An Examination of the Derivation of the Lagrange Equations of Motion.

Jeremy Dunning-Davies,
Institute for Basic Research,
Palm Harbor, Florida, U.S.A.
and
Institute for Theoretical Physics and Advanced Mathematics (IFM) Einstein-Galilei,
Via Santa Gonda,
14 - 59100 Prato,
Italy
j.dunning-davies@hull.ac.uk

Abstract.
The Lagrange equations of motion are familiar to anyone who has worked in physics. However, their range of validity is rarely, if ever, a topic for discussion. Following on an earlier examination of the consequences for these equations if the mass is not assumed constant, this note will look carefully at the other assumptions made and consider any further consequences resulting. The form of the equations applicable in electromagnetism will also be reviewed in the light of these discussions.

The Lagrange Equations of Motion.

To follow the basic outline in *Synge and Griffith* [1], suppose (x, y, z) are the Cartesian coordinates of a typical particle of a system and suppose we have a holonomic system of n degrees of freedom described by generalised coordinates q_i, $i = 1, 2, \ldots, n$. Then,

$$dx = \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} dq_i, \quad \dot{x} = \sum_{i=1}^{n} \frac{\partial x}{\partial q_1} \dot{q}_\iota$$

with similar equations for both $\dot{y}$ and $\dot{z}$.
From the second equation, it is seen immediately that

$$\frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial x}{\partial q_i}.$$

It is straightforward to show that the operators $\frac{d}{dt}$ and $\frac{\partial}{\partial q_i}$ commute. Then

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) = \frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}\right) = \ddot{x}\frac{\partial \dot{x}}{\partial \dot{q}_\iota} + \dot{x}\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial \dot{q}_i}\right)$$
$$= \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_i}\right) = \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{\partial}{\partial q_i}(\dot{x})$$
$$= \ddot{x}\frac{\partial x}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right)$$

That is

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right) = \ddot{x}\frac{\partial x}{\partial q_i}$$

with similar equations for y and z.
The next step is to multiply these equations by m, sum over all particles of the system and add the three resulting equations together to give

$$\frac{d}{dt}\frac{\partial}{\partial q_i}\left\{\sum_P \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right\} - \frac{\partial}{\partial q_i}\left\{\sum_P \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right\}$$
$$= \sum_P m\left\{\ddot{x}\frac{\partial x}{\partial q_i} + \ddot{y}\frac{\partial y}{\partial q_i} + \ddot{z}\frac{\partial z}{\partial q_i}\right\}$$

that is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \sum_P m\left\{\ddot{x}\frac{\partial x}{\partial q_i} + \ddot{y}\frac{\partial y}{\partial q_i} + \ddot{z}\frac{\partial z}{\partial q_i}\right\} = \sum_P \left\{X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i}\right\}$$

where $\sum_P$ indicates a sum over all the particles of the system, (X, Y, Z) are the components of the external forces acting on particle P and, if

$$Q_i = \sum_P \left\{X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i}\right\},$$

Q_i may be seen to be the coefficient of dq_i in the equation for virtual work.

The Lagrange equations of motion may then be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i$$

In this manipulation the mass of each particle is assumed constant and that is why the symbol m may be taken inside the differentiation signs. Also, it is important to realise that it has been assumed that the kinetic energy, T, is of the specific form $\frac{1}{2}mv^2$.

Of course, in the special case of a conservative system, a potential energy, V, which will be a function purely of position, may be introduced and is seen to satisfy the relation

$$\frac{\partial V}{\partial q_i} = -Q_i$$

Hence, in this special case, Lagrange's eqxu1ations of motion assume the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

where $L = T - V$ is the Lagrangian function.

There is nothing new in what has been written here so far except possibly the pointing out of the fact that the kinetic energy is assumed to possess a very specific form in order for the derivation of the final form of these equations of motion to proceed. It has been pointed out previously [2] that, if the mass is not assumed constant, other changes occur in the derivation and the form of the final equations of motion in that case will rely crucially on the variables on which the said mass depends. Finally, note that the Lagrangian function is introduced here under both these assumptions. In other words, the Lagrangian function as introduced here is dependent on the mass being a constant and, crucially, on the kinetic energy being of the form $\frac{1}{2}mv^2$.

Another Approach.

A variation on the above method appears in the book by Leech [3] and is worth reproducing here both for the sake of comparison and, possibly more importantly, because it leads to further useful insight into the topic as a whole. Leech begins his discussion by examining a conservative system composed of n particles under no constraints. He also confines himself initially to restricting himself to a Cartesian system. For such a system, he takes the kinetic energy to be defined by

$$T = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i \dot{x}_i{}^2$$

where i may take all values from 1 to $3n$.

