An quantum extension to inspection game

Xinyang Deng[a,b], Yong Deng[a,c,*], Zhen Wang[a,d,e,*], Qi Liu[b,f]

[a] *School of Computer and Information Science, Southwest University, Chongqing, 400715, China*
[b] *Center for Quantitative Sciences, Vanderbilt University School of Medicine, Nashville, TN, 37232, USA*
[c] *School of Engineering, Vanderbilt University, Nashville, TN, 37235, USA*
[d] *Department of Physics and Center for Nonlinear Studies, Hong Kong Baptist University, Hong Kong*
[e] *Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga-koen, Kasuga-shi, Fukuoka 816-8580, Japan*
[f] *Department of Biomedical Informatics, Vanderbilt University School of Medicine, Nashville, TN, 37232, USA*

Abstract

Quantum game theory is a new interdisciplinary field between game theory and physical research. In this paper, we extend the classical inspection game into a quantum game version by quantizing the strategy space and importing entanglement between players. The quantum inspection has various Nash equilibrium depending on the initial quantum state of the game. Our results also show that quantization can respectively help each player to increase his own payoff, but can not simultaneously improve the collective payoff in the quantum inspection game.

Keywords: Inspection game, Quantum game, Marinatto-Weber model, Quantum entanglement, Pareto improvement

*Corresponding authors: ydeng@swu.edu.cn (Y. Deng), zhenwang0@gmail.com (Z. Wang).

1. Introduction

Game theory, founded by von Neumann and Morgenstern [1] in 1940s, is a mathematical framework to explain and address the interactive decision situations where the aims, goals and preferences of the participating agents are potentially in conflict. A solution to a game, which is self enforcing and no player can gain by unilaterally deviating from it, is said to be a Nash equilibrium (NE) [2]. For example, to a two-person game, a combination of each player's strategy (s_1^*, s_2^*) is a NE of this game if

$$\begin{aligned} u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \quad \forall s_1 \in S_1 \\ u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2) \quad \forall s_2 \in S_2 \end{aligned} \tag{1}$$

where the payoffs of these two players are determined by functions $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$, respectively. Since been proposed, game theory has gradually become one of the most importance tool to be extensively used in economics, politics, biology, etc [3–13].

In recent years, an important breakthrough of game theory has taken place in quantum information theory [14]. Physicists firstly investigated the quantization of games and presented quantum games and quantum strategies. Eisert et al. [15] used quantum approaches to deal with the prisoner's dilemma, and realized Pareto efficiency if classical prisoner's dilemma were extended into the quantum domain. Meyer [16] showed that a player who implemented a quantum strategy can increase his expected payoff by using a PQ penny flipover game. A lot of researches has greatly presented that quantum games are very different from their classical counterparts [17–25].

Quantum games are opening up a new way to study the dilemmas in the classical game theory.

In this paper, we investigate the inspection game, and extend it into a quantum version. The inspection game [26] is a useful game model to represent the relationship between two individuals who have conflicting interest [27]. It is a two-person game where an individual chooses to inspect or not, the other chooses to comply or not. Since the players have conflict of interests, the game has a mixed-strategy NE which is not a Pareto efficiency. In this paper, we quantize the inspection game to explore whether the quantization could increase not only individual but also collective payoff. The results show that this presented quantum inspection game just is able to respectively increase each player's payoff, but can not realize a Pareto improvement.

2. Inspection game

Inspection game was first proposed by Dresher [26] to describe the interactions between two agents with conflicts of interest, such as inspector and smuggler, employer and worker, etc. Following previous studies on the formal description to this game [28, 29, 27], in an inspection game there are an employer who can either inspect (I) or not inspect (N), and a worker who can either work (W) or shirk (S). The payoff matrix is shown in (2). As illustrated in the payoff matrix, the employer is row player, and the worker is column player. The employer bears a cost of h from inspecting, and pays the worker a wage of w unless he find the worker is shirking. The worker bears some work-related costs g, and creates an outcome of v for the employer if he works. We assume that all variables are positive and $v > g$, $w > h$, $w > g$. Obviously, the joint payoff is maximized when the worker works and the employer does not inspect.

$$\begin{array}{c} \\ I \\ N \end{array} \begin{array}{c} \begin{array}{cc} W & S \end{array} \\ \left(\begin{array}{cc} (v-w-h, w-g) & (-h, 0) \\ (v-w, w-g) & (-w, w) \end{array}\right) \end{array} \tag{2}$$

