Unifying the Galilei and the special relativity: the Galilei electrodynamics

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Abstract

Using the concept of absolute time introduced in a previous work [7] we define two coordinate systems for spacetime, the Galilean and the Lorentzian systems. The relation between those systems allows us to develop a tensor calculus that transfer the Maxwell electrodynamics to the Galilean system. Then, by using a suitable Galilean limit, we show how this transformed Maxwell theory in the Galilei system results in the Galilei electrodynamics formulated by Levy Leblond and Le Bellac.

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1 Introduction

The Galilean Electrodynamics developed by J.-M. Levy Leblond and M. Le Bellac [1] constitutes a consistent non-relativistic limit for the Maxwell's equations. In their work they showed this non-relativistic limit is not concerned with simply taking $c \to \infty$ in the Maxwell's equations since the explicit presence of the light speed c in these equations depends on the choice of the system of units being used. In fact, in their attempt to solve those subtleties, Levy Leblond and Le Bellac argued on the convenience of the SI system in order to set a correct non-relativistic limit for the Maxwell's equations together with some restrictions on the electric and magnetic fields that encompasses two distint models, the so-called *electric* and *magnetic* limits (in fact, there is also a third model, the *General Galilean Electromagnetism*).

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One of the interests on studying the Galilean limit of the Maxwell's equations is to provide a criteria to understand which electromagnetic effects can be reasonably described by a nonrelativistic theory and then to exhibit such a theory, and which ones have their description only in a relativistic context [1]. It also sets a suitable Galilean transformation for the electromagnetic fields that corrects some low-velocity formulas given in some textbooks. Besides that, as a natural development of these ideas, some authors have continued the study of the Galilean electrodynamics considering other aspects, for instance, some applications in quantum mechanics and superconductivity [2]; the form of the electromagnetic potentials and the gauge conditions in these Galilean limits [2], [3], [4], and so on. Recently, some developments [5], [6] brought new insight into the original work of [1], where the authors considered other ways to obtain the electric and the magnetic limits. Our present work falls into this category as we intend to show how the electric, the magnetic, and the general Galilean electrodynamics obtained in [1] follow as a natural consequence of a recent scheme we proposed to unify the Galilei and the special relativity as we now describe.

In a previous work [7] we presented a method for unifying the Galilei and the special relativity into a single model. This unification was performed through the introduction of an absolute time τ that plays the role of the time variable of the Galilei relativity, together with the local time t, which is the ordinary time of the special relativity. In terms of these time variables there are two views one can employ to describe events, each one being adapted to the particularities of either the Galilei or the special relativity. Thus, events described within the realm of the Galilei relativity are defined by a coordinate set $\{\tau, \vec{x}\}$, while special relativity considers for set $\{t, \vec{x}\}$ (we assume the space coordinates to be the same in both views). As we have shown in [7], in order to combine the Galilei and the special relativity into one single model we first need to extend the previous variables set to $\{\tau, t, \vec{x}\}$. Then, given two inertial frames S, S' moving with relative velocity $\vec{v}$ we assume between the respective sets $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ the relations

$$\begin{aligned} \tau' &= \tau \\ \vec{x}\,' &= \vec{x} - \vec{v}\tau \\ c^2 t'^2 - \vec{x}\,'^2 &= c^2 t^2 - \vec{x}^2 \ . \end{aligned} \tag{1}$$

As a result, assuming a linear relation between the variables' set $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$, we obtained

$$\tau = (1-|a|)\frac{\vec{x}\cdot\vec{v}}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t = (1-|a|)\frac{\vec{x}\,'\cdot(-\vec{v})}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t' = \tau' \tag{2}$$

which provides a relation between the absolute and the local time, with a being an arbitrary parameter.

Here, in our current work, we will show how relation (2) allows us to introduce two sets of coordinates, X_G^μ, X_L^μ, which define the *Galilei* and the *Lorentz* systems, each one encoding respectively the transformation properties that are common either to the Galilei or to the special relativity. We then assume the Maxwell's equations as naturally described with respect to the Lorentz system. Then, using $\partial X_G^\mu / \partial X_L^\nu$ as transformations coefficients we transfer all fields and the Maxwell's equations to the Galilei system. In this way we introduce from the electric and the magnetic fields of the standard Maxwell theory, e.g. $\vec{E}_L$, $\vec{B}_L$ (thought as components of a tensor $\mathcal{F}_{L\mu\nu}$ or $\mathcal{F}_L^{\mu\nu}$) the corresponding Galilean analogues, $\vec{E}_G, \vec{B}_G$, and, in an similar way, we set the Galilean transformations of $\vec{E}_G$, $\vec{B}_G$ from the Lorentz transformation of $\vec{E}_L, \vec{B}_L$. Therefore, the equations satisfied by the Galilean fields are obtained directly from the Maxwell equations by replacing $\vec{E}_L, \vec{B}_L$ by their expressions in terms of $\vec{E}_G, \vec{B}_G$ together with the transformation expressing the derivatives relative to $X_{L\mu}$ in terms of the derivatives relative to $X_{G\mu}$. Once this is performed, we are ready to show how the electric and the magnetic limits of [1] arise from our corresponding Galilean form of the Maxwell equations when we take a suitable limit case. This indicates that the unification of the Galilei and the special relativity exhibited in our previous work [7], and based on the fundamental relation betwen τ, t expressed by (2), extends beyond the kinematical aspects of both relativities, reproducing the correct Galilean limit of the relativistic Maxwell electrodynamics as discovered by Levy Leblond and Le Bellac.

Our work is organized as follows. In section **2** we set our notations and review the basics aspects of the Maxwell electrodynamics that we will refer to in the subsequent sections. In section **3** we review the main elements of [7] that we will need in our work. We base our analysis on a class of transformations parameterized by a real parameter a, with $|a| > 1$, that we call *Generalized Lorentz Transformations* (GLT) and that follows from the conditions given in (1). The GLT includes the ordinary Lorentz transformation. Then, we formulate Maxwell electrodynamics as having the GLT as its invariance. This brings a modification to the form of the transformations of the electromagnetic fields but doesn't change the form of the Maxwell equations. In section **4** we explain how to perform the Galilean limit of our model. Here, contrarily to the approach of Leblond and Le Bellac, it is a characteristic of the GLT transformation that this limit doesn't require any extra condition to be imposed on the electric and the magnetic fields as it is assumed in [1], for instance, $c|\rho| \ll |\vec{\jmath}|$ and $|\vec{E}| \ll c|\vec{B}|$ in the magnetic limit, and $c|\rho| \gg |\vec{\jmath}|$ and $|\vec{E}| \gg c|\vec{B}|$ in the electric limit. Our limit is obtained as a condition that is established only on the parameter a. In section **5** we develop the Galilei electrodynamics employing the "tensor calculus" from the transformations between the two coordinate systems X_G^μ and X_L^μ. Then, we show how the three Galilean models of [1] arise by applying the Galilean limit of section **4**. In particular, we pay a special attention to the third model of [1], the general Galilean electromagnetism, that is formulated in terms of the four fields $\vec{E}_L, \vec{B}_L, \vec{D}_L, \vec{H}_L$. After defining the corresponding Galilei fields $\vec{E}_G, \vec{B}_G, \vec{D}_G, \vec{H}_G$ we show that it is possible to derive appropriate constitutive relations among the Galilean fields that preserve the Galilei invariance, a feature that is not possible in the treatment of Leblond and Le Bellac.

In our work we will use both the CGS and the SI systems of units. The former will be used in sections 2, and 3 to review the Maxwell theory and to define the Galilei fields corresponding to the electric and the magnetic fields of the standard Maxwell theory. The need for that is because we take the electric and magnetic fields as components of tensors $\mathcal{F}_{\mu\nu}$, $\mathcal{F}^{\mu\nu}$ which is suitably introduced within the CGS system. Then, after obtaining these Galilei analogues of the electric and magnetic fields we convert them to their usual expression in the SI system, where we consider the Maxwell's equations.

