$$F(xy) \geq F(x) \wedge F(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$

Therefore, $xy \in U(F;t)$. Now if $x, y \in M$ and $a \in U(F;t)$ then $F(a) \geq t$ then $F((xa)y) \geq F(a) \wedge 0.5 \geq t \wedge 0.5 = t$. Therefore $((xa)y) \in U(F;t)$ and $U(F;t)$ is an interior ideal.

Conversely assume that $U(F;t)$ is an interior ideal of M. If $x, y \in U(F;t)$ then $F(x) \geq t$ and $F(y) \geq r$ which shows $x_t \in F$ and $y_r \in F$ as $U(F;t)$ is an interior ideal so $xy \in U(F;t)$ therefore $F(xy) \geq \min\{t, r\}$ implies that $(xy)_{\min\{t,r\}} \in F$, so $(xy)_{\min\{t,r\}} \in \vee qF$. Again let $x, y \in M$ and $a \in U(F;t)$ then $F(a) \geq t$ implies that $a_t \in F$ and $U(F;t)$ is an interior ideal so $((xa)y) \in U(F;t)$ then $F((xa)y) \geq t$ implies that $((xa)y)_t \in F$ so $((xa)y)_t \in \vee qF$. Therefore F is an $(\in, \in \vee q)$-fuzzy interior ideal. ■

Definition 15 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy bi-ideal of M if for all $t, r \in (0,1]$ and $x, y, z \in M$.*
$(B1)$ *$x_t \in F$ and $y_r \in F$ implies that $(xy)_{\min\{t,r\}} \in \vee qF$.*
$(B2)$ *$x_t \in F$ and $z_r \in F$ implies $((xy)z)_{\min\{t,r\}} \in \vee qF$.*

Theorem 16 *For a fuzzy subset F of an AG-groupoid M. The conditions $(B1)$ and $(B2)$ of Definition 5, are equivalent to the following,*
$(B3)$ $(\forall x, y \in M) F(xy) \geq \min\{F(x), F(y), 0.5\}$
$(B4)$ $(\forall x, y, z \in M) F((xy)z) \geq \min\{F(x), F(y), 0.5\}$.

Proof. It is similar to proof of theorem 13. ■

Definition 17 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M if*
(i) *$F(xy) \geq \min\{F(x), F(y), 0.5\}$, for all $x, y \in M$.*
(ii) *$F((xa)(yz)) \geq \min\{F(x), F(y), F(z), 0.5\}$, for all $x, a, y, z \in M$.*

Example 18 *Let* $M = \{1, 2, 3\}$ *be a right regular modular groupoid and* " $\cdot$ " *be any binary operation defined as follows:*

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Let F *be a fuzzy subset of* M *such that*

$$F(1) = 0.6, \quad F(2) = 0.3, \quad F(3) = 0.2.$$

Then we can see easily $F(1 \cdot 3) \geq F(3) \wedge 0.5$ *that is* F *is an* $(\in, \in \vee q)$*-fuzzy left ideal but* F *is not an* $(\in, \in \vee q)$*-fuzzy right ideal.*

Theorem 19 *Every* $(\in, \in \vee q)$*-fuzzy bi-ideal is an* $(\in, \in \vee q)$*-fuzzy* $(1, 2)$ *ideal of an AG-groupoid* M*, with left identity.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy bi-ideal of M and let $x, a, y, z \in M$ then by using (4) and (1), we have

$$\begin{aligned} F((xa)(yz)) &= F(y((xa)z)) \geq \min\{F(y), F((xa)z), 0.5\} \\ &= \min\{F(y), F((za)x), 0.5\} \geq \min\{F(y), F(z), F(x), 0.5, 0.5\} \\ &= \min\{F(y), F(z), F(x), 0.5\}. \end{aligned}$$

Therefore F is an $(\in, \in \vee q)$-fuzzy $(1, 2)$ ideal of an AG-groupoid M. ■

Theorem 20 *Every* $(\in, \in \vee q)$*-fuzzy interior ideal is an* $(\in, \in \vee q)$*-fuzzy* $(1, 2)$ *ideal of an AG-groupoid* M*, with left identity* e*.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of M and let $x, a, y, z \in M$ then by using (1), we have

$$\begin{aligned} F((xa)(yz)) &\geq \min\{F(xa), F(yz), 0.5\} \geq \min\{F(xa), F(y), F(z), 0.5, 0.5\} \\ &= \min\{F((ex)a), F(y), F(z), 0.5\} = \min\{F((ax)e), F(y), F(z), 0.5\} \\ &\geq \min\{F(x), F(y), F(z), 0.5, 0.5\} = \min\{F(x), F(y), F(z), 0.5\}. \end{aligned}$$

Therefore F is an $(\in, \in \vee q)$ $-$fuzzy $(1, 2)$ ideal of an AG-groupoid M. ■

Theorem 21 *Let* $\Phi : M \longrightarrow M'$ *be a homomorphism of AG-groupoids and* F *and* G *be* $(\in, \in \vee q)$*-fuzzy interior ideals of* M *and* M'*,respectively. Then*

(i) $\Phi^{-1}(G)$ *is an* $(\in, \in \vee q)$*-fuzzy interior ideal of* M*.*

(ii) If for any subset X *of* M *there exists* $x_\circ \in X$ *such that* $F(x_\circ) = \bigvee \{F(x) \mid x \in X\}$*, then* $\Phi(F)$ *is an* $(\in, \in \vee q)$*-fuzzy interior ideal of* M' *when* Φ *is onto.*

Proof. It is same as in [14]. ■

1.4 Completely Regular AG-groupoids

Definition 22 *an AG-groupoid M is called regular, if for each $a \in M$ there exist $x \in M$ such that $a = (ax)\, a$.*

Definition 23 *an AG-groupoid M is called left (right) regular, if for each $a \in M$ there exist $z \in M$ $(y \in M)$ such that $a = za^2$ $\left(a = a^2 y\right)$.*

Definition 24 *an AG-groupoid M is called completely regular if it is regular, left regular and right regular.*

Example 25 *Let $M = \{1, 2, 3, 4\}$ and the binary operation "$\cdot$" defined on M as follows:*

$\cdot$	1	2	3	4
1	4	1	2	3
2	3	4	1	2
3	2	3	4	1
4	1	2	3	4

Then clearly $(M, \cdot)$ is a completely regular AG-groupoid with left identity 4.

Theorem 26 *If M is an AG-groupoid with left identity e, then it is completely regular if and only if $a \in (a^2M)a^2$.*

Proof. Let M be a completely regular AG-groupoid with left identity e, then for each $a \in M$ there exist $x, y, z \in M$ such that $a = (ax)a$, $a = a^2y$ and $a = za^2$, so by using $(1), (4)$ and (3), we get

$$\begin{aligned} a &= (ax)a = ((a^2y)x)(za^2) = ((xy)a^2)(za^2) = ((za^2)a^2)(xy) \\ &= ((a^2a^2)z)(xy) = ((xy)z)(a^2a^2) = a^2(((xy)z)a^2) \\ &= (ea^2)(((xy)z)a^2) = (a^2((xy)z))(a^2e) = (a^2((xy)z))((aa)e) \\ &= (a^2((xy)z))((ea)a) = (a^2((xy)z))a^2 \in (a^2M)a^2. \end{aligned}$$

Conversely, assume that $a \in (a^2M)a^2$ then clearly $a = a^2y$ and $a = za^2$, now using $(3), (1)$ and (4), we get

$$\begin{aligned} a &\in (a^2M)a^2 = \left(a^2M\right)(aa) = (aa)\left(Ma^2\right) = (aa)\,(M\,(aa)) \\ &= (aa)\,((eM))(aa)) = (aa)\,((aa)\,(Me)) \subseteq (aa)\,((aa)\,M) \\ &= (aa)\left(a^2M\right) = \left(\left(a^2M\right)a\right)a = (((aa)\,M)\,a)\,a = ((aM)\,(aa))\,a \\ &= (a\,((Ma)\,a))\,a \subseteq (aM)a. \end{aligned}$$

Therefore M is completely regular. ■

Theorem 27 *If M is a completely regular AG-groupoid with left identity e, then every $(\in, \in \vee q)$-fuzzy $(1, 2)$ ideal of M is an $(\in, \in \vee q)$-fuzzy bi-ideal of M.*

Proof. Let M be a completely regular with left identity e, and let F be an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M. Then for $x \in M$ there exists $b \in M$ such that $x = (x^2b)x^2$, so by using (1) and (4), we have

$$\begin{aligned}
F((xa)y) &= F((((x^2b)x^2)a)y) = F((ya)((x^2b)x^2)) \\
&\geq \min\left\{F(y), F(x^2b), F(x^2), 0.5\right\} \\
&\geq \min\left\{F(y), F(x^2b), F(x), F(x), 0.5, 0.5\right\} \\
&= \min\left\{F(y), F(x^2b), F(x), 0.5\right\} \\
&= \min\left\{F((xx)b), F(x), F(y), 0.5\right\} \\
&= \min\left\{F((bx)x), F(x), F(y), 0.5\right\} \\
&\geq \min\left\{F(bx), F(x), 0.5, F(x), F(y), 0.5\right\} \\
&= \min\left\{F(bx).F(x), F(y), 0.5\right\} \\
&= \min\left\{F(b((x^2b)x^2)), F(x), F(y), 0.5\right\} \\
&= \min\{F((x^2b)(bx^2)), F(x), F(y), 0.5\} \\
&= \min\left\{F(((bx^2)b)x^2), F(x), F(y), 0.5\right\}
\end{aligned}$$

$$\begin{aligned}
&= \min\left\{F((((eb)x^2)b)(xx)), F(x), F(y), 0.5\right\} \\
&= \min\left\{F((((x^2b)e)b)(xx)), F(x), F(y), 0.5\right\} \\
&= \min\left\{F(((be)(x^2b))(xx)), F(x), F(y), 0.5\right\} \\
&= \min\left\{F((x^2((be)b))(xx)), F(x), F(y), 0.5\right\} \\
&\geq \min\left\{F(x^2), F(x), F(x), 0.5, F(x), F(y), 0.5\right\} \\
&\geq \min\left\{F(x), F(x), 0.5, F(x), F(x), F(x), F(y), 0.5\right\} \\
&= \min\left\{F(x), F(y), 0.5\right\}.
\end{aligned}$$

Therefore, F is an $(\in, \in \vee q)$-fuzzy bi-ideal of M. ■

Theorem 28 *If M is a completely regular AG-groupoid with left identity e, then every $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M is an $(\in, \in \vee q)$-fuzzy interior ideal of M.*

Proof. Let M be a completely regular with left identity e, and F is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M. Then for $x \in M$ there exists $y \in M$ such that $x = (x^2y)x^2$, so by using $(4), (1), (2)$ and (3), we have

$$\begin{aligned}
F((ax)b) &= F((a((x^2y)x^2))b) = F(((x^2y)(ax^2))b) \\
&= F((b(ax^2))(x^2y)) = F((b(a(xx)))(x^2y)) \\
&= F((b(x(ax)))(x^2y)) = F((x(b(ax)))((xx)y)) \\
&= F((x(b(ax)))((yx)x)) = F((x(yx))((b(ax))x)) \\
&= F(((ex)(yx))((x(ax))b)) = F(((xx)(ye))((a(xx))b)) \\
&= F(((xx)(ye))((b(xx))a) = F(((xx)(b(xx)))((ye)a)) \\
&= F((b((xx)(xx)))((ye)a)) = F((b(ye))(((xx)(xx))a)) \\
&= F((a((xx)(xx)))((ye)b)) = F(((xx)(a(xx)))((ye)b)) \\
&= F((((ye)b)(a(xx)))(xx)) = F((((ye)b)(x(ax)))(xx)) \\
&= F((x(((ye)b)(ax)))(xx)) \geq \min\{F(x), F(x), F(x), 0.5\} \\
&= \min\{F(x), 0.5\}.
\end{aligned}$$

Therefore F is an $(\in, \in \vee q)$-fuzzy interior ideal of M. ■

Theorem 29 *Let F be an $(\in, \in \vee q)$-fuzzy bi-ideal of an AG-groupoid M. If M is a completely regular and $F(a) < 0.5$ for all $x \in M$ then $F(a) = F(a^2)$ for all $a \in M$.*

Proof. Let $a \in M$ then there exist $x \in M$ such that $a = (a^2x)a^2$, then we have

$$\begin{aligned}
F(a) &= F((a^2x)a^2) \geq \min\{F(a^2), F(a^2), 0.5\} \\
&= \min\{F(a^2), 0.5\} = F(a^2) = F(aa) \\
&\geq \min\{F(a), F(a), 0.5\} = F(a).
\end{aligned}$$

Therefore $F(a) = F(a^2)$. ■

Theorem 30 *Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of an AG-groupoid M with left identity e. If M is a completely regular and $F(a) < 0.5$ for all $x \in M$ then $F(a) = F(a^2)$ for all $a \in M$.*

Proof. Let $a \in M$ then there exists $x \in M$ such that $a = (a^2x)a^2$, using $(4), (1)$ and (3), we have

$$\begin{aligned}
F(a) &= F((a^2x)a^2) = F((a^2x)(aa)) = F(a((a^2x)a)) \\
&= F(a((ax)a^2)) = F((ea)((ax)a^2)) = F((((ax)a^2)a)e) \\
&= F(((aa^2)(ax))e) = F(((xa)(a^2a))e) = F((((a^2a)a)x)e) \\
&= F((((aa)a^2)x)e) = F(((xa^2)(aa))e) = F(((xa^2)a^2)e) \\
&\geq \min\{F(a^2), 0.5\} = F(a^2) = F(aa) \\
&\geq \min\{F(a), F(a), 0.5\} \geq \min\{F(a), 0.5\} = F(a).
\end{aligned}$$

Therefore $F(a) = F(a^2)$. ■

1.5 $(\in, \in \vee q_k)$-fuzzy Ideals in AG-groupoids

It has been given in [13] that $x_t q_k F$ is the generalizations of $x_t q F$, where k is an arbitrary element of $[0, 1)$ as $x_t q_k F$ if $F(x) + t + k > 1$. If $x_t \in F$ or $x_t q_k F$ implies $x_t \in \vee q_k F$. Here we discuss the behavior of $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy right ideal,$(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy quasi-ideal in the completely regular AG-groupoid M.

Example 31 *Let $M = \{1, 2, 3\}$ be any AG-groupoid with binary operation "$\cdot$" defined as in Example 18. Let F be a fuzzy subset of M such that*

$$F(1) = 0.7, \quad F(2) = 0.4, \quad F(3) = 0.3.$$

If we choose $k \in [0.3, 1]$, then we can see that $F(1 \cdot 3) \geq F(1) \wedge \frac{1-k}{2}$ and $F(13) \geq F(3) \wedge \frac{1-k}{2}$ by a simple calculation that is F is an $(\in, \in \vee q_k)$-fuzzy ideal but clearly F is not an $(\in, \in \vee q)$-fuzzy ideal.

Definition 32 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy subgroupoid of M if for all $x, y \in M$ and $t, r \in (0, 1]$ the following condition holds*

$$x_t \in F, y_r \in F \text{ implies } (xy)_{\min\{t,r\}} \in \vee q_k F.$$

Theorem 33 *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy subgroupoid of M if and only if $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$.*

Proof. It is similar to the proof of Theorem 13. ■

Definition 34 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of M if for all $x, y \in M$ and $t, r \in (0, 1]$ the following condition holds*

$$y_t \in F \text{ implies } (xy)_t \in \vee q_k F \;\; \left(y_t \in F \text{ implies } (yx)_t \in \vee q_k F\right).$$

Theorem 35 *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of M if and only if $F(xy) \geq \min\{F(y), \frac{1-k}{2}\} \left(F(xy) \geq \min\{F(x), \frac{1-k}{2}\}\right)$.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy left ideal of M. Suppose that there exist $x, y \in M$ such that $F(xy) < \min\left\{F(y), \frac{1-k}{2}\right\}$. Choose a $t \in (0, 1]$ such that $F(xy) < t < \min\left\{F(y), \frac{1-k}{2}\right\}$. Then $y_t \in F$ but $(xy)_t \notin F$ and $F(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $(xy)_t \overline{\in \vee q_k} F$, a contradiction. Therefore $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$.

Conversely, assume that $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$. Let $x, y \in M$ and $t \in (0, 1]$ such that $y_t \in M$ then $F(y) \geq t$. then $F(xy) \geq \min\{F(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$ then $F(xy) \geq \frac{1-k}{2}$. So $F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $(xy)_t q_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq t$. Therefore $F(xy) \geq t$ which implies that $(xy)_t \in F$. Thus $(xy)_t \in \vee q_k F$. ■

Corollary 36 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy ideal of M if and only if $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$ and $F(xy) \geq \min\{F(x), \frac{1-k}{2}\}$.*

Definition 37 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if for all $x, y, z \in M$ and $t, r \in (0, 1]$ the following conditions hold*

Definition 38 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if for all $x, y, z \in M$ and $t, r \in (0, 1]$ the following conditions hold*

(i) If $x_t \in F$ and $y_r \in M$ implies $(xy)_{\min\{t,r\}} \in \vee q_k F$,

(ii) If $x_t \in F$ and $z_r \in M$ implies $((xy)\, z)_{\min\{t,r\}} \in \vee q_k F$.

Definition 39 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M If $x_t \in F$ and $z_r \in M$ implies $((xy)\, z)_{\min\{t,r\}} \in \vee q_k F$, for all $x, y, z \in M$ and $t, r \in (0, 1]$*

Theorem 40 *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if and only if*

(i) $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, y \in M$ and $k \in [0, 1)$,

(ii) $F((xy)z) \geq \min\{F(x), F(z), \frac{1-k}{2}\}$ for all $x, y, z \in M$ and $k \in [0, 1)$.

Proof. It is similar to the proof of Theorem 13. ■

Corollary 41 *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if and only if $F((xy)z) \geq \min\{F(x), F(z), \frac{1-k}{2}\}$ for all $x, y, z \in M$ and $k \in [0, 1)$.*

Definition 42 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of M if for all $x, a, y \in M$ and $t, r \in (0, 1]$ the following conditions hold*

(i) If $x_t \in F$ and $y_r \in M$ implies $(xy)_{\min\{t,r\}} \in \vee q_k F$,

(ii) If $a_t \in M$ implies $((xa)\, y)_{\min\{t,r\}} \in \vee q_k F$.

Theorem 43 *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M if and only if*

(i) $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, y \in M$ and $k \in [0, 1)$,

(ii) $F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$ for all $x, a, y \in M$ and $k \in [0, 1)$.

Proof. It is similar to the proof of Theorem 13. ■

Lemma 44 *The intersection of any family of $(\in, \in \vee q_k)$-fuzzy interior ideals of AG-groupoid M is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy interior ideals of M and $x, a, y \in M$. Then $(\wedge_{i \in I} F_i)((xa)y) = \wedge_{i \in I}(F_i((xa)\, y)$. As each F_i is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M, so $F_i((xa)y) \geq F_i(a) \wedge \frac{1-k}{2}$ for all $i \in I$. Thus

$$\begin{aligned}(\wedge_{i\in I}F_i)\,((xa)y) &= \wedge_{i\in I}(F_i((xa)\,y) \geq \wedge_{i\in I}\left(F_i(a) \wedge \frac{1-k}{2}\right) \\ &= (\wedge_{i\in I}F_i(a)) \wedge \frac{1-k}{2} = (\wedge_{i\in I}F_i)\,(a) \wedge \frac{1-k}{2}.\end{aligned}$$

Therefore $\wedge_{i\in I}F_i$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M. ■

Definition 45 *Let f and g be a fuzzy subsets of AG-groupoid S, then the k-product of f and g is defined by*

$$(f \circ_k g)(a) = \begin{cases} \bigvee\limits_{a=bc} \min\{f(a),\ f(b),\ \frac{1-k}{2}\} & \text{if there exists } b, c \in S \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

where $k \in [0, 1)$.