For the above payoff matrix (2), it does not exist a pure-strategy NE due to $v - w > v - w - h$, $-h > -w$, $w - g > 0$ and $w > w - g$. It has a mixed NE where the employer inspects with probability $p = g/w$ and the worker works with probability $q = 1 - h/w$. In this equilibrium the employer gets a payoff of $\bar{\$}_A = v - w - hv/w$, the worker receives a payoff of $\bar{\$}_B = w - g$, and the joint payoff is $\bar{\$}_{A+B} = v - c - hv/w$. To illustrate this point, let us consider the following example [29]:

$$v = 60, g = 15, h = 8, w = 20 \tag{3}$$

Then the payoff matrix (2) takes the form

$$\begin{array}{c} \\ I \\ N \end{array} \begin{array}{c} \begin{array}{cc} W & S \end{array} \\ \left(\begin{array}{cc} (32, 5) & (-8, 0) \\ (40, 5) & (-20, 20) \end{array}\right) \end{array} \tag{4}$$

It is clear that there is no pure-strategy NE in this game. Now let us check its mixed-strategy NE. We assume that the employer inspects with probability p and the worker works with probability q. The expected payoff of each player is defined as

$$\bar{\$}_i(p,q) = pqE_i(I,W)+p(1-q)E_i(I,S)+(1-p)qE_i(N,W)+(1-p)(1-q)E_i(N,S) \tag{5}$$

where $E_i(s_1, s_2)$ is player i's payoff of strategy s_1 playing against strategy s_2. So, for employer A and worker B, their expected payoffs are

$$\begin{cases} \bar{\$}_A(p,q) = 12p + 60q - 20pq - 20 \\ \bar{\$}_B(p,q) = 20pq - 15q - 20p + 20 \end{cases} \tag{6}$$

If there does exist a mixed-strategy NE, this equilibrium can be found by calculating $\frac{\partial \bar{\$}_A(p,q)}{\partial p} = 0$ and $\frac{\partial \bar{\$}_B(p,q)}{\partial q} = 0$. The results are $p^* = 0.75$, $q^* = 0.6$. It is easy to demonstrate that (p^*, q^*) indeed is a mixed-strategy NE because

$$\begin{cases} \bar{\$}_A(p^*,q^*) \geq \bar{\$}_A(p,q^*), \quad \forall p \in [0,1] \\ \bar{\$}_B(p^*,q^*) \geq \bar{\$}_B(p^*,q), \quad \forall q \in [0,1] \end{cases} \tag{7}$$

In this NE, the expected payoffs of employer A and worker B are $\bar{\$}^*_A = 16$, $\bar{\$}^*_B = 5$, respectively. And the joint payoff of A and B is $\bar{\$}^*_{A+B} = 21$.

3. Quantum inspection game

At present, there are several schemes, for example Eisert-Wilkens-Lewenstein scheme [15] and Marinatto-Weber scheme [17], to quantize the classical strategy space so as to build a quantum game. In this section, we follow Marinatto-Weber scheme to quantize the strategy space for inspection game.

Let us define a four-dimensional Hilbert space H for the inspection game by giving its orthonormal basis vectors $H = H_A \otimes H_B = \{|IW\rangle, |IS\rangle, |NW\rangle, |NS\rangle\}$, where the first qubit is reserved to the state of employer A and the second one to that of worker B. In the initial, assuming that these two players, employer A and worker B, share the following quantum state:

$$\left|\psi_{in}\right\rangle = a\left|IW\right\rangle + b\left|IS\right\rangle + c\left|NW\right\rangle + d\left|NS\right\rangle \tag{8}$$

where $|a|^2+|b|^2+|c|^2+|d|^2 = 1$. According to state vector $\left|\psi_{in}\right\rangle$, the associated density matrix can be derived as $\rho_{in} = \left|\psi_{in}\right\rangle\left\langle\psi_{in}\right|$.

Let C be a unitary and Hermitian operator (i.e., such that $C^\dagger = C = C^{-1}$), such that

$$C\left|I\right\rangle = \left|N\right\rangle, C\left|N\right\rangle = \left|I\right\rangle, C\left|W\right\rangle = \left|S\right\rangle, C\left|S\right\rangle = \left|W\right\rangle \tag{9}$$

In the game process, employer A does nothing with probability p by using identity operator I , and performs operator C on the first qubit with probability $1-p$. Analogously, worker B does nothing with probability q and performs C on the second qubit with probability $1-q$. Then, the final density matrix for this two-qubit quantum system takes the form:

$$\begin{aligned}\rho_{fin} = pq &\left[(I_A \otimes I_B)\rho_{in}(I_A^\dagger \otimes I_B^\dagger)\right] \\ &+p(1-q)\left[(I_A \otimes C_B)\rho_{in}(I_A^\dagger \otimes C_B^\dagger)\right] \\ &+(1-p)q\left[(C_A \otimes I_B)\rho_{in}(C_A^\dagger \otimes I_B^\dagger)\right] \\ &+(1-p)(1-q)\left[(C_A \otimes C_B)\rho_{in}(C_A^\dagger \otimes C_B^\dagger)\right]\end{aligned} \tag{10}$$