2 Maxwell electrodynamics

In order to fix our notation, we will recall briefly some aspects of the standard Maxwell electrodynamics [8]. Spacetime is endowed with a metric tensor $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ and have coordinates $x^\mu \equiv (x^0, x^i) := (ct, \vec{x})$, with c being the speed of ligth in vacuum. We also write $x_\mu := \eta_{\mu\nu}x^\nu = (ct, -\vec{x})$. The electric and magnetic fields are accomodated as components of two antisymmetric tensors $\mathcal{F}^{\mu\nu}$, $\mathcal{F}_{\mu\nu}$ according to

$$\begin{aligned} B_i &= -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{F}_{jk} = -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{F}^{jk} \\ E_i &= \mathcal{F}_{0i} = -\mathcal{F}^{0i} \end{aligned} \tag{3}$$

and in terms of which the Maxwell equations in vacuo become

$$\begin{array}{l} \partial_\mu \mathcal{F}_{\nu\lambda} + \partial_\nu \mathcal{F}_{\lambda\mu} + \partial_\lambda \mathcal{F}_{\mu\nu} = 0 \\ \partial_\mu \mathcal{F}^{\mu\nu} = \frac{4\pi}{c} J^\nu \\ J^\mu = (c\rho, \vec{\jmath}) \end{array} \quad \rightleftharpoons \quad \left\{ \begin{array}{l} \vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{\jmath}\,. \end{array} \right. \tag{4}$$

In the presence of a material medium the previous Maxwell equations in vacuo must be changed due to extra contributions to the density of charge and current produced by the medium. Now, in addition to the electric and magnetic fields $\vec{E}$, $\vec{B}$, we have the fields $\vec{D}$, $\vec{H}$ that are accomodated as components of another antisymmetric tensor $\mathcal{H}^{\mu\nu}$ according to

$$\begin{aligned} H_i &= -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{H}_{jk} = -\tfrac{1}{2}\,\epsilon_{ijk}\mathcal{H}^{jk} \\ D_i &= \mathcal{H}_{0i} = -\mathcal{H}^{0i}\,. \end{aligned} \tag{5}$$

Here, Maxwell's equation in the presence of a medium becomes [8]

$$\begin{array}{l} \partial_\mu \mathcal{F}_{\nu\lambda} + \partial_\nu \mathcal{F}_{\lambda\mu} + \partial_\lambda \mathcal{F}_{\mu\nu} = 0 \\ \partial_\mu \mathcal{H}^{\mu\nu} = \frac{4\pi}{c} J^\nu \\ J^\mu = (c\rho, \vec{\jmath}) \end{array} \quad \rightleftharpoons \quad \left\{ \begin{array}{l} \vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \cdot \vec{D} = 4\pi\rho \\ \vec{\nabla} \times \vec{H} - \frac{1}{c}\partial_t \vec{D} = \frac{4\pi}{c}\vec{j}\,. \end{array} \right. \tag{6}$$

In most cases the fields $\vec{D}$, $\vec{H}$ relate to the fields $\vec{E}$, $\vec{B}$ through the polarization and magnetization vectors, $\vec{P}$ $\vec{M}$ by the constitutive relations

$$\begin{aligned} \vec{D} &:= \vec{E} + 4\pi\vec{P} \\ \vec{H} &:= \vec{B} - 4\pi\vec{M}\,. \end{aligned}$$

3 An overview of some previous results

Here, we briefly recall some of the concepts introduced in [7], which we refer the reader for details. Let S and S' be two inertial frames moving with relative velocity $\vec{v}$. Let P be an event.

According to the Galilei relativity let us assume both observers record this event as $(\tau, \vec{x})$ and $(\tau, \vec{x}\,')$. The relation between their readings is

$$\vec{x}\,' = \vec{x} - \vec{v}\tau\ . \tag{7}$$

In order to relate the Galilei relativity with the framework of special relativity we assume the existence of another time variable, the local time of the special relativity, denoted by t. Now, let us assume that in terms of these coordinates each observer has recorded the event P as $(t, \vec{x})$ and $(t', \vec{x}\,')$. Here, the fundamental relation one imposes between these variables is that

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}\,'^{\,2}\ . \tag{8}$$

Now, if we assume a linear relation between t and t' as

$$t' = at + b\vec{v}\cdot\vec{x} \tag{9}$$

with a and b arbitrary real coefficients, the fulfillment of equations (7, 8) by the set $\{\tau, \vec{x}, \vec{x}\,', t, t'\}$ and the assumptions stated in [7] gives

$$b = \sqrt{a^2-1}\frac{1}{vc} \tag{10}$$

and

$$\tau = (1-|a|)\frac{\vec{x}\cdot\vec{v}}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t = (1-|a|)\frac{\vec{x}\,'\cdot(-\vec{v})}{v^2} + \sqrt{a^2-1}\,\frac{c}{v}\,t' = \tau' \tag{11}$$

together with the so-called Generalized Lorentz Transformation (GLT)

$$\begin{cases} \vec{x}\,' = \vec{x} - (1-|a|)\frac{1}{v^2}\,\vec{x}\cdot\vec{v}\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\,c\,t\,\vec{v} \\ t' = |a|t - \sqrt{a^2-1}\frac{1}{vc}\,\vec{x}\cdot\vec{v} \end{cases} \tag{12}$$

which represents a family of transformations parameterized by a real parameter a that is assumed to depend on the relative speed v between the frames and to satisfy $|a| > 1$.

Since we are considering the absolute time τ and the physical time t we must distinguish between two relative velocities

$$\vec{v} = \frac{d\vec{x}_{SS\,'}}{d\tau}, \quad \vec{\widetilde{v}} = \frac{d\vec{x}_{SS\,'}}{dt}$$

that are related by

$$\vec{\widetilde{v}} = \vec{v}\frac{\sqrt{a^2-1}}{|a|}\frac{c}{v} \tag{13}$$

with $\vec{x}_{SS'}$ denoting the position of the origin of the frame $S\,'$ as seen by frame S. In terms of $\widetilde{v}$ the transformation given in (12) is rewritten in the form

$$\begin{aligned} \vec{x}\,' &= \vec{x} - (1-\gamma_{\widetilde{v}})\frac{\vec{x}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}} - \gamma_{\widetilde{v}}t\vec{\widetilde{v}} \\ t' &= \gamma_{\widetilde{v}}\left(t - \frac{\vec{x}\cdot\vec{\widetilde{v}}}{c^2}\right) \end{aligned} \tag{14}$$

with $\gamma_{\tilde{v}} = \frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}}$. This is the usual Lorentz transformation.

Under the transformation (12) the electromagnetic fields and the four-current transform as

$$\begin{cases} \vec{E}' = |a|\vec{E} + (1-|a|)\frac{1}{v^2}\,\vec{v}\cdot\vec{E}\,\vec{v} + \sqrt{a^2-1}\frac{1}{v}\,\vec{v}\times\vec{B} \\ \vec{B}' = |a|\vec{B} + (1-|a|)\frac{1}{v^2}\,\vec{v}\cdot\vec{B}\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\,\vec{v}\times\vec{E} \end{cases} \tag{15}$$

$$\begin{cases} \vec{D}' = |a|\vec{D} + (1-|a|)\frac{1}{v^2}\,\vec{v}\cdot\vec{D}\,\vec{v} + \sqrt{a^2-1}\frac{1}{v}\,\vec{v}\times\vec{H} \\ \vec{H}' = |a|\vec{H} + (1-|a|)\frac{1}{v^2}\,\vec{v}\cdot\vec{H}\,\vec{v} - \sqrt{a^2-1}\frac{1}{v}\,\vec{v}\times\vec{D}\,. \end{cases} \tag{16}$$

$$\begin{cases} \rho' = |a|\rho - \sqrt{a^2-1}\frac{1}{cv}\vec{v}\cdot\vec{\jmath} \\ \vec{\jmath}' = \vec{\jmath} + \Big(-\sqrt{a^2-1}\frac{c}{v}\rho - (1-|a|)\frac{1}{v^2}\,\vec{v}\cdot\vec{\jmath}\Big)\vec{v}\,. \end{cases} \tag{17}$$

Maxwell equations (4), (6) are invariant under the transformations of the coordinates (12), fields (15), (16), and 4-current (17).