The k intersection of f and g is defined by

$$(f \cap_k g)(a) = \{f(a) \wedge f(b) \wedge \frac{1-k}{2}\} \text{ for all } a \in S.$$

Definition 46 *A fuzzy subset F of an AG-groupoid M is called an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of M if following condition holds*

$$F\,(x) \geq \min\left\{(F \circ 1)\,(x)\,, (1 \circ F)\,(x), \frac{1-k}{2}\right\}.$$

where 1 is the fuzzy subset of M mapping every element of M on 1.

Lemma 47 *If M is a completely regular AG-groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy right ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy left ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy right ideal of a completely regular AG-groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, then by using (1), we have

$$\begin{aligned}F(ab) &= F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \\ &\geq F(ba^2) \wedge \frac{1-k}{2} \geq F(b) \wedge \frac{1-k}{2}.\end{aligned}$$

Conversely, assume that F is an $(\in, \in \vee q_k)$-fuzzy right ideal of M, then by using (1), we have

$$\begin{aligned}F(ab) &= F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \\ &\geq F(a^2x) \wedge \frac{1-k}{2} = F((aa)x) \wedge \frac{1-k}{2} \\ &= F((xa)a) \wedge \frac{1-k}{2} \geq F(a) \wedge \frac{1-k}{2}.\end{aligned}$$

■

Theorem 48 *If M is a completely regular AG-groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy interior ideal of a completely regular AG-groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2 x)a^2$, then by using (4) and (1), we have

$$F(ab) = F(((a^2 x)a^2)b) \geq F(aa) \geq F(a) \wedge F(a) \wedge \frac{1-k}{2}, \text{ and}$$

$$\begin{aligned} F(ab) &= F(a((b^2 y)b^2))F((b^2 y)(ab^2)) = F(((bb)y)(ab^2)) \\ &= F(((yb)b)(ab^2)) \geq F(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious. ■

Theorem 49 *If M is a completely regular AG-groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a completely regular AG-groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2 x)a^2$, then by using (4), we have

$$\begin{aligned} F(ab) &= F(((a^2 x)a^2)b) = F(((a^2 x)(aa))b) \\ &= F((a((a^2 x)a))b) \geq F(a) \wedge F(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious. ■

Theorem 50 *If M is a completely regular AG-groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of a completely regular AG-groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2 x)a^2$, then by using (1) and (4), we have

$$\begin{aligned} F(ab) &= F(((a^2 x)a^2)b) = F((((aa)x)a^2)b) = F((((xa)a)a^2)b) \\ &= F((ba^2)((xa)a)) = F((b(aa))((aa)x)) = F((a(ba))((aa)x)) \\ &= F((aa)((a(ba))x)) = F((((a(ba))x)a)a) = F((((b(aa))x)a)a) \\ &= F((((x(aa))b)a)a) = F((((a(xa))b)a)a) = F(((ab)(a(xa)))a) \\ &= F((a((ab)(xa)))a) \geq F(a) \wedge F(a) \wedge \frac{1-k}{2} = F(a) \wedge \frac{1-k}{2}. \end{aligned}$$

And, by using (4), (1) and (3),we have

$$\begin{aligned}
F(ab) &= F(a((b^2y)b^2)) = F((b^2y)(ab^2)) = F(((bb)y)(a(bb))) \\
&= F(((a(bb)y)(bb)) = F(((a(bb))(ey))(bb)) = F(((ye)((bb)a))(bb)) \\
&= F((bb)((ye)a))(bb)) \geq F(bb) \wedge \frac{1-k}{2}. \geq F(b) \wedge F(b) \wedge \frac{1-k}{2}. \\
&= F(b) \wedge \frac{1-k}{2}.
\end{aligned}$$

The converse is obvious. ■

Theorem 51 *If M is a completely regular AG-groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a completely regular AG-groupoid M, then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, then by using (1), (3) and (4), we have

$$\begin{aligned}
ab &= ((a^2x)a^2)b = (ba^2)\left(a^2x\right) = \left(xa^2\right)\left(a^2b\right) \\
&= (x\,(aa))\left(a^2b\right) = (a\,(xa))\left(a^2b\right) = \left(\left(a^2b\right)(xa)\right)a.
\end{aligned}$$

Then

$$\begin{aligned}
F(ab) &\geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\
&= \bigvee_{ab=pq} \{F(p) \wedge 1\,(q)\} \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\
&\geq F\,(a) \wedge 1\,(b) \wedge \bigvee_{ab=lm} \{F\,(l) \wedge 1(m)\} \wedge \frac{1-k}{2} \\
&= F\,(a) \wedge \bigvee_{ab=((a^2b)(xa))a} \left\{1\left(\left(a^2b\right)(xa)\right) \wedge F\,(a)\right\} \\
&\geq F\,(a) \wedge 1\left(\left(a^2b\right)(xa)\right) \wedge F\,(a) \wedge \frac{1-k}{2} = F\,(a) \wedge \frac{1-k}{2}.
\end{aligned}$$

Also by using (4) and (1), we have

$$\begin{aligned}
ab &= a\left((b^2y)\,b^2\right) = \left(b^2y\right)\left(ab^2\right) = ((bb)\,y)\left(ab^2\right) \\
&= \left(\left(ab^2\right)y\right)(bb) = b\left(\left(\left(ab^2\right)y\right)b\right).
\end{aligned}$$

Then

$$\begin{aligned}
F(ab) &\geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\
&= \bigvee_{ab=pq} \{F(p) \wedge 1\,(q)\} \wedge \bigvee_{ab=lm} \{1(l) \wedge F\,(m)\} \wedge \frac{1-k}{2} \\
&\geq F\,(b) \wedge 1(\left(\left(ab^2\right)y\right)b) \wedge 1\,(a) \wedge F\,(b) \wedge \frac{1-k}{2} = F\,(b) \wedge \frac{1-k}{2}.
\end{aligned}$$

The converse is obvious ■

Remark 52 *We note that in a completely regular AG-groupoid M with left identity, $(\in, \in \vee q_k)$-fuzzy left ideal $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$ fuzzy ideal,$(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal coincide with each other.*

Theorem 53 *If M is a completely regular AG-groupoid then $F \wedge_k G = F \circ_k G$ for every $(\in, \in \vee q_k)$- fuzzy ideal F and G of M.*

Proof. Let F is an $(\in, \in \vee q_k)$- fuzzy right ideal of M and G is an $(\in, \in \vee q_k)$-fuzzy left ideal of M, and M is a completely regular then for each $a \in M$ there exists $x \in M$ such that $a = (a^2x)a^2$, so we have

$$\begin{aligned}
(F \circ_k G)(a) &= (F \circ G)(a) \wedge \frac{1-k}{2} = \bigvee_{a=pq} \{F(p) \wedge G(q)\} \wedge \frac{1-k}{2} \\
&\geq F(a^2x) \wedge G\left(a^2\right) \wedge \frac{1-k}{2} \geq F(aa) \wedge G(aa) \wedge \frac{1-k}{2} \\
&\geq F(aa) \wedge G(aa) \wedge \frac{1-k}{2} \geq F(a) \wedge G(a) \wedge \frac{1-k}{2} \\
&= (F \wedge G)(a) \wedge \frac{1-k}{2} = (F \wedge_k G)(a).
\end{aligned}$$

Therefore $F \wedge_k G \leq F \circ_k G$, again

$$\begin{aligned}
(F \circ_k G)(a) &= (F \circ G)(a) \wedge \frac{1-k}{2} = \left(\bigvee_{a=pq} \{F(p) \wedge G(q)\}\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left\{F(p) \wedge G(q) \wedge \frac{1-k}{2}\right\} \leq \bigvee_{a=pq} \left\{(F(pq \wedge G(pq)) \wedge \frac{1-k}{2}\right\} \\
&= F(a) \wedge G(a) \wedge \frac{1-k}{2} = (F \wedge_k G)(a).
\end{aligned}$$

Therefore $F \wedge_k G \geq F \circ_k G$. Thus $F \wedge_k G = F \circ_k G$. ■

Definition 54 *an AG-groupoid M is called weakly regular if for each a in M there exists x and y in M such that $a = (ax)(ay)$.*

It is easy to see that right regular, left regular and weakly regular coincide in an AG-groupoid with left identity.

Theorem 55 *For a weakly regular AG-groupoid M with left identity, $(G \wedge_k F) \wedge_k H) \leq ((G \circ_k F) \circ_k H)$, where G is an $(\in, \in \vee q_k)$- fuzzy right ideal, F is an $(\in, \in \vee q_k)$- fuzzy interior ideal and H is an $(\in, \in \vee q_k)$- fuzzy left ideal.*

Proof. Let M be a weakly regular AG-groupoid with left identity, then for each $a \in M$ there exist x, $y \in M$ such that $a = (ax)(ay)$, then by using (3), we get $a = (ya)(xa)$, and also by using (4) and (3), we get

$$ya = y((ax)(ay)) = (ax)(y(ay)) = (ax)((ey)(ay)) = (ax)((ya)(ye))$$

Then

$$\begin{aligned}
((G \circ_k F) \circ_k H)(a) &= \bigvee_{a=pq} \{(G \circ_k F)(p) \wedge H(q)\} \geq (G \circ_k F)(ya) \wedge H(xa) \\
&\geq (G \circ_k F)(ya) \wedge H(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{ya=bc} \{G(b) \wedge F(c)\} \wedge H(a) \wedge \frac{1-k}{2} \\
&\geq (G(ax) \wedge F((ya)(ye))) \wedge H(a) \wedge \frac{1-k}{2} \\
&\geq (G(a) \wedge \frac{1-k}{2} \wedge F(a) \wedge \frac{1-k}{2}) \wedge H(a) \wedge \frac{1-k}{2} \\
&= (G(a) \wedge F(a)) \wedge H(a) \wedge \frac{1-k}{2} \\
&= ((G \wedge F) \wedge H)(a) \wedge \frac{1-k}{2} \\
&= ((G \wedge_k F) \wedge_k H)(a).
\end{aligned}$$

Therefore, $(G \wedge_k F) \wedge_k H) \leq ((G \circ_k F) \circ_k H)$. ■

Theorem 56 *For a weakly regular AG-groupoid M with left identity, $F_k \leq ((F \circ_k 1) \circ_k F)$, where F is an $(\in, \in \vee q_k)$- fuzzy interior ideal.*

Proof. Let M be a weakly regular AG-groupoid with left identity, then for each $a \in M$ there exist $x, y \in M$ such that $a = (ax)(ay)$, then by using (1) $a = ((ay)x)a$. Also by using (1) and (4), we have

$$\begin{aligned}
(ay) &= (((ax)(ay))y) = ((y(ay))(ax)) = ((a(yy))(ax)) \\
&= (((ax)(yy))a) = ((((yy)x)a)a).
\end{aligned}$$

Then

$$\begin{aligned}
((F \circ_k 1) \circ_k F)(a) &= ((F \circ 1) \circ F)(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \{(F \circ 1)(p) \wedge F(q)\} \wedge \frac{1-k}{2} \\
&\geq (F \circ 1)(((ay)x) \wedge F(a) \wedge \frac{1-k}{2} \\
&= \bigvee_{(ay)x=(bc)} \{F(b) \wedge 1(c)\} \wedge F(a) \wedge \frac{1-k}{2} \\
&\geq F(ay) \wedge 1(x) \wedge F(a) \wedge \frac{1-k}{2} \\
&= F((((yy)x)a)a) \wedge F(a) \wedge \frac{1-k}{2} \\
&\geq F(a) \wedge \frac{1-k}{2} \wedge F(a) \wedge \frac{1-k}{2} \\
&= F(a) \wedge \frac{1-k}{2} = F_k(a).
\end{aligned}$$

Therefore, $F_k \leq ((F \circ_k 1) \circ_k F)$. ■

2

Generalized Fuzzy Ideals of Abel-Grassmann groupoids

In this chapter, we investigate some characterizations of regular and intra-regular Abel-Grassmann's groupoids in terms of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy quasi-ideals.

An element a of an AG-groupoid S is called **regular** if there exist $x \in S$ such that $a = (ax)a$ and S is called **regular**, if every element of S is regular. An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular.

The following definitions for AG-groupoids are same as for semigroups in [34].

Definition 57 *(1) A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, r \in (0, 1]$, it satisfies,*

$x_t \in \delta$, $y_r \in \delta$ implies that $(xy)_{\min\{t,r\}} \in \vee q_k \delta$.

(2) A fuzzy subset δ of S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if for all $x, y \in S$ and $t, r \in (0, 1]$, it satisfies,

$x_t \in \delta$ implies $(yx)_t \in \vee q_k \delta$ ($x_t \in \delta$ implies $(xy)_t \in \vee q_k \delta$).

(3) A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following condition holds.

$y_t \in f$ implies $((xy)z)_t \in \vee q_k f$.

(4) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S if for all $x \in S$ it satisfies, $f(x) \geq \min(f \circ C_S(x), C_S \circ f(x), \frac{1-k}{2})$, where C_S is the fuzzy subset of S mapping every element of S on 1.

(5) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if $x_t \in f$ and $z_r \in S$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$, for all $x, y, z \in S$ and $t, r \in (0, 1]$.

(6) A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if for all $x, y, z \in S$ and $t, r \in (0, 1]$ the following conditions hold

(i) If $x_t \in f$ and $y_r \in S$ implies $(xy)_{\min\{t,r\}} \in \vee q_k f$,

(ii) If $x_t \in f$ and $z_r \in f$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k f$.

Theorem 58 *[34] (1) Let δ be a fuzzy subset of S. Then δ is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S if $\delta(xy) \geq \min\{\delta(x), \delta(y), \frac{1-k}{2}\}$.*

(2) A fuzzy subset δ of an AG-groupoid S is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of S if
$\delta(xy) \geq \min\{\delta(y), \frac{1-k}{2}\}$ $\left(\delta(xy) \geq \min\{\delta(x), \frac{1-k}{2}\}\right)$.

(3) A fuzzy subset f of an AG-groupoid S is an $(\in, \in \vee q_k)$-fuzzy interior ideal of S if and only if it satisfies the following conditions.
(i) $f(xy) \geq \min\left\{f(x), f(y), \frac{1-k}{2}\right\}$ *for all* $x, y \in S$ *and* $k \in [0, 1)$.
(ii) $f((xy)z) \geq \min\left\{f(y), \frac{1-k}{2}\right\}$*for all* $x, y, z \in S$ *and* $k \in [0, 1)$.

(4) Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if
(i) $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ *for all* $x, y \in S$ *and* $k \in [0, 1)$,
(ii) $f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$ *for all* $x, y, z \in S$ *and* $k \in [0, 1)$.

Here we begin with examples of an AG-groupoid.

Example 59 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\cdot$	1	2	3
1	2	2	2
2	3	3	3
3	3	3	3

Note that S has no left identity. Define a fuzzy subset $F : S \longrightarrow [0, 1]$ as follows:

$$F(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.5 \text{ for } x = 2 \\ 0.6 \text{ for } x = 3 \end{cases}$$

Then clearly F is an $(\in, \in \vee q_k)$-fuzzy ideal of S.

Example 60 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$*$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

Obviously 3 is the left identity in S. Define a fuzzy subset $G : S \longrightarrow [0, 1]$ as follows:

$$G(x) = \begin{cases} 0.8 \text{ for } x = 1 \\ 0.6 \text{ for } x = 2 \\ 0.5 \text{ for } x = 3 \end{cases}$$

Then clearly G is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.

Lemma 61 *Intersection of two ideals of an AG-groupoid is an ideal.*

Proof. It is easy. ■

Lemma 62 *If S is an AG-groupoid with left identity then $(aS)(Sa) = (aS)a$, for all a in S.*

Proof. Using paramedial law, medial law and (1), we get

$$(aS)(Sa) = [(Sa)S]a = [(Sa)(SS)]a = [(SS)(aS)]a = (a(SS))a = (aS)a.$$

■

Lemma 63 *[23]If S is an AG-groupoid with left identity, then S is intra-regular if and only if for every a in S there exist some x, y in S such that $a = (xa)(ay)$.*

Lemma 64 *Let S be an AG-groupoid. If $a = a(ax)$, for some $x \in S$, then $a = a^2y$, for some y in S.*

Proof. Using medial law, we get

$$a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y, \text{ where } y = (ax)x.$$

■

Lemma 65 *Let S be an AG-groupoid with left identity. If $a = a^2x$, for some x in S. Then $a = (ay)a$, for some y in S.*

Proof. Using medial law, left invertive law, (1), paramedial law and medial law, we get

$$\begin{aligned} a &= a^2x = (aa)x = ((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2) \\ &= a^2((xx^2)a^2)) = (((xx^2)a^2)a)a = ((aa^2)(xx^2))a = ((x^2x)(a^2a))a \\ &= [a^2\{(x^2x)a\}]a = [\{a(x^2x)\}(aa)]a = [a(\{a(x^2x)\}a)]a \\ &= (ay)a, \text{ where } y = \{a(x^2x)\}a. \end{aligned}$$

■

Using (2) and lemma 65, we get the following crucial lemma.

Lemma 66 *Every intra-regular AG-groupoid with left identity is a regular AG-groupoid with left identity.*

The converse of lemma 66 is not true in general.

Example 67 *Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Clearly S is regular because $1 = 1 \circ 1$, $2 = (2 \circ 3) \circ 2$ and $3 = (3 \circ 2) \circ 3$. But S is not intra-regular because the element 2 is not intra-regular.

Lemma 68 *Let S be an AG-groupoid with left identity, then $(aS)a^2 \subseteq (aS)a$, for all a in S.*

Proof. Using paramedial law, medial law, left invertive law and (1), we get

$$\begin{aligned}(aS)a^2 &= (aa)(Sa) = [(Sa)a]a = [(aa)(SS)]a = [(SS)(aa)]a \\ &= [a\{(SS)a\}]a \subseteq (aS)a.\end{aligned}$$

■

Lemma 69 *Let S be an AG-groupoid with left identity, then $(aS)[(aS)a] \subseteq (aS)a$, for all a in S.*

Proof. Using left invertive law and (1), we get

$$(aS)[(aS)a] = [\{(aS)a\}S]a = [(Sa)(aS)]a = [a\{(Sa)S]a \subseteq (aS)a.$$

■

Theorem 70 *Let S be an AG-groupoid with left identity, then $B[a] = a \cup a^2 \cup (aS)\, a$ is a bi-ideal of S.*

Proof. Using lemmas 62, 68, 69, left invertive law and (1), we get

$$\begin{aligned}(B[a]S)B[a] &= [\{a \cup a^2 \cup (aS)\, a\}S][a \cup a^2 \cup (aS)\, a] \\ &= [aS \cup a^2S \cup ((aS)\, a)\, S][a \cup a^2 \cup (aS)\, a] \\ &= [aS \cup a^2S \cup (Sa)\, (aS)][a \cup a^2 \cup (aS)\, a] \\ &= [aS \cup a^2S \cup a((Sa)S)][a \cup a^2 \cup (aS)\, a] \\ &\subseteq [aS \cup aS \cup aS][a \cup a^2 \cup (aS)\, a] \\ &= (aS)\left(a \cup a^2 \cup (aS)\, a\right) \\ &= (aS)\, a \cup (aS)\, a^2 \cup (aS)\, \{(aS)a\} \\ &\subseteq (aS)\, a \cup (aS)\, a^2 \cup (aS)\, (Sa) \\ &\subseteq (aS)\, a \subseteq \left(a \cup a^2 \cup (aS)\, a\right).\end{aligned}$$

Thus $a \cup a^2 \cup (aS)\, a$ is a bi-ideal. ■

Definition 71 *(1) Let f and g be fuzzy subsets of an AG-groupoid S. We define the fuzzy subsets f_k, $f \wedge_k g$ and $f \circ_k g$ of S as follows,*
(i) $f_k(a) = f(a) \wedge \frac{1-k}{2}$.
(ii) $(f \wedge_k g)(a) = (f \wedge g)(a) \wedge \frac{1-k}{2}$.
(iii) $(f \circ_k g)(a) = (f \circ g)(a) \wedge \frac{1-k}{2}$, for all $a \in S$.
(2) Let A be any subset of an AG-groupoid S, then the characteristic function $(C_A)_k$ is defined as,

$$(C_A)_k(a) = \begin{cases} \geq \frac{1-k}{2} & \text{if } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 72 *[22]The following properties hold in an AG-groupoid S.*

(i) A is an AG-subgroupoid of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy AG-subgroupoid of S.