In order to calculate the payoffs, two payoff operators are introduced:

$$\begin{aligned}P_A = &E_A(I,W)\left|IW\right\rangle\left\langle IW\right| + E_A(I,S)\left|IS\right\rangle\left\langle IS\right| \\ &+E_A(N,W)\left|NW\right\rangle\left\langle NW\right| + E_A(N,S)\left|NS\right\rangle\left\langle NS\right|\end{aligned} \tag{11}$$

$$P_B = E_B(I,W)\,|IW\rangle\,\langle IW| + E_B(I,S)\,|IS\rangle\,\langle IS| \\ +E_B(N,W)\,|NW\rangle\,\langle NW| + E_B(N,S)\,|NS\rangle\,\langle NS| \tag{12}$$

Finally, the payoff functions of A and B can be obtained as mean values of these operators:

$$\bar{\$}_A(p,q) = Tr(P_A\rho_{fin}), \quad \bar{\$}_B(p,q) = Tr(P_B\rho_{fin}). \tag{13}$$

As shown above, based on Marinatto-Weber scheme, we successfully build quantum inspection game. Next, analyses will be done to study the impact of quantization on the inspection game. In order to reduce the complexity, we specify the payoff matrix of inspection game as (4) which has been considered in the above section.

According to (4), and equations (8)-(13), the expected payoff functions for both players can be obtained as follows:

$$\begin{aligned} \bar{\$}_A(p,q) &= p[q(20|b|^2+20|c|^2-20|a|^2-20|d|^2)+(12|a|^2-12|c|^2+8|d|^2-8|b|^2)] \\ &\quad +q(60|a|^2-60|b|^2+40|c|^2-40|d|^2)+(40|b|^2-20|a|^2-8|c|^2+32|d|^2) \\ \bar{\$}_B(p,q) &= q[p(20|a|^2-20|b|^2-20|c|^2+20|d|^2)+(15|b|^2-15|a|^2+5|c|^2-5|d|^2)] \\ &\quad +p(20|c|^2-20|a|^2)+(20|a|^2+5|b|^2+5|d|^2) \end{aligned} \tag{14}$$

A NE (p^*,q^*) can be found by imposing the following two conditions:

$$\begin{aligned} \bar{\$}_A(p^*,q^*)-\bar{\$}_A(p,q^*) &= (p^*-p)[q^*(20|b|^2+20|c|^2-20|a|^2-20|d|^2) \\ &\quad +(12|a|^2-12|c|^2+8|d|^2-8|b|^2)] \geq 0, \quad \forall p \in [0,1] \\ \bar{\$}_B(p^*,q^*)-\bar{\$}_B(p^*,q) &= (q^*-q)[p^*(20|a|^2-20|b|^2-20|c|^2+20|d|^2) \\ &\quad +(15|b|^2-15|a|^2+5|c|^2-5|d|^2)] \geq 0, \quad \forall q \in [0,1] \end{aligned} \tag{15}$$

According to inequalities (15), we can find the required condition for any possible NE. There are five cases.

Case 1: let $(p^* = 1, q^* = 1)$ ***be a NE.***

Let us consider a case of $(p^* = 1, q^* = 1)$ and find the condition of it being a NE. In such a case, the conditions shown in (15) translates to

$$\begin{aligned} \bar{\$}_A(1,1) - \bar{\$}_A(p,1) &= (1-p)(12|b|^2 + 8|c|^2 - 8|a|^2 - 12|d|^2) \geq 0, \quad \forall p \in [0,1] \\ \bar{\$}_B(1,1) - \bar{\$}_B(1,q) &= (1-q)(5|a|^2 - 5|b|^2 - 15|c|^2 + 15|d|^2) \geq 0, \quad \forall q \in [0,1] \end{aligned} \tag{16}$$

The inequalities (16) require

$$C1. \quad \begin{cases} -2|a|^2 + 3|b|^2 + 2|c|^2 - 3|d|^2 \geq 0 \\ |a|^2 - |b|^2 - 3|c|^2 + 3|d|^2 \geq 0 \\ |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \end{cases} \tag{17}$$

The above condition *C1* holds, for example, when $|a|^2 = 0.6, |b|^2 = 0.4, |c|^2 = |d|^2 = 0$. In this case that $(p^* = 1, q^* = 1)$ is a NE, the corresponding payoff functions are found as follows:

$$\begin{cases} \bar{\$}_A(p^* = 1, q^* = 1) = 32|a|^2 - 8|b|^2 + 40|c|^2 - 20|d|^2 \\ \bar{\$}_B(p^* = 1, q^* = 1) = 5|a|^2 + 5|c|^2 + 20|d|^2 \end{cases} \tag{18}$$

According to condition *C1* and the above equations (18), the range of each payoff can be found easily.

$$\begin{cases} \bar{\$}_A(p^* = 1, q^* = 1) \in [-14, 19.333] \\ \bar{\$}_B(p^* = 1, q^* = 1) \in [2.5, 10.625] \\ \bar{\$}_{A+B}(p^* = 1, q^* = 1) \in [-6, 22.667] \end{cases} \tag{19}$$