In terms of the velocity $\vec{\tilde{v}}$ given in (13) the previous transformations become the usual transformations for the fields and 4-current under a Lorentz transformation

$$\begin{aligned} \vec{E}' &= \gamma\vec{E} + (1-\gamma)\frac{1}{\tilde{v}^2}\,\vec{\tilde{v}}\cdot\vec{E}\,\vec{\tilde{v}} + \frac{1}{c}\gamma\,\vec{\tilde{v}}\times\vec{B} \\ \vec{B}' &= \gamma\vec{B} + (1-\gamma)\frac{1}{\tilde{v}^2}\,\vec{\tilde{v}}\cdot\vec{B}\,\vec{\tilde{v}} - \frac{1}{c}\,\gamma\vec{\tilde{v}}\times\vec{E}\,. \\ \vec{D}' &= \gamma\vec{D} + (1-\gamma)\frac{1}{\tilde{v}^2}\,\vec{\tilde{v}}\cdot\vec{D}\,\vec{\tilde{v}} + \frac{1}{c}\gamma\,\vec{\tilde{v}}\times\vec{H} \\ \vec{H}' &= \gamma\vec{H} + (1-\gamma)\frac{1}{\tilde{v}^2}\,\vec{\tilde{v}}\cdot\vec{H}\,\vec{\tilde{v}} - \frac{1}{c}\,\gamma\vec{\tilde{v}}\times\vec{D}\,. \\ \rho' &= \gamma\Big(\rho - \frac{1}{c^2}\vec{\tilde{v}}\cdot\vec{\jmath}\Big) \\ \vec{\jmath}' &= \vec{\jmath} - (1-\gamma)\frac{1}{\tilde{v}^2}\vec{\tilde{v}}\cdot\vec{\jmath}\,\vec{\tilde{v}} - \gamma\rho\vec{\tilde{v}} \end{aligned}$$

Remark: In particular, using (7, 12) into (11) we obtain

$$t + t' = \tau\frac{v(1+|a|)}{c\sqrt{a^2-1}} \tag{18}$$

which provides an operational definition for τ in terms of the local times t and t' and without recourse to the space coordinate of the event. Since the relation between the local times t, t' and the absolute time τ given in equation (18) depends on the relative speed of the frames this suggests the local time of the special relativity is a quantity defined only with respect to a pair of frames (in order to explicitly indicate this we could have written t and t' as $t_{SS'}$, $t'_{SS'}$). As a consequence of this interpretation the local time doesn't attain the meaning of an intrinsic quantity defined uniquely with respect to a single frame. This view also follows from the idealized form on how the local time is established. In fact, the instants t and t' are recorded by clocks that are placed in the positions where the event occurred. Since these clocks are all synchronized and arranged in such way as to mark $t = t' = 0$ when the origins of the reference systems coincide, their functioning is somehow adjusted to the peculiarities of the *relative* motion between the frames. A more complete discussion on this issue is given by Horwitz, Arshansky and Elitzur in [11] (pg. 1163), which we refer the reader for details.

The absolute time τ, nonetheless, may be seen as an intrinsic quantity that is set independently for each frame in the sense that it doesn't need a pair of frames to be defined and, in particular, it has the property that each frame registers the same value for the absolute time associated to the occurrence of an event (which is indicated by the equation $\tau = \tau'$).

It is also possible to choose the parameter a in such way that we can eliminate the dependence on the relative speed v in equation (18). In fact from (18) let us introduce a function $f(v) := \frac{v(1+|a|)}{c\sqrt{a^2-1}}$, where $a \equiv a(v)$. Then, let us impose that $\frac{df}{dv} = 0$, which gives $v\frac{da}{dv} = a^2 - 1$ or

$$a = \frac{1 + k^2 v^2}{|1 - k^2 v^2|}$$

with $k \in \mathbb{R}$ an arbitrary constant. For this choice of a we have

$$t + t' = \frac{1}{kc}\tau$$

that eliminates any dependence of the local and the absolute time with respect to the relative speed v.

4 The Galilean limit

In the standard treatment, special relativity is based on the Lorentz transformation (14). In the limit $c \to \infty$ it assumes the common form of the Galilei transformation

$$\begin{aligned} \vec{x}\,' &= \vec{x} - \vec{\widetilde{v}}t \\ t' &= t\ , \end{aligned} \tag{19}$$

which differs from the form we have assumed for the Galilei transformation (7), as in this equation the time variable corresponds to the absolute time τ, which is not identified with $t = t'$, and also in the difference it exists between $\vec{v}$ and $\vec{\widetilde{v}}$ (see (13)).

In our work, instead of considering the usual Lorentz transformation, we take the GLT (12) as the basic transformation and we seek the conditions under which the Galilei transformation (7) arises as the limit case. First, we notice that (13) allows us to express $a(v)$ in terms of $\widetilde{v}$ as

$$a = \frac{1}{\sqrt{1 - \frac{\widetilde{v}^2}{c^2}}} = 1 + \frac{1}{2}\frac{\widetilde{v}^2}{c^2} + \dots \tag{20}$$

therefore, neglecting terms of order $\geq \frac{1}{c^2}$, we obtain

$$\begin{aligned}
t' &= |a|t - \sqrt{a^2-1}\frac{1}{vc}\vec{x}\cdot\vec{v} = \Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(t-\frac{\widetilde{v}^2}{c^2}\vec{x}\cdot\vec{\widetilde{v}}+\dots\Big)\\
&\simeq t\\
\vec{x}\,' &= \vec{x} - \frac{\Big(-\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)}{\big(\frac{\widetilde{v}^2}{c^2}+\dots\big)c^2}\vec{x}\cdot\vec{\widetilde{v}}\,\vec{\widetilde{v}} - \Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)t\vec{\widetilde{v}}\\
&\simeq \vec{x}-\vec{\widetilde{v}}t\\
\tau &= (1-|a|)\frac{1}{v^2}\vec{x}\cdot\vec{v}+\sqrt{a^2-1}\frac{c}{v}t = \frac{\Big(-\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)\Big(1+\frac{1}{2}\frac{\widetilde{v}^2}{c^2}+\dots\Big)}{\big(\frac{\widetilde{v}}{c}+\dots\big)c}\frac{1}{v}\,\vec{x}\cdot\vec{\widetilde{v}}+\frac{\widetilde{v}}{v}t\\
&\simeq \frac{\widetilde{v}}{v}t\,.
\end{aligned}$$

In order to have $\tau = t$ we must assume that in the Galilean limit

$$\widetilde{v} \simeq v$$

which is compatible with the Galilei limit of (13).

In order to pursue the aforementioned Galilean limit of the Maxwell equations we also need the following prescription to change from the CGS to the SI system [8]

$$\begin{array}{cc}
\text{CGS} & \text{SI} \\ \hline
c & \frac{1}{\sqrt{\epsilon_0\mu_0}} \\
\vec{E} & \sqrt{4\pi\epsilon_0}\vec{E} \\
\vec{B} & \sqrt{\frac{4\pi}{\mu_0}}\vec{B} \\
\vec{D} & \sqrt{\frac{4\pi}{\epsilon_0}}\vec{D} \\
\vec{H} & \sqrt{4\pi\mu_0}\vec{H} \\
\rho & \frac{1}{\sqrt{4\pi\epsilon_0}}\rho \\
\vec{\jmath} & \frac{1}{\sqrt{4\pi\epsilon_0}}\vec{\jmath}
\end{array} \tag{21}$$

The reason for using the SI system in connection to the Galilei limit is because in our work the Galilei limit consists on neglecting terms of an order higher or equal than $\frac{1}{c^2}$ that arise from the parameter a as given in the expansion (20). Therefore, since the Maxwell equations in the SI system don't contain the light speed c, which is replaced by $\frac{1}{\sqrt{\epsilon_0\mu_0}}$, there is no possibility to confuse the origin of the light speed in the equations when we take the limit $\frac{1}{c^2}\to 0$. In sections 5.2, 5.2, in order to indicate the passage from the CGS to the SI system, we will write the electric and magnetic fields in the SI system as $\check{\vec{E}}$ and $\check{\vec{B}}$.