(ii) A is a left (right, two-sided) ideal of S if and only if $(C_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideal of S.

(iii) A is bi-ideal (quasi-ideal) of an AG-groupoid S if and only if $(C_A)_k$ is $(\in, \in \vee q_k)$-fuzzy bi-ideal (quasi-ideal).

(iv) For any non-empty subsets A and B of S, $C_A \circ_k C_B = (C_{AB})_k$ and $C_A \wedge_k C_B = (C_{A\cap B})_k$.

2.1 Characterizations of Regular AG-groupoids with Left Identity

Theorem 73 *In AG-groupoid S, with left identity the following are equivalent.*

(i) S is regular.

(ii) $I \cap B \subseteq IB$, where I is an ideal and B is a bi-ideal.

(iii) $I[a] \cap B[a] \subseteq I[a]B[a]$, for all a in S.

Proof. $(i) \Rightarrow (ii)$

Let I and B be a two-sided ideal and a bi-ideal of a regular AG-groupoid S, respectively. Let $a \in I \cap B$, this implies that $a \in I$ and $a \in B$. Since S is regular so for $a \in S$ there exist $x \in S$ such that $a = (ax)\, a \in (IS)\, B \subseteq IB$. Thus $I \cap B \subseteq IB$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$

Since $I[a] = a \cup Sa \cup aS$ and $B[a] = a \cup a^2 \cup (aS)\, a$ are two sided ideal and bi-ideal of S generated by a. Thus using lemmas 68, 69, (1) and medial law we get

$$\begin{aligned}
(a \cup Sa \cup aS) \cap \left(a \cup a^2 \cup (aS)\, a\right) &\subseteq (a \cup Sa \cup aS)\left(a \cup a^2 \cup (aS)\, a\right) \\
&= a^2 \cup aa^2 \cup a\left((aS)\, a\right) \cup (Sa)\, a \cup (Sa)\, a^2 \\
&\cup (Sa)\left((aS)\, a\right) \cup (aS)\, a \cup (aS)\, a^2 \cup (aS)\left((aS)\, a\right) \\
&\subseteq a^2 \cup aa^2 \cup (aS)\, a^2 \cup Sa^2 \cup Sa^2 \cup a^2 S \\
&\cup (aS)\, a \cup (aS)\, a \cup (aS)\, a \cup (aS)\, a \\
&\subseteq a^2 \cup Sa^2 \cup (aS)\, a.
\end{aligned}$$

Hence by lemma 65 and (2), S is regular. ■

Corollary 74 *In AG-groupoid S, with left identity the following are equivalent.*

(i) S is regular.

(ii) $I \cap L \subseteq IL$, where I is an ideal and L is left ideal of S.

(iii) $I[a] \cap L[a] \subseteq I[a]L[a]$*, for all* a *in* S.

Corollary 75 *For an AG-groupoid* S *with left identity, the following are equivalent.*
(i) S *is regular.*
(ii) $I = I^2$*, for every ideal* I *of* S.
(iii) $I[a] = I[a]I[a]$*, for all* a *in* S.

Theorem 76 *In AG-groupoid* S*, with left identity the following are equivalent.*
(i) S *is regular.*
(ii) $f \wedge_k g \leq f \circ_k g$ *where* f *is an* $(\in, \in \vee q_k)$*-fuzzy ideal and* g *is an* $(\in, \in \vee q_k)$*-fuzzy bi-ideal of* S.

Proof. $(i) \Rightarrow (ii)$
Let f be an $(\in, \in \vee q_k)$-fuzzy ideal and g be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of a regular AG-groupoid S, respectively. Since S is regular so for $a \in S$ there exist $x \in S$ such that $a = (ax)\,a$. Thus we have,

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{a=pq} f(p) \wedge g(q) \wedge \frac{1-k}{2} \geq f(ax) \wedge g(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Thus $f \wedge_k g \leq f \circ_k g$.
$(ii) \Rightarrow (i)$
Since $I[a] = a \cup Sa \cup aS$ and $B[a] = a \cup a^2 \cup (aS)\,a$ are two sided ideal and bi-ideal of S generated by a. Therefore by lemma 72, $(C_{I[a]})_k$ and $(C_{B[a]})_k$ are $(\in, \in \vee q_k)$-fuzzy two-sided and $(\in, \in \vee q_k)$-fuzzy bi-ideals of S. Thus by (ii) and lemma 72, we have,

$$(C_{I[a] \cap B[a]})_k = (C_{I[a]})_k \wedge_k (C_{B[a]})_k \leq (C_{I[a]})_k \circ_k (C_{B[a]})_k = (C_{I[a]B[a]})_k$$

Thus $I[a] \cap B[a] \subseteq I[a]B[a]$. Hence by theorem 73, S is regular. ■

Corollary 77 *For an AG-groupoid* S *with left identity, the following are equivalent.*
(i) S *is regular.*
(ii) $f \circ_k f \geq f_k$*, for an* $(\in, \in \vee q_k)$*-fuzzy ideal* f *of* S.

Corollary 78 *In AG-groupoid* S*, with left identity the following are equivalent.*
(i) S *is regular.*
(ii) $f \wedge_k g \leq f \circ_k g$*, where* f *is an* $(\in, \in \vee q_k)$*-fuzzy ideal and* g *is an* $(\in, \in \vee q_k)$*-fuzzy left ideal of* S.

2.2 Characterizations of Intra-regular AG-groupoids with Left Identity

Example 79 *Let $S = \{a, b, c, d, e\}$, and the binary operation "$*$" be defined on S as follows:*

$*$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	1	1	1
3	1	1	4	5	6	3
4	1	1	3	4	5	6
5	1	1	6	3	4	5
6	1	1	5	6	3	4

By AG-test in [31] it is easy to verify that $(S, *)$ is an AG-groupoid. Also $1 = (1*1^2)*1$, $2 = (2*2^2)*2$, $3 = (4*3^2)*3$, $4 = (4*4^2)*4$, $5 = (3*5^2)*6$ and $6 = (3*6^2)*3$. Therefore $(S, *)$ is an intra-regular AG-groupoid. Define a fuzzy subset $f : S \longrightarrow [0, 1]$ as follows:

$$f(x) = \begin{cases} 0.9 \text{ for } x = 1 \\ 0.8 \text{ for } x = 2 \\ 0.7 \text{ for } x = 3 \\ 0.6 \text{ for } x = 4 \\ 0.5 \text{ for } x = 5 \\ 0.5 \text{ for } x = 6 \end{cases}$$

Then clearly f is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.

Lemma 80 *If I is an ideal of an intra-regular AG-groupoid S with left identity, then $I = I^2$.*

Proof. It is easy. ■

Theorem 81 *[21, 22] (1) For an intra-regular AG-groupoid S with left identity the following statements are equivalent.*
(i) A is a left ideal of S.
(ii) A is a right ideal of S.
(iii) A is an ideal of S.
(iv) A is a bi-ideal of S.
(v) A is a generalized bi-ideal of S.
(vi) A is an interior ideal of S.
(vii) A is a quasi-ideal of S.
(viii) $AS = A$ and $SA = A$.
(2) In intra-regular AG-groupoid S with left identity the following are equivalent.
(i) A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy right ideal
(ii) A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy left ideal
(iii) A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal
(iv) A fuzzy subset f of S is an $(\in, \in \vee q_k)$-fuzzy interior ideal

(*v*) *A fuzzy subset* f *of* S *is an* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal.*

Theorem 82 *For an AG-groupoid with left identity e, the following are equivalent.*
(*i*) S *is intra-regular.*
(*ii*) $Q_1 \cap Q_2 = Q_1Q_2$, *for all quasi-ideals* Q_1 *and* Q_2.
(*iii*) $Q[a] \cap Q[a] = Q[a]Q[a]$, *for all* a *in* S.

Proof. $(i) \Longrightarrow (ii)$
Let Q_1 and Q_2 be the quasi-ideals of an intra-regular AG-groupoid S. Therefore for each a in S there exists x, y in S such that $a = (xa^2)y$. Now by theorem 81, Q_1 and Q_2 become ideals of S. Now let $a \in Q_1 \cap Q_2$ this implies that $a \in Q_1$ and $a \in Q_2$. Therefore using (1) and left invertive law we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \in (S(SQ_1))Q_2 \subseteq Q_1Q_2.$$

Now by lemma 61, $Q_1 \cap Q_2$ is an ideal and $(Q_1 \cap Q_2)^2 \subseteq Q_1Q_2$. Using lemma 80, we get $(Q_1 \cap Q_2) \subseteq Q_1Q_2$. Hence $Q_1 \cap Q_2 = Q_1Q_2$.
$(ii) \Longrightarrow (iii)$ is obvious.
$(iii) \Longrightarrow (i)$
For a in S, $Q[a] = a \cup (Sa \cap aS)$ is a quasi-ideal of S generated by a. Therefore using (1), left invertive law, medial law and (iii), we get

$$\begin{aligned}
[a \cup (Sa \cap aS)] \cap [a \cup (Sa \cap aS)] &= [a \cup (Sa \cap aS)][a \cup (Sa \cap aS)] \\
&\subseteq (a \cup Sa)(a \cup Sa) \\
&= a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\
&= a^2 \cup Sa^2.
\end{aligned}$$

Therefore $a = a^2$ or $a \in Sa^2 = (Sa^2)S$. Hence S is intra-regular. ■

Theorem 83 *For an AG-groupoid with left identity e, the following are equivalent.*
(*i*) S *is intra-regular.*
(*ii*) $f \wedge_k g = f \circ_k g$, *for all* $(\in, \in \vee q_k)$*-fuzzy quasi-ideals* f *and* g.

Proof. $(i) \Longrightarrow (ii)$
Let f and g be $(\in, \in \vee q_k)$-fuzzy quasi-ideals of an intra-regular AG-groupoid S with left identity. Then by theorem 81, f and g become $(\in, \in \vee q_k)$-fuzzy ideals of S. For each a in S there exists x, y in S such that $a = (xa^2)y$ and since $S = S^2$, so for y in S there exists u, v in S such that $y = uv$. Now using paramedial law, medial law and (1), we get

$$a = (xa^2)y = (xa^2)(uv) = (vu)(a^2x) = a^2((vu)x) = (a(vu))(ax).$$

Then

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=pq} \left\{ f(p) \wedge g(q) \wedge \frac{1-k}{2} \right\} \\ &\geq \left\{ f(a(vu)) \wedge g(ax) \wedge \frac{1-k}{2} \right\} \\ &\geq f(a) \wedge g(a) \wedge \frac{1-k}{2} = f \wedge_k g(a).\end{aligned}$$

Therefore $f \circ_k g \geq f \wedge_k g$. Also

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=bc} f(b) \wedge g(c) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=bc} (f(b) \wedge \frac{1-k}{2}) \wedge (g(c) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\ &\leq \bigvee_{a=bc} f(bc) \wedge g(bc) \wedge \frac{1-k}{2} = f \wedge_k g(a).\end{aligned}$$

Therefore $f \circ_k g \leq f \wedge_k g$. Hence $(f \circ_k g)(a) = (f \wedge_k g)(a)$.
$(ii) \Longrightarrow (i)$
Assume that Q_1 and Q_2 are quasi-ideals of an AG-groupoid S with left identity and let $a \in Q_1 \cap Q_2$. Then by lemma 72, $(C_{Q_1})_k$ and $(C_{Q_2})_k$ are $(\in, \in \vee q_k)$-fuzzy quasi-ideals of S. Thus by (ii) and lemma 72, we get

$$\begin{aligned}(C_{Q_1 Q_2})_k(a) &= (C_{Q_1} \circ_k C_{Q_2})(a) = (C_{Q_1} \wedge_k C_{Q_2})(a) \\ &= (C_{Q_1 \cap Q_2})_k(a) \geq \frac{1-k}{2}.\end{aligned}$$

Therefore $a \in Q_1 Q_2$. Now let $a \in Q_1 Q_2$, then

$$(C_{Q_1 \cap Q_2})_k(a) = (C_{Q_1 Q_2})_k(a) \geq \frac{1-k}{2}.$$

Therefore $a \in Q_1 \cap Q_2$. Thus $Q_1 Q_2 = Q_1 \cap Q_2$. Hence by theorem 82, S is intra-regular. ■

Theorem 84 *For an AG-groupoid with left identity e, the following are equivalent.*
(i) *S is intra-regular.*
(ii) *$Q \cap L = QL$ $(Q \cap L \subseteq QL)$, for every quasi-ideal Q and left ideal L.*
(iii) *$Q[a] \cap L[a] = Q[a]L[a]$ $(Q[a] \cap L[a] \subseteq Q[a]L[a])$, for all a in S.*

Proof. $(i) \Longrightarrow (ii)$ is same as $(i) \Longrightarrow (ii)$ of theorem 82.
$(ii) \Longrightarrow (iii)$ is obvious.
$(iii) \Longrightarrow (i)$

For a in S, $Q[a] = a \cup (Sa \cap aS)$ $L[a] = a \cup Sa$ are quasi and left ideals of S generated by a. Therefore using (1), left invertive law, medial law and (iii), we get

$$\begin{aligned}
[a \cup (Sa \cap aS)] \cap [a \cup Sa] &= [a \cup (Sa \cap aS)][a \cup Sa] \\
&\subseteq (a \cup Sa)(a \cup Sa) \\
&= a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\
&= a^2 \cup Sa^2.
\end{aligned}$$

Hence S is intra-regular. ∎

Theorem 85 *For an AG-groupoid with left identity e, the following are equivalent.*

(i) S is intra-regular.

(ii) $f \wedge_k g = f \circ_k g$ $(f \wedge_k g \leq f \circ_k g)$, where f is any $(\in, \in \vee q_k)$ fuzzy quasi-ideal, g is any $(\in, \in \vee q_k)$ fuzzy left ideal.

(iii) $f \wedge_k g = f \circ_k g$ $(f \wedge_k g \leq f \circ_k g)$, where f and g are any $(\in, \in \vee q_k)$ fuzzy quasi-ideals.

Proof. $(i) \Longrightarrow (iii)$

Let f and g be $(\in, \in \vee q_k)$-fuzzy quasi-ideals of an intra-regular AG-groupoid S with left identity. Then by theorem 81, f and g become $(\in, \in \vee q_k)$-fuzzy ideals of S. Since S is intra-regular so for each a in S there exists x, y in S such that $a = (xa^2)y$. Now since $a = (a(vu))(ax)$. Therefore

$$\begin{aligned}
(f \circ_k g)(a) &= \bigvee_{a=pq} \left\{ f(p) \wedge g(q) \wedge \frac{1-k}{2} \right\} \\
&\geq f(a(vu)) \wedge g(ax) \wedge \frac{1-k}{2} \\
&\geq f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).
\end{aligned}$$

Thus $f \circ_k g \geq f \wedge_k g$. Also

$$\begin{aligned}
(f \circ_k g)(a) &= \bigvee_{a=bc} f(b) \wedge g(c) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=bc} (f(b) \wedge \frac{1-k}{2}) \wedge (g(c) \wedge \frac{1-k}{2}) \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=bc} f(bc) \wedge g(bc) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).
\end{aligned}$$

Therefore $f \circ_k g \leq f \wedge_k g$. Hence $(f \circ_k g)(a) = (f \wedge_k g)(a)$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

Let $a \in Q \cap L$, then by (ii) and lemma 72, we get

$$\begin{aligned}(C_{QL})_k(a) &= (C_Q \circ_k C_L)(a) = (C_Q \wedge_k C_L)(a) \\ &= (C_{Q\cap L})_k(a) \geq \frac{1-k}{2}.\end{aligned}$$

Therefore $a \in QL$. Now let $a \in QL$, then

$$(C_{Q\cap L})_k(a) = (C_{QL})_k(a) \geq \frac{1-k}{2}.$$

Therefore $a \in Q \cap L$. Thus $QL = Q \cap L$. Hence by theorem 84, S is intra-regular. ■

Theorem 86 *For an AG-groupoid with left identity, the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) *$f \wedge_k g \leq f \circ_k g$, for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g.*
(iii) *$f \wedge_k g \leq f \circ_k g$, for every $(\in, \in \vee q_k)$-fuzzy quasi-ideals f and g.*

Proof. $(i) \Longrightarrow (iii)$
Let f and g be $(\in, \in \vee q_k)$ fuzzy quasi-ideals of an intra-regular AG-groupoid with left identity S. Then by theorem 81, f and g become $(\in, \in \vee q_k)$-fuzzy ideals of S. Now since S is intra-regular. Therefore for $a \in S$ there exists x, y in S such that $a = (xa^2)y$ which yields that $a = (a(vu)(ax)$. Then

$$\begin{aligned}(f \circ_k g)(a) &= \bigvee_{a=pq} \left\{ f(p) \wedge g(q) \wedge \frac{1-k}{2} \right\} \\ &\geq f(a(vu)) \wedge g(ax) \wedge \frac{1-k}{2} = f(a) \wedge g(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Thus $f \circ_k g \geq f \wedge_k g$.
$(iii) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$
Let A and B be quasi and left ideals of S. Then by lemma 72, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy quasi and $(\in, \in \vee q_k)$-fuzzy left ideals of S and by (ii), we get

$$(C_{A\cap B})_k = C_A \wedge_k C_B \leq C_A \circ_k C_B = (C_{AB})_k.$$

Now let $a \in A \cap B$. Then $(C_{AB})_k(a) \geq (C_{A\cap B})_k(a) \geq \frac{1-k}{2}$. Therefore $a \in AB$. Thus $A \cap B \subseteq AB$. Hence by theorem 84, S is intra-regular. ■

Theorem 87 *For an AG-groupoid with left identity, the following conditions are equivalent.*
(i) *S is intra-regular.*
(ii) $Q[a] \cap L[a] \subseteq L[a]Q[a]$*, for all a in S.*
(iii) $Q \cap L \subseteq LQ$*, for every quasi-ideal Q and left ideal L of S.*
(iv) $f \wedge_k g \leq g \circ_k f$*, for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal f and every $(\in, \in \vee q_k)$-fuzzy left ideal g.*
(v) $f \wedge_k g \leq g \circ_k f$*, for every $(\in, \in \vee q_k)$-fuzzy quasi-ideals f and g.*

Proof. $(i) \Longrightarrow (v)$
Let f and g be $(\in, \in \vee q_k)$ fuzzy quasi-ideals of an intra-regular AG-groupoid S with left identity. Then by theorem 81, f and g become $(\in, \in \vee q_k)$-fuzzy ideals of S. Since S is intra-regular. Therefore for $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now using (1) and left invertive law, we get

$$a = (xa^2)y = (a(xa))y = (y(xa))a.$$

Also

$$\begin{aligned}(g \circ_k f)(a) &= \bigvee_{a=pq} \left\{ g(p) \wedge f(q) \wedge \frac{1-k}{2} \right\} \\ &\geq g(y(xa)) \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).\end{aligned}$$

Thus $g \circ_k f \geq f \wedge_k g$.
$(v) \Longrightarrow (iv)$ is obvious.
$(iv) \Longrightarrow (iii)$
Let Q and L be quasi and left ideals of S. Then by lemma 72, $(C_A)_k$ and $(C_B)_k$ are $(\in, \in \vee q_k)$-fuzzy quasi and $(\in, \in \vee q_k)$-fuzzy left ideals of S and by (ii), we get

$$(C_{Q \cap L})_k = C_Q \wedge_k C_L \leq C_L \circ_k C_Q = (C_{LQ})_k.$$

Therefore $Q \cap L \subseteq LQ$.
$(iii) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (i)$
For a in S, $Q[a] = a \cup (Sa \cap aS)$ $L[a] = a \cup Sa$ are quasi and left ideals of S generated by a. Therefore using (1), left invertive law, medial law and (iii), we get

$$\begin{aligned}[a \cup (Sa \cap aS)] \cap [a \cup Sa] &\subseteq [a \cup Sa][a \cup (Sa \cap aS)] \\ &\subseteq (a \cup Sa)(a \cup Sa) \\ &= a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa) \\ &= a^2 \cup Sa^2.\end{aligned}$$

Hence S is intra-regular. ■

Theorem 88 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) S is intra-regular.

