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On Non-integrability and the Asymptotic Breakdown of Perturbative Field Theory

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Abstract

There are several instances where non-analytic functions and non-integrable operators are deliberately excluded from perturbative Quantum Field Theory (QFT) and Renormalization Group (RG) to maintain internal consistency of both frameworks. Here we briefly review these instances and suggest that they may be a portal to an improved understanding of the asymptotic sectors of QFT and the Standard Model of particle physics (SM).

Key words: Non-integrable Hamiltonians, Non-smooth functions, Quantum Field Theory, Perturbation Theory, Renormalization Group, Cluster Decomposition Principle, Effective Field Theory, Dimensional Regularization.

1. The Feynman-Dyson integrals

It is well known that the standard formulation of perturbative quantum field theory (QFT) relies on the Feynman – Dyson series of integrals [ ]

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2...\int_{-\infty}^{\infty}dt_n\times T\{H_I(t_1)H_I(t_2)...H_I(t_n)\} \tag{1}$$

where the integrand consists of the time-ordered product of the interaction Hamiltonian $H_I(t)$. The interaction Hamiltonian is typically written as

$$H_I(t) = \int d^3x\, H(\mathbf{x},t) \tag{2}$$

in which $H(\mathbf{x},t)$ is a polynomial whose terms are local functions of the annihilation and creation fields viz.

$$\psi_l^{(+)}(x) = \int d^3p \sum_{\sigma,n} \exp(ipx)\, u_l(\mathbf{p},\sigma,n)\, a(\mathbf{p},\sigma,n) \tag{3}$$

$$\psi_l^{(-)}(x) = \int d^3p \sum_{\sigma,n} \exp(-ipx)\, v_l(\mathbf{p},\sigma,n)\ a^{+}(\mathbf{p},\sigma,\bar{n}) \tag{4}$$

$$\psi_l(x) = \sum_{\sigma,n} [\psi_l^{(+)}(x) + \psi_l^{(-)}(x)] \tag{5}$$

Here, $\psi_l^{(+)}(x)$ and $\psi_l^{(-)}(x)$ annihilate particles and create antiparticles, respectively, $\mathbf{p}$ represents the three-momentum, σ the z-projection of the spin, n and $\bar{n}$ label the number of particle and antiparticle species, respectively. Lorentz transformation properties of the fields and one-particle states, along with the constraint that fields commute at space-like separations, fix entirely the form of the coefficients u_l and v_l.

A core requirement of QFT is the *cluster decomposition principle* (CDP) which states that distant experiments yield uncorrelated outcomes. In particular, CDP protects low-energy physics from short-distance perturbations. CDP requires that the interaction Hamiltonian be formulated as a power series in the creation and annihilation operators, which are *sufficiently smooth* functions of the momenta. This condition is automatically satisfied by an interaction Hamiltonian having the form (2) [ ]. While there is widespread consensus on the compelling success of perturbative QFT in particle physics and condensed matter, restricting the analysis to sufficiently smooth Hamiltonians is likely to produce unrealistic approximations in future cases of interest. Recent years have consistently shown that many nonlinear dynamical systems display *non-smooth interactions, bifurcations, limit cycles, strange attractors, non-Gaussian noise or multifractal properties* [ ]. To give only one example, consider the class of *non-integrable systems* arising in the context of the three-body problem, chaotic oscillators, KAM theory, Henon-Heiles potential, kicked rotor, turbulent flows and so on [ ]. One is motivated to ask: *What happens if the interaction Hamiltonian is allowed to contain non-smooth contributions in the structure of creation and annihilation operators?* We discuss this topic next.

2. Non-perturbative effects of the RG flow

One plausible scenario is that the non-smooth contributions emerge at the low-energy scale of effective QFT as residual non-perturbative effects of the RG flow [ ]. To fix ideas, we follow [ ] and refer to the framework of *effective field theories* (EFT). In general, the construction of EFT is based on the so-called "momentum-shell" approach, which consists of a two-step procedure:

a) change of functional variables of integration in the path integral formulation of the theory,

b) perform partial evaluation of the modified path integral whereby short-wavelength fields are integrated out in the absence of external currents.

