A Relativistic Quantum Wave Equation That Reduces to the Schrödinger Equation in the Non-Relativistic Limit

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Abstract A new relativistic quantum wave equation has been derived by applying the quantum prescription to the momentum and the kinetic energy rather than to the momentum and the total energy, since after all it is the kinetic energy that generates the momentum. The resulting equation reduces to the Schrödinger equation in the nonrelativistic limit and to the Klein-Gordon equation for "massless particles" in the relativistic limit, i.e., if the velocity of the particle approaches that of light, *c*. For massive particles in general, the new equation deviates from the Klein-Gordon equation. The same equation is shown to decouple according to the Dirac formalism, yielding a modified form of Dirac equation. When applied to a rest particle, the modified Dirac equation is shown to avoid a negative energy solution and instead include a constant solution. The other, the time-dependent particle solution of the modified Dirac equation, has the characteristic frequency $\boldsymbol{Mc^2}/(\boldsymbol{\hbar}/2)$, i.e., twice those of the Dirac solutions, $\boldsymbol{Mc^2/\hbar}$.

Keywords Quantum Relativistic Schrödinger Klein-Gordon Dirac

1. Introduction

It is well known that neither of the Klein-Gordon equation and Dirac equation, the relativistic quantum wave equations, reduces to the non-relativistic Schrödinger equation in the low velocity, non-relativistic limit. As an example, the energy levels for a hydrogen-like atom calculated by either Klein-Gordon or Dirac equation, when higher order fine structure terms are ignored, differ from those calculated by the Schrodinger equation by the amount of Mc^2, where M is the mass of the atom and c is the speed of light [1]. In this paper, we show a new relativistic quantum wave equation emerges by requiring that it reduce to the latter. We begin with the familiar energy momentum relation for a particle [2-9],

$$E^2 = P^2c^2 + M^2c^4 \,. \tag{1}$$

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If we define $\mathcal{E} \equiv Mc^2$**,** the internal energy (many authors call this the rest energy,) we can then write $E \equiv \gamma Mc^2 = \gamma\mathcal{E}$ to be the relativistic total energy, $\mathcal{P} \equiv M\upsilon$ to be the non-relativistic momentum, and $P = \gamma M\upsilon = \gamma\mathcal{P}$ to be the relativistic momentum where $\gamma = 1\Big/\sqrt{1-\frac{\upsilon^2}{c^2}}$ and v is the velocity of the particle. The energy momentum relation, Eq. **(1)**, may then be written,

$$\mathcal{E}^2 = \mathcal{P}^2 c^2 + \frac{1}{\gamma^2} M^2 c^4 \,. \tag{2}$$

When we work with the boosted energy and momentum, *E* and *P* as in Eq. **(1)**, we will say we work in the relativistic *γ*-space, that is we work within

$$1 \le \gamma < \infty \,.$$

When we work with unboosted energy and momentum $\mathcal{E}$ and $\mathcal{P}$ as in Eq. **(2)**, we will say in the following we work in the relativistic 1/*γ*-space, that is we work within

$$0 \le 1/\gamma \le 1 \,.$$

An advantage of working in the 1/*γ*-space is that as the velocity of the particle approaches the speed of light, we avoid infinity (*γ* → ∞ when v → c) but work with zero (1/*γ* → 0 when v → c) instead. Another advantage is conceptual; if the particle velocity is *c*, we no longer have to say the mass is zero but instead simply the effect of mass is zero. The *γ* appears in the equation as parameter; for instance each of the electron's orbits in an atom has a particular angular velocity and radius hence a characteristic *γ* and 1/*γ* values. It, therefore, provides us with crucial information for the behavior of particles as will be demonstrated below.

2. Schrödinger Equation in the Low Velocity, Nonrelativistic Limit

We will first show the Schrödinger equation may be obtained in the low velocity, nonrelativistic limit both in *γ*-space and 1/*γ*-space. We will then extend the same method to derive a fully relativistic form of the quantum wave equation. We will discuss how the new equation compares with the well-tested Klein-Gordon and Dirac equations [1] [10-15].

In *γ*-Space, we rearrange Eq. **(1)** to get

$$\left(E - Mc^2\right)\left(E + Mc^2\right) = P^2 c^2 \tag{3}$$

and note that for v << c, $E + Mc^2 = \gamma Mc^2 + Mc^2 \cong 2Mc^2$. Hence for the low velocity end, we get

$$T \cong \frac{P^2}{2M} \tag{4}$$

where $T = E - Mc^2 = \gamma Mc^2 - Mc^2 \cong \frac{1}{2} M\upsilon^2$ is the relativistic kinetic energy of the particle.