(ii) $L[a] \cap Q[a] = (L[a]Q[a])L[a](L[a] \cap Q[a] \subseteq (L[a]Q[a])L[a])$, for all a in S.

(iii) $L \cap Q = (LQ)L(L \cap Q \subseteq (LQ)L)$, for left ideal L and quasi-ideal Q of S.

(iv) $f \wedge_k g = (f \circ_k g) \circ_k f(f \wedge_k g \leq (f \circ_k g) \circ_k f)$, for $(\in, \in \vee q_k)$-fuzzy left ideal f and $(\in, \in \vee q_k)$-fuzzy quasi-ideal g of S.

(v) $f \wedge_k g = (f \circ_k g) \circ_k f(f \wedge_k g \leq (f \circ_k g) \circ_k f)$, for $(\in, \in \vee q_k)$-fuzzy quasi-ideals f and g of S.

Proof. $(i) \Longrightarrow (v)$

Let f and g be $(\in, \in \vee q_k)$-fuzzy quasi-ideals of an intra-regular AG-groupoid S with left identity. Since S is intra-regular therefore for each $a \in S$ there exist $x, y \in S$ such that $a = \left(xa^2\right) y$. Then by theorem 81, f and g become $(\in, \in \vee q_k)$-fuzzy ideals of S. Then using (1), left invertive law, paramedial and medial law, we get

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y(xa))((y(xa))a) = (a(y(xa)))((xa)y). \end{aligned}$$

Now

$$\begin{aligned} ((f \circ_k g) \circ_k f)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge f(q) \wedge \frac{1-k}{2} \\ &\geq f \circ_k g(a(y(xa))) \wedge f((xa)y) \wedge \frac{1-k}{2} \\ &\geq (f \circ_k g)(a(y(xa)) \wedge f(a) \wedge \frac{1-k}{2} \\ &= \left\{ \bigvee_{a(y(xa))=uv} f(u) \wedge g(v) \right\} \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq f(a) \wedge g(y(xa))) \wedge \frac{1-k}{2} \wedge f(a) \wedge \frac{1-k}{2} \\ &\geq \left\{ f(a) \wedge g\,(a) \wedge \frac{1-k}{2} \right\} \wedge f(a) \wedge \frac{1-k}{2} \\ &= f(a) \wedge g\,(a) \wedge \frac{1-k}{2} = (f \wedge_k g)\,(a)\,. \end{aligned}$$

Therefore $f \wedge_k g \leq (f \circ_k g) \circ_k f$. Also

$$\begin{aligned}
((f \circ_k g) \circ_k f)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \{ \bigvee_{p=cd} f(c) \wedge g(d) \wedge \frac{1-k}{2} \} \wedge f(q) \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pq} \{ \bigvee_{p=cd} f(cd) \wedge g(cd) \wedge \frac{1-k}{2} \} \wedge f(pq) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \{ f(p) \wedge g(p) \wedge \frac{1-k}{2} \} \wedge f(pq) \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pq} f(pq) \wedge g(pq) \wedge f(pq) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge f(a) \wedge \frac{1-k}{2} \\
&= f(a) \wedge g(a) \wedge \frac{1-k}{2} = (f \wedge_k g)(a).
\end{aligned}$$

Therefore $f \wedge_k g \geq (f \circ_k g) \circ_k f$. Hence $f \wedge_k g = (f \circ_k g) \circ_k f$.

$(v) \Longrightarrow (iv)$ is obvious.

$(iv) \Longrightarrow (iii)$

Let L and Q be left and quasi-ideals of an AG-groupoid S. Then by lemma 72, $(C_L)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy left and quasi-ideals of S. Then using (iii), we have

$$(C_{L\cap Q})_k = (C_L \wedge_k C_Q) = (C_L \circ_k C_Q) \circ_k C_L = (C_{(LQ)L})_k.$$

This implies that $L \cap Q = (LQ)L$.

$(iii) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$

For a in S, $L[a] = a \cup Sa$ and $Q[a] = a \cup (Sa \cap aS)$ are left and quasi-ideals of S generated by a. Therefore using (1), medial law, left invertive law and (ii), we get

$$\begin{aligned}
[a \cup Sa] \cap [a \cup (Sa \cap aS)] &= ([a \cup Sa][a \cup (Sa \cap aS)])[a \cup Sa] \\
&\subseteq \{(a \cup Sa)(a \cup Sa)\}(a \cup Sa) \\
&\subseteq \{(a \cup Sa)(a \cup Sa)\}S \\
&= \{a^2 \cup a(Sa) \cup (Sa)a \cup (Sa)(Sa)\}S \\
&= \{a^2 \cup Sa^2\}S = Sa^2.
\end{aligned}$$

■

Theorem 89 *Let S be an AG-groupoid with left identity then the following conditions are equivalent.*

(i) *S is intra-regular.*
(ii) *$B \cap Q \subseteq BQ$, for every bi-ideal B and quasi-ideal Q.*
(iii) *$f \wedge_k g \leq f \circ_k g$, for every $(\in, \in \vee q_k)$-fuzzy bi-ideal f and $(\in, \in \vee q_k)$-fuzzy quasi-ideal g.*

Proof. *(i)* $\Longrightarrow$ *(iii)* Assume that S is an intra-regular AG-groupoid with left identity and f and g are $(\in, \in \vee q_k)$ fuzzy bi and quasi-ideals of S respectively. Thus, for any a in S there exist u and v in S such that $a = uv$, then

$$(f \circ_k g)(a) = \bigvee_{a=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2}.$$

Since S is intra-regular so for any a in S there exist $x, y \in S$ such that $a = (xa^2)y$. Since $S = S^2$, so for y in S there exist s and t in S such that $y = st$. By using (1), left invertive law, paramedial law and medial law, we get

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
&= [(st)(xa)]a = [(ax)(ts)]a = [\{(ts)x\}a]a = [\{(ts)x\}((xa^2)y)]a \\
&= [(xa^2)(\{(ts)x\}y)]a = [\{a(xa)\}(\{(ts)x\}y)]a \\
&= [\{(\{(ts)x\}y)(xa)\}a]a \\
&= [\{p(xa)\}a]a, \text{ where } p = ((ts)x)y
\end{aligned}$$

Now

$$\begin{aligned}
p(xa) &= p[x\{(xa^2)y\}] = p[(xa^2)(xy)] \\
&= (xa^2)[p(xy)] = [(xy)p](a^2x) \\
&= a^2([(xy)p]x) = a^2q, \text{ where } q = [(xy)p]x
\end{aligned}$$

Therefore $a = \{(a^2q)a\}a$, where $q = [(xy)p]x$ and $p = ((ts)x)y$.
Thus, we have

$$\begin{aligned}
(f \circ_k g)(a) &= \bigvee_{a=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \\
&\geq f((a^2q)a) \wedge g(a) \wedge \frac{1-k}{2} \\
&\geq (f(a^2) \wedge f(a)) \wedge g(a) \wedge \frac{1-k}{2} \\
&\geq (f(a) \wedge f(a)) \wedge g(a) \wedge \frac{1-k}{2} \\
&= (f \wedge g)(a) \wedge \frac{1-k}{2}. \\
&= (f \wedge_k g)(a).
\end{aligned}$$

This implies that $f \wedge_k g \leq f \circ_k g$.

$(iii) \Longrightarrow (ii)$ Let B be a bi-ideal and Q be a quasi ideal of S. Then, by lemma 72, $(C_B)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal and $(\in, \in \vee q_k)$-fuzzy quasi ideal of S. Then, by using lemma 72 and (ii), we get

$$(C_{B\cap Q})_k = C_B \wedge_k C_Q \leq C_B \circ_k C_Q = (C_{BQ})_k.$$

Thus $B \cap Q \subseteq BQ$.

$(ii) \Longrightarrow (i)$ Since Sa is both bi-ideal and quasi ideal of S containing a. Therefore by (ii) and using medial law, paramedial law and (1), we obtain

$$\begin{aligned} a \in Sa \cap Sa &\subseteq (Sa)(Sa) = (SS)(aa) = (aa)(SS) \\ &= S(a^2S) = (SS)(a^2S) = (Sa^2)(SS) = (Sa^2)S. \end{aligned}$$

Hence S is an intra-regular AG-groupoid. ■

3

Generalized Fuzzy Right Ideals in AG-groupoids

In this chapter, we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals in an AG-groupoid. We characterize intra-regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals.

3.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals of AG-groupoids

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, let $X^\delta_{\gamma B}$ be a fuzzy subset of X such that $X^\delta_{\gamma B}(x) \geq \delta$ for all $x \in B$ and $X^\delta_{\gamma B}(x) \leq \gamma$ otherwise. Clearly, $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that
(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now we introduce a new relation on $\mathcal{F}(X)$, denoted by "$\subseteq \vee q_{(\gamma,\delta)}$", as follows:

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover f and g are said to be (γ, δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

The above definitions can be found in [37].

Lemma 90 *[37] Let f and g be fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{g(x), \gamma\} \geq \min\{f(x), \delta\}$ *for all $x \in X$.*

Lemma 91 *[37] Let f, g and $h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

The relation "$=_{(\gamma,\delta)}$" is equivalence relation on $\mathcal{F}(X)$, see [37]. Moreover, $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 92 *Let A, B be any non-empty subsets of an AG -groupoid S with a left identity. Then we have*
(1) $A \subseteq B$ if and only if $X^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma B}$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0,1]$.
(2) $X^\delta_{\gamma A} \cap X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(A \cap B)}$.
(3) $X^\delta_{\gamma A} \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(AB)}$.

Lemma 93 *If S is an AG-groupoid with a left identity then $(ab)^2 = a^2b^2 = b^2a^2$ for all a and b in S.*

Proof. It follows by medial and paramedial laws. ∎

Definition 94 *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ we have $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.*

Theorem 95 *Let f be a fuzzy subset of an AG groupoid S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG subgroupoid of S if and only if $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, where $\gamma, \delta \in [0, 1]$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroupoid of S. Assume that there exists $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t$, this implies that $f(xy) < t \leq \gamma$, which further implies that $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t$, therefore $\min\{f(x), f(y)\} \geq t$ this implies that $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, implies that $x_t \in_\gamma f$, $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ by definition we write $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, then $\max\{f(xy), \delta\} \geq \min\{f(x), f(y), \delta\}$ this implies that $f(xy) \geq \min\{t, s, \delta\}$. Here arises two cases,

Case(a): If $\{t, s\} \leq \delta$ then $f(xy) \geq \min\{t, s\} > \gamma$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma f$.

Case(b): If $\{t, s\} > \delta$ then $f(xy) + \min\{t, s\} > 2\delta$ this implies that $(xy)_{\min\{t,s\}} q_\delta f$.

From both cases we write $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ∎

Definition 96 *A fuzzy subset f of an AG-groupoid S with a left identity is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_t \in_\gamma f$ we have $(xy)_t \in_\gamma \vee q_\delta f$ (resp $x_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$).*

Theorem 97 *A fuzzy subset f of an AG-groupoid S with a left identity is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S if and only if for all $a, b \in S$,*

$$\max\{f(ab), \gamma\} \geq \min\{f(a), \delta\}.$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Suppose that there are $a, b \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(ab), \gamma\} < t \leq \min\{f(a), \delta\}.$$

Then $\max\{f(ab),\gamma\} < t \leq \gamma$ implies that $(ab)_t \bar{\in}_\gamma f$ which further implies that $(ab)_t \overline{\in_\gamma \vee q_\delta} f$. From $\min\{f(a),\delta\} \geq t > \gamma$ it follows that $f(a) \geq t > \gamma$, which implies that $a_t \in_\gamma f$. But $(ab)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(ab),\gamma\} \geq \min\{f(a),\delta\}.$$

Conversely, assume that there exist $a, b \in S$ and $t, s \in (\gamma, 1]$ such that $a_s \in_\gamma f$ but $(ab)_t \overline{\in_\gamma \vee q_\delta} f$. Then $f(a) \geq t > \gamma$, $f(ab) < \min\{f(a),\delta\}$ and $f(ab) + t \leq 2\delta$. It follows that $f(ab) < \delta$ and so $\max\{f(ab),\gamma\} < \min\{f(a),\delta\}$, which is a contradiction. Hence $a_t \in_\gamma f$ which implies that $(ab)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ (respectively $a_t \in_\gamma f$ implies that $(ab)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$) for all a, b in S. ■

Example 98 *Consider the AG-groupoid defined by the following multiplication table on $S = \{1, 2, 3\}$:*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.4 \text{ if } x = 1, \\ 0.41 \text{ if } x = 2, \\ 0.38 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.22})$-fuzzy right ideal,*
- *f is not an $(\in, \in \vee q_{0.22})$-fuzzy right ideal, because $f(2 \cdot 3) < \min\{f(2), \frac{1-0.22}{2} = 0.39\}$.*
- *f is not a fuzzy right ideal because $f(2 \cdot 3) < f(2)$.*

Example 99 *Let $S = \{1, 2, 3\}$ and the binary operation $\circ$ be defined on S as follows:*

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.44 \text{ if } x = 1, \\ 0.6 \text{ if } x = 2, \\ 0.7 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy subgroupoid of S.*
- *f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy right ideal of S.*

Example 100 *Let $S = \{1, 2, 3\}$ and define the binary operation $\cdot$ on S as follows:*

$\cdot$	1	2	3
1	1	1	1
2	3	3	3
3	1	1	1

$(S, \cdot)$ is an AG-groupoid. Let us define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.6 \text{ if } x = 1 \\ 0.5 \text{ if } x = 2 \\ 0.55 \text{ if } x = 3 \end{cases}$$

f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S.

Lemma 101 *R is a right ideal of an AG-groupoid S if and only if $X^\delta_{\gamma R}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S.*

Proof. (i) Let $x, y \in R$, it means that $xy \in R$. Then $X^\delta_{\gamma R}(xy) \geq \delta$, $X^\delta_{\gamma R}(x) \geq \delta$ and $X^\delta_{\gamma R}(y) \geq \delta$ but $\delta > \gamma$. Thus

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = X^\delta_{\gamma R}(xy) \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma R}(xy), \gamma\} \geq \min\{X^\delta_{\gamma R}(x), \delta\}$.

(ii) Let $x \notin R$, $y \in R$

Case(a): If $xy \notin R$. Then $X^\delta_{\gamma R}(x) \leq \gamma$, $X^\delta_{\gamma R}(y) \geq \delta$ and $X^\delta_{\gamma R}(xy) \leq \gamma$. Therefore

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = X^\delta_{\gamma R}(x).$$

Hence $\max\{X^\delta_{\gamma R}(xy), \gamma\} \geq \min\{X^\delta_{\gamma R}(x), \delta\}$.

Case(b): If $xy \in R$. Then $X^\delta_{\gamma R}(xy) \geq \delta$, $X^\delta_{\gamma R}(x) \leq \gamma$ and $X^\delta_{\gamma R}(y) \geq \delta$. Thus

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = X^\delta_{\gamma R}(xy) \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = X^\delta_{\gamma R}(x).$$

Hence $\max\{X^\delta_{\gamma R}(xy), \gamma\} > \min\{X^\delta_{\gamma R}(x), \delta\}$.

(iii) Let $x \in R, y \notin R$. Then $xy \in R$. Thus $X^\delta_{\gamma R}(xy) \geq \delta$, $X^\delta_{\gamma R}(y) \leq \gamma$ and $X^\delta_{\gamma R}(x) \geq \delta$. Thus

$$\max\{X^\delta_{\gamma R}(xy), \gamma\} = X^\delta_{\gamma R}(xy) \text{ and } \min\{X^\delta_{\gamma R}(x), \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma R}(xy), \gamma\} \geq \min\{X^\delta_{\gamma R}(x), \delta\}$.

(iv) Let $x, y \notin R$, then

Case (a) Assume that $xy \notin R$. Then by definition we get $X^{\delta}_{\gamma R}(xy) \leq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = \gamma \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Therefore $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

Case (b) Assume that $xy \in R$. Then by definition we get $X^{\delta}_{\gamma R}(xy) \geq \gamma$, $X^{\delta}_{\gamma R}(y) \leq \gamma$ and $X^{\delta}_{\gamma R}(x) \leq \gamma$. Thus

$$\max\{X^{\delta}_{\gamma R}(xy), \gamma\} = X^{\delta}_{\gamma R}(xy) \text{ and } \min\{X^{\delta}_{\gamma R}(x), \delta\} = X^{\delta}_{\gamma R}(x).$$

Therefore $\max\{X^{\delta}_{\gamma R}(xy), \gamma\} > \min\{X^{\delta}_{\gamma R}(x), \delta\}$.

Conversely, let $rs \in RS$, where $r \in R$ and $s \in S$. By hypothesis $\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \min\{X^{\delta}_{\gamma R}(r), \delta\}$. Since $r \in R$, thus $X^{\delta}_{\gamma R}(r) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma R}(r), \delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma R}(rs), \gamma\} \geq \delta.$$

This implies that $X^{\delta}_{\gamma R}(rs) \geq \delta$ which implies that $rs \in R$. Hence R is a right ideal of S. ■

Here we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

Definition 102 *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if $x^2_t \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta f$ for all $x \in S$ and $t \in (\gamma, 1]$.*

Example 103 *Consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with the following multiplication table*

.	1	2	3	4	5
1	4	5	1	2	3
2	3	4	5	1	2
3	2	3	4	5	1
4	1	2	3	4	5
5	5	1	2	3	4

Clearly $(S, .)$ is intra-regular because $1 = (3.1^2).2$, $2 = (1.2^2).5$, $3 = (5.3^2).2$, $4 = (2.4^2).1$, $5 = (3.5^2).1$. *Define a fuzzy subset f on S as given:*

$$f(x) = \begin{cases} 0.7 & if\ x = 1, \\ 0.6 & if\ x = 2, \\ 0.68 & if\ x = 3, \\ 0.63 & if\ x = 4, \\ 0.52 & if\ x = 5. \end{cases}$$

Then f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.5})$-fuzzy semiprime of S.

Theorem 104 *A fuzzy subset f of an AG-groupoid S is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if and only if* $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.

Proof. Let f be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Assume that there exists $a \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(a), \gamma\} < t \leq \min\{f(a^2), \delta\}.$$

Then $\max\{f(a), \gamma\} < t$. This implies that $f(a) < t > \gamma$. Now since $\delta \geq t$, so $f(a) + t < 2\delta$. Thus $a_t \overline{\in_\gamma \vee q_\delta} f$. Also since $\min\{f(a^2), \delta\} \geq t$, so $f(a^2) \geq t > \gamma$. This implies that $a_t^2 \in_\gamma f$. Thus by definition of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime $a_t \in_\gamma \vee q_\delta f$ which is a contradiction to $a_t \overline{\in_\gamma \vee q_\delta} f$. Hence

$$\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}, \text{ for all } a \in S.$$

Conversely assume that there exist $a \in S$ and $t \in (\gamma, 1]$ such that $a_t^2 \in_\gamma f$, then $f(a^2) \geq t > \gamma$, thus $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): if $t \leq \delta$, then $f(a) \geq t > \gamma$, this implies that $a_t \in_\gamma f$.

Case(ii) : if $t > \delta$, then $f(a) + t > 2\delta$. Thus $a_t q_\delta f$.

