Unitarity does is not derive from homogeneity of space

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Abstract Symmetry information beneath wave mechanics is re-examined. Homogeneity of space is the symmetry, fundamental to the quantum free particle. The unitary information of the Canonical Commutation Relation is shown not to be implied by that symmetry.

Keywords quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space unitary, non-unitary.

1 The basic symmetry of wave mechanics: homogeneity of space

The *Canonical Commutation Relation*

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of wave mechanics. The professed significance of this relation is that it represents the homogeneity of space, and *that* is accepted by quantum theorists as unitary. In this paper, I re-examine and scrutinise the Canonical Relation's derivation and establish that the homogeneity symmetry is itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains other (unitary) information also.

Imposing homogeneity on a system is identical to imposing a null physical effect, under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *form invariance*. This is the concept, from relativity, that symmetry transformations leave (physical) formulae fixed in *form*, though *values* may alter [1]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x} \left| f_{\mathsf{x}} (x) \right\rangle = \mathsf{x} \left| f_{\mathsf{x}} (x) \right\rangle . \tag{1}$$

The san-serif x, here, is a label for f_{x} whose eigenvalue is x. The variable x (curly) is the function domain. The use of two different variables here may seem unusual and pointless. In fact, logically they are different. x is quantified existentially but x is quantified universally.

With form held fixed, as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally (Lie group and Lie algebra). To maintain the form of (1), under translation, the basis $|f_{\mathsf{x}}\rangle$ is cleverly managed: while the translation transforms the basis from $|f_{\mathsf{x}}\rangle$ to $|f_{\mathsf{x}-\epsilon}\rangle$, a similarity transformation is also applied, chosen to revert $|f_{\mathsf{x}-\epsilon}\rangle$ back to $|f_{\mathsf{x}}\rangle$. In this way $|f_{\mathsf{x}}\rangle$ is held static. The similarity transformation is a member of the one-parameter subgroup of the general linear group GL, $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}$, with the transformation parameter ϵ coinciding with the displacement parameter. We shall see later, that similarity transforms can be found only for a certain class of functions f. The overall scheme of transformations is depicted in Figure 1.

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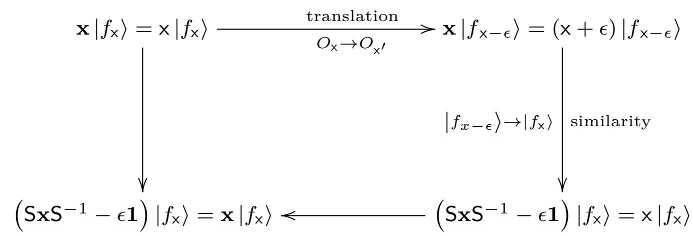

Figure 1 Scheme of transformations. The bottom left hand formula is the resulting group relation.

Now, in standard theory of quantum symmetries, textbook understanding is that $\mathsf{S}(\epsilon)$ is intrinsically and necessarily unitary. It is in *that* unitarity where the Canonical Relation finds its unitary origins. The textbook reason for that unitarity, and the purpose it serves, is the preserved existence of the scalar product and invariance of probability amplitude.

And so, because its presence is thought *intrinsically necessary*, unitarity is imposed axiomatically on the theory, *by Postulate*. The upshot is that standard theory *imposes* Hilbert space on vectors $|f_{\mathsf{x}}\rangle$. This imposed unitarity is added information, extra to the information of homogeneity. In consequence, in standard theory, the symmetry for wave mechanics is a *resultant* – unitary subgroup of homogeneity.

As an experiment, we proceed, in this paper, by treating unitarity as a purely separate issue from homogeneity and allowing $\mathsf{S}(\epsilon)$ it's widest generality, so that homogeneity is faithfully and genuinely conveyed through the theory. The experiment begins with the eigenvalue equation for position (1) being rewritten, as the eigenformula in the quantified proposition (2). From here on, all informal assumptions are to be shed and the Dirac notation is dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator $\mathbf{x}$, eigenfunctions f_{x} and eigenvalues x, seen from the reference frame O_{x}:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x) \tag{2}$$

Translation: Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through ϵ. See Figure 2. The *principle of relativity* guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_{x} in (2), thus:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_{\mathsf{x}} \mid \quad \mathbf{x} f'_{\mathsf{x}}(x') = \mathsf{x}' f'_{\mathsf{x}}(x') \tag{3}$$

A relation for $\mathbf{x}$ is to be evaluated, so $\mathbf{x}$ is held static for all reference frames. The translation transforms position, thus:

$$\forall \epsilon \forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{4}$$

and transforms the function, thus:

$$\forall \epsilon \forall x' \forall f'_{\mathsf{x}} \exists f_{\mathsf{x}} \exists x \mid \quad f_{\mathsf{x}}(x) \mapsto f'_{\mathsf{x}}(x') = f_{\mathsf{x}-\epsilon}(x-\epsilon) \tag{5}$$

Substituting (4) and (5) into (3) gives the translated formula:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}-\epsilon}(x-\epsilon) = (\mathsf{x}+\epsilon) f_{\mathsf{x}-\epsilon}(x-\epsilon)\,. \tag{6}$$

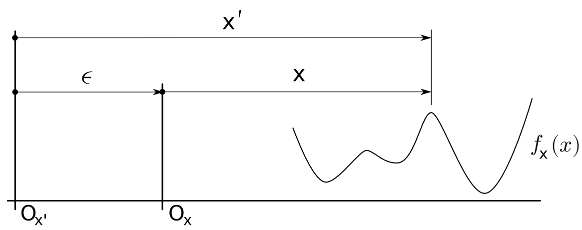

Figure 2 **Passive translation of a function** Two reference systems, O_{x} and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, individually act as reference systems for position of a function f_{x}. If the x-space is homogeneous, then regardless of the value of ϵ, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; f_{x} is non-observable and O_{x} and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

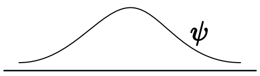

Figure 3 The linear transformations S exist only for bounded ψ_{x}.