Substituting *T* by $i\hbar\frac{\partial}{\partial t}$, *P* by $i\hbar\nabla$, and operating on a function ψ, we obtain the Schrödinger equation in the free field,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi\,. \tag{5}$$

In 1/*γ*-Space, we use the same method as in the above to rearrange Eq. **(2)**, and note that for v << c, $\mathscr{E} + \frac{Mc^2}{\gamma} \cong 2Mc^2$. Hence for the low velocity end, we obtain

$$\mathfrak{I} \approx \frac{\mathscr{P}^2}{2M} \tag{6}$$

where $\mathfrak{I} \equiv \frac{T}{\gamma} = \mathscr{E} - \frac{Mc^2}{\gamma} \cong \frac{1}{2}Mv^2$. Substituting $\mathfrak{I}$ by $i\hbar\frac{\partial}{\partial t}$ and $\mathscr{P}$ by $i\hbar\nabla$ and operating on a function ψ, we again obtain the Schrödinger equation in the free field, Eq. **(5)**.

3. Relativistic Quantum Wave Equation

The derivation of the Schrödinger equation in the above from the relativistic energy momentum relations both in *γ*-space and 1/*γ*-space presents a way of naturally extending the same method to fully relativistic cases. In addition to the momentum, we note that it is the kinetic energy that we apply the quantum prescription to, not the total energy, since after all it is the kinetic energy that generates the momentum.

3.1 Relativistic Extension of the Schrödinger Equation in *γ*-Space

We further rearrange Eq. **(1)** to read

$$\left(E - Mc^2\right)\left(E - Mc^2 + 2Mc^2\right) = P^2c^2\,. \tag{7}$$

By denoting the fully relativistic kinetic energy, $T \equiv E - Mc^2$ and substituting T by $i\hbar\frac{\partial}{\partial t}$, P by $i\hbar\nabla$ and operating on a function Φ, we obtain

$$\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + 2i\hbar M\frac{\partial}{\partial t}\right)\Phi = -\hbar^2\nabla^2\Phi \tag{8}$$

or in the tensor notation with the metric (+- - -),

$$\partial_\mu\partial^\mu\Phi = 2i\frac{Mc}{\hbar}\partial_0\Phi\;;\ \mu = 0, 1, 2, 3. \tag{9}$$

3.2 Relativistic Extension of the Schrödinger Equation in 1/*γ*-Space

We further rearrange Eq. **(2)** in the same manner as Eq. **(7)**, substitute $\mathfrak{I} = T/\gamma$ by $i\hbar\frac{\partial}{\partial t}$ and $\mathscr{P} = P/\gamma$ by $i\hbar\nabla$, and operate on a function Φ to obtain

$$\left(-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + 2i\hbar\frac{M}{\gamma}\frac{\partial}{\partial t}\right)\Phi = -\hbar^2\nabla^2\Phi \tag{10}$$

or

$$\partial_\mu\partial^\mu\Phi = 2i\frac{Mc}{\hbar\gamma}\partial_0\Phi\,. \tag{11}$$

In the following, we shall refer to the above as the Min equation in the $1/\gamma$-space or simply the Min equation. (The author proposes to use his own name not for fame but for accountability, at least until it is proven to deserve the name Schrödinger, as in "Relativistic Schrödinger Equation.") The γ-space equation, Eq. **(9)**, may be transformed to the above simply by replacing the mass term M with M/γ.

The above reduces to the Schrödinger equation if the second term in the bracket of Eq. **(10)** dominates. This would be the case if $\hbar\gamma/(Mc^2 t) << 1$, i.e., if v << c (then $\gamma \approx 1$) and $\hbar\omega = h\nu << Mc^2$ where ω is the angular velocity; for example, for the electrons in most bound-states. In the opposite extreme, i.e., if v ≈ c (then $1/\gamma \approx 0$) and $\hbar\omega = h\nu >> Mc^2$, the above reduces to the Klein-Gordon equation for "massless particles;" for example, for the photons or the electromagnetic waves. For massive particles in general, the above equation deviates from the Klein-Gordon equation. The above is a new relativistic quantum wave equation for massive particles that reduces to the Schrödinger equation in the nonrelativistic limit.