These results show that either the payoff of each player or two players' joint payoff could increase in the quantum inspection game. Moreover, from equations (19), the maximum joint payoff can be 22.667. Recalling the classical inspection game, we find that it has a unique NE where $\bar{\$}^*_A = 16$, $\bar{\$}^*_B = 5$ and $\bar{\$}^*_{A+B} = 21$. A key question is that whether the increase of joint payoff in quantum inspection game results from the simultaneous increase of each single player's payoff. If it is, the quantum version of inspection game definitely outperforms the classical one because it carries out a Pareto improvement. We give the following optimization problem to answer this question:

$$\begin{array}{ll} \max & \bar{\$}_{A+B}(p^*=1,q^*=1)=\bar{\$}_A(p^*=1,q^*=1)+\bar{\$}_B(p^*=1,q^*=1) \\ s.t. & \left\{ \begin{array}{l} \bar{\$}_A(p^*=1,q^*=1)\geq 16 \\ \bar{\$}_B(p^*=1,q^*=1)\geq 5 \\ -2|a|^2+3|b|^2+2|c|^2-3|d|^2\geq 0 \\ |a|^2-|b|^2-3|c|^2+3|d|^2\geq 0 \\ |a|^2+|b|^2+|c|^2+|d|^2=1 \end{array} \right. \end{array} \tag{20}$$

For the optimization problem (20), the maximum of object function is 21 when $|a|^2 = 0.45$, $|b|^2 = 0.3$, $|c|^2 = 0.15$, $|d|^2 = 0.1$, at that situation $\$_A(p^* = 1, q^* = 1) = 16$ and $\bar{\$}_B(p^* = 1, q^* = 1) = 5$. Therefore, in this case, the quantum inspection game increases either the payoff of employer A or the payoff of worker B, but can not simultaneously increase the payoffs of A and B, compared with the equilibrium of classical inspection game. In short, the quantum inspection game does not carry out a Pareto improvement.

Case 2: let $(p^* = 0, q^* = 0)$ ***be a NE.***

Let us examine another case of $(p^* = 0, q^* = 0)$ and find the condition of it being a NE. In such a case, the conditions shown in (15) translates to

$$\begin{aligned} \bar{\$}_A(0,0) - \bar{\$}_A(p,0) &= p(12|c|^2 - 12|a|^2 + 8|b|^2 - 8|d|^2) \geq 0, \quad \forall p \in [0,1] \\ \bar{\$}_B(0,0) - \bar{\$}_B(0,q) &= q(15|a|^2 - 15|b|^2 - 5|c|^2 + 5|d|^2) \geq 0, \quad \forall q \in [0,1] \end{aligned} \tag{21}$$

The inequalities (21) require

$$C2. \quad \begin{cases} -3|a|^2 + 2|b|^2 + 3|c|^2 - 2|d|^2 \geq 0 \\ 3|a|^2 - 3|b|^2 - |c|^2 + |d|^2 \geq 0 \\ |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \end{cases} \tag{22}$$

The above condition *C2* holds, for example, when $|a|^2 = 0.4375$, $|b|^2 = 0.375$, $|c|^2 = 0.1875$, $|d|^2 = 0$. In this case that $(p^* = 0, q^* = 0)$ is a NE, the corresponding payoff functions are found as follows:

$$\begin{cases} \bar{\$}_A(p^* = 0, q^* = 0) = -20|a|^2 + 40|b|^2 - 8|c|^2 + 32|d|^2 \\ \bar{\$}_B(p^* = 0, q^* = 0) = 20|a|^2 + 5|b|^2 + 5|d|^2 \end{cases} \tag{23}$$

According to condition *C2* and equations (23), the range of each payoff in this case are

$$\begin{cases} \bar{\$}_A(p^* = 0, q^* = 0) \in [-14, 19.333] \\ \bar{\$}_B(p^* = 0, q^* = 0) \in [2.5, 10.625] \\ \bar{\$}_{A+B}(p^* = 0, q^* = 0) \in [-6, 22.667] \end{cases} \tag{24}$$

These results are identical with the case of $(p^* = 1, q^* = 1)$. Similarly, we examine if there exists possible Pareto improvement based on the following optimization equations.

$$\max \quad \bar{\$}_{A+B}(p^*=0,q^*=0)=\bar{\$}_A(p^*=0,q^*=0)+\bar{\$}_B(p^*=0,q^*=0)$$
$$s.t. \quad \begin{cases} \bar{\$}_A(p^*=0,q^*=0)\geq 16 \\ \bar{\$}_B(p^*=0,q^*=0)\geq 5 \\ -3|a|^2+2|b|^2+3|c|^2-2|d|^2\geq 0 \\ 3|a|^2-3|b|^2-|c|^2+|d|^2\geq 0 \\ |a|^2+|b|^2+|c|^2+|d|^2=1 \end{cases} \tag{25}$$

For (25), the maximum of object function is 21 when $|a|^2=0.1$, $|b|^2=0.15$, $|c|^2=0.3$, $|d|^2=0.45$, at that situation $\$_A(p^*=0,q^*=0)=16$ and $\bar{\$}_B(p^*=0,q^*=0)=5$. Therefore, in this case of $(p^*=0,q^*=0)$ being a NE, the quantum inspection game still does not carry out a Pareto improvement compared with the NE of classical inspection game.