5 Galilei electrodynamics

5.1 The Galilean description of spacetime

Given a reference frame S we describe spacetime by means of two coordinate systems that we call the Galilei and the Lorentz systems. These systems are endowed with coordinates defined respectively by

$$X_G^\mu \equiv (X_G^0, X_G^i) := (c\tau, \vec{x}), \qquad X_L^\mu \equiv (X_L^0, X_L^i) := (ct, \vec{x}) \tag{22}$$

with c being the speed of light, considered here as a mere factor that allows us to have all coordinates X^μ with the same dimension. We have then the following diagram relating the several transformations defined so far

$$\begin{array}{ccc} (X_G^0, X_G^i) := (c\tau, \vec{x}) & \xrightarrow{G} & (X_G'^0, X_G'^i) := (c\tau', \vec{x}\,') \\ \Big\downarrow h & & \Big\downarrow h' \\ (X_L^0, X_L^i) := (ct, \vec{x}) & \xrightarrow{L} & (X_L'^0, X_L'^i) := (ct', \vec{x}\,') \end{array} \tag{23}$$

with

$$h : \begin{cases} X_L^0 = \frac{v}{\sqrt{a^2-1}}\left\{\frac{1}{c}X_G^0 - (1-|a|)\frac{1}{v^2}\vec{X}_G\cdot\vec{v}\right\} \\ \vec{X}_L = \vec{X}_G \end{cases} \tag{24}$$

$$h' : \begin{cases} X_L'^0 = \frac{v}{\sqrt{a^2-1}}\left\{\frac{1}{c}X_G'^0 + (1-|a|)\frac{1}{v^2}\vec{X}_G'\cdot\vec{v}\right\} \\ \vec{X}_L' = \vec{X}_G' \end{cases} \tag{25}$$

$$G : \begin{cases} \frac{1}{c}X_G'^0 = \frac{1}{c}X_G^0 \\ \vec{X}_G' = \vec{X}_G - \frac{1}{c}X_G^0\vec{v} \end{cases} \tag{26}$$

$$L : \begin{cases} X_L'^0 = |a|X_L^0 - \sqrt{a^2-1}\,\frac{1}{v}\,\vec{X}_L\cdot\vec{v} \\ \vec{X}_L' = \vec{X}_L - (1-|a|)\frac{1}{v^2}\,\vec{X}_L\cdot\vec{v}\,\vec{v} - \sqrt{a^2-1}\,\frac{1}{v}X_L^0\vec{v}\ . \end{cases} \tag{27}$$

It is immediate to check the diagram (23) is commutative, e.g.

$$h'\circ G = L\circ h\ .$$

The transformation h written in (24) relates two coordinates systems within the same frame S, therefore instead of the velocity $\vec{v}$ we can conceive the transformation h in terms of an arbitrary parameter $\vec{\beta}$. The same applies to the transformation h' defined in (25) that relates two coordinates systems within the same frame S' that can be defined in terms of another parameter $\vec{\beta}'$. Here, among all choices of parameters $\vec{\beta}$, $\vec{\beta}'$ it is the one with $\vec{\beta} = \vec{v} = -\vec{\beta}'$ that determines the Lorentz transformation between two frames moving with relative velocity $\vec{v}$ as having the form $L = h\circ G\circ h^{-1}$, or equivalently, that makes the diagram (23) commutative.

We will now establish the transformation properties that arises from the use of one and another of those systems taking for our starting point equation (24).

In the Lorentzian system let $\mathcal{F}_{L\mu\nu}$ and $\mathcal{F}_L^{\mu\nu}$ be the electromagnetic field strengths with $\vec{E}$ and $\vec{B}$ given as in (3). In the Galilean system we introduce the corresponding field strengths $F_{G\mu\nu}$, $F_G^{\mu\nu}$ by the relations

$$\mathcal{F}_G^{\mu\nu} = \frac{\partial x_G^\mu}{\partial x_L^\alpha}\frac{\partial x_G^\nu}{\partial x_L^\beta}\,\mathcal{F}_L^{\alpha\beta} \qquad\qquad \mathcal{F}_{G\mu\nu} = \frac{\partial x_L^\alpha}{\partial x_G^\mu}\frac{\partial x_L^\beta}{\partial x_G^\nu}\,\mathcal{F}_{L\alpha\beta}$$

whose components are

$$\mathcal{F}_G^{0i} = -\sqrt{a^2-1}\,\frac{c}{v}\,E_{Li} + (1-|a|)\,\frac{c}{v^2}\,(\vec{v}\times\vec{B}_L)_i \tag{28}$$

$$\mathcal{F}_G^{ij} = -\epsilon_{ijk}B_{Lk} \tag{29}$$

$$\mathcal{F}_{G0i} = \frac{1}{\sqrt{a^2-1}}\,\frac{v}{c}\,E_{Li} \tag{30}$$

$$\mathcal{F}_{Gij} = -\frac{(1-|a|)}{\sqrt{a^2-1}}\frac{1}{v}\,(v_iE_{Lj}-v_jE_{Li}) - \epsilon_{ijk}B_{Lk}\ . \tag{31}$$

Contrarily to the Lorentzian case, here we have $\mathcal{F}_G^{0i} \neq -\mathcal{F}_{G0i}$, $\mathcal{F}_G^{ij} \neq \mathcal{F}_{Gij}$, which doesn't provide a unique way to identify the Galilean counterpart of the electric and magnetic fields as it was done in (3).

In the Lorentz system the contravariant four current is defined as $J_L^\mu \equiv (J_L^0, J_L^i) := (c\rho_L, \vec{j}_L)$, and we denote the corresponding Galilean contravariant four-current as $J_G^\mu \equiv (J_G^0, J_G^i) \equiv (c\rho_G, \vec{j}_G)$, which is defined as

$$J_G^\mu := \frac{\partial X_G^\mu}{\partial X_L^\alpha}J_L^\alpha \tag{32}$$

and whose components are

$$\begin{aligned} J_G^0 &:= \sqrt{a^2-1}\frac{c}{v}J_L^0 + (1-|a|)\frac{c}{v^2}v_iJ_L^i \\ J_G^i &:= J_L^i\ . \end{aligned} \tag{33}$$

In the same Lorentz system the covariant four-current is defined as $J_{L\mu} \equiv (J_{L0}, J_{Li}) := (c\rho_L, -\vec{j}_L)$, and we denote the corresponding covariant four-current in the Galilei system as $J_{G\mu} \equiv (J_{G0}, J_{Gi}) \equiv (c\rho_G, -\vec{j}_G)$ which is defined as

$$J_{G\mu} := \frac{\partial X_L^\alpha}{\partial X_G^\mu}J_{L\alpha} \tag{34}$$

and whose components are

$$\begin{aligned} J_{G0} &= \frac{1}{\sqrt{a^2-1}}\,\frac{v}{c}J_{L0} \\ J_{Gi} &= -\frac{(1-|a|)}{\sqrt{a^2-1}}\,\frac{1}{v}v_iJ_{L0} + J_{Li}\ . \end{aligned} \tag{35}$$

5.2 The covariant model and the magnetic limit

This model is defined by taking the covariant tensor $\mathcal{F}_{G\mu\nu}$ and the covariant four-current $J_{G\mu}$ as the main element of analysis. Here, we define the electric and magnetic fields as

$$(\vec{E}_G;\vec{B}_G) := (\mathcal{F}_{G0i};-\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{Gjk}) \ . \tag{36}$$

From (30, 31) we have

$$\begin{aligned} \vec{E}_G &= \frac{1}{\sqrt{a^2-1}}\,\frac{v}{c}\,\vec{E}_L \\ \vec{B}_G &= \frac{(1-|a|)}{\sqrt{a^2-1}}\,\frac{1}{v}\,\vec{v}\times\vec{E}_L+\vec{B}_L \ . \end{aligned} \tag{37}$$

The covariant four-current $J_{G\mu}=(c\rho_G,-\vec{\jmath}_G)$ is given by (35) and it assumes the form

$$\begin{aligned} \rho_G &= \frac{1}{\sqrt{a^2-1}}\frac{v}{c}\rho_L \\ \vec{\jmath}_G &= \frac{(1-|a|)}{\sqrt{a^2-1}}\frac{c}{v}\rho_L\vec{v}+\vec{\jmath}_L \ . \end{aligned} \tag{38}$$

The transformation of the Galilei fields and the four-current is obtained from

$$\begin{aligned} \mathcal{F}'_{G\mu\nu} &= \frac{\partial X^\alpha_G}{\partial X'^\mu_G}\frac{\partial X^\beta_G}{\partial X'^\nu_G}\mathcal{F}_{G\alpha\beta} \\ J'_{G\mu} &= \frac{\partial X^\alpha_G}{\partial X'^\mu_G}J_{G\alpha} \ , \end{aligned} \tag{39}$$

which gives

CGS	SI
$\vec{E}'_G=\vec{E}_G+\frac{1}{c}\vec{v}\times\vec{B}_G$	$\check{\vec{E}}'_G=\check{\vec{E}}_G+\vec{v}\times\check{\vec{B}}_G$
$\vec{B}'_G=\vec{B}_G$	$\check{\vec{B}}'_G=\check{\vec{B}}_G$
$\rho'_G=\rho_G-\frac{1}{c^2}\vec{v}\cdot\vec{\jmath}_G$	$\check{\rho}'_G=\check{\rho}_G-\epsilon_0\mu_0\vec{v}\cdot\check{\vec{\jmath}}_G$
$\vec{\jmath}'_G=\vec{\jmath}_G$	$\check{\vec{\jmath}}'_G=\check{\vec{\jmath}}_G$.