We define a unit four vector

$$I^{\mu} = I_0^{\mu} + I_1^{\mu} + I_2^{\mu} + I_3^{\mu} \tag{12}$$

where

$$I^{\mu} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix},\ I_0^{\mu} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ I_1^{\mu} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ I_2^{\mu} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ I_3^{\mu} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{13}$$

When applied to the four derivative, it is understood that

$$\begin{aligned} I^{\mu}\partial_{\mu} &= \partial_0 - \partial_1 - \partial_2 - \partial_3 \\ I_0^{\mu}\partial_{\mu} &= \partial_0;\ I_1^{\mu}\partial_{\mu} = -\partial_1;\ \text{etc.} \end{aligned} \tag{14}$$

This allows the above Min equation to be rewritten in a more maneuverable form,

$$\partial_{\mu}\partial^{\mu}\Phi = 2i\frac{Mc}{\hbar\gamma} I_0^{\mu}\partial_{\mu}\Phi . \tag{15}$$

We see the left hand side term of the above in both the Maxwell's equations and the Klein-Gordon equation and the right hand side time derivative term in both the Schrödinger equation and the Dirac equation.

The above Min equation may be decoupled into the bispinor equations by deploying the Dirac formalism as following.

3.3 A Modified Dirac Equation

We will rewrite the energy-momentum equation, Eq. **(1)**, in a tensor form,

$$P^{\mu}P_{\mu} - M^2c^2 = 0 \tag{16}$$

where μ = 0, 1, 2, 3, and

$$P^{\mu} = (P^0, P^1, P^2, P^3) = (P^0, P^i) = (E/c, P^i) \tag{17}$$

where i = 1,2,3. Following Dirac [11, 12, 15], the above may be factored into two 4x4 linear algebraic matrix equations

$$P^{\mu}P_{\mu} - M^2c^2 = (\gamma^{\kappa}P_{\kappa} + Mc)(\gamma^{\kappa}P_{\kappa} - Mc) \tag{18}$$

where the Dirac matrices, γ^μ, are defined

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = (\gamma^0, \gamma^i) \tag{19}$$

with

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{20}$$

We note $P^0 = \frac{E}{c} = \frac{\gamma \mathscr{E}}{c} = \frac{\gamma M c^2}{c}$ and denote $(P_1, P_2, P_3) \equiv \vec{P}$ and $(\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3) \equiv \vec{\mathscr{P}}$, hence $\vec{P} = \gamma \vec{\mathscr{P}}$,

I (or simply 1) is a 2x2 unit matrix, and σ^i are 2 x 2 Pauli matrices. From Eq. **(16)** and **(18)**, we get a factored equation

$$\gamma^\kappa P_\kappa - Mc = 0 \tag{21}$$

and

$$\gamma^\kappa P_\kappa + Mc = 0\,. \tag{22}$$

From the first set of the factored equations, Eq. **(21)**, we get

$$\gamma^0 \mathscr{E} - \frac{M}{\gamma} c^2 - c \sum_i \gamma^i \mathscr{P}^i = 0 \tag{23}$$

which may be further rearranged

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathscr{E} - \frac{M}{\gamma} c^2 - c \sum_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \mathscr{P}^i = 0\,, \tag{24}$$

and finally to

$$\begin{pmatrix} \mathscr{E} - \frac{M}{\gamma} c^2 & 0 \\ 0 & -(\mathscr{E} - \frac{M}{\gamma} c^2) - 2\frac{M}{\gamma} c^2 \end{pmatrix} = c \sum_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \mathscr{P}^i\,. \tag{25}$$

Substituting $\mathscr{T} \equiv \mathscr{E} - Mc^2/\gamma = T/\gamma$ by $i\hbar\partial_t$ and $\mathscr{P} = P/\gamma$ by $i\hbar\nabla$ where $\partial_t \equiv \frac{\partial}{\partial t}$, and operating on spinors ψ_A and ψ_B, defined by

$$\Psi \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix};\ \Psi_A \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},\ \Psi_B \equiv \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} \tag{26}$$

we obtain,

$$\begin{pmatrix} i\hbar\partial_t & 0 \\ 0 & -i\hbar\partial_t - 2\frac{M}{\gamma} c^2 \end{pmatrix} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = ic\hbar \sum_{i=1,2,3} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \partial_i \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} \tag{27}$$