Case 3: let $(p^*=1,q^*=0)$ ***be a NE.***

Now let us consider the case of $(p^*=1,q^*=0)$ to find the condition of it being a NE. In this case, the conditions shown in (15) translates to

$$\begin{aligned} &\bar{\$}_A(1,0)-\bar{\$}_A(p,0)=(1-p)(12|a|^2-8|b|^2-12|c|^2+8|d|^2)\geq 0, \quad \forall p\in[0,1] \\ &\bar{\$}_B(1,0)-\bar{\$}_B(1,q)=q(-5|a|^2+5|b|^2+15|c|^2-15|d|^2)\geq 0, \quad \forall q\in[0,1] \end{aligned} \tag{26}$$

The inequalities (26) require

$$C3. \quad \begin{cases} 3|a|^2-2|b|^2-3|c|^2+2|d|^2\geq 0 \\ -|a|^2+|b|^2+3|c|^2-3|d|^2\geq 0 \\ |a|^2+|b|^2+|c|^2+|d|^2=1 \end{cases} \tag{27}$$

Condition *C3* holds, for example, when $|a|^2 = 0.75$, $|b|^2 = 0$, $|c|^2 = 0.25$, $|d|^2 = 0$. In this case of $(p^* = 1, q^* = 0)$ being a NE, the corresponding payoff functions are shown as follows:

$$\begin{cases} \bar{\$}_A(p^* = 1, q^* = 0) = -8|a|^2 + 32|b|^2 - 20|c|^2 + 40|d|^2 \\ \bar{\$}_B(p^* = 1, q^* = 0) = 5|b|^2 + 20|c|^2 + 5|d|^2 \end{cases} \tag{28}$$

Based on condition *C3* and equations (28), we can verify that the range of each payoff is completely same with that of above cases, namely

$$\begin{cases} \bar{\$}_A(p^* = 1, q^* = 0) \in [-14, 19.333] \\ \bar{\$}_B(p^* = 1, q^* = 0) \in [2.5, 10.625] \\ \bar{\$}_{A+B}(p^* = 1, q^* = 0) \in [-6, 22.667] \end{cases} \tag{29}$$

Similarly, we find that in this case of $(p^* = 1, q^* = 0)$ being a NE, the quantum version of inspection game is still not able to implement a Pareto improvement on the mixed-strategy NE of classical inspection game. The maximum joint payoff of A and B is 21 when $|a|^2 = 0.3$, $|b|^2 = 0.45$, $|c|^2 = 0.1$, $|d|^2 = 0.15$, at the same time $\$_A(p^* = 1, q^* = 0) = 16$ and $\bar{\$}_B(p^* = 1, q^* = 0) = 5$.

Case 4: let $(p^* = 0, q^* = 1)$ ***be a NE.***

Let us explore the case of $(p^* = 0, q^* = 1)$ being a NE. In such a case, the conditions shown in (15) translates to

$$\begin{aligned} &\bar{\$}_A(0,1) - \bar{\$}_A(p,1) = p(8|a|^2 - 12|b|^2 - 8|c|^2 + 12|d|^2) \geq 0, \quad \forall p \in [0,1] \\ &\bar{\$}_B(0,1) - \bar{\$}_B(0,q) = (1-q)(-15|a|^2 + 15|b|^2 + 5|c|^2 - 5|d|^2) \geq 0, \quad \forall q \in [0,1] \end{aligned} \tag{30}$$

The inequalities (30) require

$$C4. \quad \begin{cases} 2|a|^2 - 3|b|^2 - 2|c|^2 + 3|d|^2 \geq 0 \\ -3|a|^2 + 3|b|^2 + |c|^2 - |d|^2 \geq 0 \\ |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \end{cases} \tag{31}$$

The condition *C4* holds, for example, when $|a|^2 = |c|^2 = 0$, $|b|^2 = |d|^2 = 0.5$. In this case of $(p^* = 0, q^* = 1)$ being a NE, the corresponding payoff functions are shown as follows:

$$\begin{cases} \bar{\$}_A(p^* = 0, q^* = 1) = 40|a|^2 - 20|b|^2 + 32|c|^2 - 8|d|^2 \\ \bar{\$}_B(p^* = 0, q^* = 1) = 5|a|^2 + 20|b|^2 + 5|c|^2 \end{cases} \tag{32}$$