(40)

From (37), expressing $\vec{E}_L,\vec{B}_L$ in terms of $\vec{E}_G$, $\vec{B}_G$, and from (38) expressing ρ_L, $\vec{\jmath}_L$ in terms of $\rho_G,\vec{\jmath}_G$ and replacing them all in (4) and converting to the SI system we obtain the Maxwell's equations in the Galilean system

$$\begin{aligned} &\vec{\nabla}_G\times\check{\vec{E}}_G+\partial_\tau\check{\vec{B}}_G=0 \\ &\vec{\nabla}_G\cdot\check{\vec{B}}_G=0 \\ &\vec{\nabla}_G\cdot\check{\vec{E}}_G+(1-|a|)\frac{1}{v^2}\vec{v}\cdot\partial_\tau\check{\vec{E}}_G=\frac{1}{\epsilon_0}\check{\rho}_G \\ &\vec{\nabla}_G\times\check{\vec{B}}_G-(1-|a|)\frac{1}{v^2}\vec{\nabla}_G(\vec{v}\cdot\check{\vec{E}}_G)+2(1-|a|)\frac{1}{v^2}(\vec{v}\cdot\vec{\nabla}_G)\check{\vec{E}}_G+2(1-|a|)\frac{1}{v^2}\partial_\tau\check{\vec{E}}_G=\mu_0\check{\vec{\jmath}}_G \ . \end{aligned} \tag{41}$$

The equations in (41) assume an awkward form due to the presence of the relative velocity $\vec{v}$ between the frames S, S', which originates from the fact that the transformation h given in (24) (and employed to derive relations (41)) is given in terms of $\vec{v}$. This situation is somehow similar to the relation involving $t, \vec{x}$, and τ shown in (11), which also contains the relative velocity. However, the confusion is only apparent if we recall that equation (41) represents the same Maxwell equations, the difference in form arises because they are expressed relative to the Galilei system of coordinates. Here, from the form of the equations in the Galilei system we can rightly say that the description of Classical Electrodynamics is simpler and more meaningful when written in the Lorentzian system. However, in the Galilean limit $\frac{1}{c^2} \to 0$ the equations (41) become

$$\begin{aligned}
&\vec{\nabla}_G \times \check{\vec{E}}_G + \partial_\tau \check{\vec{B}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{B}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{E}}_G = \frac{1}{\epsilon_0} \check{\rho}_G \\
&\vec{\nabla}_G \times \check{\vec{B}}_G = \mu_0 \check{\vec{\jmath}}_G
\end{aligned} \tag{42}$$

and these equations are left invariant by transformations (40). Together, they correspond to the same equations obtained in the so-called *magnetic limit* of [1]. However, in their work this form of the Maxwell's equation is obtained imposing that $c|\rho_L| \ll |j_L|$, $|\vec{E}_L| \ll c|\vec{B}_L|$, a condition that is not necessary to assume in our model.

Remark: As we mentioned in subsection 5.1, we may consider the transformation h (24) written in terms of any arbitrary parameter $\vec{\beta}$. In this case, the transformations given in (37, 38) would have $\vec{\beta}$ replacing the relative velocity $\vec{v}$.

Let us advance this interpretation a little further. In defining the Galilean fields and the 4-current in frame S let us assume our basic element for Galilean electrodynamics is $\mathcal{K}_G := (\vec{E}_G, \vec{B}_G, \rho_G, \vec{\jmath}_G, \vec{\beta})$. In terms of these basic fields we can express Lorentzian fields inverting the previous relations (37, 38), for example

$$\begin{aligned}
&\vec{E}_L = \sqrt{a(\beta)^2 - 1}\,\frac{c}{\beta} \vec{E}_G \\
&\vec{B}_L = \vec{B}_G - (1 - |a(\beta)|) \frac{c}{\beta^2} \vec{\beta} \times \vec{E}_G \\
&\rho_L = \sqrt{a(\beta)^2 - 1}\,\frac{c}{\beta} \rho_G \\
&\vec{\jmath}_L = \vec{\jmath}_G - (1 - |a(\beta)|) \frac{c^2}{\beta^2} \rho_G \vec{\beta}\ .
\end{aligned} \tag{43}$$

Similarly, let us assume another frame S' with basic element $\mathcal{K}'_G := (\vec{E}'_G, \vec{B}'_G, \rho'_G, \vec{\jmath}'_G, \vec{\beta}')$ with

$$\begin{aligned}
\vec{E}'_L &= \sqrt{a(\beta')^2 - 1}\,\frac{c}{\beta'}\vec{E}'_G \\
\vec{B}'_L &= \vec{B}'_G - (1 - |a(\beta')|)\frac{c}{\beta'^2}\vec{\beta}' \times \vec{E}'_G \\
\rho'_L &= \sqrt{a(\beta')^2 - 1}\,\frac{c}{\beta'}\rho'_G \\
\vec{\jmath}'_L &= \vec{\jmath}'_G - (1 - |a(\beta')|)\frac{c^2}{\beta'^2}\rho'_G\vec{\beta}' \;.
\end{aligned} \tag{44}$$

The Lorentzian fields in expressions (43, 44) correspond to a particular reparametrization of the Galilean fields in terms of the parameters $\vec{\beta}$, $\vec{\beta}'$. In order to fix the transformation between the Lorentzian fields (relative to the frames S and S') from the transformations of the corresponding basic Galilean fields (43, 44), we first must adjust the parameters $\vec{\beta}$ and $\vec{\beta}'$ as

$$\vec{\beta} = \vec{v} = -\vec{v}' = -\vec{\beta}'$$

and assume the transformation between the Galilean fields and the 4-current as given in table (40). This process of "adjustment" ressembles the setting of the clocks $t = t' = 0$ when the origins of the frames S, S' coincide, in the sense they are necessary in order to set the transformations of the fields, in much the same way as we do $t = t' = 0$ as a choice to derive the standard Lorentz transformations. Therefore, for the magnetic field we have, for example,

$$\begin{aligned}
\vec{B}'_L &= \vec{B}'_G - (1 - |a(\beta')|)\frac{c}{\beta'^2}\vec{\beta}' \times \vec{E}'_G \\
&= \vec{B}_G - (1 - |a(\beta)|)\frac{c}{\beta^2}(-\vec{\beta}) \times (\vec{E}_G + \frac{1}{c}\vec{\beta} \times \vec{B}_G) \qquad [\vec{\beta} = \vec{v}] \\
&= |a|\vec{B}_G + (1 - |a|)\frac{c}{v^2}\vec{v} \times \vec{E}_G + (1 - |a|)\frac{1}{v^2}(\vec{v} \cdot \vec{B}_G)\vec{v} \\
&= |a|\vec{B}_L + (1 - |a|)\frac{1}{v^2}\vec{v} \cdot \vec{B}_L\vec{v} - \sqrt{a^2 - 1}\frac{1}{v}\vec{v} \times \vec{E}_L \;.
\end{aligned}$$