Hence from (i) and (ii) we write $a_t \in_\gamma \vee q_\delta f$. Hence f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. ∎

Theorem 105 *For a right ideal R of an AG-groupoid S with a left identity, the following conditions are equivalent:*

(i) R is semiprime.

(ii) $X_{\gamma R}^{\delta}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. (i) $\Rightarrow$ (ii) Let R be a semiprime ideal of an AG-groupoid S. Let a be an arbitrary element of S such that $a \in R$. Then $a^2 \in R$. Hence $X_{\gamma R}^{\delta}(a) \geq \delta$ and $X_{\gamma R}^{\delta}(a^2) \geq \delta$ which implies that $\max\{X_{\gamma R}^{\delta}(a), \gamma\} \geq \min\{X_{\gamma R}^{\delta}(a^2), \delta\}$.

Now let $a \notin R$. Since R is semiprime, we have $a^2 \notin R$. This implies that $X_{\gamma R}^{\delta}(a) \leq \gamma$ and $X_{\gamma R}^{\delta}(a^2) \leq \gamma$. Therefore, $\max\{X_{\gamma R}^{\delta}(a), \gamma\} \geq \min\{X_{\gamma R}^{\delta}(a^2), \delta\}$. Hence, $\max\{X_{\gamma R}^{\delta}(a), \gamma\} \geq \min\{X_{\gamma R}^{\delta}(a^2), \delta\}$ for all $a \in S$.

(ii) $\Rightarrow$ (i) Let $X_{\gamma R}^{\delta}$ be fuzzy semiprime. If $a^2 \in R$, for some a in S, then $X_{\gamma R}^{\delta}(a^2) \geq \delta$. Since $X_{\gamma R}^{\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime, we have $\max\{X_{\gamma R}^{\delta}(a), \gamma\} \geq \min\{X_{\gamma R}^{\delta}(a^2), \delta\}$. Therefore $\max\{X_{\gamma R}^{\delta}(a), \gamma\} \geq \delta$. But $\delta > \gamma$, so $X_{\gamma R}^{\delta}(a) \geq \delta$. Thus $a \in R$. Hence R is semiprime. ∎

3.2 Intra-Regular AG-groupoids

Theorem 106 *Let S be an AG- groupoid with a left identity. Then the following conditions are equivalent:*

(i)S is intra-regular.

(ii)For a right ideal R of an AG-groupoid S, $R \subseteq R^2$ and R is semiprime.

(iii)For an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$ and f is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S with a left identity. Since S is intra-regular, for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using (1), medial law, paramedial law and left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = [(x(aa))y] = [(a(xa))y] = [(y(xa))a] \\ y(xa) &= [y(x((xa^2)y))] = [y((xa^2)(xy))] = [(xa^2)(xy^2)] = [(y^2x)(a^2x)] \\ &= [a^2((y^2x)x)] = [\{(a(y^2x)\}(ax)]. \end{aligned}$$

For any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned} \max\{(f \circ f)(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{\{f(p) \wedge f(q)\}, \gamma\} \right\} \\ &\geq \max\left\{\min\{\{f(a(y^2x)), f(ax)\}, \gamma\right\} \\ &\geq \max\{\min\{f(a(y^2x)), f(ax)\}, \gamma\} \\ &= \min\{\max\{f(a(y^2x)), \gamma\}, \max\{f(ax), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\} \\ &= \min\{(f)(a), \delta\}. \end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$. Next we show that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Since $S = S^2$, for each y in S there exist y_1, y_2 in S such that $y = y_1 y_2$. Thus using medial law, paramedial law and (1), we get

$$\begin{aligned} a &= (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) \\ &= a^2[(y_2y_1)x] = a^2t, \text{ where } [(y_2y_1)x] = t. \end{aligned}$$

Then

$$\begin{aligned} \max\{f(a), \gamma\} &= \max\{f(a^2t), \gamma\} \\ &\geq \min\{f(a^2), \delta\}. \end{aligned}$$

$(iii) \Rightarrow (ii)$ Let R be any right ideal of an AG-groupoid S. By (iii), $X^\delta_{\gamma R}$ is semiprime so R is semiprime. Now using (iii), we get

$$X^\delta_{\gamma R} = X^\delta_{\gamma R \cap R} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma R^2}.$$

Hence we get $R \subseteq R^2$.

$(ii) \Rightarrow (i)$ Since Sa^2 is a right ideal containing a^2, using (ii) we get $a \in Sa^2 \subseteq (Sa^2)^2 = (Sa^2)(Sa^2) \subseteq (Sa^2)S$. Hence S is intra-regular. ∎

Theorem 107 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*
(i) S is intra-regular.
(ii) For any right ideal R and any subset A of S, $R \cap A \subseteq RA$ and R is a semiprime ideal.
(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, $f \cap g \subseteq \vee q_{(\gamma,\delta)}(f \circ g)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra regular AG-groupoid S. Since S is intra-regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using (1), medial law, paramedial law and left invertive law, we get

$$a = (xa^2)y = [(x(aa))y] = [(a(xa))y] = [y(xa)]a.$$

$$\begin{aligned} y(xa) &= [y\{x((xa^2)y)\}] = [y\{(xa^2)(xy)\}] = [(xa^2)(xy^2)] \\ &= [(y^2x)(a^2x)] = [a^2(y^2x^2)]. \end{aligned}$$

Thus $a = (a^2t)a$, where $(y^2x^2) = t$.

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{f(b) \wedge g(c)\}, \gamma \right\} \\ &\geq \max\{\min\{f(a^2t), g(a)\}, \gamma\} \\ &= \min\{\max\{f(a^2t), \gamma\}, \max\{g(a), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$. The rest of proof is similar as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be an right ideal and A be a subset of S. By (iii), we get

$$X^\delta_{\gamma(R\cap A)} =_{(\gamma,\delta)} X^\delta_{\gamma R} \cap X^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma R} \circ X^\delta_{\gamma A} =_{(\gamma,\delta)} X^\delta_{\gamma RA}.$$

Thus $R \cap A \subseteq RA$. The rest of the proof is similar as in Theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore $a \in Sa^2 \cap Sa \subseteq (Sa^2)(Sa) \subseteq (Sa^2)S$. Hence S is intra-regular. ■

Theorem 108 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*
(i) S is intra regular.
(ii) For any right ideal R and any subset A of S, $R \cap A \subseteq AR$ and R is a semiprime ideal.
(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, $f \cap g \subseteq \vee q_{(\gamma,\delta)}(g \circ f)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular AG-groupoid S. Since S is intra-regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using left invertive law, we get

$$\begin{aligned} a &= (xa^2)y = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] = [x(y_2y_1)]a^2 \\ &= a[\{x(y_2y_1)\}a] = a[\{x(y_2y_1)\}\{(xa^2)y\}] = a[(xa^2)[\{x(y_2y_1)\}y]] \\ &= a[[y\{x(y_2y_1)\}](a^2x)] = a[a^2([y\{x(y_2y_1)\}]x)] = a[(x[y\{x(y_2y_1)\}])a^2] \\ &= a[(x[y\{x(y_2y_1)\}])(aa)] = a[a((x[y\{x(y_2y_1)\}])a)] \\ &= a(au), \text{ where } u = (x[y\{x(y_2y_1)\}])a. \end{aligned}$$

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{g(b) \wedge f(c)\}, \gamma \right\} \\ &\geq \max\{\min\{g(a), f(au)\}, \gamma\} \\ &= \min[\max\{g(a), \gamma\}, \max\{f(au), \gamma\}] \\ &\geq \min[\min\{g(a), \delta\}, \min\{f(a), \delta\}] \\ &= \min[g(a), f(a), \delta\}] \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Then we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} g \circ f$. The rest of the proof is similar as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A be a subset of S. By (iii), we get

$$X^\delta_{\gamma(R\cap A)} = X^\delta_{\gamma(A\cap R)} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma AR}.$$

By Lemma 92, $R \cap A \subseteq AR$. The rest of the proof is similar as in Theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa^2 \cap Sa \subseteq (Sa)(Sa^2) = (a^2S)(aS) = [(aa)(SS)](aS) \\ &= [(SS)(aa)](aS) \subseteq (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 109 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) S is intra-regular.

(ii) For any subset A and any right ideal R of S, $A \cap R \subseteq AR$ and R is a semiprime.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g of S, $f \cap g \subseteq \vee q_{(\gamma,\delta)} (f \circ g)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial law, paramedial law and (1), we get

$$\begin{aligned} a &= (xa^2)y = [(xa^2)(y_1y_2)] = [(y_2y_1)(a^2x)] \\ &= [a^2((y_2y_1)x)] = [(x(y_2y_1))(aa)] = [a\{(x(y_2y_1))a\}] \\ &= a(ta), \text{ where } x(y_2y_1) = t, \text{ and} \\ ta &= t[(xa^2)y] = (xa^2)(ty) = (yt)(a^2x) = a^2[(yt)x]. \end{aligned}$$

Thus $a = a(a^2v)$, where $(yt)x = v$ and $x(y_2y_1) = t$.

For any a in S there exist s and t in S such that $a = st$. Then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{ \bigvee_{a=st} \{f(s) \wedge g(t)\}, \gamma \right\} \\ &\geq \max\{\min\{f(a), g(a^2v)\}, \gamma\} \\ &= \min\{\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Thus $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$. The rest of the proof is similar as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A be a subset of S. By (iii), we get

$$X^\delta_{\gamma(A\cap R)} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma AR}.$$

Thus $A \cap R \subseteq AR$. The rest of the proof is similar as in Theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right ideal containing a^2. By (ii), it is semiprime. Therefore

$$\begin{aligned} a &\in Sa \cap Sa^2 \subseteq (Sa)(Sa^2) = (a^2S)(aS) \subseteq (a^2S)S \\ &= [a^2(SS)]S = [(SS)a^2]S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 110 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) *S is intra-regular.*

(ii) *For any subsets A, B and for any right ideal R of S, $A \cap B \cap R \subseteq (AB)R$ and R is a semiprime ideal.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, g and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal h, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and h is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.*

Proof. $(i) \Rightarrow (iii)$ Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and h be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Since S is intra-regular then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned} a &= (xa^2)y = (y_2y_1)(a^2x) = a^2[(y_2y_1)x] = [x(y_2y_1)]a^2 \\ &= a[\{x(y_2y_1)\}a] = a(pa), \text{ where } x(y_2y_1) = p \text{ and} \\ pa &= p[(xa^2)y] = (xa^2)(py) = [(yp)(a^2x)] \\ &= a^2[(yp)x] = [x(yp)](aa) = a[\{x(yp)\}a] \\ &= [a(qa)], \text{ where } x(yp) = q, \text{ and} \\ qa &= q[(xa^2)y] = (xa^2)(qy) = (yq)(a^2x) = a^2[(yq)x]. \end{aligned}$$

Thus $a = a[a(a^2c)] = a[a^2(ac)] = a^2[a(ac)]$, where $(yq)x = c$ and $x(yp) = q$ and $(y_2y_1) = p$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned} \max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma\right\} \\ &\geq \max\{\min\{(f \circ g)(aa), h(a(ac))\}, \gamma\} \\ &= \max\left\{\bigvee_{aa=pq} \{\{f(p) \wedge g(q)\}, h(a(ac))\}, \gamma\right\} \\ &\geq \max[\min\{f(a), g(a), h(a(ac))\}, \gamma] \\ &= \min[\max\{f(a), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a(ac)), \gamma] \\ &\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &= \min[\min\{f(a), g(a), h(a)\}, \delta] \\ &= \min\{(f \cap g \cap h)(a), \delta\}. \end{aligned}$$

Thus $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)} (f \circ g) \circ h$. The rest of the proof is similar as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by (iii), we get

$$X^\delta_{\gamma(A \cap B) \cap R} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma B} \cap X^\delta_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} (X^\delta_{\gamma A} \circ X^\delta_{\gamma B}) \circ X^\delta_{\gamma R} =_{(\gamma,\delta)} X^\delta_{\gamma (AB) R}.$$

Then we get $(A \cap B) \cap R \subseteq (AB)R$. The rest of the proof is similar as in Theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$Sa \cap Sa \cap Sa^2 \subseteq [(Sa)(Sa)](Sa^2) = [(SS)(aa)](Sa^2) \subseteq (Sa^2)S.$$

Hence S is intra-regular. ■

Theorem 111 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*

(i) S is intra-regular.

(ii) For any subsets A, B and for any right ideal R of S, $A \cap R \cap B \subseteq (AR)B$ and R is a semiprime.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, h and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.

Proof. $(i) \Rightarrow (iii)$ Let f, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S. Now since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned}
a &= [(x(aa))y] = [(a(xa))y] = [(y(xa))a], \\
y(xa) &= y[x((xa^2)y)] = y[(xa^2)(yx)] = [(xa^2)(xy^2)] = (y^2x)(a^2x) \\
&= a^2(y^2x^2) = (aa)(y^2x^2) = (x^2y^2)(aa) = a[(x^2y^2)a], \\
(x^2y^2)a &= (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) = a^2[\{(y(y^2x^2)\}x]
\end{aligned}$$

Thus $a = [a(a^2v)]a$, where $\{y(y^2x^2)\}x = v$.

For any a in S there exist p and q in S such that $a = pq$. Then

$$\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{(f \circ g)(p) \wedge h(q)\}, \gamma \right\} \\
&\geq \max\{\min\{(f \circ g)(a(a^2v)), h(a)\}, \gamma\} \\
&= \max\left\{ \bigvee_{a(a^2v)=cd} \{\{f(c) \wedge g(d)\}, h(a)\}, \gamma \right\} \\
&\geq \max[\min\{f(a), g(a^2v), h(a)\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(a^2v), \gamma\}, \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

Thus $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rest of the proof is same as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by (iii), we get

$$X^\delta_{\gamma(A\cap R)\cap B} =_{(\gamma,\delta)} X^\delta_{\gamma A} \cap X^\delta_{\gamma R} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)}(X^\delta_{\gamma A} \circ X^\delta_{\gamma R}) \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(AR)B}.$$

Hence we get $(A \cap R) \cap B \subseteq (AR)B$. The rest of the proof is same as in Theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$\begin{aligned} a &\in Sa \cap Sa^2 \cap Sa \subseteq [(Sa)(Sa^2)](Sa) \subseteq [S(Sa^2)]S \\ &= [S(Sa^2)](SS) = (SS)[(Sa^2)S] \\ &= S[(Sa^2)S] = (Sa^2)(SS) = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Theorem 112 *Let S be an AG- groupoid with a left identity. Then the following conditions equivalent:*
(i) S is intra-regular.
(ii) For any subsets A, B and for any right ideal R of S, $R \cap A \cap B \subseteq (RA)B$ and R is a semiprime.
(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets g, h, $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g, h be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets of an intra-regular AG-groupoid S. Now since S is intra-regular it follows that for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using medial, paramedial laws and (1), we get

$$\begin{aligned} a &= (x(aa))y = (a(xa))y = (y(xa))a, \\ y(xa) &= y[x\{(xa^2)y\}] = y[(xa^2)(yx)] = (xa^2)(xy^2) \\ &= (y^2x)(a^2x) = a^2[(y^2x)x] = [(x^2y^2)a]a \text{ and} \\ (x^2y^2)a &= (x^2y^2)[(xa^2)y] = (xa^2)[(x^2y^2)y] = [y(y^2x^2)](a^2x) \\ &= a^2[\{y(y^2x)\}x] = a^2v. \end{aligned}$$

Thus $a = [(a^2v)a]a$, where $[y(y^2x)]x = v$.

For any a in S there exist b and c in S such that $a = bc$. Then

$$\begin{aligned} \max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma \right\} \\ &\geq \max\{\min\{(f \circ g)((a^2v)a), h(a)\}, \gamma\} \\ &= \max\left\{ \bigvee_{(a^2v)a=pq} \{\{f(p) \wedge g(q)\}, h(a)\}, \gamma \right\} \\ &\geq \max[\min\{f(a^2v), g(a), h(a)\}, \gamma] \\ &= \min\left\{\max\{f(a^2v), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a), \gamma\}\right\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}\} \\ &= \min\{\min\{f(a), g(a), h(a)\}, \delta\} \\ &= \min\{(f \cap g \cap h)(a), \delta\}. \end{aligned}$$

Thus $(f \cap g) \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$. The rest of the proof is similar as in Theorem 106.

$(iii) \Rightarrow (ii)$ Let R be a right ideal and A, B be any subsets of S. Then by (iii), we get

$$X^{\delta}_{\gamma(R\cap A)\cap B} =_{(\gamma,\delta)} X^{\delta}_{\gamma R} \cap X^{\delta}_{\gamma A} \cap X^{\delta}_{\gamma B} \subseteq \vee q_{(\gamma,\delta)}(X^{\delta}_{\gamma R} \circ X^{\delta}_{\gamma A}) \circ X^{\delta}_{\gamma B} =_{(\gamma,\delta)} X^{\delta}_{\gamma(RA)B}.$$

Hence we get $(R \cap A) \cap B \subseteq (RA)B$. The rest of the proof is similar as in theorem 106.

$(ii) \Rightarrow (i)$ Sa^2 is a right-ideal of an AG-groupoid S containing a^2. By (ii), it is semiprime. Thus (ii), we get

$$\begin{aligned} a \in Sa^2 \cap Sa \cap Sa &\subseteq [(Sa^2)(Sa)](Sa) \subseteq [(Sa^2)S]S \\ &= (SS)(Sa^2) = (SS)[(SS)(aa)] = (SS)[(aa)(SS)] \\ &= (SS)(a^2S) = (Sa^2)(SS) = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

4

Generalized Fuzzy Quasi-ideals of Abel Grassmann's Groupoids

In this chapter, we introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals in AG-groupoids. We characterize intra-regular AG-groupoids by the properties of these ideals.

Definition 113 *A fuzzy subset f of an AG- groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy-quasi ideal of S. if it satisfies,* $\max\{f(x), \gamma\} \geq \min\{(f \circ S)(x), (S \circ f)(x), \delta\}$, *for all $x \in S$, $\gamma, \delta \in [0, 1]$, by S we mean $X^{\delta}_{\gamma S}$, where $\gamma = 0$ and $\delta = 1$.*

Example 114 *Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 & \text{if } x = 1 \\ 0.44 & \text{if } x = 2 \\ 0.42 & \text{if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$-fuzzy quasi-ideal,*
- *f is not an $(\in, \in \vee q_{0.11})$-fuzzy quasi-ideal.*

Lemma 115 *If Q is a quasi-ideal of an AG-groupoid S if and only if the fuzzy subset $X^{\delta}_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy quasi-ideal of S.*

Proof. Suppose that Q is a quasi-ideal of an AG-groupoid S. Let $X^{\delta}_{\gamma Q}$ be the fuzzy subset of Q. Suppose that $x \in S$.

Case(a):

If $x \notin Q$, then $x \notin SQ$ or $x \notin QS$.

(i) If $x \notin SQ$, then by definition $X^{\delta}_{\gamma Q}(x) \leq \gamma$ and $(S \circ X^{\delta}_{\gamma Q})(x) \leq \gamma$. Thus,

$\max\{X^{\delta}_{\gamma Q}(a),\gamma\} = \gamma$ and $\min\{(S\circ X^{\delta}_{\gamma Q})(x),\delta\} = (S\circ X^{\delta}_{\gamma Q})(x)$. This clearly implies that $\max\{X^{\delta}_{\gamma Q}(a),\gamma\} \geq \min\{(S\circ X^{\delta}_{\gamma Q})(x),\delta\}$ but $\min\{(S\circ X^{\delta}_{\gamma Q})(x),\delta\} \geq \min\{(S\circ X^{\delta}_{\gamma Q})(x),(X^{\delta}_{\gamma Q}\circ S)(x),\delta\}$. Therefore,

$$\max\{X^{\delta}_{\gamma Q}(a),\gamma\} \geq \min\{(S\circ X^{\delta}_{\gamma Q})(x),(X^{\delta}_{\gamma Q}\circ S)(x),\delta\}.$$

(ii) Similarly, if $x \notin QS$, then by above procedure we can prove the required result.