Depending on condition *C4* and equations (32), we can obtain the range of each payoff

$$\begin{cases} \bar{\$}_A(p^* = 0, q^* = 1) \in [-14, 19.333] \\ \bar{\$}_B(p^* = 0, q^* = 1) \in [2.5, 10.625] \\ \bar{\$}_{A+B}(p^* = 0, q^* = 1) \in [-6, 22.667] \end{cases} \tag{33}$$

By examining, we again find that in this case there is not Pareto improvement even adopting the quantum inspection game. The maximum joint payoff of A and B, which equals to 21, is calculated when $|a|^2 = 0.15$, $|b|^2 = 0.1$, $|c|^2 = 0.45$, $|d|^2 = 0.3$, and $\$_A(p^* = 1, q^* = 0) = 16$, $\bar{\$}_B(p^* = 1, q^* = 0) = 5$ corresponding to that set of values.

Case 5: let (p^*, q^*) ***be a mixed-strategy NE.***

In this case, let (p^*, q^*) become a mixed-strategy NE. Here, we first consider the situation that $p^*, q^* \in (0, 1)$. For p^* and q^* differing from 0 or 1, the inequalities (15) are required to be satisfied. Since the factors $(p^* - p)$ and

$(q^* - q)$ may be positive or negative for different values of p and q, the only way to fulfil conditions shown in (15) is to make the coefficients of $(p^* - p)$ and $(q^* - q)$ become zeros. So, the NE can be obtained as follows.

$$p^* = \frac{3|a|^2 - 3|b|^2 - |c|^2 + |d|^2}{4|a|^2 - 4|b|^2 - 4|c|^2 + 4|d|^2} \quad , \quad q^* = \frac{3|a|^2 - 2|b|^2 - 3|c|^2 + 2|d|^2}{5|a|^2 - 5|b|^2 - 5|c|^2 + 5|d|^2} \tag{34}$$

The above mixed-strategy NE is constrained by several self-evident conditions denoted as $C5$: (i) $p^*, q^* \in (0,1)$; (ii) $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Corresponding to the pair of p^* and q^*, the payoff functions are

$$\begin{cases} \bar{\$}_A(p^*, q^*) = \frac{16|a|^4 - 16|b|^4 - 16|c|^4 + 16|d|^4 + 12|a|^2|d|^2 - 12|b|^2|c|^2}{|a|^2 - |b|^2 - |c|^2 + |d|^2} \\ \bar{\$}_B(p^*, q^*) = \frac{5|a|^4 - 5|b|^4 - 5|c|^4 + 5|d|^4 + 20|a|^2|d|^2 - 20|b|^2|c|^2}{|a|^2 - |b|^2 - |c|^2 + |d|^2} \end{cases} \tag{35}$$

Likewise, the bound of each payoff is found as follows.

$$\begin{cases} \bar{\$}_A(p^*, q^*) \in [11, 16] \\ \bar{\$}_B(p^*, q^*) \in [5, 7.5] \\ \bar{\$}_{A+B}(p^*, q^*) \in [18.5, 21] \end{cases} \tag{36}$$

It is clear that there is not Pareto improvement in the quantum inspection game because the maximin joint payoff of A and B does not exceed 21. On the contrary, in this case the import of quantization decreases the collective payoffs of all players. Moreover, in this case the payoff of employer A never exceeds 16, and that of worker B is never below 5. Different from cases mentioned above, in this case of (p^*, q^*) being a NE where $p^*, q^* \in (0,1)$, it is definitely beneficial for worker B, and harmful for employer A, compared with the classical inspection game.

In addition, a mixed-strategy NE may also be $(p^*, 0)$, $(p^*, 1)$, $(0, q^*)$ or $(1, q^*)$, where $p^*, q^* \in (0, 1)$. For these cases, we have examined these possibilities one by one, and still found that Pareto improvement did not occur.

4. NEs of some typical quantum states

In this section, to illustrate the features of quantum inspection game, some typical quantum states will be set as initial states. These examples are helpful to further understand the quantum version of inspective game.

Example 1: $|\psi_{in}\rangle = |IW\rangle$**.**

Let $|a|^2 = 1$, $|b|^2 = |c|^2 = |d|^2 = 0$, a simple initial state $|\psi_{in}\rangle = |IW\rangle$ is obtained. According to Marinatto and Weber's description [17], this state is a factorizable quantum state. Based on equations (8) - (15), the NE of this game is readily found. It has a unique NE of $(p^* = 0.75, q^* = 0.6)$. And the payoff of each player is $\bar{\$}_A(0.75, 0.6) = 16$, $\bar{\$}_B(0.75, 0.6) = 5$. These results are identical with the classical inspection game, which shows that the quantum game has reproduced the results of classical game theory in the case of factorizable quantum state.