This last form represents the generalized Lorentz transformation for the magnetic field as we have seen in (15). The same reasoning applied to the other objects will reproduce the expected Lorentz transformation given in (15, 16, 17). Then, we notice that the general scheme shown in (23) also extends to the basic elements $\mathcal{K}_G = (\vec{E}_G, \vec{B}_G, \rho_G, \vec{\jmath}_G, \vec{\beta})$, $\mathcal{K}_L := (\vec{E}_L, \vec{B}_L, \rho_L, \vec{\jmath}_L, \vec{\beta})$, $\mathcal{K}'_G = (\vec{E}'_G, \vec{B}'_G, \rho'_G, \vec{\jmath}'_G, \vec{\beta}')$, $\mathcal{K}'_L := (\vec{E}'_L, \vec{B}'_L, \rho'_L, \vec{\jmath}'_L, \vec{\beta}')$ when we identify $\vec{\beta} = \vec{v} = -\vec{v}' = -\vec{\beta}'$, in the sense that the following diagram is commutative

$$\begin{tikzcd}
(\vec{E}_G, \vec{B}_G, \rho_G, \vec{\jmath}_G, \vec{v}) = \mathcal{K}_G \arrow[r, "Galilei"] \arrow[d, "(43)"] & \mathcal{K}'_G = (\vec{E}'_G, \vec{B}'_G, \rho'_G, \vec{\jmath}'_G, \vec{v}') \arrow[d, "(44)"] \\
(\vec{E}_L, \vec{B}_L, \rho_L, \vec{\jmath}_L, \vec{v}) = \mathcal{K}_L \arrow[r, "Lorentz"] & \mathcal{K}'_L = (\vec{E}'_L, \vec{B}'_L, \rho'_L, \vec{\jmath}'_L, \vec{v}')
\end{tikzcd}$$

with the transformations given in (43), (44) playing the role of the transformations h, h' shown in diagram (23).

5.3 The contravariant model and the electric limit

This model is defined by taking the contravariant tensor $\mathcal{F}_G^{\mu\nu}$ and the contravariant four-current J_G^{μ} as the main elements of analysis. We define

$$(\vec{E}_G; \vec{B}_G) := (-\mathcal{F}_G^{0i}; -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_G^{jk}) \,. \tag{45}$$

From (28) and (29) we have

$$\begin{aligned} &\vec{E}_G := \sqrt{a^2-1}\,\frac{c}{v}\,\vec{E}_L - (1-|a|)\,\frac{c}{v^2}\,(\vec{v}\times\vec{B}_L) \\ &\vec{B}_G := \vec{B}_L \,. \end{aligned} \tag{46}$$

The four-current in the Lorentz system is $J_L^{\mu} \equiv (J_L^0, J_L^i) := (c\rho_L, \vec{\jmath}_L)$, and we denote the corresponding Galilean contravariant four-current as $J_G^{\mu} \equiv (J_G^0, J_G^i) \equiv (c\rho_G, \vec{\jmath}_G)$, which is defined as

$$J_G^{\mu} := \frac{\partial X_G^{\mu}}{\partial X_L^{\alpha}} J_L^{\alpha} \tag{47}$$

and whose components are

$$\begin{aligned} &\rho_G := \sqrt{a^2-1}\frac{c}{v}\rho_L + (1-|a|)\frac{1}{v^2}\vec{v}\cdot\vec{\jmath}_L \\ &\vec{\jmath}_G := \vec{\jmath}_L \,. \end{aligned} \tag{48}$$

The transformation of the Galilei fields and the four current is now obtained from

$$\begin{aligned} \mathcal{F}_G'^{\mu\nu} &= \frac{\partial X_G'^{\mu}}{\partial X_G^{\alpha}}\frac{\partial X_G'^{\nu}}{\partial X_G^{\beta}}\mathcal{F}_G^{\alpha\beta} \\ J_G'^{\mu} &= \frac{\partial X_G'^{\mu}}{\partial X_G^{\alpha}} J_G^{\alpha} \end{aligned} \tag{49}$$

and it gives

$$\begin{array}{ll} \text{CGS} & \text{SI} \\ \hline \vec{E}'_G = \vec{E}_G & \check{\vec{E}}'_G = \check{\vec{E}}_G \\ \vec{B}'_G = \vec{B}_G - \frac{1}{c}\vec{v}\times\vec{E}_G & \check{\vec{B}}'_G = \check{\vec{B}}_G - \epsilon_0\mu_0\vec{v}\times\check{\vec{E}}_G \\ \rho'_G = \rho_G & \check{\rho}'_G = \check{\rho}_G \\ \vec{\jmath}\,'_G = \vec{\jmath}_G - \frac{1}{c}\vec{v}\rho_G & \check{\vec{\jmath}}\,'_G = \check{\vec{\jmath}}_G - \vec{v}\check{\rho}_G \end{array} \tag{50}$$

In the Galilei system the Maxwell equations become

$$\begin{aligned} &\epsilon_0\sqrt{\mu_0}\; v\vec{\nabla}_G\times\check{\vec{E}}_G - \frac{2}{\sqrt{\mu_0}}(1-|a|)\frac{1}{v}\partial_\tau\check{\vec{B}}_G - \frac{1}{\sqrt{\mu_0}}(1-|a|)\frac{1}{v}(\vec{v}\cdot\vec{\nabla}_G)\check{\vec{B}}_G + \\ &+\epsilon_0\sqrt{\mu_0}(1-|a|)\frac{1}{v}\vec{v}\times\partial_\tau\check{\vec{E}}_G = 0 \\ &\vec{\nabla}_G\cdot\check{\vec{B}}_G + (1-|a|)\frac{1}{v^2}\vec{v}\cdot\partial_\tau\check{\vec{B}}_G = 0 \\ &\vec{\nabla}_G\cdot\check{\vec{E}}_G = \frac{1}{\epsilon_0}\check{\rho}_G \\ &\vec{\nabla}_G\times\check{\vec{B}}_G - \epsilon_0\mu_0\partial_\tau\check{\vec{E}}_G = \mu_0\check{\vec{\jmath}}_G \end{aligned} \tag{51}$$

and the Galilean limit of these equations become

$$
\begin{aligned}
&\vec{\nabla}_G \times \check{\vec{E}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{B}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{E}}_G = \frac{1}{\epsilon_0}\check{\rho}_G \\
&\vec{\nabla}_G \times \check{\vec{B}}_G - \epsilon_0\mu_0\partial_\tau \check{\vec{E}}_G = \mu_0 \check{\vec{j}}_G \ .
\end{aligned}
\tag{52}
$$

It is straightforward to check that equations (52) are left invariant by the transformations (50) and together they correspond to the electric limit of [1]. This limit was obtained in [1] assuming that $c|\rho_L| \gg |\vec{j}_L|$ and $E_L \gg cB_L$, a condition that we didn't need to assume.

5.4 The General Galilean model

Our third model combines some aspects of the two previous ones. Here, we consider electrodynamics in a medium and introduce the fields $(\vec{E}_G, \vec{B}_G, \vec{D}_G, \vec{H}_G)$ defined in terms of the Galilean fields strengths $\mathcal{F}_{G\mu\nu}$, $\mathcal{H}_G^{\mu\nu}$ as follows

$$
\begin{aligned}
E_{Gi} &:= \mathcal{F}_{G0i} \\
B_{Gi} &:= -\frac{1}{2}\epsilon_{ijk}\mathcal{F}_{Gjk} \\
D_{Gi} &:= -\mathcal{H}_G^{0i} \\
H_{Gi} &:= -\frac{1}{2}\epsilon_{ijk}\mathcal{H}_G^{jk}
\end{aligned}
\tag{53}
$$

where, in addition to (28, 29), we also consider

$$
\begin{aligned}
\mathcal{H}_G^{0i} &= -\sqrt{a^2-1}\,\frac{c}{v}\,D_{Li} + (1-|a|)\,\frac{c}{v^2}\,(\vec{v}\times\vec{H}_L)_i \\
\mathcal{H}_G^{ij} &= -\epsilon_{ijk}H_k \ .
\end{aligned}
\tag{54}
$$