Also, the equations of motion are

$$F_i = \frac{d}{dt}(m_i \dot{x}_i)$$

Combining these two equations gives

$$F_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}$$

However, for a conservative system, a potential energy, which depends only on position, may be defined by

$$F_i = -\frac{\partial V}{\partial x_i}$$

and these final two equations may be combined to give

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = -\frac{\partial V}{\partial x_i}$$

As noted already though, this equation holds for Cartesian coordinates only. Ideally, an equation in terms of generalised coordinates would be desirable. Hence consider such a system with the generalised coordinate q_i given by

$$q_i = q_i(x_i, t)$$

where the t dependence is included since moving coordinate systems are then covered in what follows. It may be assumed that an inverse relation

$$x_i = x_i(q_i, t)$$

exists. Such an assumption is found to hold true in examples of physical interest but it must be remembered always that it is still an assumption. It follows that

$$\dot{x}_j = \frac{dx_j}{dt} = \sum_i \frac{\partial x_j}{\partial q_i}\dot{q}_i + \frac{\partial x_j}{\partial t}$$

from which it follows that

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \frac{\partial x_j}{\partial q_i}.$$

The above expression for the kinetic energy then leads to

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_j m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \sum_j m_j \dot{x}_j \frac{\partial x_j}{\partial q_i}$$

using the previous result.

Taking the time derivative of this expression leads to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \sum_j \left\{ m_j \ddot{x}_j \frac{\partial x_j}{\partial q_i} + m_j \dot{x}_j \frac{d}{dt}\frac{\partial x_j}{\partial q_i} \right\}$$

However, it may be noted that

$$\frac{d}{dt}\frac{\partial x_j}{\partial q_i} = \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial x_j}{\partial q_i}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial x_j}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\sum_k \frac{\partial x_j}{\partial q_k}\dot{q}_k + \frac{\partial x_j}{\partial t}\right) = \frac{\partial \dot{x}_j}{\partial q_i}$$

Combining these latter two equations leads to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \sum_j \left\{ F_j \frac{\partial x_j}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{1}{2} m_j \dot{x}_j^{\,2}\right) \right\} = Q_i + \frac{\partial T}{\partial q_i}$$

where the $Q_i = \sum_j F_j \frac{\partial x_j}{\partial q_i}$ may be termed the components of generalised force. If the system is conservative though then

$$Q_i = -\sum_j \frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial q_i} = -\frac{\partial V}{\partial q_i}.$$

and the equation becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i},$$

the normal form of the Lagrange equations of motion.

Hence the introduction of generalised coordinates has introduced an additional term, $\frac{\partial T}{\partial q_i}$, when this equation is compared with that for the situation involving Cartesian coordinates specifically. As Leech points out, this is the generalised form of such terms as centrifugal and Coriolis forces. It might be noted also that this approach shows just how Mandelker [4] reasons as he does but here the full background becomes apparent and the reliance of the first set of results in this section on Cartesian coordinates is completely clear.

An Electromagnetic Example.

It seems of interest to consider the case of a moving point charge of constant mass m. The total force acting on such a charge is

$$e\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}x\boldsymbol{H}\right)$$

where e is the charge on the particle, $\boldsymbol{E}$ the electric field, and $\boldsymbol{H}$ the magnetic field.

Following Ferraro [5], the Lagrangian may then be found as is now shown. Denote the usual scalar and vector potentials by ϕ and $\boldsymbol{A}$ respectively, where ϕ will be a function of position only. Further, since

$$\boldsymbol{E}=-\frac{1}{c}\frac{\partial A}{\partial t}-\nabla\phi,\quad \boldsymbol{H}=\nabla x\boldsymbol{A},$$

the equation of motion of the charge may be written

$$m\frac{\partial^2 x_\alpha}{\partial t^2}=-\frac{1}{c}\frac{\partial A_\alpha}{\partial t}-\frac{\partial\phi}{\partial x_\alpha}-\frac{1}{c}\left(\sum_{\beta=1}^{3}\frac{\partial x_\beta}{\partial t}\frac{\partial}{\partial x_\beta}\right)A_\alpha+\frac{1}{c}\sum_{\beta=1}^{3}\frac{\partial x_\beta}{\partial t}\frac{\partial A_\alpha}{\partial x_\beta},$$

where A_α are the components of $\boldsymbol{A}$.

Now

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_\beta\frac{\partial x_\beta}{\partial t}\frac{\partial}{\partial x_\beta},$$

and so, the above equation of motion may be written in the form

$$\text{q}i\,\frac{d}{dt}\left(\frac{m}{e}\frac{dx_\alpha}{dt}+\frac{1}{c}A_\alpha\right)=-\frac{\partial\phi}{\partial x_\alpha}+\frac{1}{c}\sum_\beta\frac{\partial x_\beta}{\partial t}\frac{\partial A_\alpha}{\partial x_\beta},$$

where α,, β = 1, 2, 3.