Example 2: $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IW\rangle + \sqrt{\frac{1}{2}}|NS\rangle$ ***or*** $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IS\rangle + \sqrt{\frac{1}{2}}|NW\rangle$**.**

These two quantum states $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IW\rangle + \sqrt{\frac{1}{2}}|NS\rangle$ and $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IS\rangle + \sqrt{\frac{1}{2}}|NW\rangle$ are known as Bell states in quantum mechanics, showing that these two players are entangled in the initial. In these two states, the quantum inspection game has a same and unique NE of $(p^* = 0.5, q^* = 0.5)$, where $\bar{\$}_A(0.5, 0.5) = 11$, $\bar{\$}_B(0.5, 0.5) = 7.5$. These results show that worker B benefits from the entanglement, but employer A does not. Moreover, the entanglement decreases the joint payoff of A and B.

Example 3: $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IW\rangle + \sqrt{\frac{1}{2}}|IS\rangle$**.**

In this example, the quantum inspection game has countless NEs denoted as $\{(1, q^*)|q^* \in [0, 1]\}$. Payoffs of players are $\bar{\$}_A(1, q^*) = 12$, $\bar{\$}_B(1, q^*) = 2.5$, $\forall q^* \in [0, 1]$. Compared the NE of classical inspection game, both of players' payoffs have been decreased.

Example 4: $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|NW\rangle + \sqrt{\frac{1}{2}}|NS\rangle$**.**

The results of this example are given as follows, which are similar with Example 3.

NEs: $\{(0, q^*)|q^* \in [0, 1]\}$.

$\bar{\$}_A(0, q^*) = 12$, $\bar{\$}_B(0, q^*) = 2.5$, $\forall q^* \in [0, 1]$.

Example 5: $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IW\rangle + \sqrt{\frac{1}{2}}|NW\rangle$**.**

In this example, the payoff of employer A becomes negative, and that of worker B is a relatively high value.

NEs: $\{(p^*, 0)|p^* \in [0, 1]\}$.

$\bar{\$}_A(p^*, 0) = -14$, $\bar{\$}_B(p^*, 0) = 10$, $\forall p^* \in [0, 1]$.

Example 6: $|\psi_{in}\rangle = \sqrt{\frac{1}{2}}|IS\rangle + \sqrt{\frac{1}{2}}|NS\rangle$**.**

Here, similar with Example 5, no matter what strategy employer A chooses, his payoff is always negative.

NEs: $\{(p^*, 1)|p^* \in [0, 1]\}$.

$\bar{\$}_A(p^*, 1) = -14$, $\bar{\$}_B(p^*, 1) = 10$, $\forall p^* \in [0, 1]$.

Example 7: $|\psi_{in}\rangle = \sqrt{\frac{1}{4}}|IW\rangle + \sqrt{\frac{1}{4}}|IS\rangle + \sqrt{\frac{1}{4}}|NW\rangle + \sqrt{\frac{1}{4}}|NS\rangle$**.**

For this example, the results are shown as follows.

NEs: $\{(p^*, q^*)|p^*, q^* \in [0, 1]\}$.

$\bar{\$}_A(p^*, q^*) = 11$, $\bar{\$}_B(p^*, q^*) = 7.5$, $\forall p^*, q^* \in [0, 1]$.

It presents very interesting results. In this setting of initial quantum state, the strategies of players do not affect the equilibrium of this quantum game. The joint payoff of A and B, which equals to 18.5, is still lower than that of the NE of classical game.

5. Conclusions

In this paper, the classical inspection game has been extended into a quantum version, by using Marinatto-Weber quantum game model. The conditions of establishment for various NEs have been discussed in the quantum inspection game. Based on classical game theory, there is only a unique mixed-strategy NE in the inspection game. However, if quantizing this game and setting a suitable initial quantum state for the two-qubit system, either pure-strategy NEs or mixed-strategy NEs can be found.

Compared with the classical inspection game, the quantum form of the game has two main characteristics. At first, in the quantum inspection game, either employer or worker has ways to obtain more than that receiving from NE of classical inspection game. At second, the quantization can not bring

Pareto improvement to the classical inspection game. In the quantum version of inspection game, the employer or worker who wants to improve his benefit has to harm the interest of the other. The results show that there may exist irreconcilable conflict between employer and worker in the inspection game, which is different from prisoner's dilemma where Pareto efficiency can be carried out by using quantum strategies [15]. This work brings a new insight to the inspection game and quantum game theory.

Acknowledgment

The work is partially supported by China Scholarship Council, National Natural Science Foundation of China (Grant No. 61174022), Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20131102130002), R&D Program of China (2012BAH07B01), National High Technology Research and Development Program of China (863 Program) (Grant No. 2013AA013801), the open funding project of State Key Laboratory of Virtual Reality Technology and Systems, Beihang University (Grant No.BUAA-VR-14KF-02).