The four-current in this general model follows the same definition given in section 5.3 and shown in equations (47, 48). We obtain

$$
\begin{aligned}
\vec{E}_G &= \frac{1}{\sqrt{a^2-1}}\,\frac{v}{c}\,\vec{E}_L \\
\vec{B}_G &= \vec{B}_L + \frac{1-|a|}{\sqrt{a^2-1}}\frac{1}{v}\,\vec{v}\times\vec{E}_L \\
\vec{D}_G &= \sqrt{a^2-1}\,\frac{c}{v}\vec{D}_L - (1-|a|)\,\frac{c}{v^2}\,\vec{v}\,\times\vec{H}_L \\
\vec{H}_G &= \vec{H}_L \ . \\
\rho_G &= \sqrt{a^2-1}\frac{c}{v}\rho_L + (1-|a|)\frac{1}{v^2}\vec{v}\cdot\vec{j}_L \\
\vec{j}_G &= \vec{j}_L \ .
\end{aligned}
\tag{55}
$$

The transformations of the fields and the four-current follows the same pattern as before and assume the form

$$
\begin{array}{ll}
\text{CGS} & \text{SI} \\
\hline
\vec{E}'_G = \vec{E}_G + \frac{1}{c}\vec{v} \times \vec{B}_G & \check{\vec{E}}'_G = \check{\vec{E}}_G + \vec{v} \times \check{\vec{B}}_G \\
\vec{B}'_G = \vec{B}_G & \check{\vec{B}}'_G = \check{\vec{B}}_G \\
\vec{D}'_G = \vec{D}_G & \check{\vec{D}}'_G = \check{\vec{D}}_G \\
\vec{H}'_G = \vec{H}_G - \frac{1}{c}\vec{v} \times \vec{D}_G & \check{\vec{H}}'_G = \check{\vec{H}}_G - \vec{v} \times \check{\vec{D}}_G \\
\rho'_G = \rho_G & \check{\rho}'_G = \check{\rho}_G \\
\vec{\jmath}\,'_G = \vec{\jmath}_G - \frac{1}{c}\vec{v}\rho_G & \check{\vec{\jmath}}\,'_G = \check{\vec{\jmath}}_G - \vec{v}\check{\rho}_G
\end{array}
\tag{56}
$$

In the Galilei system the Maxwell's equations (6) become

$$
\begin{aligned}
&\vec{\nabla}_G \times \check{\vec{E}}_G + \partial_\tau \check{\vec{B}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{B}}_G = 0 \\
&\vec{\nabla}_G \cdot \check{\vec{D}}_G = \check{\rho}_G \\
&\vec{\nabla}_G \times \check{\vec{H}}_G - \partial_\tau \check{\vec{D}}_G = \check{\vec{\jmath}}_G
\end{aligned}
\tag{57}
$$

and they assume the same form as the Maxwell's equation in the Lorentz system. Contrarily to what we have seen in the covariant and contravariant models, this form of the Maxwell equations in the Galilei system shown in (57) are invariant under the Galilei transformations of the fields and the four-current (40).

Now, let us analyze the constitutive relations between the fields. In the SI system the constitutive relations assume the form [8]

$$
\begin{aligned}
\check{\vec{D}}_L &= \epsilon_0 \check{\vec{E}}_L + \check{\vec{P}} \\
\check{\vec{H}}_L &= \frac{1}{\mu_0}\check{\vec{B}}_L - \check{\vec{M}} \ .
\end{aligned}
\tag{58}
$$

When we assume the vacuum we have $\check{\vec{P}} = 0$, $\check{\vec{M}} = 0$ in (58), which determines the following relation between the Galilean fields

$$
\begin{aligned}
\check{\vec{D}}_G &= -\frac{2}{\mu_0}(1-|a|)\frac{1}{v^2}\check{\vec{E}}_G - \frac{1}{\mu_0}(1-|a|)\frac{1}{v^2}\vec{v} \times \check{\vec{B}}_G + \frac{1}{\mu_0}(1-|a|)^2\frac{1}{v^4}(\vec{v} \cdot \check{\vec{E}}_G)\vec{v} \\
\check{\vec{H}}_G &= \frac{1}{\mu_0}\check{\vec{B}}_G - \frac{1}{\mu_0}(1-|a|)\frac{1}{v^2}\vec{v} \times \check{\vec{E}}_G \ .
\end{aligned}
\tag{59}
$$

Here, our results differ considerably from those obtained by Levy Leblond and Le Bellac. In their model there is only one set of fields, which corresponds to the Lorentzian fields of our work, $\check{\vec{E}}_L, \check{\vec{B}}_L, \check{\vec{D}}_L, \check{\vec{H}}_L$. Therefore, in their work the transformations of what we call Lorentzian fields would assume the form given in table (40) that are not consistent with the constitutive relations in vacuum $\check{\vec{D}}_L = \epsilon_0 \check{\vec{E}}_L$, $\check{\vec{H}}_L = \frac{1}{\mu_0}\check{\vec{B}}_L$. In our model, there is no such inconsistency since the transformations given in (40) are imposed on another set, the Galilean fields $\check{\vec{E}}_G, \check{\vec{B}}_G, \check{\vec{D}}_G, \check{\vec{H}}_G$, that are related to the Lorentzian fields through equations (55) (in the CGS system). Nonetheless, even though the constitutive relations don't contradict the transformations of the Galilean fields we notice that the application of the Galilean limit in (59) is not consistent since it would give $\check{\vec{D}}_G = 0$, contradicting one of the Maxwell's equations, $\vec{\nabla} \cdot \check{\vec{D}}_G = \check{\rho}_G$.

This violation of one of the Maxwell's equation may be seen as an attempt to endow Galilean spacetime with a wrong metric. In fact, both the Maxwell's equation and the constitutive relations of the Galilean fields can be thought as arising from the formulation of Maxwell's equation in a Riemannian manifold in the following way. Using CGS units the Maxwell's equation in a Riemannian space is given by the equations

$$\mathcal{F}_{\alpha\beta;\gamma} + \mathcal{F}_{\beta\gamma;\alpha} + \mathcal{F}_{\gamma\alpha;\beta} = 0, \qquad \mathcal{H}^{\alpha\beta}{}_{;\beta} = -\frac{4\pi}{c} J^\alpha$$

where

$$\mathcal{H}^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} \mathcal{F}_{\mu\nu} \tag{60}$$

determines the constitutive relations. This last equation links the constitutive relation with the geometry of the spacetime through the metric tensor $g^{\alpha\beta}$ [12]. Here, if we take spacetime having a Lorentzian metric

$$\eta_L^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we have the corresponding object

$$\eta_G^{\mu\nu} = \frac{\partial X_G^\mu}{\partial X_L^\alpha} \frac{\partial X_G^\nu}{\partial x_L^\beta} \eta_L^{\alpha\beta} = \begin{pmatrix} \frac{2c^2}{v^2}(|a|-1) & \frac{c}{v^2}(|a|-1)v_i \\ \frac{c}{v^2}(|a|-1)v_i & -\delta_{ij} \end{pmatrix}.$$

Assuming $g^{\alpha\beta} = \eta_G^{\alpha\beta}$ in (60) we obtain the constitutive relations for the Galilean fields as

$$\begin{aligned} \vec{D}_G &= 2(|a|-1)\frac{c^2}{v^2}\vec{E}_G + (|a|-1)\frac{c}{v^2}\vec{v} \times \vec{B}_G + (1-|a|)^2 \frac{c^2}{v^4} \vec{v} \cdot \vec{E}_G \vec{v} \\ \vec{H}_G &= \vec{B}_G + (|a|-1)\frac{c}{v^2}\vec{v} \times \vec{E}_G \end{aligned}$$

that corresponds to conditions (59) written in the CGS system. This shows the link between the constitutive relatons for the Galilean fields and a metric $\eta_G^{\alpha\beta}$ for spacetime. Now, the line element of spacetime maybe written either in the Lorentzian or the Galilean system as

$$ds^2 = \eta_{L\mu\nu} dX_L^\mu dX_L^\nu = \eta_{G\mu\nu} dX_G^\mu dX_G^\nu$$

with

$$\eta_{G\mu\nu} = \begin{pmatrix} \frac{1}{a^2-1}\frac{v^2}{c^2} & \frac{1}{|a|+1}\frac{v_i}{c} \\ \frac{1}{|a|+1}\frac{v_i}{c} & -\delta_{ij} + \frac{|a|-1}{|a|+1}\frac{v_i v_j}{v^2} \end{pmatrix}.$$

Taking the Galilean limit we have

$$n_{G\mu\nu} \rightarrow \begin{pmatrix} 1 & \frac{1}{2c}v_i \\ \frac{1}{2c}v_i & -\delta_{ij} \end{pmatrix}$$

which gives the following expression for the line element in the Galilei system

$$ds^2 = c^2 d\tau^2 + \vec{v}\cdot d\vec{x}\, d\tau - d\vec{x}^2$$

therefore, we may argue that the constitutive relations (58) are not consistent since it is equivalent to assuming a non-suitable metric for the Galilei spacetime.