Case(b):

If $x \in Q$, then $x \in SQ\cap QS$ or $x \notin SQ\cap QS$.

(i) If $x \in SQ\cap QS$ that is $x \in SQ$ and $x \in QS$. Then, by definition $X^{\delta}_{\gamma Q}(x) \geq \delta$ and $(S\circ X^{\delta}_{\gamma Q})(x) \geq \delta$ and $(X^{\delta}_{\gamma Q}\circ S) \geq \delta$. Thus,

$$\max\{X^{\delta}_{\gamma Q}(a),\gamma\} \geq \min\{(S\circ X^{\delta}_{\gamma Q})(x),(X^{\delta}_{\gamma Q}\circ S)(x),\delta\}.$$

(ii) Similarly, we can prove the result for $x \notin SQ\cap QS$.

Conversely, assume that $X^{\delta}_{\gamma Q}$ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy-quasi ideal of S. Then,

$$\begin{aligned}
\max\{X^{\delta}_{\gamma Q}(a),\gamma\} &\geq \min\{(X^{\delta}_{\gamma Q}\circ S)(x),(S\circ X^{\delta}_{\gamma Q})(x),\delta\}\\
&= \min\{(X^{\delta}_{\gamma Q}\circ X^{1}_{0S})(x),(X^{1}_{0S}\circ X^{\delta}_{\gamma Q})(x),\delta\}\\
&= \min\{X^{\delta}_{\gamma QS}(x),X^{\delta}_{\gamma SQ}(x),\delta\}\\
&= \min\{X^{\delta}_{\gamma QS\cap SQ}(x),\delta\}\\
&\geq \min\{X^{\delta}_{\gamma QS\cap SQ}(x),\delta\}, \text{ for all } x\in S.
\end{aligned}$$

If $x \in SQ\cap QS$. Then, $\max\{X^{\delta}_{\gamma Q}(a),\gamma\} \geq \min\{X^{\delta}_{\gamma QS\cap SQ}(x),\delta\} = \delta$. Thus, $X^{\delta}_{\gamma Q}(x) \geq \delta$, which implies that $x \in Q$. Hence, Q is a quasi-ideal of S. ■

Definition 116 *A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma ,\in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if it satisfies $y_t \in_\gamma f, (xy)_t \in_\gamma \vee q_\delta f$ $(x_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f)$, for all $t,s \in (0,1]$, and $\gamma,\delta \in [0,1]$.*

Theorem 117 *A fuzzy subset f of an AG-groupoid S is called $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy left (respt. right) ideal if and only if $\max\{f(ab),\gamma\} \geq \min\{f(b),\delta\}$, (respt. $\max\{f(ab),\gamma\} \geq \min\{f(a),\delta\}$) for all $a,b \in S$.*

Proof. It is easy. ■

Theorem 118 *Every $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy-quasi ideal of S.*

Proof. Let f be an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S. Suppose that $x \in S$. If for any x in S there exist y and z in S such that

$x = yz$, then

$$\begin{aligned}
\min\{(S \circ f)(x), \delta\} &= \min\left\{\bigvee_{x=yz}\{S(y) \wedge f(z)\}, \delta\right\} \\
&= \min\{\bigvee_{x=yz} \min\{S(y), f(z)\}, \delta\} \\
&= \min\{\bigvee_{x=yz} \min\{1, f(z)\}, \delta\} \\
&= \bigvee_{x=yz} \min\{f(z), \delta\} \\
&\leq \bigvee_{x=yz} \{\max\{f(yz), \gamma\}\} \\
&= \bigvee_{x=yz} \{\max\{f(x), \gamma\}\} \\
&= \max\{f(x), \gamma\}.
\end{aligned}$$

Hence, $\max\{f(x), \gamma\} \geq \min\{(S \circ f)(x), \delta\} \geq \min\{(S \circ f)(x), (f \circ S)(x), \delta\}$. Thus, f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of an AG-groupoid S. ■

Theorem 119 *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy-quasi ideal of S.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an AG-groupoid S. Let $a \in S$. If for any a in S there exist b and c in S such that $a = bc$, then

$$\begin{aligned}
\min\{(g \circ S)(a), \delta\} &= \min\left\{\bigvee_{a=bc}\{g(b) \wedge S(c)\}, \delta\right\} \\
&= \bigvee_{a=bc} \min\{\min\{g(b), S(c), \delta\}\} \\
&= \bigvee_{a=bc} \min\{g(b), \delta\} \\
&\leq \bigvee_{a=bc} \max\{g(bc), \gamma\} \\
&= \max\{g(a), \gamma\}.
\end{aligned}$$

Thus, $\max\{g(a), \gamma\} \geq \min\{(g \circ S)(a), \delta\} \geq \min\{(g \circ S)(a), (S \circ g)(a), \delta\}$. Hence, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of an AG-groupoid S. ■

4.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Quasi-ideals of Intra-regular AG-groupoids

Example 120 *Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.*

$\cdot$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.7 & \text{if } x = 1 \\ 0.64 & \text{if } x = 2 \\ 0.51 & \text{if } x = 3. \end{cases}$$

Then, f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$-fuzzy quasi-ideal.

Theorem 121 *For an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid f of an intra regular AG- groupoid S with left identity, we have $f \subseteq \vee q_{(\gamma,\delta)} S \circ f$.*
Proof. *Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of an intra regular AG-groupoid S with left identity. Then, for any a in S, there exists x and y in S such that*

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = \acute{x}a, \text{ where } y(xa) = \acute{x}.$$

If for any a in S there exist b and c in S such that $a = bc$, then

$$\begin{aligned} \min\{(S \circ f)(a), \delta\} &= \min\left\{\bigvee_{a=bc} \{(S(b) \wedge f(c)), \delta\}\right\} \\ &= \bigvee_{a=bc} \min\{\min\{S(\acute{x}), f(a)\}, \delta\} \\ &= \bigvee_{a=bc} \min\{\min\{1, f(a)\}, \delta\} \\ &= \bigvee_{a=bc} \min\{f(a), \delta\} \\ &= \min\{f(a), \delta\} \\ &\leq \max\{(f(a), \gamma). \end{aligned}$$

Hence $f \subseteq \vee q_{(\gamma,\delta)} S \circ f$. ■

Theorem 122 *For an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid g of an intra regular AG- groupoid S with left identity, we have $g \subseteq \vee q_{(\gamma,\delta)} g \circ S$.*
Proof. *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of an intra regular AG-groupoid S with left identity. Then, for any x in S, there exist y and*

z in S such that

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\
y(xa) &= y(x((xa^2)y))) = y((xa^2)(xy)) = (xa^2)(xy^2) = (y^2x)(a^2x) \\
&= a^2(y^2xx) = (aa)(y^2x^2) = a[(y^2x^2)a] = at, \text{ where } [(y^2x^2)a] = t.
\end{aligned}$$

If for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\min\{(g \circ S)(a), \delta\} &= \min\left\{ \bigvee_{a=pq} \{(g(p) \wedge S(q)), \delta\} \right\} \\
&= \min\left\{ \bigvee_{a=pq} \min\{g(a), S(t)\}, \delta \right\} \\
&= \bigvee_{a=pq} \min\left\{\min\{g(a), 1\}, \delta\right\} \\
&= \bigvee_{a=pq} \min\{(g(a), \delta) \\
&= \min\{(g(a), \delta) \\
&\leq \max\{g(a), \gamma\}.
\end{aligned}$$

Hence $g \subseteq \vee q_{(\gamma,\delta)} g \circ S$. ■

Theorem 123 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent.*
(i) S is intra regular.
(ii) For every quasi ideal Q of S, $Q \subseteq Q^2$.
(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy quasi ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of an intra regular AG-groupoid S with left identity. Since S is intra regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Now, by using (1), para medial law, medial law we obtain

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a. \\
y(xa) &= y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(xy^2) = (y^2x)(a^2x), \\
(aa)(y^2x^2) &= [(x^2y^2)a]a, \text{ so} \\
a &= [\{(x^2y^2)a\}a]a = (ua)a, \text{ where } (x^2y^2)a = u.
\end{aligned}$$

If for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\max\{(f \circ f)(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{f(p) \wedge f(q)\}, \gamma \right\} \\
&\geq \max[\min\{f(ua), f(a)\}, \gamma] \\
&= \min[\max\{f(ua), \gamma\}, \max\{f(a), \gamma\}] \\
&\geq \min[\min\{S \circ f(ua), \delta\}, \min\{f(a), \delta\} \\
&= \min[\min\{ \bigvee_{ua=xy} S(x) \wedge f(y), \delta\}, \min\{f(a), \delta\} \\
&\geq \min[\min\{S(u) \wedge f(a), \delta\}, \min\{f(a), \delta\} \\
&= \min\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\} \\
&= \min\{f(a), \delta\}.
\end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$.

$(iii) \Rightarrow (ii)$ Suppose that Q is a quasi-ideal of S. Then, by (iii), we get

$$X^{\delta}_{\gamma Q} = X^{\delta}_{\gamma Q \cap Q} =_{(\gamma,\delta)} X^{\delta}_{\gamma Q} \cap X^{\delta}_{\gamma Q} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma Q} \circ X^{\delta}_{\gamma Q} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma Q^2}.$$

Hence we get $Q \subseteq Q^2$.

$(ii) \Rightarrow (i)$ Since S be an AG-groupoid. Therefore, $a \in Sa \cap Sa = Sa^2 = (Sa^2)S$. Hence, S is intra regular. ■

Example 124 *In example* 2*, S is obviously intra-regular because* $3 = (3 \cdot 3^2) \cdot 3$*,* $2 = (1 \cdot 2^2) \cdot 1$ *and* $1 = (2 \cdot 1^2) \cdot 2$*. Now we will satisfy* $\max\{(f \circ f)(a), \gamma\} \geq \min\{f(a), \delta\}$ *for intra-regularity as given in theorem 123, where f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$-fuzzy quasi-ideal as defined in this example. Obviously*

$$(f \circ f)(a) = \max\{(f \circ f)(a), 0.2\} > \min\{f(a), 0.3\} = 0.3.$$

Clearly $(f \circ f)(a) > 0.3$*, for all a in S.*

Theorem 125 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent .*

(i) S is intra regular.

(ii) For every quasi ideal Q and any subset A of S, $Q \cap A \subseteq QA$.

(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy quasi ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g, we have $f \cap g \subseteq \vee q_{(\gamma,\delta)} (f \circ g)$.

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra regular AG-groupoid S. Since S is intra regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now, by using medial, para medial laws and (1), we obtain

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

$$\begin{aligned} y(xa) &= y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(xy^2) \\ &= (y^2x)(a^2x) = a^2(y^2x^2) = (aa)(y^2x^2) = ((y^2x^2)a)a. \end{aligned}$$

Thus,

$$a = [((y^2x^2)a)a]a = (ta)a, \text{ where, } (y^2x^2)a = t.$$

If for any a in S there exist b and c in S such that $a = bc$, then

$$\begin{aligned} \max\{(f\circ g)(a),\gamma\} &= \max\left\{\bigvee_{a=bc}\{f(b)\wedge g(c)\},\gamma\right\} \\ &\geq \max\{\min\{f(ta),g(a)\},\gamma\} \\ &= \min[\max\{f(ta),\gamma\},\max\{g(a),\gamma\}] \\ &\geq \min[\min\{(S\circ f)(ta),\delta\},\{g(a),\delta\}] \\ &= \min[\min\{\bigvee_{ta=xy}(S(x)\wedge f(y),\delta\},\{g(a),\delta\}] \\ &\geq \min[\min\{(S(t)\wedge f(a),\delta\},\{g(a),\delta\}] \\ &= \min\{\min\{f(a),\delta\},\min\{g(a),\delta\}\} \\ &= \min\{(f\cap g)(a),\delta\}. \end{aligned}$$

Thus we have $f\cap g \subseteq \vee q_{(\gamma,\delta)} f\circ g$.

$(iii) \Rightarrow (ii)$ Let Q be an quasi ideal and A any subset of S. Then, by (iii), we get

$$X^{\delta}_{\gamma(Q\cap A)} =_{(\gamma,\delta)} X^{\delta}_{\gamma Q}\cap X^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma Q}\circ X^{\delta}_{\gamma A} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma QA}.$$

Hence we get $Q\cap A\subseteq QA$.

$(ii) \Rightarrow (i)$ Since Sa is a quasi ideal containing a. Therefore, $a\in Sa\subseteq (Sa)(Sa) = Sa^2 = (Sa^2)S$. Hence, S is intra regular. ■

Example 126 *Consider f as defined in example 2, where S is intra-regular. Now let us define g as follows:*

$$g(x) = \begin{cases} 0.8 & \text{if } x = 1 \\ 0.7 & \text{if } x = 2 \\ 0.6 & \text{if } x = 3. \end{cases}$$

Then clearly g is $(\in_{0.2}, \in_{0.2}\vee q_{0.3})$-fuzzy subset of S because

$$g(a) = \max\{g(a),0.2\} > \min\{g(a),0.3\} = 0.3, \text{ for } a \text{ in } S.$$

Now we will satisfy the condition $\max\{(f\circ g)(a),\gamma\}\geq \min\{(f\cap g)(a),\delta\}$. *Clearly*

$$(f\circ g)(a) = \max\{(f\circ g)(a),0.2\} > \min\{(f\cap g)(a),0.3\} = 0.3.$$

Obviously $(f\circ g)(a) > 0.3$ for all a of S.

Theorem 127 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent .*
(i) S is intra regular.
(ii) For any subset A and for every quasi ideal Q of S, we have $A\cap Q\subseteq AQ$.
(iii) For any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$ fuzzy subset f and for every $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-quasi ideal g of S, we have $f\cap g\subseteq \vee q_{(\gamma,\delta)}(f\circ g)$.

Proof. $(i)\Rightarrow(iii)$ Let f be any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy subset and g be any $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$-fuzzy quasi ideal of an intra regular AG-groupoid S. Since S is intra regular, then for any a in S there exist x,y in S such that $a=(xa^2)y$. Now, by using medial, para medial laws, (1), we obtain

$$(xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a.$$

$$\begin{aligned} y(xa) &= y[x\{(xa^2)y\}]=y[(xa^2)(xy)]=(xa^2)(xy^2)\\ &= (y^2x)(a^2x)=a^2(y^2x^2)=a^2u,\text{ where }u=y^2x^2\end{aligned}$$

Thus,

$$\begin{aligned} a &= (a^2u)a=(au)(aa)=a[(au)a]\\ &= a(ta),\text{ where }t=au\text{ and }u=y^2x^2.\end{aligned}$$

If for any a in S there exist p and q in S such that $a=pq$, then

$$\begin{aligned} \max\{(f\circ g)(a),\gamma\} &= \max\left\{\bigvee_{a=pq}\{f(p)\wedge g(q)\},\gamma\right\}\\ &\geq \max\{\min\{f(a),g(ta)\},\gamma\}\\ &= \min[\max\{f(a),\gamma\},\max\{g(ta),\gamma\}]\\ &\geq \min[\min\{f(a),\delta\},\min\{(S\circ g)(ta),\delta\}]\\ &= \min[\min\{f(a),\delta\},\min\bigvee_{ta=rs}\{S(r)\wedge g(s)\}\},\delta]\\ &\geq \min[\min\{f(a),\delta\},\min\{S(t)\wedge g(a)\}\},\delta]\\ &= \min[\min\{f(a),\delta\},\min\{g(a),\delta\}]\\ &= \min\{f(a),g(a),\delta\}\}\\ &= \min\{(f\cap g)(a),\delta\}.\end{aligned}$$

Thus $f\cap g\subseteq \vee q_{(\gamma,\delta)}f\circ g$.
$(iii)\Rightarrow(ii)$ Let A be any subset and Q be a quasi ideal of S. Then (iii), we get

$$X^{\delta}_{\gamma(A\cap Q)}=_{(\gamma,\delta)}X^{\delta}_{\gamma A}\cap X^{\delta}_{\gamma Q}\subseteq \vee q_{(\gamma,\delta)}X^{\delta}_{\gamma A}\circ X^{\delta}_{\gamma Q}=_{(\gamma,\delta)}\vee q_{(\gamma,\delta)}X^{\delta}_{\gamma AQ}.$$

Hence we get $A \cap Q \subseteq AQ$.

$(ii) \Rightarrow (i)$ Since Sa is a quasi ideal containing a. Therefore, $a \in Sa \subseteq (Sa)(Sa) = Sa^2 = (Sa^2)S$. Hence, S is intra regular. ■

Theorem 128 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent .*

(i) S is intra regular.

(ii) For any subsets A, B and quasi ideal Q of S, we have $Q \cap A \cap B \subseteq (QA)B$.

(iii) For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets g, h and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal f, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ S \circ f)$.

Proof. $(i) \Rightarrow (iii)$ Let f, g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy any subsets and h be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of intra regular AG-groupoid S with left identity. Since S is intra regular, then for any a in S there exist x, y in S such that $a = (xa^2)y$. Now, by using medial law, left inversive law, para medial law, (1) we get

$$\begin{aligned} a &= a(ta) = [a(ta)](ta) = (at)[(ta)a] = (at)(a^2t) = [(a^2t)t]a \\ &= (t^2a^2)a = (a^2t^2)a = [(t^2a)a]a. \end{aligned}$$

If for any a in S there exist b and c in S such that $a = bc$, then

$$\begin{aligned} \max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{ \bigvee_{a=bc} \{(f \circ g)(b) \wedge h(c)\}, \gamma \right\} \\ &\geq \max\{\min\{(f \circ g)((t^2a)a), h(a)\}, \gamma\} \\ &= \max\{\min\{ \bigvee_{(t^2a)a=rs} f(r) \wedge g(s), h(a)\}, \gamma\} \\ &\geq \max\{\min\{f(t^2a) \wedge g(a), h(a)\}, \gamma\} \\ &= \max\{\min\{f(t^2a), g(a), h(a)\}, \gamma\} \\ &= \min[\max\{f(t^2a), \gamma\}, \max\{g(a), \gamma\}, \max\{h(a), \gamma\}] \\ &\geq \min[\min\{S \circ f(t^2a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &= \min[\min\{ \bigvee_{t^2a=rs} S(r) \wedge f(s), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &\geq \min[\min\{S(t^2) \wedge f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &= \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\ &= \min[\min\{f(a), g(a), h(a)\}, \delta] \\ &= \min\{(f \cap g \cap h)(a), \delta\}. \end{aligned}$$

Thus we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(iii) \Rightarrow (ii)$ Let Q be an quasi-ideal and A, B be any subsets of S. Then (iii), we get

$$X^\delta_{\gamma Q \cap A \cap B} =_{(\gamma,\delta)} X^\delta_{\gamma Q} \cap X^\delta_{\gamma A} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma Q} \circ X^\delta_{\gamma A} \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^\delta_{\gamma (QA)B}.$$

Hence we get $Q \cap A \cap B \subseteq (QA)B$.