The core assumption of both CDP and EFT holds that the Lagrangian built from the remaining "coarse-grained" fields supplies exact results for the *n*-point amplitudes. Let $\varphi_n,\ n = 1, 2, ...N$ denote the complete set of short-wavelength fields characterizing the dynamics of the theory at some running high-energy scale $\Lambda < \Lambda_{UV}$, in which Λ_{UV} stands for the ultraviolet cutoff. The new set $m = 1, 2, 3 ... M$ of "coarse-grained" fields are defined through

$$\Phi_m = f_m(\varphi_n; \Lambda) \tag{6}$$

where $M < N$ and the "coarse-graining" functions $f_m(...)$ are typically non-invertible. If $L(\varphi_n)$ represents the Euclidean Lagrangian associated with the short-wavelength fields, the effective Lagrangian corresponding to the "coarse-grained" fields (6) takes the form

$$\exp[-\int d^4 L_{eff}(\Phi_m)] = \int \prod_n D\varphi_n \, \delta[\Phi_m - f_m(\varphi_n; \Lambda)] \exp[-\int d^4 L(\varphi_n)] \tag{7}$$

The meaning of (7) is that the original "microscopic" Lagrangian can be safely factored out when computing the *n*-point amplitudes of Φ_m in the presence of external currents J_m. This is because the generating functional for Φ_m can be expressed in a form that does not preserve any memory of the microscopic fields, that is,

$$Z(J_m) = \int \prod_m D\Phi_m \exp[-\int d^4 x L_{eff}(\Phi_m) + \int d^4 x \, J_m(x) \Phi_m(x)] \tag{8}$$

The functions $f_m(...)$ are required to be smooth in order for the effective Lagrangian (7) to be expanded in multi-monomials of local products of Φ_m. It is also readily seen from (7) that the effective Lagrangian becomes ill-defined if $L(\varphi_n)$ is either non-smooth or non-integrable. One cannot arbitrarily discard this possibility in the near or far Terascale sector of high-energy physics *or* in a dynamic environment that no longer comply with the conditions of equilibrium statistical physics [ ]. Likely plausible is the case where "coarse graining" is partially successful, only part of the EFT survives and some residual non-smooth contributions continue to persist at the EFT scale.

3. The damping function in the "momentum-shell" integration scheme

These instances can also directly impact the basis of the "momentum-shell" approach. The "momentum-shell" approach turns out to be invalid from an analytical point of view as sharp momentum scale yields *singular terms* in taking derivatives in the RG flow equations [ ]. To correct this deficiency, it is necessary to introduce a suitable *damping function* $D(k^2/\Lambda^2)$ whose role is to "blur" the sharp momentum scale and to suppress the loop integrals arisen from internal propagators that exceed this scale ($|k| > \Lambda$). The damping function "coarse-grains" the free part of the effective Lagrangian in momentum space viz.

$$S_0[\Phi,\Lambda] = \int \frac{1}{2}\Phi(k)(k^2+m^2)D(\frac{k^2}{\Lambda^2})\Phi(-k)\frac{d^4k}{(2\pi)^4} \tag{9}$$

The damping function is required to be strictly *analytic* in order to maintain the locality property of the theory: in particular, it has to ensure that the effective Lagrangian at any scale can be expanded into an infinite sum of local terms, where each term includes products of fields and their derivatives defined at *single* space-time locations.

We close these brief remarks with a key observation on dimensional regularization. The generating functional describing the physics at an arbitrary observation scale $\mu < \Lambda$ in exactly four dimensions is given by the path integral

$$Z_{\mu}[j_{\nu}] = \int D\varphi \exp\{-d^4x[L_0(\varphi,\mu) + L_{\text{int}}(\varphi,\mu) + \int d^4(x)\, j_{\nu}(x)\varphi(x)]\} \tag{10}$$

A frequently cited drawback of dimensional regularization is that, unlike the momentum cutoff scheme, it cannot be extrapolated beyond perturbation theory. The mainstream viewpoint is that there is no realistic way of replicating the path integral (10) in *non-integer dimensions* (that is, for $\varepsilon = 4 - D \neq 0$), whereby the dynamics is specified by an effective Lagrangian expanded in local terms. Thus, according to the CDP, a non-perturbatively valid construction of a local QFT rooted in dimensional regularization appears to be impossible. However, as we have repeatedly pointed out [ ], allowing the underlying space-time to have *arbitrarily small* deviations from four dimensions (ε << 1), is able to overcome these objections. It provides *the only sensible* solution of working in a region that asymptotically matches all the conditions mandated by local QFT and the SM [ ].