By using the first of the following relationships: $\gamma^0 - 1 = 2\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and $\gamma^0 + 1 = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the above may be written,

$$i\hbar\gamma^{\mu}\partial_{\mu}\Psi+\left(\gamma^{0}-1\right)\frac{Mc}{\gamma}\Psi=0\,. \tag{28}$$

In the following, we will call the above the Modified Dirac equation. In a decoupled form, it reads from Eq **(27)**,

$$\begin{aligned}&\partial_0\Psi_A=\sigma^i\partial_i\Psi_B\\&\left(\partial_0-2i\frac{Mc}{\hbar\gamma}\right)\Psi_B=\sigma^i\partial_i\Psi_A\end{aligned} \tag{29}$$

The first is a massless, electromagnetic interaction between the spinors Ψ_A and Ψ_B. The second is a massive interaction between the two as long as $1/\gamma > 0$. If v = c, then $1/\gamma = 0$, and both are massless interactions. Massless in the latter means not $M = 0$, but $1/\gamma = 0$.

From the second set of factored equations, Eq. **(22)**, we get

$$i\hbar\gamma^{\mu}\partial_{\mu}\Psi+\left(\gamma^{0}+1\right)\frac{Mc}{\gamma}\Psi=0\,. \tag{30}$$

We then get the exact same set of equations as Eq. **(29)** except Ψ_A and Ψ_B are interchanged.

Hence, the modified Dirac equation, Eq. **(28)**, is derived from the Min equation which reduces to the Schrödinger equation in the low velocity, non-relativistic limit. Conversely, we can easily show the modified Dirac spinors Ψ_A and Ψ_B satisfy Min equation when combined. We can say that the Modified Dirac equation describes the interaction of two spinor functions, one electromagnetically and the other through mass.

This compares with the Dirac equation

$$i\hbar\gamma^{\mu}\partial_{\mu}\Psi-Mc\Psi=0 \tag{31}$$

which may be decoupled to

$$\begin{aligned}&\left(\partial_0+i\frac{Mc}{\hbar}\right)\Psi_A=\sigma^i\partial_i\Psi_B\\&\left(\partial_0-i\frac{Mc}{\hbar}\right)\Psi_B=\sigma^i\partial_i\Psi_A\end{aligned}\,. \tag{32}$$

Both in the pair of the above decoupled Dirac equations describe the interaction of the two spinors though mass, compared to Eq. **(29)** where one of the pair is a massless interaction.

3.4 Simple Solutions for the Modified Dirac Equation

3.4.1 A Particle at Rest

If Ψ is independent of position, we get

$$\frac{\partial\Psi}{\partial x}=\frac{\partial\Psi}{\partial y}=\frac{\partial\Psi}{\partial z}=0 \tag{33}$$

i.e., $P_x = P_y = P_z = 0$, or zero momentum and zero velocity with $1/\gamma=1$. The Modified Dirac equation, Eq. **(28)**, then reduces to

$$\frac{i\hbar}{c}\gamma^{0}\frac{\partial\Psi}{\partial t}+\left(\gamma^{0}-1\right)Mc\Psi=0 \tag{34}$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \Psi_A}{\partial t} \\ \dfrac{\partial \Psi_B}{\partial t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} i \frac{Mc^2}{\hbar} \begin{pmatrix} \Psi_A \\ \Psi_B \end{pmatrix} = 0 . \tag{35}$$

We then get

$$\begin{aligned} \frac{\partial \Psi_A}{\partial t} &= 0 \\ \frac{\partial \Psi_B}{\partial t} &= i \frac{2Mc^2}{\hbar} \Psi_B \end{aligned} \tag{36}$$

or

$$\begin{aligned} \Psi_A &= \text{constant} \\ \Psi_B &= e^{i \frac{Mc^2}{\hbar/2} t} \Psi_B(0) \end{aligned} \tag{37}$$

The solutions of Ψ_A and Ψ_B may be interchanged due to the second set of equations, Eq. **(30)**. The above pair of particles-at-rest solutions compare with those of Dirac equation which reads [15],

$$\begin{aligned} \Psi_A &= e^{-i \frac{Mc^2}{\hbar} t} \Psi_A(0) \\ \Psi_B &= e^{i \frac{Mc^2}{\hbar} t} \Psi_B(0). \end{aligned} \tag{38}$$

One of the above Dirac pair is a negative energy solution representing antiparticles. The modified pair, Eq. **(37)**, avoids a negative energy solution and instead contains a constant solution. This constant solution does not exist in the Dirac solutions, Eq. **(38)**. The constant solution may be viewed as being consistent with the idea of Dirac Sea, representing the vacuum state. The other, the time-dependent particle solution, has the characteristic frequency $Mc^2/(\hbar/2)$ in the modified Dirac solution, i.e., twice that of the Dirac solutions where it is $\pm Mc^2/\hbar$.