Now consider the function

$$L=\frac{1}{2}m\left\{\left(\frac{dx_1}{dt}\right)^2+\left(\frac{dx_2}{dt}\right)^2+\left(\frac{dx_3}{dt}\right)^2\right\}-e\phi+\frac{e}{c}\left(A_1\frac{dA_1}{dt}+A_2\frac{dA_2}{dt}+A_3\frac{dA_3}{dt}\right).$$

From this expression, it follows that

$$\frac{\partial L}{\partial \dot{x_\alpha}}=m\dot{x}_\alpha+\frac{e}{c}A_\alpha,\quad \frac{\partial L}{\partial x_\alpha}=-e\frac{\partial\phi}{\partial x_\alpha}+\frac{e}{c}\sum_\beta\dot{x_\beta}\frac{\partial A_\beta}{\partial x_\alpha},$$

and the above equation of motion may then be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x_\alpha}}-\frac{\partial L}{\partial x_\alpha}=0.$$

Therefore, the Lagrangian for the charged particle is seen to be given by

$$L=\frac{1}{2}mv^2-e\phi+\frac{e}{c}(\boldsymbol{v}.\boldsymbol{A}).$$

It might be noted that, in accordance with the assumptions outlined above, the kinetic energy is of the form $\frac{1}{2}mv^2$and, although it hasn't been stated explicitly, the mass m is assumed constant. Hence, here is a case outside normal Newtonian mechanics where an acceptable Lagrangian is found which does conform to the restrictions imposed during the original basic derivation within fundamental Newtonian mechanical ideas.

It might be noted that Leech[3] also considers this case but takes a slightly different approach.

An Alternative Derivation of the Lagrange Equations.

Recourse to the methods of the Variational Calculus provides an alternative method for deriving the Lagrange equations of motion when it is realised that those very equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

are actually the condition for the integral

$$\int Ldt$$

to possess a stationary value or, in the language of variational calculus, the Lagrange equations imply

$$\delta \int Ldt = 0.$$

This final expression is a statement of Hamilton's principle. As is pointed out by Leech [3], in the usual derivation this ultimately represents a deduction from Newton's laws but an alternative is to regard it as a basic principle. If such is the case then Lagrange's equations and the remainder of mechanics might be thought to stem from it. However, is this alternative view really valid? The crucial question to be considered at this point relates to the meaning of the symbol L, the so-called Lagrangian. In the straightforward derivation of Lagrange's equations from more fundamental Newtonian mechanics, it is seen – as noted above – that the Lagrangian function is the difference between the kinetic and potential energies where the kinetic energy is specifically of the form $\frac{1}{2}mv^2$. When approached via Hamilton's principle alone, it is difficult to see how the actual mechanical content is introduced. It might be noted, for example, that simply *assuming* $L = T - V$ is not really adequate since such an assumption might be felt to imply that any expression for the kinetic energy is acceptable but, as seen already, such is not the case.

Conclusions.

Nothing in the above is new. In fact, everything included is summarising what has been known for many years. However, what is new is the emphasis on noting what has been assumed in these derivations. It is crucial to note all assumptions so that, in future, when it is desired to consider specific physical examples, it will be clear whether or not the seemingly general results discussed here really are applicable. The specific important assumptions made *here* are:

(i) the mass is assumed constant,

(ii) the kinetic energy is assumed of the specific form $\frac{1}{2}mv^2$,

(iii) in the alternative approach, it is assumed that the relation linking the Cartesian coordinates with the generalised coordinates may be inverted.

All these assumptions are apparently very basic and simple and, as such, it is often forgotten that they are, in fact, assumptions. However, it is important here, as in other areas of physics, to remember all assumptions made in the theory as they can be important and ignoring or forgetting them can often lead to errors. This is, indeed, one of the problems with mathematical theory. Often the assumptions made of necessity in a piece of theory are forgotten, if they are even noted. The above provides a good example and should be noted.

References.

[1] J. L. Synge & B. A. Griffith, *Principles of Mechanics*, McGraw-Hill, New York, 1959.

[2] J. Dunning-Davies, "Some Results in Classical Mechanics for the Case of a Variable Mass" http://viXra.org/abs/1309.0210

[3] J. W. Leech, C*lassical Mechanics*, Methuen, London, 1958.

[4] J. Mandelker, *Matter Energy Mechanics*, Philosophical Library, New York, 1954.

[5] V. C. A. Ferraro, *Electromagnetic Theory*, Athlone Press, Univ. of London, 1956.