References

[1] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior (60th Anniversary Commemorative Edition), Princeton university press, 2007.

[2] J. F. Nash, et al., Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36 (1) (1950) 48–49.

[3] M. A. Nowak, R. M. May, Evolutionary games and spatial chaos, Nature 359 (6398) (1992) 826–829.

[4] G. Szabó, A. Szolnoki, Congestion phenomena caused by matching pennies in evolutionary games, Physical Review E 91 (3) (2015) 032110.

[5] A. Szolnoki, M. Perc, Conformity enhances network reciprocity in evolutionary social dilemmas, Journal of The Royal Society Interface 12 (103) (2015) 20141299.

[6] Z. Wang, L. Wang, M. Perc, Degree mixing in multilayer networks impedes the evolution of cooperation, Physical Review E 89 (5) (2014) 052813.

[7] C. Gracia-Lazaro, J. Gomez-Gardenes, L. M. Floria, Y. Moreno, Intergroup information exchange drives cooperation in the public goods game, Physical Review E 90 (4) (2014) 042808.

[8] X. Deng, Z. Wang, Q. Liu, Y. Deng, S. Mahadevan, A belief-based evolutionarily stable strategy, Journal of Theoretical Biology 361 (2014) 81–86.

[9] X. Deng, Q. Liu, R. Sadiq, Y. Deng, Impact of roles assignation on heterogeneous populations in evolutionary dictator game, Scientific Reports 4 (2014) 6937.

[10] F. C. Santos, J. M. Pacheco, Scale-free networks provide a unifying framework for the emergence of cooperation, Physical Review Letters 95 (9) (2005) 098104.

[11] Z. Wang, S. Kokubo, J. Tanimoto, E. Fukuda, K. Shigaki, Insight into the so-called spatial reciprocity, Physical Review E 88 (4) (2013) 042145.

[12] Z. Wang, C.-Y. Xia, S. Meloni, C.-S. Zhou, Y. Moreno, Impact of social punishment on cooperative behavior in complex networks, Scientific reports 3 (2013) 3055.

[13] S. Boccaletti, G. Bianconi, R. Criado, C. Del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendina-Nadal, Z. Wang, M. Zanin, The structure and dynamics of multilayer networks, Physics Reports 544 (1) (2014) 1–122.

[14] E. Klarreich, Playing by quantum rules, Nature 414 (6861) (2001) 244–245.

[15] J. Eisert, M. Wilkens, M. Lewenstein, Quantum games and quantum strategies, Physical Review Letters 83 (15) (1999) 3077.

[16] D. A. Meyer, Quantum strategies, Physical Review Letters 82 (5) (1999) 1052.

[17] L. Marinatto, T. Weber, A quantum approach to static games of complete information, Physics Letters A 272 (5) (2000) 291–303.

[18] A. Iqbal, A. Toor, Evolutionarily stable strategies in quantum games, Physics Letters A 280 (5) (2001) 249–256.

[19] C. Bo, M. Ying-Jun, L. Gui-Lu, Quantum game with restricted matrix strategies, Communications in Theoretical Physics 40 (6) (2003) 655.

[20] J. Du, H. Li, X. Xu, X. Zhou, R. Han, Entanglement enhanced multi-player quantum games, Physics Letters A 302 (5) (2002) 229–233.

[21] S. C. Benjamin, P. M. Hayden, Multiplayer quantum games, Physical Review A 64 (3) (2001) 030301.

[22] Q. Li, A. Iqbal, M. Chen, D. Abbott, Evolution of quantum strategies on a small-world network, European Physical Journal B 85 (2012) 376.

[23] H. Situ, A quantum approach to play asymmetric coordination games, Quantum information processing 13 (3) (2014) 591–599.

[24] Ł. Pawela, J. Sładkowski, Quantum prisoner's dilemma game on hyper-graph networks, Physica A: Statistical Mechanics and its Applications 392 (4) (2013) 910–917.

[25] A. Li, X. Yong, Entanglement guarantees emergence of cooperation in quantum prisoner's dilemma games on networks, Scientific Reports 4 (2014) 6286.

[26] M. Dresher, A sampling inspection problem in arms control agreements: A game-theoretic analysis, Tech. rep., Memorandum RM-2972-ARPA, The RAND Corporation, Santa Monica, California (1962).

[27] S. M. Kamal, Y. Al-Hadeethi, F. A. Abolaban, F. M. Al-Marzouki,

M. Perc, An evolutionary inspection game with labour unions on small-world networks, Scientific Reports 5 (2015) 8881.

[28] D. Fudenberg, J. Tirole, Game theory, MIT Press, Cambridge, Massachusetts, 1991.

[29] D. Nosenzo, T. Offerman, M. Sefton, A. van der Veen, Encouraging compliance: bonuses versus fines in inspection games, Journal of Law, Economics, and Organization 30 (3) (2014) 623–648.