Remark: Another set of constitutive relations

In equations (58) we have considered constitutive equations for the Lorentzian fields in the case the medium is the vacuum (then with $\vec{P}=0, \vec{M}=0$) and as a result we obtained the corresponding constitutive relations for the Galiean fields (59). Now, we focus on the inverse process and we will consider constitutive relations for the Galilean fields and investigate the relations they impose on the Lorentzian fields.

Lets us assume Σ is a frame relative to which the Galilean fields in vacuum satisfy the relation

$$\begin{aligned} \check{\vec{E}}_{G\Sigma} &= \frac{1}{\epsilon_0}\check{\vec{D}}_{G\Sigma} \\ \check{\vec{H}}_{G\Sigma} &= \frac{1}{\mu_0}\check{\vec{B}}_{G\Sigma}\ . \end{aligned} \tag{61}$$

For another frame S moving with velocity $\vec{v}$ relative to Σ we have the fields $\check{\vec{D}}_{GS}, \check{\vec{B}}_{GS}, \check{\vec{E}}_{GS}, \check{\vec{H}}_{GS}$ that relates to the Galilean fields in frame Σ by (56). Then, from (61) the constitutive relations of the Galilean fields in frame S become

$$\begin{aligned} \check{\vec{E}}_{GS} &= \frac{1}{\epsilon_0}\check{\vec{D}}_{GS} + \vec{v}\times\check{\vec{B}}_{GS} \\ \check{\vec{H}}_{GS} &= \frac{1}{\mu_0}\check{\vec{B}}_{GS} - \vec{v}\times\check{\vec{D}}_{GS}\ . \end{aligned} \tag{62}$$

Now, taking the relations between the Galilean and the Lorentzian fields given in (55) we obtain from (62) the constitutive relations for the Lorentzian fields

$$\begin{aligned} \check{\vec{D}}_{LS} &= \frac{\epsilon_0^2\mu_0}{(a^2-1)}v^2\check{\vec{E}}_{LS} + \left[\frac{(1-|a|)}{\sqrt{a^2-1}}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{1}{v} - \frac{(2-|a|)}{\sqrt{a^2-1}}\epsilon_0\sqrt{\epsilon_0\mu_0}\,v\right]\vec{v}\times\check{\vec{B}}_{LS} + \\ &\quad + \left[-\frac{(1-|a|)}{(1+|a|)}\frac{\epsilon_0}{v^2} + \frac{(3-|a|)}{(1+|a|)}\epsilon_0^2\mu_0\right]\vec{v}\times(\vec{v}\times\check{\vec{E}}_{LS}) \\ \check{\vec{H}}_{LS} &= \frac{1}{\mu_0}\check{\vec{B}}_{LS} + \left[\frac{(1-|a|)}{\sqrt{a^2-1}}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{1}{v} - \frac{(2-|a|)}{\sqrt{a^2-1}}\epsilon_0\sqrt{\epsilon_0\mu_0}\,v\right]\vec{v}\times\check{\vec{E}}_{LS} + \epsilon_0\,\vec{v}\times(\vec{v}\times\check{\vec{B}}_{LS})\ . \end{aligned}$$

This expression differs considerably from the ordinary expression of the Lorentzian theory, see for example equations (11) of [10].

6 Conclusion

In our work we employ a kind of tensor calculus defined from the relation between two coordinate systems for spacetime, the Galilei and the Lorentz systems defined in (22). Here, we considered the standard Maxwell electrodynamics as defined relative to the Lorentz system by means of the tensor $\mathcal{F}_{L\mu\nu}$, or $\mathcal{F}_L^{\mu\nu}$. The relation between the covariant and contravariant components of these tensors corresponds at most to an overall minus sign as this is implicit in the assumption that spacetime has a metric signature of the type diag$(+,-,-,-)$. Therefore, the setting of Maxwell's equation is immaterial either we employ $\mathcal{F}_{L\mu\nu}$ or $\mathcal{F}_L^{\mu\nu}$.

Our approach to derive the Galilei electrodynamics involved first to transform the Maxwell theory to the Galilei system. Contrarily to the model of the Galilei electrodynamics of [1] that is based on a priori imposition of relations of the type $c|\rho| \ll |\vec{\jmath}|$ and $|\vec{E}| \ll c|\vec{B}|$ in the magnetic limit, and $c|\rho| \gg |\vec{\jmath}|$ and $|\vec{E}| \gg c|\vec{B}|$ in the electric limit, the main features of our treatment depends on the particular roles played by the electromagnetic tensors $\mathcal{F}_{G\mu\nu}$, $\mathcal{F}_G^{\mu\nu}$ and the corresponding relations that are established between the Galilei and the Lorentzian fields (28-31). Therefore, we concluded from sections 5.2, 5.3 that the choice of one or another of these Galilean tensors determine different models of the Galilei electrodynamics upon taking the Galilean limit.

The general Galilean model of section 5.4 mixes both constructions as it accommodates the electric and the magnetics fields $\vec{E}_G, \vec{B}_G$ as components of a covariant tensor $\mathcal{F}_{G\mu\nu}$, while the electric and the magnetic excitations $\vec{D}_G, \vec{H}_G$ are accommodated as components of a contravariant tensor $\mathcal{H}_G^{\mu\nu}$. The remarkable aspect of this third model is that it produces the same set of equations for the Galilean and the Lorentzian fields, this time with the absolute time replacing the local time in the Galilei electrodynamics. Here, in our model we still have problems to conciliate realistic constitutive relations for the Galilean fields in the case the medium considered is the vacuum. Perhaps we could try to set the parameter a in the GLT transformation in a way that would guarantee the consistency of the constitutive relation with the equations obeyed by the Galilean fields without contradicting $\check{\vec{\nabla}} \cdot \check{\vec{D}}_G = \check{\rho}_G$.

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References

[1] Le Bellac, M., Lévy-Leblond, J.M., *Galilei electromagnetism*, Nuovo Cimento 14 (1973) 217.

[2] Montigny, M., Rouseaux, G., *On some applications of Galilean electrodynamics of moving bodies*, Am. J. Phys. 75 (2007) 984.

[3] Niederle, J., Nikitin, A.G., *More on Galilean Electromagnetism*, 5th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics (MPHYS5) Belgrade, Serbia, 2008.

[4] Rousseaux, G., *Lorentz or Coulomb in Galilean electromagnetism*, Europhys. Lett. 71 (2005) 15.

[5] Manfredi, G., *Non-relativistic limits of Maxwell's equations*, Eur. J. Phys. 34 (2013) 859.

[6] Heras, J., *The Galilean limits of Maxwell's equations*, Am. J. Phys. 78 (2010) 1048.

[7] Carvalho, M.; Oliveira, L. A., *Unifying the Galilei Relativity and the Special Relativity*, ISRN Mathematical Physics, vol. 2013, Article ID 156857, 17 pages, 2013. doi:10.11552013156857

[8] Jackson, J. D., *Classical Electrodynamics*, John Wiley & Sons, NJ, 1998.

[9] Russakoff, G., *A derivation of the macroscopic Maxwell equations*, Am. J. Phys. 38 (1970) 1188.

[10] Rousseaux, G., *Forty Years of Galilean Electromagnetism (1973-2013)*, Eur. Phys. J. Plus (2013)128:81; DOI 10.1140/epjp/i2013-13081-5.

[11] Horwitz, L.P., Arshansky, R.I., Elitzur, A.C.: *On the two Aspects of Time: The Distinction and Its Implications*, Found. Phys. 18 (1988) 1159.

[12] Landau, L. D., Lifschitz, E. M.: *The Classical Theory of Fields*, Butterworth Heinemann, 1980.