Similarity: Now applying the similarity transformation. This involves the (one parameter) linear operator $\mathsf{S}_{(\epsilon)}$. Such an $\mathsf{S}_{(\epsilon)}$ exists only if there exists a space of functions ψ_{x}, that is complete, normalisable, not restricted to separable[1] functions, and is a subset of the (translatable) functions f_{x}. Logically, the act of assuming such an $\mathsf{S}_{(\epsilon)}$ hypothesises that such a class of functions does indeed exist. No such function space is guaranteed. Accordingly, the assertion of proposition (7) is newly assumed information entering the system.

$$\forall x \forall \epsilon \forall \psi_{\mathsf{x}-\epsilon} \exists \mathsf{S} \exists \psi_{\mathsf{x}} \mid \quad \mathsf{S}_{(\epsilon)}^{-1} \psi_{\mathsf{x}}(x) = \psi_{\mathsf{x}-\epsilon}(x-\epsilon) . \tag{7}$$

In standard theory, $\mathsf{S}_{(\epsilon)}$ is set unitary by the mathematician. Doing that restricts the space of functions ψ_{x} to the Hilbert space L^2. Here, $\mathsf{S}_{(\epsilon)}$ is a member of the one parameter subgroup of the infinite dimensional, (non-unitary) general linear group $\mathsf{GL}(\infty)$. This restricts ψ_{x} not to the Hilbert space L^2 but to the Banach space L^1.

The similarity transformation is formed, thus:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} \psi_{\mathsf{x}}(x) = (\mathsf{x}+\epsilon)\, \psi_{\mathsf{x}}(x) .$$

Introducing the trivial eigenformula: $\forall \psi_{\mathsf{x}} \forall x \forall \epsilon \mid \epsilon \mathbf{1} \psi_{\mathsf{x}}(x) = \epsilon \psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbf{1} \right) \psi_{\mathsf{x}}(x) = \mathsf{x} \psi_{\mathsf{x}}(x) . \tag{8}$$

Now comparing the original position eigenformula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathbf{x} \psi_{\mathsf{x}}(x) = \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} - \epsilon \mathbf{1} \right) \psi_{\mathsf{x}}(x) . \tag{9}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(\infty)$, there exists a unique linear operator $\mathbf{g}$ for real parameters ϵ, such that:

$$\forall \mathsf{S} \exists \mathbf{g} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \tag{10}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extracted, thus: $\forall \mathbf{g} \forall \eta \exists \mathbf{k} : \ \mathbf{g} = \eta \mathbf{k}$, implying:

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta \epsilon \mathbf{k}} \tag{11}$$

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)}^{-1} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta \epsilon \mathbf{k}} \tag{12}$$

Substitution of (11) and (12) into (9) gives:

$$\begin{aligned}
&\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \exp(+\eta \epsilon \mathbf{k})\, \mathbf{x} \exp(-\eta \epsilon \mathbf{k})\, \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{1} + \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \mathbf{x} \left[\mathbf{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} + \mathcal{O}\left(\epsilon^2\right)\right] \left[\mathbf{1} - \eta \epsilon \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta \epsilon \mathbf{k} \mathbf{x} - \eta \epsilon \mathbf{x} \mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon \mathbf{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow &\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && [\mathbf{k} \mathbf{x} - \mathbf{x} \mathbf{k}]\, \psi_{\mathsf{x}}(x) = \left[\eta^{-1} \mathbf{1} - \mathcal{O}(\epsilon)\right] \psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{k}, \mathbf{x}]\, \psi_{\mathsf{x}}(x) = \eta^{-1} \mathbf{1} \psi_{\mathsf{x}}(x) \tag{13}$$

And by a similar proof, conditional on the existence of eigenfunctions $\chi(k)$, of $\mathbf{k}$:

$$\forall k \forall \eta \exists \mathsf{k} \exists \chi_{\mathsf{k}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{x}, \mathbf{k}]\, \chi_{\mathsf{k}}(k) = \eta^{-1} \mathbf{1} \chi_{\mathsf{k}}(k) . \tag{14}$$

Importantly, we see (13) and (14) is $\forall \eta$, rather than the particular case of $\eta^{-1} = -i$ that we see in the unitary subalgebra we know as the Canonical Commutation Relation:

$$[\mathbf{k}, \mathbf{x}] = -i \mathbf{1} \quad \text{or} \quad [\mathbf{p}, \mathbf{x}] = -i \hbar \mathbf{1} \tag{15}$$

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

Conclusion

The above establishes that unitary information in the Canonical relation is not implicit in the homogeneity symmetry.

References

1. S357 Course Team; chair: Ray Mackintosh, *Space, time and cosmology unit 3*, no. 1997, The Open University Milton Keynes UK, 1997.