$(ii) \Rightarrow (i)$ We know Sa is a quasi-ideal of an AG-groupoid S containing a. Therefore, $a \in Sa \cap Sa = ((Sa)(Sa))Sa = (Sa^2)(Sa) \subseteq (Sa^2)S$. Hence, S is intra regular. ■

Theorem 129 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent.*

(i) *S is intra regular.*

(ii) *For any subsets A, B and quasi ideal Q of S, we have $A \cap Q \cap B \subseteq (AQ)B$.*

(iii) *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets f, h and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal g, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ S \circ f)$.*

Proof. $(i) \Rightarrow (iii)$ Let f, h be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy any subsets and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of intra regular AG-groupoid S with left identity. Since S is intra regular, so for any a in S there exist x, y in S such that $a = (xa^2)y$. Now, by using medial law, para medial law, (1) we get

$$a = (t^2a^2)a = [t^2(aa)]a = [a(t^2a)]a.$$

If for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\max\{(f \circ g \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a=pq}\{(f \circ g)(p) \wedge h(q)\}, \gamma\right\} \\
&\geq \max[\min\{(f \circ g)(a(t^2a)), h(a)\}, \gamma] \\
&= \max[\min\{\bigvee_{a(t^2a)=rs} f(r) \wedge g(s), h(a), \gamma] \\
&\geq \max[\min\{f(a) \wedge g(t^2a)\}, h(a), \gamma] \\
&= \max[\min\{f(a), g(t^2a)\}, h(a)\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(t^2a), \gamma\}, \max\{h(a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{S \circ g(t^2a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), \delta\}, \min\{\bigvee_{(t^2a)=vw} S(v) \wedge g(w), \delta\}, \min\{h(a), \delta\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{S(t^2) \wedge g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min\{\min\{f(a), g(a), h(a)\}, \delta\} \\
&= \min\{(f \cap g \cap h)(a), \delta\}.
\end{aligned}$$

Thus we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(iii) \Rightarrow (ii)$ Let A, B be any subsets and Q be a quasi-ideal of S. Then (iii), we get

$$\begin{aligned}
X^\delta_{\gamma A \cap Q \cap B} &= {}_{(\gamma,\delta)}X^\delta_{\gamma A} \cap X^\delta_{\gamma Q} \cap X^\delta_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma Q} \circ X^\delta_{\gamma B} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^\delta_{\gamma(AQ)B}.
\end{aligned}$$

Hence we get $A \cap Q \cap B \subseteq (AQ)B$.

$(ii) \Rightarrow (i)$ Since Sa is a quasi-ideal of an AG-groupoid S containing a, so $a \in Sa \cap Sa = ((Sa)(Sa))Sa = (Sa^2)(Sa) \subseteq (Sa^2)S$. Hence, S is intra regular. ■

Theorem 130 *Let S be an AG- groupoid with left identity. Then, the following conditions are equivalent.*

(i) *S is intra regular.*

(ii) *For a quasi-ideal Q and any subsets A, B of S, we have $A \cap B \cap Q \subseteq (AB)Q$.*

(iii) *For an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets g and h, we have $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}((f \circ g) \circ h)$.*

Proof. $(i) \Rightarrow (iii)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy-quasi ideal and g, h be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets of intra regular AG-groupoid S with left identity. Since S is intra regular, so for any a in S there exist x, y in S such that $a = (xa^2)y$. Now, by using medial, paramedial laws, (1) we get

$$a = (a^2t^2)a = (at^2)a^2 = a^2(t^2a).$$

If for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\max\{((f \circ g) \circ h)(a), \gamma\} &= \max\left\{\bigvee_{a=pq} \{(f \circ g)(p) \wedge h(q)\}, \gamma\right\} \\
&\geq \max\{\min\{(f \circ g)(aa), h(t^2a)\}, \gamma\} \\
&= \max\left\{\bigvee_{aa=rs} \{\{f(r) \wedge g(s)\}, h(t^2a)\}, \gamma\right\} \\
&\geq \max[\min\{f(a), g(a), h(t^2a)\}, \gamma] \\
&= \min[\max\{f(a), \gamma\}, \max\{g(a), \gamma\}, \max\{h(t^2a), \gamma\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{S \circ h(t^2a), \delta\}] \\
&= \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{\bigvee_{t^2a=cd} S(c) \wedge h(d), \delta\}] \\
&\geq \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{S(t^2) \wedge h(a), \delta\}] \\
&= \min[\min\{f(a), \delta\}, \min\{g(a), \delta\}, \min\{h(a), \delta\}] \\
&= \min[\min\{f(a), g(a), h(a)\}, \delta] \\
&= \min[(f \cap g \cap h)(a), \delta].
\end{aligned}$$

Thus we obtain $f \cap g \cap h \subseteq \vee q_{(\gamma,\delta)}(f \circ g) \circ h$.

$(iii) \Rightarrow (ii)$ Let A, B be any subsets and Q be a quasi-ideal of S. Then (iii), we get

$$\begin{aligned}
X^\delta_{\gamma A \cap B \cap Q} &= {}_{(\gamma,\delta)}X^\delta_{\gamma A} \cap X^\delta_{\gamma B} \cap X^\delta_{\gamma Q} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma A} \circ X^\delta_{\gamma B} \circ X^\delta_{\gamma Q} \\
&= {}_{(\gamma,\delta)} \vee q_{(\gamma,\delta)} X^\delta_{\gamma (AB)Q}.
\end{aligned}$$

Hence we get $A \cap B \cap Q \subseteq (AB)Q$.

$(ii) \Rightarrow (i)$ Since Sa is a quasi-ideal of an AG-groupoid S containing a. Therefore, $a \in Sa \cap Sa = ((Sa)(Sa))Sa = (Sa^2)(Sa) \subseteq (Sa^2)S$. Hence, S is intra regular. ■

5

Generalized Fuzzy Prime and Semiprime Ideals of Abel Grassmann Groupoids

In this chapter we introduce $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime (semiprime) ideals in AG-groupoids. We characterize intra regular AG-groupoids using the properties of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideals.

Lemma 131 *If A is an ideal of an AG-groupoid S if and only if $X^\delta_{\gamma A}$ is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.*

Proof. (i) Let $x, y \in A$ which implies that $xy \in A$. Then by definition we get $X^\delta_{\gamma A}(xy) \geq \delta$, $X^\delta_{\gamma A}(x) \geq \delta$ and $X^\delta_{\gamma A}(y) \geq \delta$ but $\delta > \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= X^\delta_{\gamma A}(xy) \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y)\} = \delta. \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(ii) Let $x \notin A$ and $y \in A$, which implies that $xy \notin A$. Then by definition $X^\delta_{\gamma A}(x) \leq \gamma$, $X^\delta_{\gamma R}(y) \geq \delta$ and $X^\delta_{\gamma R}(xy) \leq \gamma$. Therefore

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= X^\delta_{\gamma A}(x). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(iii) Let $x \in A, y \notin A$ which implies that $xy \notin A$. Then by definition, we get $X^\delta_{\gamma A}(xy) \leq \gamma$, $X^\delta_{\gamma A}(y) \leq \gamma$ and $X^\delta_{\gamma A}(x) \geq \delta$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= X^\delta_{\gamma A}(y). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma A}(xy), \gamma\} \geq \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\}$.

(iv) Let $x, y \notin A$ which implies that $xy \notin A$. Then by definition we get such that $X^\delta_{\gamma A}(xy) \leq \gamma$, $X^\delta_{\gamma A}(y) \leq \gamma$ and $X^\delta_{\gamma A}(x) \leq \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma A}(xy), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma A}(x), X^\delta_{\gamma A}(y), \delta\} &= \{X^\delta_{\gamma A}(x), X^\delta_{\gamma R}(y)\} = \gamma. \end{aligned}$$

Hence $\max\{X^{\delta}_{\gamma A}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma A}(x), X^{\delta}_{\gamma A}(y), \delta\}$.

Converse, let $(xy) \in AS$ where $x \in A$ and $y \in S$, and $(xy) \in SA$ where $y \in A$ and $x \in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma A}(xy), \gamma\} \geq \min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y), \delta\}$. Since $x \in A$, therefore $X^{\delta}_{\gamma A}(x) \geq \delta$, and $y \in A$ therefore $X^{\delta}_{\gamma A}(y) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma A}(x), X^{\delta}_{\gamma A}(y), \delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma A}(xy), \gamma\} \geq \delta.$$

This clearly implies that $X^{\delta}_{\gamma A}(xy) \geq \delta$. Therefore $xy \in A$. Hence A is an ideal of S. ■

Example 132 *Let $S = \{1, 2, 3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Then $(S, \cdot)$ is an AG-groupoid. Define a fuzzy subset $f : S \to [0, 1]$ as follows.

$$f(x) = \begin{cases} 0.31 \text{ for } x = 1 \\ 0.32 \text{ for } x = 2 \\ 0.30 \text{ for } x = 3 \end{cases}$$

Then clearly

- *f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$-fuzzy ideal of S,*
- *f is not an $(\in, \in \vee q_{0.3})$-fuzzy ideal of S, because $f(1 \cdot 2) < f(2) \wedge \frac{1-0.3}{2}$,*
- *f is not a fuzzy ideal of S, because $f(1 \cdot 2) < f(2)$.*

Definition 133 *A fuzzy subset f of an AG-groupoid S is called an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of S if for all x, y and $z \in S$ and $t, s \in (\gamma, 1]$, the following conditions holds.*

(1) if $x_t \in_{\gamma} f$ and $y_s \in_{\gamma} f$ implies that $(xy)_{\min\{t,s\}} \in_{\gamma} \vee q_{\delta} f$.

(2) if $x_t \in_{\gamma} f$ and $z_s \in_{\gamma} f$ implies that $((xy)z)_{\min\{t,s\}} \in_{\gamma} \vee q_{\delta} f$.

Theorem 134 *A fuzzy subset f of an AG-groupoid S is $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of S if and only if*

(I) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

(II) $\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof. $(1) \Leftrightarrow (I)$ is the same as theorem 95.

$(2) \Rightarrow (II)$. Assume that $x, y \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t$ which implies that $f((xy)z) < t \leq \gamma$ this implies that $((xy)z)_t \overline{\in}_{\gamma} f$ which further implies that $((xy)z)_t \overline{\in_{\gamma} \vee q_{\delta}} f$. Also

$\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$ but $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction. Hence

$$\max\{f((xy)z), \gamma\} \geq \min\{f(x), f(z), \delta\}.$$

$(II) \Rightarrow (2)$ Assume that x, y, z in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ by definition we can write $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, then $\max\{f((xy)z), \delta\} \geq \min\{f(x), f(y), \delta\}$ this implies that $f((xy)z) \geq \min\{t, s, \delta\}$. We consider two cases here,

Case(i): If $\{t, s\} \leq \delta$ then $f((xy)z) \geq \min\{t, s\} > \gamma$ this implies that $((xy)z)_{\min\{t,s\}} \in_\gamma f$.

Case(ii): If $\{t, s\} > \delta$ then $f((xy)z) + \{t, s\} > 2\delta$ this implies that $((xy)z)_{\min\{t,s\}} q_\delta f$.

From both cases we write $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y, z in S. ■

Lemma 135 *A subset B of an AG-groupoid S is a bi-ideal if and only if $X^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. (i) Let B be a bi-ideal and assume that $x, y \in B$ then for any a in S we have $(xa)y \in B$, thus $X^\delta_{\gamma B}((xa)y) \geq \delta$. Now since $x, y \in B$ so $X^\delta_{\gamma B}(x) \geq \delta$, $X^\delta_{\gamma B}(y) \geq \delta$ which clearly implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \geq \delta$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = X^\delta_{\gamma B}((xa)y) \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

(ii) Let $x \in B, y \notin B$, then $(xa)y \notin B$, for all a in S. This implies that $X^\delta_{\gamma B}((xa)y) \leq \gamma$, $X^\delta_{\gamma B}(x) \geq \delta$ and $X^\delta_{\gamma B}(y) < \gamma$. Therefore

$$\begin{aligned} \max\{X^\delta_{\gamma B}((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} &= X^\delta_{\gamma B}(y). \end{aligned}$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

(iii) Let $x \notin B, y \in B$ implies that $(xa)y \notin B$, for all a in S. This implies that $X^\delta_{\gamma B}((xa)y) \leq \gamma$, $X^\delta_{\gamma B}(x) \leq \gamma$, $X^\delta_{\gamma B}(y) \geq \delta$ then

$$\begin{aligned} \max\{X^\delta_{\gamma B}((xa)y), \gamma\} &= \gamma, \text{ and} \\ \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} &= X^\delta_{\gamma B}(x) \end{aligned}$$

Therefore

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}.$$

(iv) Let $x, y \notin B$ which implies that $(xa)y \notin B$, for all a in S. This implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma$, $X^\delta_{\gamma B}((xa)y) \leq \gamma$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = \gamma \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma.$$

Hence $\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\}$.

If $(xa)y \in B$, then $\min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y)\} \geq \delta$, $X^{\delta}_{\gamma B}((xa)y) \geq \delta$. Thus

$$\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} = X^{\delta}_{\gamma B}((xa)y) \text{ and}$$
$$\min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\} = \delta.$$

Hence $\max\{X^{\delta}_{\gamma B}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(x), X^{\delta}_{\gamma B}(y),\delta\}$.

Converse, let $(b_1 s)b_2 \in (BS)B$, where $b_1, b_2 \in B$ and $s \in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma B}((b_1 s)b_2),\gamma\} \geq \min\{X^{\delta}_{\gamma B}(b_1), X^{\delta}_{\gamma B}(b_2),\delta\}$. Since $b_1, b_2 \in B$, therefore $X^{\delta}_{\gamma B}(b_1) \geq \delta$ and $X^{\delta}_{\gamma B}(b_2) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma B}(b_1), X^{\delta}_{\gamma B}(b_2),\delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma B}((b_1 s)b_2),\gamma\} \geq \delta.$$

This clearly implies that $X^{\delta}_{\gamma B}((b_1 s)b_2) \geq \delta$. Therefore $(b_1 s)b_2 \in B$. Hence B is a bi-ideal of S. ■

Definition 136 *A fuzzy AG-subgroupoid f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if for all $x, y, z \in S$ and $t, r \in (\gamma, 1]$ the following conditions holds.*

(I) $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

(II) $y_t \in_\gamma f$ implies $((xy)z)_t \in_\gamma \vee q_\delta f$.

Lemma 137 *A fuzzy subset f of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of an AG-groupoid S if and only if it satisfies the following conditions.*

(III) $\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\}$ for all $x, y \in S$ and $\gamma, \delta \in [0,1]$.

(IV) $\max\{f(xyz),\gamma\} \geq \min\{f(y),\delta\}$ for all $x, y, z \in S$ and $\gamma, \delta \in [0,1]$.

Proof. $(I) \Rightarrow (III)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. Let (I) holds. Let us consider on contrary. If there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy),\gamma\} < t \leq \min\{f(x), f(y),\delta\}.$$

Then $\max\{f(xy),\gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ again implies that $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(x), f(y),\delta\} \geq t > \gamma$ this implies that $f(x) \geq t > \gamma$ and $f(y) \geq t > \gamma$ implies that $x_t \in_\gamma f$ and $y_t \in_\gamma f$.

But $(xy)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction .Thus

$$\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\}.$$

$(III) \Rightarrow (I)$ Assume that $x, y,$ in S and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$ and $y_s \in_\gamma f$. Then $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, $\max\{f(xy),\gamma\} \geq \min\{f(x), f(y),\delta\} \geq \min\{t, s,\delta\}$. We consider two cases here,

Case(1): If $\{t, s\} \leq \delta$ then $\max\{f(xy),\gamma\} \geq \min\{t, s\} > \gamma$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma f$.

Case(2): If $\{t,s\} > \delta$ then $f(xy) + \min\{t,s\} > 2\delta$ this implies that $(xy)_{\min\{t,s\}} q_\delta f$.

Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

$(II) \Rightarrow (IV)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. Let (II) holds. Let us consider on contrary. If there exists $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f((xy)z), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f((xy)z), \gamma\} < t \leq \gamma$ this implies that $((xy)z)_t \bar{\in}_\gamma f$ further implies that $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ this implies that $f(y) \geq t > \gamma$ implies that $y_t \in_\gamma f$. But $(xyz)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction according to definition. Thus (IV) is valid

$$\max\{f((xy)z), \gamma\} \geq \min\{f(y), \delta\}$$

$(IV) \Rightarrow (II)$ Assume that x, y, z in S and $t, s \in (\gamma, 1]$ such that $y_t \in_\gamma f$. Then $f(y) \geq t > \gamma$, by (IV) we write $\max\{f((xy)z), \gamma\} \geq \min\{f(y), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): If $t \leq \delta$ then $f((xy)z) \geq t > \gamma$ this implies that $((xy)z)_t \in_\gamma f$.

Case(ii): If $t > \delta$ then $f((xy)z) + t > 2\delta$ this implies that $((xy)z) q_\delta f$.

From both cases $((xy)z)_t \in_\gamma \vee q_\delta f$. Hence f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. ■

Lemma 138 *If I is a interior ideal of an AG-groupoid S if and only if $X^\delta_{\gamma I}$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy interior ideal of S.*

Proof. (i) Let $x, a, y \in I$ which implies that $(xa)y \in I$. Then by definition we get $X^\delta_{\gamma I}((xa)y) \geq \delta$ and $X^\delta_{\gamma I}(a) \geq \delta$, but $\delta > \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma I}((xa)y), \gamma\} &= X^\delta_{\gamma I}((xa)y) \text{ and} \\ \min\{X^\delta_{\gamma I}(a), \delta\} &= \delta. \end{aligned}$$

Hence $\max\{ X^\delta_{\gamma I}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma I}(a), \delta\}$.

(ii) Let $x \notin I$, $y \notin I$ and $a \in I$, which implies that $(xa)y \in I$. Then by definition $X^\delta_{\gamma I}((xa)y) \geq \delta$ and $X^\delta_{\gamma I}(a) \geq \delta$. Therefore

$$\begin{aligned} \max\{X^\delta_{\gamma I}((xa)y), \gamma\} &= X^\delta_{\gamma I}((xa)y), \text{ and} \\ \min\{X^\delta_{\gamma I}(a), \delta\} &= \delta. \end{aligned}$$

Hence $\max\{X^\delta_{\gamma I}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma I}(a), \delta\}$.

(iii) Let $x \in I, y \in I$ and $a \notin I$ which implies that $(xa)y \notin I$. Then by definition, we get $X^\delta_{\gamma I}((xa)y) \leq \gamma$, $X^\delta_{\gamma I}(a) \leq \gamma$. Thus

$$\begin{aligned} \max\{X^\delta_{\gamma I}((xa)y), \gamma\} &= \gamma \text{ and} \\ \min\{X^\delta_{\gamma I}(a), \delta\} &= X^\delta_{\gamma I}(a). \end{aligned}$$

Hence $\max\{X^{\delta}_{\gamma I}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma I}(a),\delta\}$.

(iv) Let $x,a,\ y \notin I$ which implies that $(xa)y \notin I$. Then by definition we get such that $X^{\delta}_{\gamma I}((xa)y) \leq \gamma$, $X^{\delta}_{\gamma I}(a) \leq \gamma$. Thus

$$\begin{aligned} \max\{X^{\delta}_{\gamma I}((xa)y),\gamma\} &= \gamma \text{ and} \\ \min\{X^{\delta}_{\gamma I}(a),\delta\} &= X^{\delta}_{\gamma I}(a). \end{aligned}$$

Hence $\max\{X^{\delta}_{\gamma I}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma I}(a),\delta\}$.

Conversely, let $(xa)y \in (SI)S$, where $a \in I$ and $x,y \in S$. Now by hypothesis $\max\{X^{\delta}_{\gamma I}((xa)y),\gamma\} \geq \min\{X^{\delta}_{\gamma I}(a),\delta\}$. Since $a \in I$, therefore $X^{\delta}_{\gamma I}(a) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma I}(a),\delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma I}((xa)y),\gamma\} \geq \delta.$$

This clearly implies that $X^{\delta}_{\gamma I}((xa)y) \geq \delta$. Therefore $(xa)y \in I$. Hence I is an interior ideal of S. ■

Example 139 *Consider an AG-groupoid $S = \{1,2,3\}$ in the following multiplication table.*

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 \text{ if } x = 1, \\ 0.44 \text{ if } x = 2, \\ 0.42 \text{ if } x = 3. \end{cases}$$

Then, we have

- *f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$-fuzzy quasi-ideal,*
- *f is not an $(\in, \in \vee q_{0.11})$-fuzzy quasi-ideal.*

Definition 140 *An $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy subset f of an AG-groupoid S is said to be prime if for all a,b in S and $t \in (\gamma,1]$. It satisfies,*

$(1)(ab)_t \in_{\gamma} f$ implies that $(a)_t \in_{\gamma} \vee q_{\delta} f$ or $(b)_t \in_{\gamma} \vee q_{\delta} f$.