3.4.2 Plane Wave Solutions

For the Modified Dirac equations, Eq. **(28)**, we now look for the plane wave solution of the type

$$\Psi(x) = a e^{\pm i \kappa \cdot x} u(\kappa) \tag{39}$$

Note that

$$\partial_\mu \Psi(x) = \pm i \kappa_\mu \Psi(x)$$

Note also

$$\gamma^\mu \kappa_\mu = \gamma^0 \kappa^0 - \vec{\gamma} \cdot \vec{\kappa} = \begin{pmatrix} \kappa^0 & -\vec{\kappa} \cdot \vec{\sigma} \\ \vec{\kappa} \cdot \vec{\sigma} & -\kappa^0 \end{pmatrix} . \tag{40}$$

The Modified Dirac equation, Eq. **(28)**, becomes

$$\left[\mp\hbar\gamma^{\mu}\kappa_{\mu}+\left(\gamma^{0}-1\right)\frac{Mc}{\gamma}\right]u=0 \tag{41}$$

or

$$\begin{pmatrix}\mp\hbar\kappa^{0} & \pm\hbar\vec{\kappa}\cdot\vec{\sigma}\\ \mp\hbar\vec{\kappa}\cdot\vec{\sigma} & \pm\hbar\kappa^{0}-2\dfrac{Mc}{\gamma}\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix}=0\,, \tag{42}$$

so we get,

$$\begin{aligned}u_A&=\frac{\vec{\kappa}\cdot\vec{\sigma}}{\kappa^{0}}u_B\\ u_B&=\frac{\vec{\kappa}\cdot\vec{\sigma}}{\kappa^{0}\mp\dfrac{Mc}{(\hbar/2)\gamma}}u_A\,.\end{aligned} \tag{43}$$

In the above, u_A and u_B may be interchanged owing to Eq. **(30)**. By the use of the relationship $P^{\mu}=\gamma\mathscr{P}^{\mu}=\gamma\hbar\kappa^{\mu}\equiv\hbar k^{\mu}$, and $\mathscr{P}^{0}=E/c=Mc$, we then get

$$\begin{aligned}u_A&=\frac{\mathscr{P}\cdot\vec{\sigma}}{\mathscr{P}^{0}}u_B\\ u_B&=\frac{\mathscr{P}\cdot\vec{\sigma}}{\mathscr{P}^{0}\mp\dfrac{2Mc}{\gamma}}u_A\,.\end{aligned} \tag{44}$$

We can carry the above further to obtain canonical expressions for u_A and u_B, which, however, we will not pursue here. It suffices to note that the two bispinors u_A and u_B interact in the purely electromagnetic manner on one hand and through mass on the other and for the limiting case v = c or $1/\gamma=0$, both are electromagnetic interactions.

4. Conclusion

It is shown that the Min equation is a relativized Schrödinger equation in the sense that the equation reduces to the latter in the low velocity, non-relativistic limit. When the relativistic effect dominates, the equation approaches the Klein-Gordon equation for massless particles. The same equation is shown to decouple into a modified form of the Dirac equation describing spin one-half particles.

The new equation is the result of applying the quantum prescription to the momentum and the kinetic energy rather than to the momentum and the total energy. This is justified since after all it is the kinetic energy that generates the momentum. The equation is written in both the relativistic γ-space as well as the $1/\gamma$-space with the latter shown to give us an advantage of avoiding infinity as the velocity of the particle approaches c and in addition provide us with crucial information for the behavior of particles.

When applied to a rest particle, the modified Dirac equation avoids a negative energy solution and instead includes a constant solution. This constant solution does not exist in the Dirac solutions. The other, the time-dependent particle solution, has the characteristic frequency $\boldsymbol{Mc^2}/(\boldsymbol{\hbar}/2)$ in the modified Dirac solution, i.e., twice that of the Dirac solutions where it is $\boldsymbol{Mc^2/\hbar}$. The plane wave solutions show the bispinors in the modified Dirac equation interact through mass on the one hand and in the purely electromagnetic manner on the other.

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