Theorem 141 *An $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy prime ideal f of an AG-groupoid S if for all a,b in S, and $t \in (\gamma,1]$.It satisfies*

(2) $\max\{f(a), f(b),\gamma\} \geq \min\{f(ab),\delta\}$.

Proof. Let f be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy prime ideal of an AG-groupoid S. If there exists a,b in S and $t \in (\gamma,1]$, such that $\max\{f(a), f(b),\gamma\} < t \leq \min\{f(ab),\delta\}$ then $\min\{f(ab),\delta\} \geq t$ implies that $f(ab) \geq t > \gamma$ and $\min\{f(a), f(b),\gamma\} < t$ this implies that $f(a) < t \leq \gamma$ or $f(b) < t \leq \gamma$

again implies that $(a)_t\bar{\in}_\gamma f$ or $(b)_t\bar{\in}_\gamma f$ i.e. $(ab)_t \in_\gamma f$ but $(a)_t\overline{\in_\gamma \vee q_\delta}f$ or $(b)_t\overline{\in_\gamma \vee q_\delta}f$, which is a contradiction. Hence (2) is valid.

Conversely, assume that (2) is holds. Let $(ab)_t \in_\gamma f$. Then $f(ab) \geq t > \gamma$ and by (2) we have $\max\{f(a), f(b), \gamma\} \geq \min\{f(ab), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(a): If $t \leq \delta$, then $f(a) \geq t > \gamma$ or $f(b) \geq t > \gamma$ this implies that $(a)_t \in_\gamma f$ or $(b)_t \in_\gamma f$.

Case(b): If $t > \delta$, then $f(a) + t > 2\delta$ or $f(b) + t > 2\delta$ this implies that $(a)_t q_\delta f$ or $(b)_t q_\delta f$. Hence f is prime. ■

Theorem 142 *Let I be an non empty subset of an AG-groupoid S with left identity. Then*

(i) I is a prime ideal.

(ii) $\chi^\delta_{\gamma I}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime ideal of S.

Proof. $(i) \Rightarrow (ii)$. Let I be a prime ideal of an AG-groupoid S. Let $(ab) \in I$ then $\chi^\delta_{\gamma I}(ab) \geq \delta$, this implies that so $ab \in I$ and I is prime, so $a \in I$ or $b \in I$, by definition we can get $\chi^\delta_{\gamma I}(a) \geq \delta$ or $\chi^\delta_{\gamma I}(b) \geq \delta$, therefore

$$\begin{aligned} \min\{\chi^\delta_{\gamma I}(ab), \delta\} &= \delta \text{ and} \\ \max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} &= \max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b)\} \geq \delta. \end{aligned}$$

which implies that $\max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} \geq \min\{\chi^\delta_{\gamma I}(ab), \delta\}$. Hence $\chi^\delta_{\gamma I}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime ideal of S.

$(ii) \Rightarrow (i)$. Assume that $\chi^\delta_{\gamma I}$ is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S, then I is prime. Let $(ab) \in I$ by definition we can write $\chi^\delta_{\gamma I}(ab) \geq \delta$, therefore,by given condition we have $\max\{\chi^\delta_{\gamma I}(a), \chi^\delta_{\gamma I}(b), \gamma\} \geq \min\{\chi^\delta_{\gamma I}(ab), \delta\} = \delta$. this implies that $\chi^\delta_{\gamma I}(a) \geq \delta$ or $\chi^\delta_{\gamma I}(b) \geq \delta$ this implies that $a \in I$ or $b \in I$. Hence I is prime. ■

Example 143 *Let $S = \{1, 2, 3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Then $(S, \cdot)$ is an intra-regular AG-groupoid with left identity 1. Define a fuzzy subset $f : S \to [0, 1]$ as follows.

$$f(x) = \begin{cases} 0.34 \text{ for } x = 1 \\ 0.36 \text{ for } x = 2 \\ 0.35 \text{ for } x = 3 \end{cases}$$

Then clearly

- *f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.22})$-fuzzy prime ideal,*
- *f is not an $(\in, \in \vee q_{0.22})$-fuzzy prime ideal,*

- *f is not fuzzy prime ideal.*

Theorem 144 *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset f of an AG-groupoid S is prime if and only if $U(f,t)$ is prime in AG-groupoid S, for all $0 < t \leq \delta$.*

Proof. Let us consider an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset f of an AG-groupoid S is prime and $0 < t \leq \delta$. Let $(ab) \in_\gamma U(f,t)$ this implies that $f(ab) \geq t > \gamma$. Then by theorem 141 $\max\{f(a), f(b), \gamma\} \geq \min\{f(ab), \delta\} \geq \min\{t, \delta\} = t$, so $f(a) \geq t > \gamma$ or $f(b) \geq t > \gamma$, which implies that $a \in_\gamma U(f,t)$ or $b \in_\gamma U(f,t)$. Therefore $U(f,t)$ is prime in AG-groupoid S, for all $0 < t \leq \delta$.

Conversely, assume that $U(f,t)$ is prime in AG-groupoid S, for all $0 < t \leq \delta$. Let $(ab)_t \in_\gamma f$ implies that $ab \in_\gamma U(f,t)$, and $U(f,t)$ is prime, so $a \in_\gamma U(f,t)$ or $b \in_\gamma U(f,t)$, that is $a_t \in_\gamma f$ or $b_t \in_\gamma f$. Thus $a_t \in_\gamma \vee q_\delta f$ or $b_t \in_\gamma \vee q_\delta f$. Therefore f must be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy prime in AG-groupoid S. ■

Definition 145 *A fuzzy subset f of an AG-groupoid S is said to be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime for all $s, t \in (\gamma, 1]$ and $a \in S$. it satisfies*

(1) $a_t^2 \in_\gamma f$ implies that $a_t \in_\gamma \vee q_\delta f$.

Theorem 146 *A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime if and only if it satisfies*

(2) $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$ for all $a \in S$.

Proof. (1) $\Rightarrow$ (2) Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime of S. Assume that there exists $a \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(a), \gamma\} < t \leq \min\{f(a^2), \delta\}.$$

Then $\max\{f(a), \gamma\} < t$ this implies that $f(a) < t \leq \gamma$, implies that , $f(a) + t < 2t \leq 2\delta$ this implies that $a_t \overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(a^2), \delta\} \geq t$ this implies that $f(a^2) \geq t > \gamma$, further implies that $a_t^2 \in_\gamma f$ but $a_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence (2) is valid,

$$\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}, \text{ for all } a \in S.$$

(2) $\Rightarrow$ (1). Assume that there exist $a \in S$ and $t \in (\gamma, 1]$ such that $a_t^2 \in_\gamma f$, then $f(a^2) \geq t > \gamma$, thus by (2), we have $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\} \geq \min\{t, \delta\}$. We consider two cases here,

Case(i): if $t \leq \delta$, then $f(a) \geq t > \gamma$, this implies that $a_t \in_\gamma f$.

Case(ii) : if $t > \delta$, then $f(a) + t > 2\delta$, that is $a_t q_\delta f$. From (i) and (ii) we write $a_t \in_\gamma \vee q_\delta f$. Hence f is semiprime for all $a \in S$. ■

Theorem 147 *For a non empty subset I of an AG-groupoid S with left identity the following conditions are equivalent.*

(i) *I is semiprime.*

(ii) *$\chi_{\gamma I}^{\delta}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. $(i) \Rightarrow (ii)$ Let I is semiprime of an AG-groupoid S.

Case(a): Let a be any element of S such that $a^2 \in I$. Then I is semiprime, so $a \in I$. Hence $\chi^{\delta}_{\gamma I}(a^2) \geq \delta$ and $\chi^{\delta}_{\gamma I}(a) \geq \delta$. Therefore

$$\begin{aligned} \max\{\chi^{\delta}_{\gamma I}(a), \gamma\} &= \chi^{\delta}_{\gamma I}(a) \text{ and} \\ \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\} &= \delta. \end{aligned}$$

which implies that $\max\{\chi^{\delta}_{\gamma I}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\}$.

Case(b): Let $a \notin I$, since I is semiprime therefore $a^2 \notin I$. This implies that $\chi^{\delta}_{\gamma I}(a) \leq \gamma$ and $\chi^{\delta}_{\gamma I}(a^2) \leq \gamma$, such that

$$\begin{aligned} \max\{\chi^{\delta}_{\gamma I}(a), \gamma\} &= \gamma \text{ and} \\ \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\} &= \chi^{\delta}_{\gamma I}(a^2). \end{aligned}$$

Therefore $\max\{\chi^{\delta}_{\gamma I}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\}$. Hence in both cases

$$\max\{\chi^{\delta}_{\gamma I}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\} forallainS.$$

$(ii) \Rightarrow (i)$ Let $\chi^{\delta}_{\gamma I}$ be an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy semiprime. Let $a^2 \in I$ for some a in S. Then $\chi^{\delta}_{\gamma I}(a^2) \geq \delta$. Therefore $\max\{\chi^{\delta}_{\gamma I}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma I}(a^2), \delta\} = \delta$ this implies that $\chi^{\delta}_{\gamma I}(a) \geq \delta$ again this implies that $a \in I$. Hence I is semiprime. ■

Example 148 *Let $S = \{1, 2, 3\}$, and the binary operation "$\cdot$" be defined on S as follows.*

$\cdot$	1	2	3
1	3	2	3
2	3	3	3
3	3	3	3

Then $(S, \cdot)$ is an AG-groupoid. Define a fuzzy subset $f : S \to [0, 1]$ as follows.

$$f(x) = \begin{cases} 0.41 \text{ for } x = 1 \\ 0.39 \text{ for } x = 2 \\ 0.42 \text{ for } x = 3 \end{cases}$$

Then clearly

- *f is $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy semiprime,*
- *f is not $(\in, \in \vee q_{0.2})$-fuzzy semiprime,*
- *f is not fuzzy semiprime.*

5.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy semiprime Ideals of Intra-regular AG-groupoids

Lemma 149 *If f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an intra-regular AG-groupoid S, then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime in S.*

Proof. Let S be a intra regular AG-groupoid. Then for any $a \in S$ there exists some $x, y \in S$ such that $a = (xa^2)y$. Now

$$\max\{f(a), \gamma\} = \max\{f(xa^2)y, \gamma\} \geq \min\{f(a^2), \delta\}.$$

Hence f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime in S. ■

Theorem 150 *Let S be an AG- groupoid then the following conditions are equivalent.*
(i) S is intra regular.
(ii) For every ideal A of S, $A \subseteq A^2$ and A is semiprime.
(iii) For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy ideal f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$, and f is fuzzy semiprime.

Proof. $(i) \Rightarrow (iii)$. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an intra regular AG-groupoid S with left identity. Now since S is intra regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Now using paramedial law, medial law and left invertive law, we get

$$a = (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a.$$

Let for any a in S there exist p and q in S such that $a = pq$, then

$$\begin{aligned}
\max\{(f \circ f)(a), \gamma\} &= \max\left\{\bigvee_{a=pq}\{\{f(p) \wedge f(q)\}, \gamma\}\right\} \\
&\geq \max\{\min\{f(y(xa)), f(a)\}, \gamma\} \\
&\geq \max\{\min\{f(y(xa)), f(a)\}, \gamma\} \\
&= \min\{\max\{f(y(xa)), \gamma\}, \max\{f(a), \gamma\}\} \\
&\geq \min\{\min\{f(a), \delta\}, \min\{f(a), \delta\}\} \\
&= \min\{f(a), \delta\}.
\end{aligned}$$

Thus $f \subseteq \vee q_{(\gamma,\delta)} f \circ f$.

Now we show that f is a fuzzy semiprime ideal of intra-regular AG-groupoid S, Since S is intra-regular therefore for any a in S there exist x, y in S such that $a = (xa^2)y$. Then

$$\begin{aligned}
\max\{f(a), \gamma\} &= \max\{f((xa^2)y), \gamma\} \\
&\geq \min\{f(a^2), \delta\}.
\end{aligned}$$

$(iii) \Rightarrow (ii)$. Suppose A be any ideal of S. Then by (iii), we get

$$\chi_{\gamma A}^{\delta} = \chi_{\gamma A\cap A}^{\delta} = \chi_{\gamma A}^{\delta} \cap \chi_{\gamma A}^{\delta} \subseteq \vee q_{(\gamma,\delta)} \chi_{\gamma A}^{\delta} \circ \chi_{\gamma A}^{\delta} =_{(\gamma,\delta)} X_{\gamma A^2}^{\delta}.$$

Hence we get $A \subseteq A^2$. Now we show that A is semiprime. Let A is an ideal then $(\chi_{\gamma A}^{\delta})$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Let $a^2 \in A$, then since $\chi_{\gamma A}^{\delta}$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an AG-groupoid S, hence by (iii), $\max\{\chi_{\gamma A}^{\delta}(a), \gamma\} \geq \min\{\chi_{\gamma A}^{\delta}(a^2), \delta\} = \delta$ this implies that $\chi_{\gamma A}^{\delta}(a) \geq \delta$. Thus $a \in A$. This implies that A is semiprime.

$(ii) \Rightarrow (i)$. Assume that every ideal is semiprime of S. Since Sa^2 is a ideal of an AG-groupoid S generated by a^2. Therefore

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is intra regular. ■

Lemma 151 *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of an AG-groupoid S, is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. Let S be an AG-groupoid then for any $a, x, y \in S$ and f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal. Now

$$\begin{aligned} \max\{f((xa)y), \gamma\} &\geq \max\{f(xa), \gamma\} \\ &\geq \min\{f(a), \delta\}. \end{aligned}$$

Hence f is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S. ■

Theorem 152 *For an AG-groupoid S with left identity the following are equivalent.*

(i) *S is intra regular.*

(ii) *Every two sided ideal is semiprime.*

(iii) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal f of S is fuzzy semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal f of S is fuzzy semiprime.*

(v) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized interior ideal f of S is semiprime.*

Proof. $(i) \Rightarrow (v)$ Let S be an intra-regular and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized interior ideal of an AG-groupoid S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$. We have

$$\begin{aligned} \max\{f(a), \gamma\} &= \max\{f((xa^2)y), \gamma\} \\ &\geq \min\{f(a^2), \delta\}. \end{aligned}$$

$(v) \Rightarrow (iv$ is obvious.

$(iv) \Rightarrow (iii)$ it is obvious by lemma 151.

$(iii) \Rightarrow (ii)$. Let A be a two sided ideal of an AG-groupoid S, then by theorem 131, $(\chi_{\gamma A}^{\delta})$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Let

$a^2 \in A$, then since $\chi^{\delta}_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal therefore $\chi^{\delta}_{\gamma A}(a^2) \geq \delta$, thus by (iii) $\max\{\chi^{\delta}_{\gamma A}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma A}(a^2), \delta\} = \delta$ this implies that $\chi^{\delta}_{\gamma A}(a) \geq \delta$. Thus $a \in A$. Hence A is semiprime.

$(ii) \Rightarrow (i)$. Assume that every two sided ideal is semiprime and since Sa^2 is a two sided ideal contain a^2. Thus

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is an intra-regular. ■

Theorem 153 *Let S be an AG-groupoid with left identity, then the following conditions equivalent*

(i) *S is intra-regular.*

(ii) *Every ideal of S is semiprime.*

(iii) *Every bi-ideal of S is semiprime.*

(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f of S is semiprime.*

(v) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f of S is semiprime.*

Proof. $(i) \Rightarrow (v)$. Let S be an intra-regular and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-generalized bi-ideal of S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$.

$$\begin{aligned}
a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\
&= \{y(x((xa^2)y)))\}a = \{x(y((xa^2)y))\}a = \{x((xa^2)y^2))\}a \\
&= \{(xa^2)(xy^2)\}a = \{x^2(a^2y^2)\}a = \{a^2(x^2y^2)\}a = \{a(x^2y^2)\}a^2 \\
&= \{((xa^2)y)(x^2y^2)\}a^2 = \{(y^2y)(x^2(xa^2))\}a^2 = \{(y^2x^2)(y(xa^2))\}a^2 \\
&= \{(y^2x^2)((y_1y_2)(xa^2))\}a^2 = \{(y^2x^2)((a^2y_2)(xy_1))\}a^2 = \{(y^2x^2)((a^2x)(y_2y_1))\}a^2 \\
&= \{(y^2x^2)(((y_2y_1)x)(aa))\}a^2 = \{(y^2x^2)(a^2(x(y_2y_1)))\}a^2 = \{(aa)(x((x^2y^2))(y_2y_1))\}a^2 \\
&= \{a^2\{(x(x^2y^2)(y_2y_1))\}\}a^2 = (a^2t)a^2, \text{where } t = (x(x^2y^2)(y_2y_1)).
\end{aligned}$$

we have

$\max\{f(a), \gamma\} = \max\{f(a^2t)a^2, \gamma\} \geq \max\{\min\{f(a^2), f(a^2)\}, \delta\} = \min\{f(a^2), \delta\}$.

Therefore $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.

$(v) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (iii)$. Let B is a bi-ideal of S, then $\chi^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an AG-groupoid S. let $a^2 \in B$ then since $\chi^{\delta}_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal therefore $\chi^{\delta}_{\gamma B}(a^2) \geq \delta$, thus by (iv), $\max\{\chi^{\delta}_{\gamma B}(a), \gamma\} \geq \min\{\chi^{\delta}_{\gamma B}(a^2), \delta\} = \delta$ this implies that $\chi^{\delta}_{\gamma B}(a) \geq \delta$. Thus $a \in B$. Hence B is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal of S is semiprime and since Sa^2 is an ideal containing a. Thus

$$a \in (Sa^2) \subseteq (SS)a^2 \subseteq (a^2S)S = ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S.$$

Hence S is an intra-regular. ■

Theorem 154 *Let S be an AG-groupoid with left identity, then the following conditions equivalent*
(i) *S is intra-regular.*
(ii) *Every ideal of S is semiprime.*
(iii) *Every quasi-ideal of S is semiprime.*
(iv) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f of S is semiprime.*

Proof. $(i) \Rightarrow (iv)$. Let S be an intra-regular AG-groupoid with left identity and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of S. Then for all $a \in S$ there exists x, y in S such that $a = (xa^2)y$. Now using left invertive law and medial law, then

$$a = (xa^2)(y_1y_2) = (y_2y_1)(a^2x) = a^2((y_2y_1)x) = a^2t, \text{ where } t = (y_2y_1)x.$$

we have
$\max\{f(a), \gamma\} = \max\{f(a^2t), \gamma\} \geq \min\{f(a^2), \delta\}$.
Therefore $\max\{f(a), \gamma\} \geq \min\{f(a^2), \delta\}$.

$(iv) \Rightarrow (iii)$. let Q be an quasi ideal of S, then $\chi^\delta_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal of an AG-groupoid S. let $a^2 \in Q$ then since $\chi^\delta_{\gamma Q}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi ideal as then $\chi^\delta_{\gamma Q}(a^2) \geq \delta$ therefore by (iv), $\max\{\chi^\delta_{\gamma Q}(a), \gamma\} \geq \min\{\chi^\delta_{\gamma Q}(a^2), \delta\} = \delta$ this implies that $\chi^\delta_{\gamma Q}(a) \geq \delta$. Thus $a \in Q$. Hence Q is semiprime.

$(iii) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (i)$ Assume that every ideal of S is semiprime and since Sa^2 is an ideal containing a^2. Thus

$$a \in (Sa^2) \subseteq (Sa)(Sa) = (SS)(aa) = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular. ■

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In this book, we introduce the concept of $(\in, \in \vee q)$-fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in a non-associative algebraic structures called Abel-Grassmann's groupoid, discuss several important features of a regular AG-grupoid, investigate some characterizations of regular and intra-regular AG-grupoids, and so on.

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